LAST TIME:

I- G connected, simply-connected, reductive over C

$\mathfrak{g}$ Lie algebra, $\mathfrak{h}$ Cartan, $\mathfrak{h}^* \cong \mathfrak{h}$

Fix λ (Integral and hyperbolic)

$\mathrm{ad}(\lambda)$ Induces a grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ $\quad \mathfrak{g}(i) = \{x \in \mathfrak{g} : \mathrm{ad}(\lambda)(x) = i\, x\}$

Set $G(0)$ with Lie algebra $\mathfrak{g}(0)$ and consider

$$S_{\mathbb{H}} = \{(\mathcal{O}, \mathcal{L}) \quad \mathcal{O} \in G(0)\backslash \mathfrak{g}(-1) \quad \mathcal{L} \text{ of Springer Type}\}$$

RELEVANCE OF THIS SET:

Let IH_1 be the affine graded Hecke algebra.

(As vector space this is C(W) $\otimes$ Sym($\mathfrak{h}$ ^*)
The generators are {t_{s_alpha} alpha simple}$\cup${ nu in $\mathfrak{h}^*$ }
The commuting relations $\nu t_{s_\alpha} = t_{s_\alpha} s_\alpha(\nu) - \langle \omega, \check{\alpha} \rangle.$

- Standard and Irred $\mathrm{mod}_\lambda(\mathbb{H}_1)$ $\longleftrightarrow$ S
parametrized by

- Moreover, via Kazhdan-Lusztig proof of Langlands-Deligne Conjecture (building from work by Borel)

$S_{\mathbb{H}}$ parametrizes Irred. Iwahori-spherical rep of $\check{G}(F)$

with central ch(λ) .

II- [ABV]

$\mathrm{ad}(\lambda) \rightsquigarrow P(\lambda) = P = \bigoplus_{i \geq 0} \mathfrak{g}(i)$

$(\lambda$ integral$) \rightsquigarrow$ Generalized Flag G/P.

$\lambda \rightsquigarrow y(\lambda) = \exp(\pi i \lambda) \rightsquigarrow \theta_\lambda$ $\qquad K = G^{\theta_\lambda}$
involution on G

$$S_{[ABV]} = \{(Q, \mathcal{V}) \quad Q \in K\backslash G/P,\ \mathcal{V}\ \text{L.S.}\}$$

RELEVANCE OF. $S_{[ABV]}$.

The choice of $\mathcal{O}_\lambda$ $\rightsquigarrow$ Block of $\check{G}(\mathbb{R})$ - rep.

Irred ; Standard in Block $\longleftrightarrow$ parametrized $S_{[ABV]}$.

QUESTIONS

A- On the Hecke Alg side, Lusztig (Cuspidal Local Systems and affine graded Hecke ALg., II)

$$(\mathcal{O},\mathcal{L}) \in S_{\mathbb{H}_1} \rightsquigarrow \mathrm{stand}_{\mathbb{H}_1}(\mathcal{O},\mathcal{L})$$

$$\mathrm{Stand}_{\mathbb{H}_1}(\mathcal{O},\mathcal{L}) = \sum_{(\mathcal{O}',\mathcal{L}')} m^{\mathbb{H}_1}_{(\mathcal{O}',\mathcal{L}')}(\mathcal{O},\mathcal{L}) \quad \mathrm{irred}_{\mathbb{H}_1}(\mathcal{O}',\mathcal{L}')$$

$$m^{\mathbb{H}_1}_{(\mathcal{O}',\mathcal{L}')}(\mathcal{O},\mathcal{L}) = \sum_i (-1)^i \left[\mathcal{L} : \mathcal{H}^i(IC(\overline{\mathcal{O}'},\mathcal{L}')|_{\mathcal{O}} \right]$$

On the `Real" side, [ABV]

$$(Q,\nu) \in S_{[ABV]} \rightsquigarrow \mathrm{Stand}_{\mathbb{R}}(Q,\nu)$$

$$\mathrm{Stand}_{\mathbb{R}}(Q,\nu) = \sum_{(Q',\nu')} m^{\mathbb{R}}_{(Q',\nu')}(Q,\nu) \quad \mathrm{irred}_{\mathbb{R}}(Q',\nu')$$

$$m^{\mathbb{R}}_{(Q',\nu')}(Q,\nu) = \sum_i (-1)^i \left[\nu : \mathcal{H}^i(IC(\overline{Q}',\nu')|_{Q} \right].$$

ATLAS computes all these numbers.

Can we compute the multiplicities in the Hecke Algebra side in terms
Of the multiplicities in the Real side?

B- Is it possible to relate KL polynomials to KLV-polynomials?

Last time (a) I spoke about the regular case and ended up with experimental data
Involving some singular parameters.
(b) In the regular case the answer involved some iterated bundles .
(An analogous construction will play a role in what we do today.)

(I). The choice of K >>>>>>>> closed K orbit Q_0 on G/P
(II) The iterated bundle >>>>>>> Q_{max} a K-orbit dim = Q_0 + dim(g(-1))
Other orbits in the closure of Q_max}

Remark: The construction allows to define a map

$$\Phi : \mathfrak{g}(-1) \longrightarrow \overline{Q}_{max} \subset G/P$$ (<u>normally non-singular inclusion</u>).

TODAY

In the singular case, we need to handle local systems.
We ask if we can relate S_{IH} to S_{[ABV]}.

Where do the relevant {(Q, L)} come from?

THE ANSWER MIGHT BE IN

Lusztig, Cuspidal Local Systems and Affine Graded Hecke ALg., II

Chriss and Ginzburg, Chapter 8

PART OF THE ISSUE IS TO UNDERSTAND (mentioned last time)

$\blacksquare$ $$\mathrm{Stand}_{\mathbb{H}_1}(\theta, \delta) = \sum_{(\theta', \delta')} m(\theta, \delta) \quad \mathrm{irred}_{\mathbb{H}_1}(\theta', \delta')$$

These are not easy papers. There is a lot that I do not understand.

The point is

(A) to isolate key results in this paper, imitate their work (when possible)
in order to get similar results on the `Real side''

Identity $\blacksquare$ encodes a geometric object and an Algebra $(\mathcal{H})$ action on that object.

$\mathcal{H}$ Is first defined " abstractly'', then it is identified as a convolution algebra.

Hence, translating the result to the Real picture involves

(A_1) describing an analogous ''geometric object'' in the [ABV] parameter space.

(I will indicate how we plan to do this in an example for G = GL(n, C))

(A_2) defining an analogous algebra $\mathcal{H}^{\mathbb{R}}$ (Abstractly)

(What is $\mathcal{H}^{\mathbb{R}}$? I would like to avoid answering directly this question.

Instead, I would like to "get by'' by trying to relate $\mathcal{H}^{\mathbb{R}}$ to $\mathcal{H}$ and just

use Lusztig's understanding of $\mathcal{H}$)

(Even this `short-cut'' plan requires a deep understanding of the reference.
My short comings on this point will be obvious as I speak.)

(B) A critical component in the reference is to ''relate'' the category of finite Dimensional modules for the convolution algebra to the category of finite Dimensional representations of IH_1.

As a vector space IH_1 = C[W] $\otimes$ $\mathrm{Sym}(\mathfrak{h}^*)$ with generators $\{t_{s_\alpha} \mid \alpha \text{ simple}\} \cup \{w \in \mathfrak{h}^*\}$

+ relations.

Where does W come from??? Roughly form the { G(0) orbits on g(-1) }

Lusztig parametrizes this set of orbit by a set of equivalence classes of
`Good parabolics''

$$\theta \to P_\theta = L_\theta U_\theta \quad : \quad \mathrm{Lie}(U_\theta) = \mathfrak{u}_\theta$$

$$\bar{\theta} = \overline{G(0).\,(\mathfrak{g}(-1) \cap \mathfrak{u}_\theta)}$$

Rough

P_θ will not quite contain B defined by λ

but will be conjugate to a parabolic

Containing B + with the same Levi ...

$\rightsquigarrow W_\theta \rightsquigarrow W$

I am not ready to talk about this.

IN SUMMARY:

1- I will talk about A and how we want to translate those ideas to the `Real picture''

2- I will explain what I understand about

3- I will indicate an attempt at translating the λ Action to the ''Real'' picture.

THE GEOMETRIC SPACE ON WHICH $\mathcal{H}$ ACTS

- $\mathcal{N}$ nilpotent cone $\qquad \tilde{\mathcal{N}} = T^*\mathcal{B}$

λ inf. hyperbolic $\rightsquigarrow$ σ s.s. : $\mathcal{N}^\sigma \cong \mathfrak{g}(-1)$

$$\begin{array}{ccc} M = G(0)\times_{B(0)} \mathfrak{g}(-1) \subset \tilde{\mathcal{N}}^\sigma & & \tilde{\mathcal{N}} \\ \text{open} \qquad\qquad \downarrow \mu & \text{Any restrict} & \downarrow \\ \mathfrak{g}(-1) & & \mathcal{N} \end{array}$$

Springer Resolution

Let $\mathbb{C}_{\tilde{\mathcal{N}}^\sigma}$ complex: $\mathbb{C}_{\tilde{\mathcal{N}}^\sigma}\big|_M = \mathbb{C}_M[\dim M]$

DECOMPOSITION THEOREM

- $$\mu_*(C_M[\dim M]) = \bigoplus_{k\in\mathbb{Z}} {}^pH^k(\mu_*(\mathbb{C}_M[\dim M]))[-k]$$

$$\underset{\text{Lusztig's notation}}{=} \bigoplus \bar{B}^k[-k] = \bar{B}$$

$$\bigoplus_k {}^pH^k(\mu_* C_M[\dim M]) = \bigoplus \bar{B}^k = \bar{B}' \quad \text{is s.s Perverse.}$$

Write {. P_j } for the set of irreducible Pervese sheaves that occur in $\bar{B}$’

- Given $(\lambda,\ \mathcal{O} = G(0).y,\ \mathcal{L})$, form $A(y,\lambda) = Z_G(y,\lambda)/Z_G(y,\lambda)_0$

$$\mathcal{L} \longleftrightarrow \rho \in \widehat{A(y,\lambda)}.$$

$$\mathcal{H}^l(i_y^*\,\mu_*(\mathbb{C}_M[\dim M])) \quad \text{local system.}$$

The relevant geometric object is

$$[\mathcal{L}, \bigoplus_{\ell} \mathcal{H}^{\ell}(i_y^* \mu_*(\mathbb{C}_M[\dim M])]$$

$$= \mathrm{Hom}_{A(y,\lambda)}(\rho, \bigoplus_{\ell} \mathcal{H}^{\ell}(i_y^* \mu_*(\mathbb{C}_M[\dim M]))).$$

THE MODULE STRUCTURE

Preliminaries

X algebraic variety $\qquad \mathcal{D}^b(X)$ bounded derived category

- $\mathcal{D}(X)$ the full subcategory whose objects are complexes with constructible cohomology sheaves.
- $[n]: \mathcal{D}(X) \to \mathcal{D}(X)$ shift functor
  $(\mathcal{H}^j(A[n]) = \mathcal{H}^{j+n}(A))$
- $\mathrm{Hom}^j_{\mathcal{D}(X)}(A, B)$ as vector space is

$$\mathrm{Hom}_{\mathcal{D}(X)}(A, B[j]) = \mathrm{Hom}_{\mathcal{D}(X)}(A[i], B[i+j]) = \cdots$$

$$\mathrm{Hom}^{\bullet}_{\mathcal{D}(X)}(A, B) = \sum_j \mathrm{Hom}^j_{\mathcal{D}(X)}(A, B)$$

(One needs to also consider $\mathcal{D}_G(X)$ ---)

The relevant Algebra:

Recall:

$A, B, C \in \mathcal{D}_G(X)$

$$\mathrm{Hom}_{\mathcal{D}_G X}(B, A) \times \mathrm{Hom}_{\mathcal{D}_G X}(A, B) \to \mathrm{Hom}_{\mathcal{D}_G X}(A, C)$$

$$h \qquad\qquad h' \quad \rightsquigarrow \quad h \circ h'$$

$$h \in \mathrm{Hom}^j_{\mathcal{D}_G X} \qquad h' \in \mathrm{Hom}^k_{\mathcal{D}_G(X)} \qquad h\circ h' \in \mathrm{Hom}^{j+k}_{\mathcal{D}_G(X)}$$

$$h: B[k] \to C[k+j] \qquad A \to B[k] \rightsquigarrow \quad A \to C[k+j]$$

$\rightsquigarrow$ (we need is $\mathrm{Hom}_{\mathcal{D}_G(X)}(B, B)$ is a unital Algebra)

- $A = \mathrm{Hom}_{\mathcal{D}(g(-1))}\left(\bigoplus_n \bar{B}^n[-n], \bigoplus \bar{B}^n[-n]\right) = \mathrm{Hom}_{\mathcal{D}(g(-1))}(\bar{B}, \bar{B})$

  $B^n = {}^pH^n(\mu_*(\mathbb{C}_M[\dim M]))$

- A is a graded algebra.

  $A = \bigoplus_s A_s \qquad A_s = \bigoplus_{\substack{n, n', k \\ n-n'+k=s}} \mathrm{Hom}_{\mathcal{D}}(\bar{B}^n[-n], \bar{B}^{n'}[-n'+k])$

  $A_s \cdot A_{s'} \subset A_{s+s'}$; $A_s = 0 \quad s<0$

  $A_0 = \bigoplus_{n,n'} \mathrm{Hom}^0_{\mathcal{D}(g(-1))}(\bar{B}^n[-n], \bar{B}^{n'}[-n]) = \bigoplus_{n n'} \mathrm{Hom}^0_{\mathcal{D}(g(-1))}(\bar{B}^n, \bar{B}^{n'})$

  $= \bigoplus \mathrm{Hom}^0_{\mathcal{D}(g(-1))}(\bar{B}', \bar{B}')$ s.s alg.

- $A \xrightarrow[\text{Proj.}]{} A_0$ alg. homomorphism.

- Thm 8.14

The category $\mathrm{mod}_x(\mathbb{H}_1)$ is equivalent to the category of finite dim. $\mathfrak{h}$- modules on which some power of "$I^x = \ker \chi_\lambda$" acts by zero.

(As already said, I will not go over why is this so.)

THE STANRD AND IRREDUCIBLE MODULE ATTACHED TO $(\mathcal{O}, \mathcal{L})$.

Let $\{P_j\}$ be the set of irred. Perverse sheaves that occur in $\bigoplus_k {}^pH^k(\mu_*(\mathbb{C}_M[\dim M]))$.

- Irred. Perverse sheaf P_j $\xleftrightarrow{1-1}$ Simple $\mathfrak{h}$- module. $\Pi_j = \mathrm{Hom}_{\mathcal{D}(\mathfrak{g}(M))}(P_j, \bar{B}')$

  Π_j is a left $\mathfrak{h}_0$-module (by composition) can be regarded as $\mathfrak{h}$-mod. all distinct

- Standard Module $(\mathcal{O}, \mathcal{L})$

  $\mathcal{O} = G(0).y$ $\rightsquigarrow$ $M(i_y^* \bar{B})$ (Lusztig's notation for)

$$\bigoplus_l \mathcal{H}^l(i_y^* \mu_*(\mathbb{C}_M[\dim M]))$$
$$= \bigoplus_{lk} \mathcal{H}^{l-k}(i_y^* \bar{B}^k) \qquad \bar{B}^k = {}^pH^k(\mu_* \mathbb{C}_M[\dim M])$$

$*_1$ Endow $M(i_y^* \bar{B})$ with a $\mathrm{Hom}_{\mathcal{D}_y}(i_y^* \bar{B}, i_y^* \bar{B})$ - module structure via

$$\begin{array}{ccccc} (\phi & , & \xi) & \longmapsto & \phi(\xi) \\ \cap & & \cap & & \cap \\ \mathrm{Hom}^j_{\mathcal{D}_y}(\ ) & & \mathcal{H}^l(\ ) & & \mathcal{H}^{l+j}(\ ) \end{array}$$

$*_2$ $i_y^* \mathcal{A} = \mathrm{Hom}_{\mathcal{D}(y(w))}(\bar{B}, \bar{B}) \longrightarrow \mathrm{Hom}_{\mathcal{D}_y}(i_y^* \bar{B}, i_y^* \bar{B})$ alg homo

$\rightsquigarrow$ $M(i_y^* \bar{B})$ can be regarded as a $\mathcal{A}$-module.

$*_3$

$A(y,\lambda)$ acts on $M(i_y^! \bar{B})$ (action commutes with $\mathcal{A}$-action)

$\rightsquigarrow$ $\mathrm{Hom}_{A(y,\lambda)}(\rho, M(i_y^! \bar{B})) = M(i_y^! \bar{B})^{\rho}$ (in Lusztig's notation)

$= \mathrm{Hom}_{(y,\lambda)}(\rho, \bigoplus_\ell \mathcal{H}^\ell(\mu_* \mathbb{C}_M[\dim M]))$

$\mathcal{A}$-module.

A SUBTLE POINT

- Both $M(i_y^* \bar{B}) = \bigoplus_\ell \mathcal{H}^\ell(i_y^! \mu_* (\mathbb{C}_M[\dim M]))$ and $\mathcal{A}$ are graded

$M(i_y^! \bar{B}) = \bigoplus M_s(i_y^! \bar{B})$ $\qquad$ $\mathcal{A} = \bigoplus_{s'} \mathcal{A}_{s'}$

$M_s(i_y^! \bar{B}) = \bigoplus_k \mathcal{H}^s(i_y^! \bar{B}^n)$ $\qquad$ $\mathcal{A}_{s'} = \bigoplus_{\substack{n,n',k \\ n' = n+k-s}} \mathrm{Hom}_{\mathcal{D}(X)}(\bar{B}^n[-n], \bar{B}^{n'}[-n'+k])$

$$\mathcal{A}_{s'} \cdot M_s(i_y^! \bar{B}) \subset M_{s+s'}(i_y^! \bar{B}).$$

$\rightsquigarrow$ $gr(M(i_y^! \bar{B}))$ becomes a $\mathcal{A}_0$-module (i.e. $\bigoplus_{s>0} \mathcal{A}_s$ act by zero).

The formula for $\mathrm{stand}(\theta,\delta)$ in terms of $\mathrm{irred}(\theta,\delta)$ (in the Grothendieck group of f.g. $\mathcal{A}$ modules) is obtained by studying $(gr(M i_y^* \bar{B}), \mathcal{A}_0)$.

DECOMPOSITION OF STANDARD MODULES IN TERMS OF IRREDUCIBLE MODULES

The statement is (Proposition 10.5)

In the Grothendieck group $(gr(M(i_\gamma \bar{B}) \cong M(i_\gamma^* \bar{B}))$.

Formally (*) $\bigoplus_\ell \mathcal{H}^\ell(i_\gamma^* P_j) \otimes \Pi_j := \bigoplus \mathcal{H}^\ell(i_\gamma^* P_j) \otimes \mathrm{Hom}^0_{\mathcal{D}(X)}(P_j, \bar{B}')$

$$= \bigoplus_\ell \mathcal{H}^\ell\Big(i_\gamma^* \Big[\sum_j P_j \otimes \mathrm{Hom}^0_{\mathcal{D}(X)}(P_j, \bar{B}')\Big]\Big) = \bigoplus_\ell \mathcal{H}^\ell(i_\gamma^* \bar{B}')$$

The point is that a A_0 - module (*) $\cong gr(M(i_\gamma^* \bar{B}))$

As the A_0 and $A(y,\lambda)$ actions commute

$$gr(M(i_\gamma^* \bar{B}))^\rho = \bigoplus_j \Big[\bigoplus_\ell \mathcal{H}^\ell(i_\gamma^* P_j)\Big]^\rho \otimes \Pi_j$$

as A_0-module.

so mult. Π_j in $gr(M(i_\gamma^* \bar{B})^\rho)$ is

$\dim\Big(\bigoplus_\ell \mathcal{H}^\ell(i_\gamma^* P_j)^\rho\Big)$

Finish by using that in the Grothendieck group

$M(i_\gamma^* \bar{B})^\rho = gr(M(i_\gamma^* \bar{B})^\rho)$ .....

HOW DO WE EXPECT TO ANSWER QUESTION A?

EXAMPLE: $G = \check{G} = GL(n, \mathbb{C})$

$$\Lambda = [\underbrace{a_1 \cdots a_1}_{t_1} \ \underbrace{a_2 \cdots a_2}_{t_2} \ \cdots \ \underbrace{a_k \cdots a_k}_{t_k}]$$

Assume $a_{i+1} - a_i = 1 \qquad a_i \in \mathbb{N}$.

Define $P(\Lambda) = G(0)\, U_\Lambda = GL(t_1) \times \cdots \times GL(t_k) \times U_\Lambda$

$\cup$

B (upper triangular)

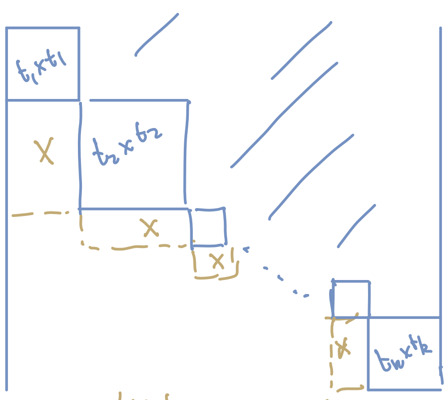
$$\bullet - \circ - \ \bullet - \circ - \bullet - \bullet - \bullet - \circ \ \cdots$$

$i_c \quad i_1 \quad i_c \quad i_1$

Fix closed $K = GL(p) \times GL(q)$ - orbit on $G / P(\Lambda)$

$$Q_0 = \pi_\Lambda (\underbrace{+ - +}_{t_1} \ \underbrace{- - -}_{t_2} \ \underbrace{+ + - +}_{t_3} \ - - - - \ \cdots)$$

where $\pi_\Lambda : G/B \to G/P(\Lambda)$



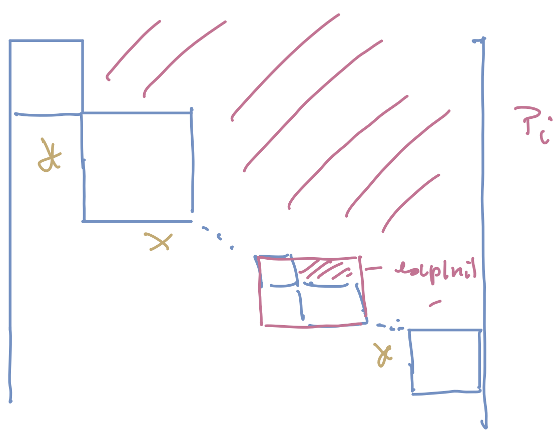
$\{x\} \to \mathfrak{g}(-1)$

Definition: Let $\{P_j \ j = 1 \cdots k-1\}$ be the set of Parabolic subg.:

$\cdot_1$ $P_j = L_j U_j \supset P(\Lambda)$

$\cdot_2$ $L_j \cap P(\Lambda) = G(0) \exp(n_i) \qquad n_i \subset \mathfrak{g}(1)$

$\cdot_3$ minimal with properties $\cdot_1$ $\cdot_2$.



Imitating Gelfand-McPherson define :

$$RS = K \times_{P\cap K} P_1 \times_P P_2 \times_P \cdots \times_P P_{k-1} / P \qquad P = P(\lambda)$$

- RS is smooth of dimension = $\dim \mathfrak{g}(-1) + \dim Q_0$
- $\tau : RS \longrightarrow G/P$

$$[k, y_1, y_2, \cdots y_{n-1}] \longmapsto k\, y_1 \cdot y_2 \cdots y_{k-1} \cdot \mathfrak{p}(\lambda)$$

is a well defined projective map.

- An argument similar to the one in Dec 9. lecture gives

$$\tau(RS) = \overline{Q}_{max} \qquad \dim Q_{max} = \dim RS$$

- Take $\mathbb{C}_{RS}[\dim RS]$ and immitate Lusztig's argument.

$$\tau_*(\mathbb{C}_{RS}[\dim RS]) = \bigoplus_k {}^pH^k(\tau_*(\mathbb{C}_{RS}[\dim RS]))[-k]$$

$$= \bigoplus_k \overline{B}^k_{\mathbb{R}}[-k] = \overline{B}_{\mathbb{R}}.$$

Set $B'_{\mathbb{R}} = \bigoplus_k \overline{B}^k_{\mathbb{R}}$. ss Perverse.

$\leadsto$ Irreducibles that occur $\subset \{ IC(\overline{Q}_j, \nu_j) : \overline{Q}_j \subset \overline{Q}_{max} \}$

If $(Q = K.z, \nu)$, arguing as Lusztig does

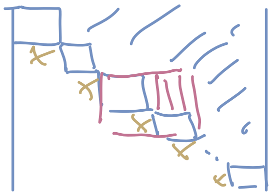
$$[\nu : \bigoplus_\ell \mathcal{H}^\ell(i_z^*(\tau_*(\mathbb{C}_{RS}[\dim RS])))] \text{ is a}$$

$$A^{\mathbb{R}} = \operatorname{Hom}_{\mathcal{D}\,G/P}(\overline{B}_{\mathbb{R}}, \overline{B}_{\mathbb{R}}) \text{ - module.}$$

What is $A^{\mathbb{R}}$?? 😳

CAN WE RELATE THE HECKE ALGEBRA WORLD TO THE REAL WORLD?🛠️🪜🪚

Recall:



$\{x\} \longrightarrow \mathfrak{g}(-1)$

$L_i \rightsquigarrow \square \qquad L_i \cap P(\lambda) = G(\lambda)\exp(n_i)$

$\mathfrak{g}(-1) \simeq \bar{n}_1 + \bar{n}_2 + \cdots + \bar{n}_k.$

$y \rightsquigarrow (y_1, y_2, \quad y_k)$

Define:

$P(\lambda) = G(\lambda)\,\mathcal{U}(\lambda)$

$$\Phi : \mathfrak{g}(-1) \longrightarrow G/P(\lambda)$$

$$y = (y_1, \ldots, y_k) \longmapsto \exp(y_1)\exp(y_2)\cdots\exp(y_k)\cdot P(\lambda).$$

$$\tilde{\Phi} : G(0) \times_{B(0)} (\mathfrak{g}(-1)) \longrightarrow K \times_{P\cap K} P_1 \times_P P_2 \times_P \cdots \times_P P_k \,/P$$

$$[g_0, y] \longmapsto [1, \mathrm{Ad}(g_0)y_1, \ldots, \mathrm{Ad}(g_0)y_n]$$

Prop. Φ is a normally non-singular inclusion of codimension $= \dim(\bar{U}(\lambda) \cap K)$

(Argument similar to Dec. 9 notes).

- Tubular neighborhood $\simeq \bar{U}(\lambda)\cap K \times \Phi(\mathfrak{g}(-1))$

$\rightsquigarrow$ Base change holds

$$\begin{array}{ccc} G(0)\times_{B(0)}\mathfrak{g}(-1) & \xrightarrow{\tilde{\Phi}} & K \times_{P\cap K} P_1 \times_P \cdots \times_P P_k/P \\ \downarrow \mu & & \downarrow \\ \mathfrak{g}(-1) & \xrightarrow{\Phi} & Q_{max} \end{array}$$

$\phi^*(\pi_*(\mathbb{C}_{RS}[\dim RS])) = \mu_*(\mathbb{C}_M[\dim M])$

$\phi^*({}^pH^k(\pi_*\mathbb{C}_{RS}[\dim RS])[\dim \bar{U}\cap K]) \cong {}^pH^k(\mu_*(\mathbb{C}_M[\dim M])).$ for each k

Localization $Q = K\Phi(y)$

$$\phi^*(i^*_{y(s)}\,\pi_*\mathbb{C}_{RS}[\dim RS]) = i^*_y\,\mu_*(\mathbb{C}_M[\dim M]).$$

In the GL(n,C) case all local systems are trivial. As G(0) is a subgroup of K:

🔷 $\rightsquigarrow \phi^{*} \operatorname{Hom}_{A(\phi(s))}\left(\mathbb{1}, \bigoplus_{\ell} \mathcal{H}^{\ell}\left(i^{*}_{\phi(s)} \gamma_{*} \mathbb{C}_{RS}[\dim RS]\right)\right) = \operatorname{Hom}_{A(e,s)}\left(\mathbb{1}, \bigoplus \mathcal{H}^{\ell}\left(i^{*}_{\gamma} \mu_{*} \mathbb{C}_{M}[\dim M]\right)\right)$

CAN WE MATCH THE MODULE STRUCTURES? 🤔 🧐

The right hand side of 🔷 is a $\mathcal{A} = \operatorname{Hom}_{\mathcal{D}(g(-1))}\left(\bigoplus \bar{B}^{k}[-k], \bigoplus \bar{B}^{k'}[-k']\right)$ module via

$$i^{*}_{\gamma} : \mathcal{A} \to \operatorname{Hom}_{\mathcal{D}_{\gamma}}\left(i^{*}_{\gamma} \bigoplus \bar{B}^{k}[-k], i^{*}_{\gamma} \bigoplus \bar{B}^{k'}[-k']\right).$$

gr(right hand side of 🔷) is a. $\mathcal{A}_{0} = \operatorname{Hom}^{0}_{\mathcal{D}(g(-1))}\left(\bigoplus \bar{B}^{k}, \underbrace{\bigoplus \bar{B}^{k}}_{\text{Perverse.}}\right)$ module

On the `Real side" of 🔷 we act via $\mathcal{A}^{\mathbb{R}} = \operatorname{Hom}_{\mathcal{D}G/P}\left(\bigoplus \bar{B}^{k}_{\mathbb{R}}[-k], \bigoplus B^{k'}_{\mathbb{R}}[-k']\right)$

gr(Real side) is a. $\mathcal{A}_{0}^{\mathbb{R}} = \operatorname{Hom}^{0}_{\mathcal{D}(G/P}\left(\bigoplus \bar{B}^{k}_{\mathbb{R}}, \bar{B}^{k'}_{\mathbb{R}}\right).$

🟢 Base Change (using the normally non-singular inclusion :

$$\Phi : g(-1) \longrightarrow \bar{Q}_{max} \subset G/P(\lambda)$$

of codimension $= \dim RS - \dim M$

$= \dim \bar{U}(w) \cap K = c$

tells us $\quad \Phi^{*} \bar{B}^{k}_{\mathbb{R}}[-c] = \bar{B}^{k}$

Hecke side.

Perverse.

• $\mathcal{A}_{0} = \operatorname{Hom}^{0}_{\mathcal{D}(g(-1))}\left(\bigoplus_{k} \bar{B}^{k}, \bigoplus_{k'} \bar{B}^{k'}\right) = \operatorname{Hom}^{0}_{\mathcal{D}g(-1)}\left(\Phi^{*} \bigoplus_{k} \bar{B}^{k}_{\mathbb{R}}[-c], \Phi^{*} \bigoplus_{k'} B^{k'}_{\mathbb{R}}[-c]\right)$

$\doteq \operatorname{Hom}^{0}_{\mathcal{D}g(-1)}\left(\Phi^{!} \bigoplus_{k} \bar{B}^{k}_{\mathbb{R}}, \Phi^{!} \bigoplus_{k'} \bar{B}^{k'}_{\mathbb{R}}\right) \overset{?}{\doteq} \operatorname{Hom}^{0}_{\mathcal{D}(G/P)}\left(\bigoplus_{k} B^{k}_{\mathbb{R}}, \Phi_{!}\Phi^{!} \bigoplus_{k'} B^{k'}_{\mathbb{R}}\right)$

$\bar{B}^{k}_{\mathbb{R}}$ CLC

As: $\phi_! \, \Phi^! \; K - K$ $\overset{?}{\text{is}}$ $\mathfrak{t}_0 \to \mathfrak{t}_0^{\mathbb{R}}$ (homo of algebras?) an iso?

- Similar questions when trying to relate $\mathfrak{t}$ $\mathfrak{t}^{\mathbb{R}}$. ....

- Does Φ^* intertwines the $\mathfrak{t}^{\mathbb{R}}$ and $\mathfrak{t}$-actions?

HOW TO IDENTIFY THE RELEVANT ORBITS? (Combinatorial aspect of the problem)

EXAMPLE: $GL(12,\mathbb{C})$ $\lambda = [444\ 3333\ 22\ 111]$

$G(0) = GL(3) \times GL(4) \times GL(2) \times GL(3)$

$K = GL(5) \times GL(7)$

$Q_0 \leadsto +++ \ ---- \ ++ \ ---$

$$\begin{array}{cccc} + & - & + & - \\ + & - & + & - \\ + & - & & - \\ & - & & \\ V_1 & V_2 & V_3 & V_4 \end{array}$$

$g(-1) \simeq Hom(V_1 V_2) + Hom(V_2, V_3) + Hom(V_3, V_4)$

$\dim g(-1) = 12 + 8 + 6 = 26.$

$P_1 = L_1 V_1$ $\quad L_1 \simeq GL(3) \times GL(4)$ $\quad P(\lambda) \subset P_1$

$P_2 = L_2 V_2$ $\quad L_2 \simeq GL(4) \times GL(2)$ $\quad P(\lambda) \subset P_2$

$P_3 = L_3 V_3$ $\quad L_3 \simeq GL(2) \times GL(3)$ $\quad P(\lambda) \subset P_3$

$\bar{Q}_{max} = ?$

Step 1 (Base case Dec 9) P_1

$+++ \ ----$

$w_1 = (s_3 s_2 s_1 s_4 s_5 s_6)(s_3 s_2 s_4 s_5)(s_3 s_4)$ $\quad l(w_1) = 12$

$w_1^{-1} \circ Q_0 = 123 \sim 321 \ ++ \ ---$ Yamamoto length 12

Step 2 P_2

$\underbrace{---- \ ++}_{s_7}$

$w_2 = (s_7 s_8 s_6 s_5 s_4)(s_7 s_6 s_5)$ $\quad l(w_2) = 8$

$w_2^{-1} \circ w_1^{-1} \circ Q_0 = 1234 + 4321 ---$ $\quad$ length $= 8 + 12 = 20$

Step 3 P_3 $\leadsto$ same manner $\quad w_3^{-1} \circ w_2^{-1} \circ w_1^{-1} \ Q_0 = \pi_{GLp}(1234 + 43 -- 21).$

(1) Builds from a Base case (! simple non compact root).

(2) $\lambda = [a_1 \ldots a_1 \ a_2 \ldots a_2 \ \ldots]$ Assumes $a_i - a_{i+1} = 1.$

A DIFFICULT EXAMPLE

Using Jeff's Example F_4 file (available on his web-page)

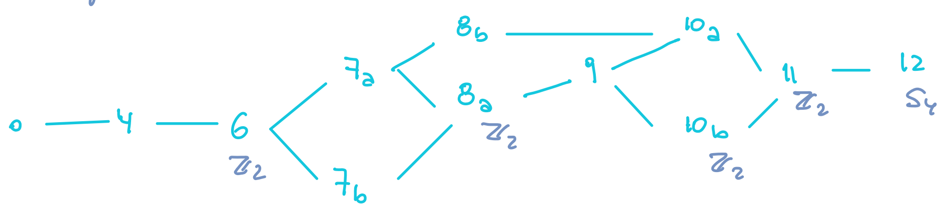
F_4 $\lambda = (3\,1\,1\,1)$

Dan's notation.

$\alpha = [1\ -1\ -1\ -1]$ $\beta = [0\ 0\ 0\ 2]$

$\gamma = [0\ 0\ 1\ -1]$ $\delta = [0\ 1\ -1\ 0]$

$\dim \mathfrak{g}(2) = 12$



0	#3 (3, 228)	$[i_c\, i_1\ i_c\, i_c]$
4	#89 (83, 193)	$[c^-\ c^+\ i_c\ c^-]$
6_t	#134 (116, 157)	$[c^-\ c^+\ c^-\ r_2]$
6_s	#133 (116, 156)	
7_a	#168 (140, 135)	$[c^-\ c^+\ c^-\ c^-]$
7_b	#153 (132, 146)	$[c\ c^-\ c^+\ r_1\ r_2]$
8_{at}	#186 (153, 128)	$[r_2\ c^+\ c^-\ r_2]$ ✓
8_{as}	#184 (153, 126)	
8_b	#208 (169, 110)	$[i_c\, i_1\ i_c\ c^-]$
9	#234 (186, 90)	$[c^-\ c^+\ c^-\ c^-]$
10_{bt}	#250 (195, 73)	$[r\ c^+\ r_2\ c^-]$
10_{bs}	#252 (195, 75)	

😁 Up to this point KL coincide.

😵😵 $10_{a\,t}$ No {KLV poly. (Q, 0)} matches Dan's computations.

$$P^{Dan}_{10at,\,0} = P^{KLV}_{385,\,0} \quad \ldots$$

11_t # 287 (213, 46) $[c^-\ c^+\ c^-\ c^-]$ ☹️ 11 admits only one local system

11_s ?

$12_{(22)}$ # 298 (213, 34) $[c^- \; c^+ \; r_2 \; c^-]$ ✓

$12_{(31)}$ # 299 (213, 35) ✓

$12_{(211)}$ # 296 (213, 32) ✓

$12_{(4)}$ $P_{12,4-}(q) = P_{297,-}(q) - P_{298,-}(q)$