# Matrices almost of order two

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# Outline

# Adeles



Arithmetic problems $\leftrightsquigarrow$ matrices over $\mathbb{Q}$.

Example: count $\left\{ v \in \mathbb{Z}^2 \;\middle|\; {}^t v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v \le N \right\}$.

Hard: no analysis, geometry, topology to help.

Possible solution: use $\mathbb{Q} \hookrightarrow \mathbb{R}$.

Example: find area of $\left\{ v \in \mathbb{R}^2 \;\middle|\; {}^t v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v \le N \right\}$.

Same idea with $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ leads to

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p = \prod_{v \in \{p, \infty\}}{}' \mathbb{Q}_v,$$

locally compact ring $\supset \mathbb{Q}$ discrete subring.

Arithmetic $\leftrightsquigarrow$ analysis on $GL(n, \mathbb{A})/GL(n, \mathbb{Q})$.

# Background about $GL(n, \mathbb{A})/GL(n, \mathbb{Q})$



Gelfand: analysis re $G \leftrightsquigarrow$ irr (unitary) reps of $G$.

$$\begin{aligned} &\text{analysis on } GL(n, \mathbb{A})/GL(n, \mathbb{Q}) \\ \leftrightsquigarrow\ &\text{irr reps } \pi \text{ of } \prod_{v \in \{p, \infty\}}{}' GL(n, \mathbb{Q}_v) \\ \leftrightsquigarrow\ &\pi = \bigotimes_{v \in \{p, \infty\}}{}' \pi(v), \qquad \pi(v) \in \widehat{GL(n, \mathbb{Q}_v)} \end{aligned}$$

Building block for harmonic analysis is one irr rep $\pi(v)$ of $GL(n, \mathbb{Q}_v)$ for each $v$.

Contributes to $GL(n, \mathbb{A})/GL(n, \mathbb{Q}) \leftrightsquigarrow$ tensor prod has $GL(n, \mathbb{Q})$-fixed vec.

# Local Langlands

Big idea from Langlands unpublished[1] 1973 paper:

$$\widehat{GL(n,\mathbb{Q}_v)} \overset{?}{\leftrightsquigarrow} n\text{-diml reps of } \mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v). \qquad \text{(LLC)}$$

Big idea actually goes back at least to 1967; 1973 paper proves it for $v = \infty$.

Caveat: need to replace Gal by Weil-Deligne group.

Caveat: "Galois" reps in (LLC) not irr.

Caveat: Proof of (LLC) for finite $v$ took another 25 years (finished by Harris[2] and Taylor 2001).

Conclusion: irr rep $\pi$ of $GL(n,\mathbb{A}) \leftrightsquigarrow$ one $n$-diml rep $\sigma(v)$ of $\mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v)$ for each $v$.

[1] Note to Impressionable Youth: "big idea" and "unpublished" go together in the career of R. Langlands. Are you R. Langlands?

[2] Not that one, the other one.

# Background about arithmetic

$\{\mathbb{Q}_2, \mathbb{Q}_3, \ldots, \mathbb{Q}_\infty\}$ loc cpt fields where $\mathbb{Q}$ dense.

If $E/\mathbb{Q}$ Galois, $\Gamma = \mathrm{Gal}(E/\mathbb{Q})$

$$E_v =_{\mathrm{def}} E \otimes_{\mathbb{Q}} \mathbb{Q}_v \quad \curvearrowleft \Gamma$$

is a direct sum of Galois extensions of $\mathbb{Q}_v$.

$\Gamma$ transitive on summands.

Choose one summand $E_\nu \subset E \otimes_{\mathbb{Q}} \mathbb{Q}_v$, define

$$\Gamma_v = \mathrm{Stab}_\Gamma(E_\nu) = \mathrm{Gal}(E_\nu/\mathbb{Q}_v) \subset \Gamma.$$

$\Gamma_v \subset \Gamma$ closed, unique up to conjugacy.

$$E \otimes_{\mathbb{Q}} \mathbb{Q}_v = \sum_{\overline{\sigma} \in \Gamma/\Gamma_v} \sigma \cdot E_\nu$$

Conclusion: $n$-diml $\sigma$ of $\Gamma \rightsquigarrow n$-diml $\sigma(v)$ of $\Gamma_v$.

Čebotarëv: know almost all $\sigma(v) \rightsquigarrow$ know $\sigma$.

# Global Langlands conjecture

Write $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \supset \mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v) = \Gamma_v$.

analysis on $GL(n,\mathbb{A})/GL(n,\mathbb{Q})$

$\leftrightsquigarrow$ irr reps $\pi$ of $\prod'_{v\in\{p,\infty\}} GL(n,\mathbb{Q}_v)$ $\quad \pi^{GL(n,\mathbb{Q})} \ne 0$

$\leftrightsquigarrow \pi = \bigotimes'_{v\in\{p,\infty\}} \pi(v), \qquad \pi^{GL(n,\mathbb{Q})} \ne 0$

$\overset{\text{LLC}}{\leftrightsquigarrow}$ $n$-diml rep $\sigma(v)$ of $\Gamma_v$, each $v$ $\quad$ which $\sigma(v)$??

GLC: $\pi^{GL(n,\mathbb{Q})} \ne 0$ if reps $\sigma(v)$ of $\Gamma_v \rightsquigarrow$ one $n$-diml representation $\sigma$ of $\Gamma$.

If $\Gamma$ finite, most $\Gamma_v = \langle g_v \rangle$ cyclic, all $g_v$ occur.

Arithmetic prob: how does conj class $g_v$ vary with $v$?

# Starting local Langlands for $GL(n, \mathbb{R})$

All that was why it's interesting to understand

$$\widehat{GL(n,\mathbb{R})} \overset{\text{LLC}}{\leftrightsquigarrow} n\text{-diml reps of } \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$
$$\leftrightsquigarrow n\text{-diml reps of } \mathbb{Z}/2\mathbb{Z}$$
$$\leftrightsquigarrow \left\{ n \times n \text{ cplx } y,\ y^2 = \mathrm{Id} \right\} / GL(n,\mathbb{C}) \text{ conj}$$

Langlands: more reps of $GL(n, \mathbb{R})$ (Galois $\rightsquigarrow$ Weil).

But what have we got so far?

$y \rightsquigarrow m, \quad 0 \le m \le n \quad (\dim(-1 \text{ eigenspace}))$

$$\rightsquigarrow \text{unitary char } \xi_m \colon B \to \{\pm 1\}, \quad \xi_m(b) = \prod_{j=1}^{m} \mathrm{sgn}(b_{jj})$$

$$\rightsquigarrow \text{unitary rep } \ \pi(y) = \mathrm{Ind}_B^{GL(n,\mathbb{R})} \xi_m.$$

This is all irr reps of infl char zero.

# Integral infinitesimal characters



Infinitesimal char for $GL(n,\mathbb{R})$ is unordered tuple

$$(\gamma_1,\ldots,\gamma_n), \qquad (\gamma_i \in \mathbb{C}).$$

Assume first $\gamma$ integral: all $\gamma_i \in \mathbb{Z}$. Rewrite

$$\gamma = (\underbrace{\gamma_1,\ldots,\gamma_1}_{m_1 \text{ terms}},\ldots,\underbrace{\gamma_r,\ldots,\gamma_r}_{m_r \text{ terms}}) \qquad (\gamma_1 > \cdots > \gamma_r).$$

A flat of type $\gamma$ consists of

1. flag $\mathcal{V} = (V_0 \subset V_1 \subset \cdots \subset V_r = \mathbb{C}^n)$, $\quad \dim V_i/V_{i-1} = m_i$;
2. and the set of linear maps

$$\mathcal{F} = \{T \in \mathrm{End}(V) \mid TV_i \subset V_i,\ T|_{V_i/V_{i-1}} = \gamma_i \mathrm{Id}\}.$$

Such $T$ are diagonalizable, eigenvalues $\gamma$.

Each of $\mathcal{V}$ and $\mathcal{F}$ determines the other (given $\gamma$).

Langlands param of infl char $\gamma =$ pair $(y,\mathcal{F})$ with $\mathcal{F}$ a flat of type $\gamma$, $y$ $n \times n$ matrix with $y^2 = Id$.

# Integral local Langlands for $GL(n,\mathbb{R})$



$$\gamma = (\underbrace{\gamma_1,\ldots,\gamma_1}_{m_1 \text{ terms}},\ldots,\underbrace{\gamma_r,\ldots,\gamma_r}_{m_r \text{ terms}}) \quad (\gamma_1 > \cdots > \gamma_r) \text{ ints.}$$

Langlands parameter of infl char $\gamma =$ pair $(y,\mathcal{V})$, $y^2 = \mathrm{Id}$, $\mathcal{V} = (V_i)$ flag, $\dim V_i/V_{i-1} = m_i$.

$\pi \in \widehat{GL(n,\mathbb{R})}$, infl char $\gamma \overset{\text{LLC}}{\leftrightsquigarrow} \{(y,\mathcal{V}\}/\text{conj by } GL(n,\mathbb{C})$.

So what are these $GL(n,\mathbb{C})$ orbits?

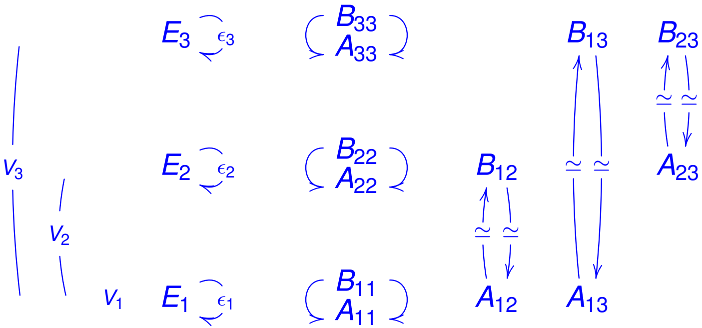
Proposition Suppose $y^2 = \mathrm{Id}_n$ and $\mathcal{V}$ is a flag in $\mathbb{C}^n$. Have subspaces $E_i$, $A_{ij}$, $B_{ij}$ $(i \le j)$, signs $\epsilon_i$ s.t.

1. $y|_{E_i} = \epsilon_i$.
2. $y\colon A_{ij} \underset{\sim}{\overset{\sim}{\rightleftarrows}} B_{ij}$.
3. $V_i = \sum_{i' \le i} E_{i'} + \sum_{i' \le i,j} A_{i',j} + \sum_{j \le i' \le i} B_{j,i'}$
4. $e_i = \dim E_i$, $a_{ij} = \dim A_{ij} = \dim B_{ij}$ depend only on $GL(n,\mathbb{C}) \cdot (y,\mathcal{V})$.

# Action of involution $y$ on a flag

Last $i$ rows represent subspace $V_i$ in flag.

Arrows show action of $y$.




Represent diagram symbolically (Barbasch)

$$\Big(\underbrace{\gamma_1^{\epsilon_1},\ldots}_{\text{dim } E_1 \text{ terms}},\ldots,\underbrace{\gamma_r^{\epsilon_r},\ldots}_{\text{dim } E_r \text{ terms}},\underbrace{(\gamma_1\gamma_1),\ldots}_{\text{dim } A_{11} \text{ terms}},\underbrace{(\gamma_1\gamma_2),\ldots}_{\text{dim } A_{12} \text{ terms}},\ldots,\underbrace{(\gamma_r\gamma_r),\ldots}_{\text{dim } A_{r,r} \text{ terms}}\Big)$$

This is involution in $S_n$ plus some signs.

# General infinitesimal characters



Recall infl char for $GL(n,\mathbb{R})$ is unordered tuple

$$(\gamma_1,\ldots,\gamma_n), \qquad (\gamma_i \in \mathbb{C}).$$

Organize into congruence classes mod $\mathbb{Z}$:

$$\gamma = \Big(\underbrace{\gamma_1,\ldots,\gamma_{n_1}}_{\text{cong mod } \mathbb{Z}}, \underbrace{\gamma_{n_1+1},\ldots,\gamma_{n_1+n_2}}_{\text{cong mod } \mathbb{Z}},\ldots,$$

$$\underbrace{\gamma_{n_1+\cdots+n_{s-1}+1},\ldots,\gamma_n}_{\text{cong mod } \mathbb{Z}}\Big),$$

then in decreasing order in each congruence class:

$$\gamma = (\underbrace{\underbrace{\gamma_1^1,\ldots,\gamma_1^1}_{m_1^1 \text{ terms}},\ldots,\underbrace{\gamma_{r_1}^1,\ldots,\gamma_{r_1}^1}_{m_{r_1}^1 \text{ terms}}}_{n_1 \text{ terms}},\ldots,\underbrace{\underbrace{\gamma_1^s,\ldots,\gamma_1^s}_{m_1^s \text{ terms}},\ldots,\underbrace{\gamma_{r_s}^s,\ldots,\gamma_{r_s}^s}_{m_{r_s}^1 \text{ terms}}}_{n_s \text{ terms}})$$

$$\gamma_1^1 > \gamma_2^1 > \cdots > \gamma_{r_1}^1, \quad \cdots \quad \gamma_1^s > \gamma_2^s > \cdots > \gamma_{r_s}^s.$$

# Nonintegral flats



Start with general infinitesimal character

$$\gamma = (\underbrace{\underbrace{\gamma_1^1, \ldots, \gamma_1^1}_{m_1^1 \text{ terms}}, \ldots, \underbrace{\gamma_{r_1}^1, \ldots, \gamma_{r_1}^1}_{m_{r_1}^1 \text{ terms}}}_{n_1 \text{ terms}}, \ldots, \underbrace{\underbrace{\gamma_1^s, \ldots, \gamma_1^s}_{m_1^s \text{ terms}}, \ldots, \underbrace{\gamma_{r_s}^s, \ldots, \gamma_{r_s}^s}_{m_{r_s}^1 \text{ terms}}}_{n_s \text{ terms}})$$

A flat of type $\gamma$ consists of

1a. direct sum decomp $\mathbb{C}^n = V^1 \oplus \cdots \oplus V^s$, dim $V^k = n_k$;

1b. flags $\mathcal{V}^k = \{V_0^k \subset \cdots \subset V_{r_k}^k = V^k\}$, dim $V_i^k / V_{i-1}^k = m_i^k$;

2. and the set of linear maps
$\mathcal{F}(\{\mathcal{V}^k\}, \gamma) = \{T \in \mathrm{End}(\mathbb{C}^n) \mid TV_i^k \subset V_i^k, T|_{V_i^k/V_{i-1}^k} = \gamma_i^k \mathrm{Id}\}.$

Such $T$ are diagonalizable, eigenvalues $\gamma$.

Each of (1) and (2) determines the other (given $\gamma$).

invertible operator $e(T) =_{\text{def}} \exp(2\pi i T)$ depends only on flat: eigenvalues are $e(\gamma_i^k)$ (ind of $i$), eigenspaces $\{V^k\}$.

Langlands param of infl char $\gamma =$ pair $(y, \mathcal{F})$ with $\mathcal{F}$ a flat of type $\gamma$, $y$ $n \times n$ matrix with $y^2 = e(T)$.

# Langlands parameters for $GL(n, \mathbb{R})$



Infl char $\gamma = (\gamma_1, \ldots, \gamma_n)$ ($\gamma_i \in \mathbb{C}$ unordered).

Recall Langlands parameter $(y, \mathcal{F})$ is

1. direct sum decomp of $\mathbb{C}^n$, indexed by $\{\gamma_i + \mathbb{Z}\}$;
2. flag in each summand
3. $y \in GL(n, \mathbb{C})$, $y^2 = e(\gamma_i)$ on summand for $\gamma_i + \mathbb{Z}$.

Proposition $GL(n, \mathbb{C})$ orbits of Langlands parameters of infl char $\gamma$ are indexed by

1. pair some $(\gamma_i, \gamma_j)$ with $\gamma_i - \gamma_j \in \mathbb{Z} - 0$;
2. label unpaired $\gamma_k$ with sign $\epsilon_k$; and
3. require $\epsilon_j = \epsilon_k$ if $\gamma_j = \gamma_k$.

Example infl char $(3/2, 1/2, -1/2)$:

$$[(3/2, 1/2), (-1/2)^{\pm}], \qquad \text{two params}$$

$$[(3/2, -1/2), (1/2)^{\pm}], \qquad \text{two params}$$

$$[(1/2, -1/2), (3/2)^{\pm}, \qquad \text{two params}$$

$$[(3/2)^{\pm}, (1/2)^{\pm}, (-1/2)^{\pm}] \quad \text{eight params}$$

$[(\gamma_1, \gamma_2)] \leftrightsquigarrow$ disc ser, HC param $\gamma_1 - \gamma_2$ of $GL(2, \mathbb{R})$

$[\gamma^{+ \text{ or } -}] \leftrightsquigarrow$ character $t \mapsto |t|^{\gamma} (\text{sgn } t)^{0 \text{ or } 1}$ of $GL(1, \mathbb{R})$.

# Other reductive groups



$G(\mathbb{R})$ real red alg group, ${}^\vee G$ dual (cplx conn red alg).

Semisimple conj class $\mathcal{H} \subset {}^\vee\mathfrak{g} \leftrightsquigarrow$ infl char. for $G$.

For semisimple $\gamma \in {}^\vee\mathfrak{g}$ and integer $k$, define

$$\mathfrak{g}(k,\gamma) = \{X \in {}^\vee\mathfrak{g} \mid [\gamma, X] = kX\}.$$

Say $\gamma \sim \gamma'$ if $\gamma' \in \gamma + \sum_{k>0} \mathfrak{g}(k,\gamma)$.

Flats in ${}^\vee\mathfrak{g}$ are the equivalence classes (partition each semisimple conjugacy class in ${}^\vee\mathfrak{g}$).

Exponential $e(\gamma) = \exp(2\pi i\gamma) \in {}^\vee G$ const on flats.

If $G(\mathbb{R})$ split, Langlands parameter for $G(\mathbb{R})$ is $(y, \mathcal{F})$ with $\mathcal{F} \subset {}^\vee\mathfrak{g}$ flat, $y \in {}^\vee G$, $y^2 = e(\mathcal{F})$.

Theorem (LLC—Langlands, 1973) Partition $\widehat{G(\mathbb{R})}$ into finite $L$-packets $\leftrightsquigarrow$ ${}^\vee G$ orbits of $(y, \mathcal{F})$.

Infl char of $L$-packet is ${}^\vee G \cdot \mathcal{F}$.

Future ref: $(y, \mathcal{F}) \rightsquigarrow$ involution $w(y, \mathcal{F}) \in W$.

$G(\mathbb{R})$ possibly not split: twisted involution $w(y, \mathcal{F})$.

# and now for something completely different. . .

$G$ cplx conn red alg group.

Problem: real forms of $G/$(equiv)?

Soln (Cartan): $\leftrightsquigarrow \{x \in \mathrm{Aut}(G \mid x^2 = 1\}/$conj.

Details: given aut $x$, choose cpt form $\sigma_0$ of $G$ s.t. $x\sigma_0 = \sigma_0 x =_{\text{def}} \sigma$.

Example.

$G = GL(n, \mathbb{C})$, $x_{p,q}(g) =$ conj by $\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$.

Choose $\sigma_0(g) = {}^t\overline{g}^{-1}$ (real form $U(n)$).

$$\sigma_{p,q}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} {}^t\overline{A} & -{}^t\overline{C} \\ -{}^t\overline{B} & {}^t\overline{D} \end{pmatrix}^{-1},$$

real form $U(p,q)$.

Another case of matrices almost of order two.

# Cartan involutions



$G$ cplx conn red alg group.

cartan parameter is $x \in G$ s.t. $x^2 \in Z(G)$.

$\theta_x = \mathrm{Ad}(x) \in \mathrm{Aut}(G)$ Cartan involution.

Say $x$ has central cochar $z = x^2$.

$G = SL(n, \mathbb{C})$, $x_{p,q} = \begin{pmatrix} e(-q/2n)I_p & 0 \\ 0 & e(p/2n)I_q \end{pmatrix}$.

$x_{p,q} \leftrightsquigarrow$ real form $SU(p,q)$, central cocharacter $e(p/n)I_n$.

Theorem (Cartan) Surjection $\{$cartan params$\} \rightsquigarrow$ $\{$equal rk real forms of $G(\mathbb{C})\}$.

$G = SO(n, \mathbb{C})$, $x_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ allowed iff $q = 2m$ even.

$x_{n-2m,m} \leftrightsquigarrow$ real form $SO(n-2m, 2m)$, central cochar $I_n$.

$G = SO(2n, \mathbb{C})$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $x_J = n$ copies of $J$ on diagonal.

$x_J \leftrightsquigarrow$ real form $SO^*(2n)$, central cochar $-I_{2n}$.

# Imitating Langlands

Since cartan param $\leftrightsquigarrow$ part of Langlands param, why not complete to a whole "Langlands param"?

Start with $z \in Z(G)$

Choose reg ss class $\mathcal{G} \subset \mathfrak{g}$ so $e(g) = z$ ($g \in \mathcal{G}$).

Define Cartan parameter of infl cochar $\mathcal{G}$ as pair $(x, \mathcal{E})$, with $\mathcal{E} \subset \mathcal{G}$ flat, $x \in G(\mathbb{C})$, $x^2 = e(\mathcal{E})$.

Equivalently: pair $(x, \mathfrak{b})$ with $\mathfrak{b} \subset \mathfrak{g}$ Borel.

As we saw for Langlands parameters for $GL(n)$,

$$\text{Cartan param } (x, \mathcal{E}) \rightsquigarrow \text{involution } w(x, \mathcal{E}) \in W;$$

const on $G \cdot (x, \mathcal{E})$; $w(x, \mathfrak{b}) =$ rel pos of $\mathfrak{b}, x \cdot \mathfrak{b}$.

Langlands params $\leftrightsquigarrow$ repns.

Cartan params $\leftrightsquigarrow$ ???

# What Cartan parameters count

Fix reg ss class $\mathcal{G} \subset \mathfrak{g}$ so $e(g) \in Z(G)$ ($g \in \mathcal{G}$).

Define Cartan parameter of infl cochar $\mathcal{G} = (x, \mathcal{E})$, with $\mathcal{E} \subset \mathcal{G}$ flat, $x \in G$, $x^2 = e(\mathcal{E})$.

Theorem Cartan parameter $(x, \mathcal{E}) \leftrightsquigarrow$

1. real form $G(\mathbb{R})$ (with Cartan inv $\theta_x = \mathrm{Ad}(x)$;
2. $\theta_x$-stable Cartan $T(\mathbb{R}) \subset G(\mathbb{R})$;
3. Borel subalgebra $\mathfrak{b} \supset \mathfrak{t}$.

That is: $\{(x, \mathcal{E})\}/(G\,\text{conj})$ in 1-1 corr with $\{(G(\mathbb{R}), T(\mathbb{R}), \mathfrak{b})\}/(G\,\text{conj})$.

Involution $w = w(x, \mathcal{E}) \in W \leftrightsquigarrow$ action of $\theta_x$ on $T(\mathbb{R})$.

Conj class of $w \in W \leftrightsquigarrow$ conj class of $T(\mathbb{R}) \subset G(\mathbb{R})$.

How many Cartan params over involution $w \in W$?

Same question: # Langlands params over $w \in W$?

Answer uses structure theory for reductive gps...

# Counting Cartan params



Max torus $T \subset G \rightsquigarrow$

cowgt lattice $X_*(T) =_{\text{def}} \mathrm{Hom}(\mathbb{C}^\times, T)$.

Weyl group $W \simeq N_G(T)/T \subset \mathrm{Aut}(X_*)$.

Each $w \in W$ has Tits representative $\sigma_w \in N(T)$.

Lie algebra $\mathfrak{t} \simeq X_* \otimes_{\mathbb{Z}} \mathbb{C}$, so $W$ acts on $\mathfrak{t}$.

$\mathfrak{g}_{ss}/G \simeq \mathfrak{t}/W$; $\mathcal{G}$ has unique dom rep $g \in \mathfrak{t}$.

Theorem Fix dom rep $g$ for $\mathfrak{G}$, involution $w \in W$.

1. Each $G$ orbit of Cartan params over $w$ has rep $e((g-\ell)/2)\sigma_w$, $\ell \in X_*$ s.t. $(w-1)(g-\rho^\vee-\ell) = 0$.
2. Two such reps are $G$-conj iff $\ell' - \ell \in (w+1)X_*$.
3. set of orbits over $w$ is

$$\begin{cases} \text{princ homog/ } X_*^w/(w+1)X_* & (w-1)(g-\rho^\vee) \in (w-1)X_* \\ \text{empty} & (w-1)(g-\rho^\vee) \notin (w-1)X_* \end{cases}$$

If $g \in X_* + \rho^\vee$, get canonically

Cartan params of infl cochar $\mathcal{G} \simeq X_*^w/(w+1)X_*$.

# Integer matrices of order 2



$X_*$ lattice ($\mathbb{Z}^n$), $w \in \mathrm{Aut}(X_*)$, $w^2 = 1$. 3 examples...

$$X_* = \mathbb{Z}, \quad w_+ = (1),$$
$$X_* = \mathbb{Z}, \quad w_- = (-1),$$
$$X_* = \mathbb{Z}^2, \quad w_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note: $-w_s$ differs from $w_s$ by chg of basis $e_1 \mapsto -e_1$.

**Theorem** Any $w \simeq_{\mathbb{Z}}$ sum of copies of $w_+$, $w_-$, $w_s$.

$$X_*^w/(1+w)X_* = \begin{cases} \mathbb{Z}/2\mathbb{Z} & w = w_+ \\ 0 & w = w_- \\ 0 & w = w_s \end{cases}.$$

**Corollary** If $w = (w_+)^p \oplus (w_-)^q \oplus (w_s)^r$, then

$$\mathrm{rk}\, X_*^w = p + r$$
$$\mathrm{rk}\, X_*^{-w} = q + r$$
$$\dim_{\mathbb{F}_2} X_*^w/(1+w)X_* = p.$$

So $p$, $q$, and $r$ determined by $w$; decomp of $X_*$ is not.

# Putting it all together

So suppose $G$ cplx reductive alg, ${}^\vee G$ dual.

Fix infl char (semisimple ${}^\vee G$ orbit) $\mathcal{H} \subset {}^\vee\mathfrak{g}$, infl cochar (reg integral ss $G$ orbit) $\mathcal{G} \subset \mathfrak{g}$.

Definition. Cartan param $(x, \mathcal{E})$ and Langlands param $(y, \mathcal{F})$ said to match if $w(x, \mathcal{E}) = -w(y, \mathcal{F})$

Example of matching:
$w(y, \mathcal{F}) = 1 \iff$ rep is principal series for split $G$;
$w(x, \mathcal{E}) = -1 \iff T(\mathbb{R})$ is split Cartan subgroup.

Theorem. Irr reps (of infl char $\mathcal{H}$) for real forms (of infl cochar $\mathcal{G}$) are in 1-1 corr with matching pairs $[(x, \mathcal{E}), (y, \mathcal{F})]$ of Cartan and Langlands params.

Corollary. L-packet for Langlands param $(y, \mathcal{F})$ is (empty or) princ homog space for $X_*^{-w}/(1-w)X_*$, $w = w(y, \mathcal{F})$.

# What did I leave out?

Included cool slides called **Background about arithmetic** and **Global Langlands conjecture** discussed assembling local reps to make global rep, and when the global rep should be automorphic.

Omitted two slides called **Background about rational forms** and **Theorem of Kneser *et al.***, about ratl forms of each $G/\mathbb{Q}_v \rightsquigarrow$ ratl form $G/\mathbb{Q}$.

Omitted interesting extensions of local results over $\mathbb{R}$ to study of unitary reps.