Setting

- G complex, connected, simply-connected reductive Lie group.
- $\mathfrak{g}$ Lie algebra, $\mathfrak{h}$ Cartan subalgebra; $\mathfrak{h} \cong \mathfrak{h}^*$
- $\lambda \in \mathfrak{h}^*$ (hyperbolic, integral)
- $ad(\lambda) \rightsquigarrow \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ $\quad \mathfrak{g}(i) = \{ x \in \mathfrak{g} : ad(\lambda) x = i x \}$
- $G(0)$ (Lie algebra $\mathfrak{g}(0)$) acts on $\mathfrak{g}(i)$ with finitely many orbits.
- $(G(0), \mathfrak{g}(-1))$ is a pre-homogeneous vector space ; $\{ \mathcal{O} \}$ $G(0)$-orbits

$$\mathcal{O} = G(0) \cdot e \qquad A(e,\lambda) = Z_G(e,\lambda)/Z_G(e,\lambda)_0$$

$$\rho \in \hat{A}(e,\lambda) \rightsquigarrow \mathcal{L}_\rho \text{ local system on } \mathcal{O}.$$

- Set $S = \{ (\mathcal{O}, \mathcal{L}) \quad \mathcal{O} \subset \mathfrak{g}(-1) \}$; then

$$S = \{ (\mathcal{O}, \mathcal{L}) \quad \mathcal{O} \subset \mathfrak{g}(-1) \} \longleftrightarrow \text{Irr Perv.}(\mathfrak{g}(-1))$$

$$r = (\mathcal{O}, \mathcal{L}) \longrightarrow \text{Per}(r).$$

- (Lusztig) $\text{Per}(\mathfrak{g}(-1))$ decomposes into blocks

blocks $\underset{\text{Parametrized}}{\rightsquigarrow}$ ($\underset{\text{Levi}}{M}$, $\underset{\text{with grading}}{\mathfrak{m}}$, $(\mathcal{O}, \mathcal{L})$)
cuspidal local system on $\mathfrak{m}(-1)$

[Parametrization encodes info on how to determine the block. The Decomposition Thm. plays a key role.] , $M_{\mathbb{R}}$

- We focus on $\text{Block}(h, \{0\}, \mathbb{C})$

Fact: $\qquad \text{Block}(h, \{0\}, \mathbb{C}) \leftrightarrow \{ (\mathcal{O}, \mathcal{L}) \ \mathcal{L} \text{ of "Springer Type".} \}$
[more later]

Springer Type means: $\mathcal{L} \to \rho \in \hat{A}(e,\lambda)$ is the restriction from $A(e) \to A(e, \lambda)$ of a rep. that occurs in the Springer Correspondence.

• Affine Graded Hecke Algebras (Part 1)

Lusztig defined a finite set of A.G.H algebras

* $\{ \mathbb{H}_1 \ \mathbb{H}_2 \ \cdots \ \mathbb{H}_n \}$

(See for e.g. " Cuspidal local systems and Graded Hecke Alg. II " Lusztig and reference within.
(Canadian paper for short)

* Irreducible $\mathbb{H}_k$-mod. are f. dim.

* λ fixed $\quad S = \{ (\theta, \mathcal{L}) \ \ \theta \subset \mathfrak{g}(2) \ \ \mathcal{L}$ l. system $\} \leftrightarrow \bigcup_k$ Irred $\mathbb{H}_k$-mod.

We focus on: Category of f.d $\mathbb{H}_1$-mod with central ch. λ .

See Lusztig [J.AMS, 1989]
or " [canadian]

$$\mathfrak{g} \supset \mathfrak{h} \qquad \mathfrak{b} = \mathfrak{h} + \mathfrak{n} \ \leftrightarrow \ \Delta^+ \supset \Pi_{\text{simple}}$$

As vector space

$$\mathbb{H}_1 = \mathbb{C}[W] \otimes \mathrm{Sym}(\mathfrak{h}^*)$$

It is generated by $\{ t_{s_\alpha} \ \ \alpha \in \Pi \} \cup \{ \omega \in \mathfrak{h}^* \}$.
(under our assumptions)

$$\omega \, t_{s_\alpha} = t_{s_\alpha} \, s_\alpha(\omega) + \langle \omega, \alpha^\vee \rangle$$

★ The statement 'with central ch." is not obvious.
Lusztig described $Z(\mathbb{H}_1)$ and proved that
acts on irred by a ch.

• The link with $\mathrm{Per}(\mathfrak{g}(-1))$.

Kazhdan - Lusztig

$$\text{Both std, irred } \mathbb{H}_1\text{-mod.} \quad \mathrm{mod}_\lambda(\mathbb{H}_1) \leftrightarrow \{ (\theta, \mathcal{L}) \ \ \theta \subset \mathfrak{g}(-1), \ \mathcal{L} \text{ Springer type} \} = S_\lambda^{\mathbb{H}_1}$$

$$\{(\theta, \mathcal{L})\}$$

We'll get back to this.

The Parameter Space $S_q^{H,}$

As before G complex, connected, simply connected Lie group

$\check{G}$ Langlands dual.

Let F be a p-adic field of ch 0.

$\mathcal{O}$ be the ring of integers $F = \mathbb{Q}_p$ $\mathcal{O} = \mathbb{Z}_p$

$\mathcal{P}$! prime ideal, e.g. $F = \mathbb{Q}_p$ $\mathcal{P} = p\,\mathbb{Z}_p$.

$\mathcal{O}/\mathcal{P} = k_{q_F}$ finite field. (residue field)

Assume $\check{G}$ is defined over F. Set $K = \check{G}(\mathcal{O})$ and $\pi : \check{G}(\mathcal{O}) \to \check{G}(F)$

$\cup$

$\check{B}(F)$

Borel

$\tilde{I}_F$ Iwahori-subg, the preimage of $\check{B}(F)$.

$$1 \to I_F \to \mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(\bar{k}_{q_F}/k_{q_F}) \to 1$$

$\supset \mathbb{Z}$ dense.

W_F = preimage of $\mathbb{Z}$

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1$$

$$w \to n \qquad \|w\| = q^n$$

$$W_F / I_F = \langle \tilde{w} \rangle$$

Langlands-Deligne conj

$*_1$ Admissible rep should be a union of packets

$*$ Packets $\leftrightarrow$ $\phi : W_F \times SL(2,\mathbb{C}) \to G$ "admissible"

$*_3$ $\{\phi$ adm. : Packet$_\phi$ consist of $\tilde{I}_F$-spherical$\}$

$\updownarrow$

$\phi : \underbrace{W_F / I_F}_{\langle \tilde{w} \rangle} \times SL(2,\mathbb{C}) \to G$ admiss.

adm $\to$ $\sigma = \phi(\tilde{w})$ ss

$e\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $h\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $u = \phi(e)$ unipotent $\sigma u \sigma^{-1} = u^{q}$

(1987) Kazhdan-Lusztig. proved the conjecture, building from Borel's equivalence of categorie.

Borel : Category $\{\tilde{I}_F$-spherical adm. $\check{G}(F)$ rep$\}$ $\leftrightarrow$ Category $\{$f.d. rep of Iwahori-Hecke Alg$\}$

K-L $\bullet$ $(u,\sigma) \longrightarrow$ Packet consisting of $\vec{\mathcal{I}}_F$-sph.

$\bullet$ $\rho \in \widehat{A(u,\sigma)} = \widehat{Z_G(u,\sigma)/Z_G(u,\sigma)_0}$ $\leftrightarrow$ elements in the packet.

K-Theory $(u,\sigma,\rho) \longrightarrow$ std(u,σ,ρ) has ! irred. quotient.

when $\sigma_{ss} = \sigma_{hyp} \cdot \cancel{\sigma_{ellip}}$ $\quad$ "1

$\bullet$ std$(u,\sigma,\rho) = \mathrm{Hom}_{A(u,\sigma)}(\rho, H_\bullet(\underbrace{B_u^\sigma}))$ $\mathcal{H}_W$-module

Affine Hecke Alg

where B_u springer fiber; B_u^σ "σ-stable" part.

* Springer type condition?

(1995) Lusztig. $\lambda \to \sigma_\lambda$

- Irred mod$_{\sigma_\lambda}(\mathcal{H}_W) \longleftrightarrow$ Irred mod$_\lambda(\mathbb{H}_1)$

K-theory $\rightsquigarrow$ Intersection Cohomology.

$[(u,\sigma)]$ eq. classes $\longrightarrow$ $[\lambda,\ ;\ G(0)\cdot e = \mathcal{O}]$

$\rightsquigarrow$ Packets of $\vec{\mathcal{I}}_F$-sph.

$(\lambda, \mathcal{O}, \mathcal{L})$ Springer Type $\longmapsto$ std$(\lambda,\mathcal{O},\rho_{\mathcal{L}})$ $\mathbb{H}_1$-standard.

Irr$(\lambda,\mathcal{O},\rho) \longleftrightarrow$ Irred Perverse sheaf.

In particular (σ hyperbolic, λ fixed)

$\{(\mathcal{O},\mathcal{L})\ \mathcal{L}$ Springer type$\}$ = "Parameter Space" $\{$Adm. $\vec{\mathcal{I}}_F$ - spherical $\check{G}(F)$ -rep.$\}$

In Canadian paper, (more genl than here);

$$\mathrm{std}_{\mathbb{H}_1}(\mathcal{O},\mathcal{L}) = \sum_{\mathcal{O}',\mathcal{L}'} m^{\mathbb{H}_1}_{\mathcal{O}',\mathcal{L}'}(\mathcal{O},\mathcal{L})\ \mathrm{irred}_{\mathbb{H}_1}(\mathcal{O}',\mathcal{L}')$$

$$m^{\mathbb{H}_1}_{\mathcal{O}',\mathcal{L}'}(\mathcal{O},\mathcal{L}) = \sum_i (-1)^i\ [\mathcal{L} : \mathcal{H}^i(IC(\bar{\mathcal{O}}',\mathcal{L}')|_{\mathcal{O}})].$$

(we'll get back to this)

(1995) Adv. paper $\rightsquigarrow$ Algorithm to compute $m_{\delta',\delta'}(\sigma,\delta)$
[2006] Lusztig

(2008) Ciubotaru $\rightsquigarrow$ used the Alg. to compute, in example
$m_{\sigma',\delta'}(\sigma,\delta)$. [Notably F_4].
[He computed KL-polynomials]

Further references: Vogan, Local Langlands Conj., section 4
Barbasch–Moy, Inv. Math 98 (1989)

The other side of the story.

Keep the assumptions introduced above. (Following [ABV])

- $ad(\lambda) \rightsquigarrow P(\lambda) = \mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$ $\quad \lambda$ integral.
- $y(\lambda) = exp(\pi i \lambda)$
  $e(\lambda) = exp(2\pi i \lambda)$ $\quad G(\lambda) = Z_G(e(\lambda)) = G$ (λ integral)

  $y(\lambda) \rightsquigarrow \theta_{y(\lambda)}$ on $G(\lambda)$ $\quad G(\lambda)^{\theta(\lambda)} = K(\lambda)$
- Set $\{(Q, \chi) \quad Q \in K(\lambda)/G/P(\lambda);\ \chi$ local system on $Q\} = \mu$
- $K(\lambda) \rightsquigarrow$ block $\check{G}(\mathbb{R})$

  standard and irred mod. on block $\check{G}(\mathbb{R}) \leftrightarrow \mu$.
- If $\gamma = (Q, \gamma)$ $\quad \gamma' = (Q', \gamma')$; in the Grothendieck group

$$\mathrm{stand}(\gamma) = \sum_{\gamma'} m^{\mathbb{R}}_{\gamma',\gamma} \quad \mathrm{irred}(\gamma')$$

$$m^{\mathbb{R}}_{\gamma',\gamma} = \sum_i (-1)^i \left[\gamma : \mathcal{H}^i(IC(\bar{Q}', \gamma'))|_Q \right].$$

ATLAS COMPUTES $m^{\mathbb{R}}_{\gamma',\gamma}$.

The Questions: (with Peter)

(A) Is there a canonical injection

$\{ \eta = (\theta, \mathcal{L}) \ \theta \in \mathfrak{g}(-1), \mathcal{L} \text{ Springer type} \}$ H_1-side Parameter space

$\downarrow \psi$

$\{ \gamma(Q, \nu) \quad Q \in K \backslash G/P, \nu \text{ local syst.} \}$ [ABV] Parameter space

$\rightsquigarrow$ $m^{H_1}(\eta, \eta')$ can be computed

in terms of (maybe various) $m^{\mathbb{R}}(\gamma, \gamma')$

$\gamma \subset$ Image of ψ?

More ambitious

(B) Can we compute KL-polynomials in the H_1-side in terms of KLV-polynomials?

(we need to match shifts in definition of irred. perverse sheaves.)

Remarks.

1. [ABV] (Chriss-Ginzburg) follows Beilinson-Bernstein-Deligne to define irred. Perverse Sheaves. (If $Z \hookrightarrow X$ $d = \dim Z$, a local system ν on Z is placed at degree $-d$.) This is also what Lusztig does in the "Canadian"-paper. This does not seem to be the shift he uses in Adv. 95. (These changes do not affect (A))

2. When using normal slice arguments, we need to be careful with shifts.

3. The KLV poly. of [ABV] differ from those in [Lusztig-Vogan] by a shift.

The simplest case : $G = \check{G} = GL(n, \mathbb{C})$; $\lambda = \rho$

$K = GL([\frac{n}{2}], \mathbb{C}) \times GL(\lceil \frac{n}{2} \rceil, \mathbb{C})$

T diagonal Cartan

$\underset{\text{to } \theta\text{-stable}}{b} \rightsquigarrow \Delta^+ \supset \Pi = \{\alpha_i\}$ are all i.

Matching orbits

$\{T\text{-orbits on } \mathfrak{g}(-1)\}$ and $\{\{-\alpha_j\} \; \alpha_j \in \Pi\}$

$\# \{T\text{-orbits on } \mathfrak{g}(-1)\} = 2^{\text{rank } \mathfrak{g}}$.

Each $\{\alpha_j\} \in \Pi \rightsquigarrow P_j \supset B$ parabolic subgroup.

<u>Define</u> • $K \times_B P_1 \times_B \times_B P_2 \cdots \times_B P_{n-1}/B$ the quotient of

$K \times P_1 \times P_2 \times \cdots P_{n-1}$ by the action

$(b_1, b_2 \quad b_n)(k, x_1, x_2 \quad x_{n-1}) = (k b_1, b_1^{-1} x_1 b_2, b_2^{-1} x_2 b_3 \cdots)$

• $\rho : K \times_B P_1 \times_B \cdots \times_B P_{n-1}/B \longrightarrow G/B$

$[k, x_1, \cdots, x_n] \longmapsto k x_1 x_2 \; x_{n-1}.B$

(well defined, independent of representation)

Image $\rho = \underset{\substack{K\text{-orbit.} \\ \{\alpha_1 \cdots \alpha_n\}}}{\bar{Q}_{max}}$; $\underset{\{\emptyset\}}{K/B = Q_0}$ $\underset{\text{Closed}}{+ - + - + -}$

$= s_{n-1} \circ s_{n-2} \cdots s_1 \; Q_0$ where

$s_j \circ Q$ = ! dense in $\pi_j^{-1}(\pi_j(Q))$ $\pi_{\alpha_j} : G/B \to G/P_j$

• $(s_1 (+ - + - \; +) = (\underset{c^+}{11} + - + \cdots))$; $s_2 (1 1 + - + \cdot) = (1 + 1 \cdots)$

$\rightsquigarrow$ $\dim \bar{Q}_{max} = \dim(K \times_B P_1 \times \cdots)$

$= \dim Q_0 + \dim \mathfrak{g}(-1)$.

• RS: $K \times_B P_1 \times_B \cdots \times_B P_{n-1}/B$ is a R. Singularities of $\bar{Q}_{max}$.

- RS has $2^{\operatorname{rank} \mathfrak{g}}$ cells.

$I \subseteq \Pi$ $\rightsquigarrow$ $\mathrm{cell}_I = \{ [k x_1 \cdots x_{n-1}] : x_j = 1 \quad j \notin I \}$

Here we have an injective map.

$2^{\operatorname{rank} \mathfrak{g}}$ T-orbits $\longrightarrow$ $K \backslash G / B$

$\emptyset \longmapsto Q_0 = + - + - + - \cdots$ closed orbit

$I \longmapsto \beta(\mathrm{cell}_I)$ $\quad I \subsetneq \Pi$

$\Pi \longmapsto Q$ max.

orbit closure inclusion
$\Updownarrow$
inclusion of set of simple roots $\longmapsto$ respects orbit closure inclusion

- Comments

$*$ $\{\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_t}\}$ $\longmapsto K\, \overline{[\exp X_{-\alpha_{i_1}} \exp X_{-\alpha_{i_2}} \cdots \exp X_{-\alpha_{i_t}} B)}$

ordered $i_1 < i_2 \cdots$ $= K\, \overline{(I + \Sigma X_{\alpha_{i_j}})}. b$

we obtain C.T map.

$(\exp E_{ij} = I + E_{ij} \; ; \; \exp E_{21} \exp E_{32} = I + E_{21} + E_{32})$

One could use instead of $K \times_B P_1 \times_B \cdots \times_B P_{n-1} / B$,

$K \times_B P_{n-1} \times_B P_{n-2} \quad \times_B P_1 / B$.

$*$ $\dim (K. \exp X_{-\alpha_{i_1}} \cdots \exp X_{-\alpha_{i_t}}. b) = \dim Q_0 + t = \dim Q_0 + \dim \mathcal{O}_{i_1 \cdots i_t}$
$= \dim (\bar{n} \cap k) + \dim \mathcal{O}_{i_1 \cdots i_t}$.

$*$ Technical comment: b is θ-stable
If $\mathfrak{g} = k \oplus s$ is Cartan decomposition

$\bar{N} = \bar{N} \cap K \exp(\bar{n} \cap s)$.

Set $z_{i_1 i_2 \cdots i_t} = \exp(X_{-\alpha_{i_1}}) \cdots \exp X_{-\alpha_{i_t}} = \underset{\bar{N} \cap K}{k_{i_1 \cdots i_t}} \cdot \underset{\exp(\bar{n} \cap s)}{\exp(s)_{i_1 \cdots i_t}}$

$$K.\overline{\exp(X_{-\alpha_{i_1}}) \cdots \exp(X_{-\alpha_{i_t}}).b} = K\,\overline{\exp(s)_{i_1 \cdots i_t}.b} = \bar{Q}_{i_1 \cdots i_t}$$

where $\exp(2 s_{i_1 \cdots i_t}) = (\theta z_{i_1 \cdots i_t})^{-1} z_{i_1 \cdots i_t}$.

By Hausdorff–Campbell

$$= \exp\left(2\,[X_{-\alpha_{i_1}} + \cdots + X_{-\alpha_{i_t}}] + \sum_k h^k_{i_1 \cdots i_t}\right)$$

$h^k_{i_1 \cdots i_t} \in g(-(2k+1))$ c.s.

Claim I: $\bar{N} \cap K.\,[T \underbrace{\exp(s_{i_1 \cdots i_t})}_{\exp(\bar{n} \cap s)}.b]$ is open in $\bar{Q}_{i_1 \cdots i_t}$.

Why?

(a) $\dim(\bar{N} \cap K.\,[T\exp(s_{i_1 \cdots i_t})b]) =$

$\dim(\bar{N} \cap K) + \dim[T \cdot \exp(s_{i_1 \cdots i_t}).b]$

(b) Since T is connected and preserves

each $g(-(2k+1))$ $\dim(T.\exp(s_{i_1 \cdots i_t})b) \geq \dim(O_{i_1 \cdots i_t})$

$\dim(\bar{N} \cap K) + \dim(O_{i_1 \cdots i_t}) \leq \dim \bar{N} \cap K.\,[T\exp(s_{i_1 \cdots i_t}).b] \leq \dim_{\mathbb{R}} Q_{i_1 \cdots i_t}$

$\dim \bar{N} \cap K + \dim(O_{i_1 \cdots i_t})$

[We can also prove this statement by using the explicit description $g(-1) \hookrightarrow G/B$, $N \to I + N.\,F_{\text{based flag}}$ in C.T.

C.T. do no use RS. Their map correspond to choosing $\bar{Q}_{max} \xleftarrow{\beta} K \times_\beta P_1 \times_\beta P_2 \times_\beta \cdots P_n / B$. The detailed proof takes more than 2 pages]

Claim II:

Let $\phi : g(-1) \longmapsto G/B$

$$\sum \lambda_i X_{-\alpha_i} \longmapsto \exp(\lambda_1 X_{-\alpha_1}) \cdots \exp(\lambda_m X_{-\alpha_m}).b.$$

Then, $\tilde{\Phi}: \bar{B} \cap K \times_T \mathfrak{g}(-1) \longrightarrow \bar{N} \cap K \cdot \phi(\mathfrak{g}(-1))$

$[b, z] \longmapsto b \cdot \Phi(z)$ is an isomorphism of v.

Sketch: $\boxed{1}$ $\mathfrak{g}(-1) \simeq \prod_i \exp(\lambda_i X_{-\alpha_i})$ (in any order)

[Linear Alg. gps., Borel Prop. 14.4]

$\boxed{2}$ $\bar{N} \cap K \times \mathfrak{g}(-1) \hookrightarrow G/B$

$(n, z) \mapsto n \cdot \phi(z) \cdot b$ is

an open embedding.

(1) ϕ is a normally non-singular inclusion of codimension $\dim(\bar{n} \cap k)$; ie there is a neighborhood V of $\Phi(\mathfrak{g}(-1))$ and a retraction $V \to \Phi(\mathfrak{g}(-1))$ locally homeo to a proj.

(2) $m^{H}(\theta, \theta') = m^{\mathbb{R}}(Q_\theta, Q'_\theta)$

where Q_θ is the K-orbit that corresponds to θ.

How? One way: Use ([ABV], Prop. 7.14(e)) applied to $\bar{B} \cap K \times_T \mathfrak{g}(-1)$.

(3) $\phi(\mathfrak{g}(-1))$ is a normal slice to the closed orbit Q_0:

$(\Phi(\mathfrak{g}(-1)), \{\Phi(\mathfrak{g}(-1)) \cap Q_j : \theta_j \subset \bar{Q}_{max}\})$ is

iso (as stratified space) to $(\mathfrak{g}(-1), \{\theta_i\})$.

$\rightsquigarrow P_{Q_{\theta_i}, Q_0} \simeq P_{\theta_i, 0}$ [Provided irred Perv. Sheaves are defined with compatible shifts]

(4) What about $P_{Q_{\theta_i}, Q_{\theta_j}}$?

In gral (Bialynicki-Birula?) Needs care...

We have $\pi: RS \to \bar{Q}_{max}$ and $Q_{\theta_j} \subset \bar{Q}_{max}$.

(Normal Slice)

In matching we essentially pass $N_{Q_{\theta_j}} \hookrightarrow \pi^{-1}(N_{Q_{\theta_j}})$ the cells I described do not always stratify $\pi^{-1}(N_{Q_{\theta_j}})$. (I think ok here $s_{\alpha} \cdots s_{\alpha_i}$ no repeated roots)...

$G = \check{G} = GL(4, \mathbb{C})$ (Example 1)

$\lambda = [4\ 3\ 2\ 1]$

$K(\lambda) = GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$

$y(\lambda) = \exp(\pi i \lambda)$ $\begin{pmatrix} + & & & \\ & - & & \\ & & + & \\ & & & - \end{pmatrix}$

T diag. torus

$\{T \text{ orbits on } \mathfrak{g}(-1)\} \to \{\text{sets } \{-\alpha_i\}\ \alpha_i \text{ simple}\}$ web interface.

$1 + - 1$ # $\quad$ $\ell = 3$

not in the web

$\uparrow s_3$

$\{\alpha_2 \alpha_3\}$ $+ 1 - 1$ #13 $\quad$ $\{\alpha_1 \alpha_2\}$ $1 + 1 -$ # 15 $\quad$ $\{\alpha_1, \alpha_3\}$ $1 1 2 2$ #12 $\quad$ $\ell = 2$

$s_3 \uparrow$ $\quad$ $\uparrow s_2$ $\quad$ not in web

$\{\alpha_2\}$ $+ 1 1 -$ #8 $\quad$ $\{\alpha_1\} \to 1 1 + -$ # 10 $\quad$ $\{\alpha_3\}$ $+ - 1 1$ # 6 $\quad$ $\ell = 1$

$\nwarrow s_2$ $\quad$ $\uparrow s_1$ $\quad$ $\nearrow s_3$

$0 \leftrightarrow + - + -$ #0 $\quad$ $\ell = 0$

All KLV polynomials $P_{a', a} = \begin{cases} 0 \\ 1 \end{cases}$ $a' \subset \bar{a}$. as they should be.

$K \times_B P_3 \times_B P_2 \times_P P_1 / B$ $\sim$ to this computations

$K \times_B P_1 \times_B P_2 \times_B B_2 / B$ $\sim$ another set of orbits that also satisfy $P_{a'\, a} = \begin{cases} 0 \\ \text{or} \\ 1 \end{cases}$ when they should.

What about other classical groups?

Example 2. $\check{G} = SO(7, \mathbb{C})$ dual group $Sp(6, \mathbb{C})$ - where the Geometric P. live.

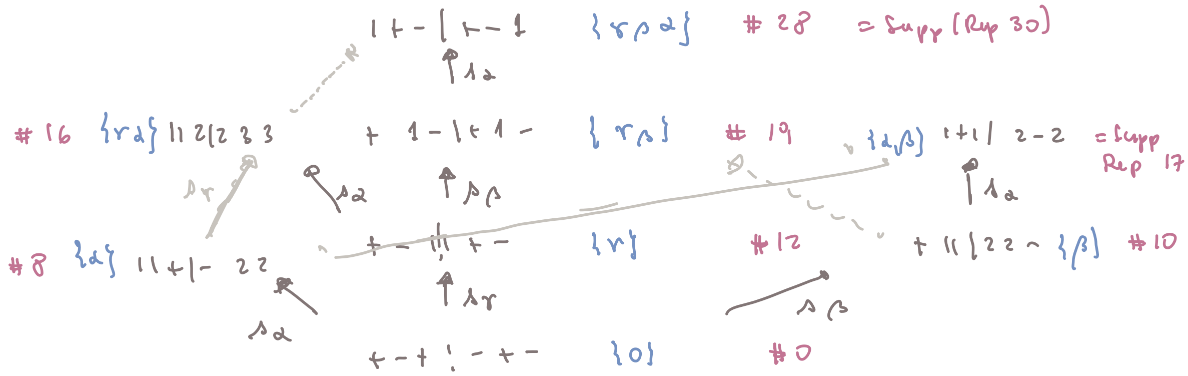
$\underset{\alpha}{\circ} - \underset{\beta}{\circ} \Rightarrow \underset{\gamma}{\circ}$

$G(0) = T$.

orbits $\leftrightarrow$ {set of simple roots}

How to match orbits on $\mathfrak{g}(-1)$ $\rightsquigarrow$ K orbits on $Sp(6,\mathbb{C})/B$?
(web interface)

Choice 1:



orbit closure inclusion

$P_{30,0} = P_{30,19} = P_{30,17} = P_{30,16} = P_{30,12} = P_{30,8} = P_{30,10} = 1$

$P_{17,0} = P_{17,10} = P_{17,8} = 1$

$P_{19,0} = P_{19,12} = P_{19,10} = 1$ 0 at other orbits - - ..

The relevant resolution of $Q_{\#28}$ $K \times_B P_\alpha \times_B P_\beta \times_B P_\gamma / B$.

What if we use $K \times_B P_\gamma \times_B P_\beta \times_B \times P_\alpha / B$?

Next page
↓

2^{nd} choice

$1 + 2 \mid 2 - +$ $\{\alpha\beta\gamma\}$ Rep 34

$\uparrow s_\gamma$

#16 $\{\alpha\gamma\}$ $11\ 22\ 33$ $\qquad$ $1+1 \mid 2-2$ $\{\alpha\beta\}$ #17 $\qquad$ $+ 12 \mid 12 -$ $\{\beta\gamma\}$ #23

$\uparrow s_\beta \qquad\qquad \uparrow s_\gamma$

#12 $\{\gamma\}$ $+-11 \mid +-$ $\xrightarrow{s_\gamma}$ $\qquad$ $11 + \mid - 22$ $\{\alpha\}$ #8 $\qquad$ $+ 11 \mid 22 -$ $\{\beta\}$ #10

$\uparrow s_\alpha$

s_γ $\qquad$ $+-+ \mid -+-$ #0 $\qquad$ s_β

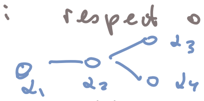
KLV poly: $P_{a',a} = \begin{cases} 0 & a' \not\subset \bar{a} \\ 1 & a' \subset \bar{a}. \end{cases}$ ✓

Corresponds $K \times_B P_\gamma \times_B P_\beta \times_B P_\gamma / B$

(more than one solution ....)

<u>Example 3</u>: $G = Spin(4,4)$ (All relevant local syst. are trivial)

Possible matches of orbits: respect orbit closure inclusion



orbits on $g(-1)$	web int.	orbit	web interface.
ϕ	#0	$\{\alpha_1 \alpha_2\}$	#42
$\{\alpha_1\}$	#24	$\{\alpha_1 \alpha_3\}$	#32
		$\{\alpha_1 \alpha_4\}$	#30
$\{\alpha_2\}$	#16	$\{\alpha_2 \alpha_3\}$	#38
		$\{\alpha_2 \alpha_4\}$	#34
$\{\alpha_3\}$	#20	$\{\alpha_3 \alpha_4\}$	#28
$\{\alpha_4\}$	#12		

$\{\alpha_1 \alpha_2 \alpha_3\}$ $\Big\langle$ $s_1 \times 28 = s_2 \times 42 = 61$ or $s_2 \times 32 = 51$

$\{\alpha_1 \alpha_2 \alpha_4\}$ $\Big\langle$ $s_1 \times 34 = s_4 \times 42 = 57$; $s_2 \times 30 = 47$

$\{\alpha_1\ \alpha_2\ \alpha_4\}$ ↗ $s_4 \times 30 = s_3 \times 34 = 53$

↘ $s_2 \times 28 = 49$

$\{\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\}$ $\underset{\text{either}}{\rightsquigarrow}$ 65; 66; 74; 78

All possible KLV (respecting orbit relations) do the job, i.e. $P_{a',a} = \begin{cases} 0 & a' \not\subset \bar{a} \\ 1 & a' \subset \bar{a}. \end{cases}$

(no "i_2" in root a_i)

Remark: • Lusztig's work assumes G s.c.

(I) • If λ regular, G s-c, $\check{c}$ $\mathbb{R}$ split $\rightsquigarrow$ $P_{a',a} = \begin{cases} 0 \\ 1 \end{cases}$

(II) Assume $G = \overset{\vee}{G} = SO(8, \mathbb{C})$
$G(\mathbb{R}) = SO(4,4)$
λ regular

Attach 0-orbit $\rightsquigarrow$ k.b $= a_0$
all simple roots $= i_1$
$(+-+-\ 0\ -+-+)$

If we assign to

• $\{\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\}$ $\rightsquigarrow$ $s_3 \circ s_2 \circ s_1 \circ s_4\ a_0 = a$ #37 (web interface)
(1 2 3 1 0 4 2 3 4)

s_4 #0 = #6; s_1 #6 = #16 $\ni$ supp (Rep 18) s_2 #16 = #25 supp (Rep 28)

s_3 #25 = 37 $\ni$ supp 43 $[c^+\ i_2\ c^-\ c^+]$

$P_{43,0} = 2+q$

• If $\{\alpha_2, \alpha_3, \alpha_4\}$ $\sim$ $\underbrace{s_3 \circ s_4 \circ \overbrace{s_2\ a_0}^{\#8}}_{\#18\ \cdot\ \text{supp (Rep 20)}} = \#29 = \text{Supp}(34)$

$P_{34,0} = 2.$

First list of examples with singular λ

Ex 1: $GL(3,\mathbb{C})$ $\lambda = [1\,1\,1\,0\,0]$

$\circ - \circ - \bullet - \circ$ ($\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$)

$G(\lambda) = GL(3,\mathbb{C}) \times GL(2,\mathbb{C})$;

$G(\lambda)$-orbits

$$0 \longrightarrow \pi_{G/P}(+++--) \quad Q_0$$

$$[1\,0] \rightsquigarrow \pi_{G/P}(1++-1) = \pi_{G/P}(s_1 s_2 s_4 s_3 (+++--))$$

$$[1\,0]\,[1\,0] \longrightarrow \pi_{G/P}(12+21) = \pi_{G/P}(s_2 \circ s_3 (s_1 s_2 s_4 s_3)(+++--)).$$

$$\begin{array}{cc} + & - \\ + & - \\ + & \\ V_1 & V_0 \end{array}$$

Multisegment Notation

Ex. 2: $\check{G} = Sp(4,\mathbb{C})$ $\lambda = (1,1)$

$\circ \Leftarrow \bullet$ ($\underbrace{\varepsilon_1 - \varepsilon_2}_{\alpha}$, ε_2)

(web interface is used)

$P = P_\alpha$

Dan's computations

	0	2	3_t	3_s
0	1	1	1	q
2	0	1	1	0
3_t	0	0	1	0
3_s	0	0	0	1

$0 \rightsquigarrow \pi_{P_\alpha}(++--) = \pi_{P_\alpha}(\#2)$

$2 \rightsquigarrow \pi_{P_\alpha}(1+-1) = \pi_{P_\alpha}(\#7)$

$3 \rightsquigarrow \pi_{P_\alpha}(1221) = \pi_{P_\alpha}(\#10)$

#10 supp (Reps 10, 11)

Web interface $\rightsquigarrow$ $P_{10,2} = P_{10,7} = P_{10,10} = 1$

$P_{11,2} = q$; $P_{11,7} = P_{11,10} = 0$

$P_{7,2} = 1$

😊

Ex 3.

$\check{G} = G_2$ $\qquad$ $\circ \equiv 0$ $\qquad$ λ = middle element of sl_2
$G_2(z_1)$

Dan's computations

	0	2	3	4_s	4_t
0	1	1	$q+1$	1	q
2	0	1	1	1	0
3	0	0	1	1	1
4_s	0	0	0	1	0
4_t	0	0	0	0	1

4 dim orbit admits a third local system (cuspidal) that does not contribute to this block.

Web interface

	$l=0$ #2 $[i_c i_1]$	$l=2$ #5 $[c^- c^+]$	$l=3$ #7 $[c^- i_2]$	$l=4$ #9 $[r_{2_1} r_2]$	$l(4)$ # 11 Rep. $[r_n, r_2]$?
#2	1	1	$q+1$	1	q
#5	0	1	1	1	0
#7	0	0	1	1	1
#9	0	0	0	1	0
🧐 ? #11 supp q	0	0	0	0	1

	#1 $[i_1 i_c]$	#6 $[c^+ c^-]$	#8 $[r_2 c^-]$	#9 $[r_{2_1} r_2]$	#10 $[r_{2_1} r_n]$?
#1	1	1	$q+1$	1	q
#6	0	1	1	1	0
#8	0	0	1	1	1
#9	0	0	0	1	0
🧐 ? #10	0	0	0	0	1

How do we compute?

Comments on Notation:

In this exposition, in describing the relevant parameter space I use the notation $\{(\mathcal{O}, \mathcal{L})$ $\mathcal{O} \subset G/P(\lambda) \ldots\}$. Usually ones writes $\check{G}/\check{P}(\lambda)$. I made the change to simplify notation (all the action is on the parameter space side.) In this slide I will use traditional notation. Lie groups are denoted by H, $\check{H}$ Langlands dual

Brief Comments on the computations.

(David addressed related issues in a previous ATLAS meeting.)

Assume λ_0 is integral dominant but singular.

$\lambda_0 \quad \rightsquigarrow \quad \check{P} = \check{L}\check{U} \subset \check{H}$
$\check{\mathfrak{l}} = \{\check{\alpha} : \langle \lambda_0, \check{\alpha} \rangle = 0\}$

Dual side

$\{\text{irred } \tilde{J}_0\}$ — $\text{Block}_{\lambda_0}(\text{Lie}(H), K_H)$

$\uparrow$ Zuckerman Translation functor

$\{\text{irred } J$, $\tau(J)$ contains none of the simple roots in $\mathfrak{l}\}$ — $\text{Block}_{\lambda}(\text{Lie}(H), K_H)$, λ regular

$\langle \check{J} : \tau(\check{J}) \supset \text{all simple roots in } \check{\mathfrak{l}} \rangle$

Subcategory $(\text{Block}_{\check{\lambda}}(\text{Lie}(\check{H}), \check{K}_H))$

$= \check{K}_H$-equivariant $\mathcal{D}$-mod $\check{H}/\check{P}$.

If $\pi : \check{H}/\check{B} \rightarrow \check{H}/\check{P}$ and

$\cup$

equivariant $\mathcal{D}$-module with support on the largest $\check{K}_H$-orbit $\subset \pi^{-1}(S)$ $\xleftarrow{\text{pull back}}$ $\check{S}$ = support of a equivariant $\mathcal{D}$-module.

How do we compute?

1. Identify $\{\check{J}$ irred. Block $(\check{H}, \check{K}_H)$ $\tau(\check{J}) \supset$ all the simple roots in $\check{l}\}$

2. Normal slice or Normally non-singular inclusion $\}$ arguments ... If you want $\theta \mapsto Q$, $\hat{g}(-1)$, G/P. Seek Rep. (Q, ν) : length Q = dim θ.

3. Check your work by asking ATLAS to compute KLV-pol.

A couple of examples when G = F_4

Dan takes $\alpha_1 = [1,-1,-1,-1]$; $\alpha_2 = [0,0,0,2]$; $\alpha_3 = [0,0,-1,-1]$

$\alpha_4 = [0,1,-1,0]$

$$\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} \Rightarrow \underset{\alpha_3}{\circ} - \underset{\alpha_4}{\circ}$$

(1) $\lambda = [2,2,0,0]$ middle element triple $\tilde{A}_2$

Here $\langle \lambda, \alpha_1 \rangle = \langle \lambda, \alpha_2 \rangle = \langle \lambda, \alpha_3 \rangle = 0$.

Dan's computations

	0	7	8
0	1	q^3+1	1
7	0	1	1
8	0	0	1

(web-interface)

In the block $F_4(B_4) \mid F_4(\tilde{A}_2)$

0 $\rightsquigarrow$ #2 (2, 228) $\quad l = 0$

7 $\rightsquigarrow$ #13 (13, 133) $\quad l = 7$

8 $\rightsquigarrow$ #14 (14, 141) $\quad l = 8$

reproduces Dan's table

[you will find no such match if you look in the top block].

(2) $\lambda = [7,3,1,1]$ $\quad F_4(a)$

$$\underset{n}{\circ} - \underset{n}{\circ} \Rightarrow \underset{c}{\circ} - \underset{n}{\circ}$$

	Rep	supp
0	4	
1	24	
2_a	34	
2_b	29	
3_a	60	
3_b	49	
3_c	43	
3_{dt}	46	45
3_{ds}	47	45
4_a	81	
4_b	65	
4_c	73	
4_{at}	76	71
4_{as}	77	71
5_a	103	
5_{bt}	96	89
5_{bs}	97	89

6_b	135	117
6_s	136	117

all polynomials match.

Here I used web-interface.