

# Inflatable mathematics

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Gaussian  
elimination

Bruhat order

Schubert varieties

Calculating  
with(out) Schubert  
varieties

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# Outline

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mathematics

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Building up from simple pieces

Ideas from linear algebra

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# The main idea

Begin with **linear algebra**: solving systems of linear equations by Gaussian elimination.

Idea: **reduce number of coordinates by one.**

Relate to **geometry**: arranging lines and planes.

Idea: **reduce to geometry of one dimension less.**



Use same idea for more complicated geometry.

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# Gaussian elimination: easy cases

System of three equations in three unknowns is

$$\begin{array}{rccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & c_1 \\ a_{21}'x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & c_2' \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & c_3 \end{array}$$

I'll assume always the system has just one solution.

Easiest case is diagonal system: **divide each equation by a constant** to solve.

Next easiest is lower triangular: **add multiples of some eqns to later ones** to make diagonal.

Suppose lower triangular EXCEPT one coefficient  $a_{12} \neq 0$ . **Add multiple of 1st eqn to second** to get. . .

This system is nearly lower triangular, except that the first two equations are interchanged.

# Gaussian elimination: typical case

“Typical” system of equations in three unknowns is

$$\begin{array}{rclclcl} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & c_1 \\ a_{21}'x_1 & + & a_{22}'x_2 & + & a_{23}x_3 & = & c_2' \\ a_{31}''x_1 & + & a_{32}'x_2 & + & a_{33}x_3 & = & c_3'' \end{array}$$

where “typically”  $a_{13} \neq 0$ . **Add multiple of 1st equation to each later eqn to get...**

Now “typically”  $a_{22}' \neq 0$ . **Add multiple of 2nd eqn to last to get...**

Again this last system is nearly lower triangular, except that order of the three eqns is reversed.

To say what happens in general, use matrix notation  **$A\mathbf{x} = \mathbf{c}$** . Here  $A = (a_{ij})$  is  $n \times n$  coeff matrix, and  $\mathbf{x} = (x_j)$  is the column vector of  $n$  unknowns.

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# Theorem for Gaussian elimination

## Theorem

*Suppose  $A$  is an invertible  $n \times n$  matrix, and  $\mathbf{c}$  is an  $n$ -tuple of constants. Consider the system of  $n$  equations in  $n$  unknowns*

$$\mathbf{Ax} = \mathbf{c}.$$

*Using the two operations*

- 1. dividing an equation by a non-zero constant, and*
- 2. adding a multiple of one equation to a later one,*

*we can transform this system into a new one*

$$\mathbf{A}'\mathbf{x} = \mathbf{c}'.$$

*The new system, after reordering the equations, is lower triangular.*

# Possibilities for three unknowns

$$\begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (321)$$

$$a_{13} \neq 0, \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \neq 0$$

$$\begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (312)$$

$$a_{13} = 0, a_{12} \neq 0, a_{23} \neq 0$$

$$\begin{pmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (213)$$

$$a_{13} = a_{23} = 0, a_{12} \neq 0$$

$$\begin{pmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (231)$$

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = 0, a_{13} \neq 0$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (132)$$

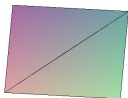
$$a_{12} = a_{13} = 0, a_{23} \neq 0$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (123)$$

$$a_{12} = a_{23} = a_{13} = 0$$

# From algebra to geometry

A **flag** in 3 dimensions is a (straight) **line** through the origin, contained inside a **plane** through the origin:



One flag not so interesting. What's interesting is how many different flags there are, and how they're related.

System of equations =  $3 \times 3$  matrix  $\rightsquigarrow$  **flag**:

**line** = multiples of first row, **plane** = span of first two rows.

*Two matrices give same flag if and only if differ by*

- *multiply row by constant*
- *add multiple of one row to later row.*



# Possible flags $L \subset P$

these flags  
are **generic**

$$\begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (321)$$

$$P_{xy} \not\supset L \subset P \not\supset L_x$$



$$\begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (312)$$

these flags  
**restrict L**

$$P_{xy} \supset L \subset P \not\supset L_x$$



these flags  
**restrict P**

$$\begin{pmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (231)$$

$$P_{xy} \not\supset L \subset P \supset L_x$$



$$\begin{pmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (213)$$

these **restrict P more**

$$L_x \neq L \subset P = P_{xy}$$



these **restrict L more**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (132)$$

$$L = L_x \subset P \neq P_{xy}$$



this is the  
**base flag**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (123)$$

$$L_x \subset P_{xy}$$

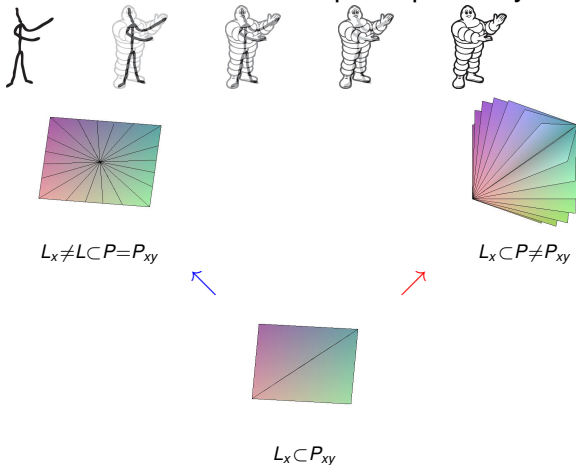
# Geometric picture

Moving up  $\rightsquigarrow$  more complicated geometry.

up one **blue step**: fixed line  $\rightsquigarrow$  variable line in a plane.

up one **red step**: fixed plane  $\supset L \rightsquigarrow$  variable plane  $\supset L$ .

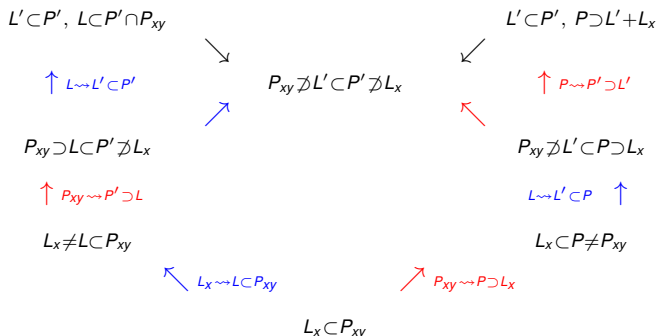
**Inflatable mathematics**: replace points by circles.



# What's a Schubert variety?

Divided flags (in three dimensions) into six “Bruhat cells” by relation with standard flag  $L_x \subset P_{xy}$ .

**Schubert variety** is one cell and everything below it:

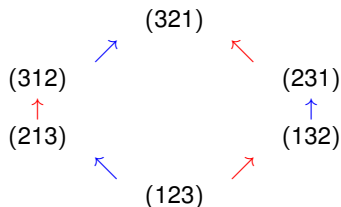


What's almost true: **each Schubert variety “inflated” from a smaller one, replacing each point by a circle.**

Fails only at the top. . .

# Mathematics on a need-to-know basis

To compute with Schubert varieties, need only **arrangement of blue and red arrows**, describing how small Schubert varieties are inflated:



Permutations recorded which rows had pivots in Gaussian elimination. Now they're just symbols.

Rules for making diagram:

1. One entry for each permutation of  $\{1, 2, 3\}$ .
2. Exchange 1 ... 2: blue arrow up.
3. Exchange 2 ... 3: red arrow up.

# As many dimensions as you want

Rules in  $n$  dimensions:

1. One entry for each permutation of  $\{1, 2, \dots, n\}$ .
2. Exchange  $i \dots i + 1$ :  
arrow up of color  $i$ .

Counting problems in this picture  $\leftrightarrow$  geometry of Schubert varieties.

There are **lots** of counting games to play...

height of a permutation =  $\#\{ \text{pairs } (i, j) \text{ out of order} \}$ .

$\#\{\text{permutations at height } d\}$  = coefficient of  $x^d$  in polynomial

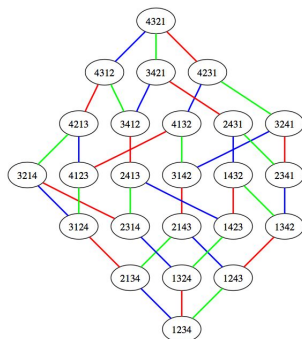
$$(1)(1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}).$$

$\#\{\text{ascending paths bottom to top}\} =$

$$\binom{n}{2}! / 1^{n-1} 3^{n-2} 5^{n-2} \dots (2n-5)^2 (2n-3)$$

**Stanley's formula**

(Formula says 16 ascending paths bottom to top in this picture.)



# More complicated groups

Picture just described (with  $n!$  vertices) is for invertible  $n \times n$  matrices. This is the basic example of a **real reductive Lie group**. Mathematicians and physicists look at lots of other reductive groups.

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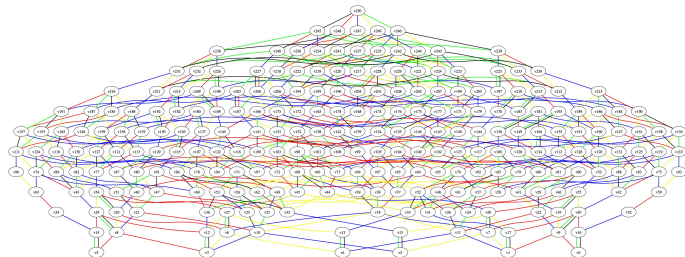
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Each reductive group has a finite diagram describing how its big Schubert varieties are “inflated” from smaller ones. This one is for a 45-dimensional group called  $SO(5, 5)$ .

For this group there are 251 Schubert varieties, but each arrow still means **replace points by circles**.

# What do you do with the pretty pictures?

## Where we started:

systems of  $n$  linear eqns  $\overset{\text{Gauss elim}}{\longleftrightarrow}$  group  $GL(n) \longleftrightarrow$  Schubert varieties  $\longleftrightarrow$  graph with  $n!$  vertices, arrows of  $n - 1$  colors.

Graph tells what cases can happen during Gaussian elimination; how Gaussian elimination changes with the system of equations; even which cases are most common.

## Similarly:

math or physics problem  $\overset{\text{repn theory}}{\longleftrightarrow}$  reductive group  $G \longleftrightarrow$  Schubert varieties for  $G \longleftrightarrow$  finite graph for inflating.

1979: David Kazhdan (Harvard) and George Lusztig (MIT) showed how to answer questions about representation theory by calculating in the finite graph.

Defined **Kazhdan-Lusztig polynomial**  $P_{x,y}$  for  $x$  and  $y$  in the graph. Polynomial in  $q$ , non-neg integer coeffs.

Polynomial is non-zero only if  $y$  is above  $x$  in graph. Calculated by a recursion based on knowing all  $P_{x',y'}$  for  $y'$  smaller than  $y$ .

# How the computation works

Now fixing a reductive group  $G$  and its graph of Schubert varieties.

- ▶ For each pair  $(x, y)$  of graph vertices, want to compute KL polynomial  $P_{x,y}$ .
- ▶ Induction: start with  $y$ 's on bottom of graph, work up. For each  $y$ , start with  $x = y$ , work down.

$x'$

|

- ▶ Seek line **up**  $x$  same color as some line **down**  $y$ .

|

$y'$

If it's there, then  $P_{x,y} = P_{x',y}$  (known by induction).

If not,  $(x, y)$  is **primitive**: no color down from  $y$  goes up from  $x$ .

- ▶ One hard calculation for each primitive pair  $(x, y)$ .



# What to do for primitive pair $(x, y)$

- ▶ graph vertex  $y \leftrightarrow$  big Schubert variety  $F_y$ .
  - ▶ lower vertex  $x \leftrightarrow$  little Schubert variety  $F_x$ .
- $P_{x,y}$  describes how  $F_y$  looks near  $F_x$ .
- ▶ Pick line **down**  $y$ ; means  $F_y \approx$  inflated from  $F_{y'}$ .

|  
 $y'$

- ▶ **Primitive** means red line  $x$  is also **down** from  $x$ .

|  
 $x'$

- ▶ Geometry translates to algebra  $P_{x,y} \approx P_{x',y'} + qP_{x,y'}$ . Precisely:

$$P_{x,y} = P_{x',y'} + qP_{x,y'} - \sum_{x' \leq z < y'} \mu(z, y') q^{(l(y') - l(z) - 1)/2} P_{x',z}.$$

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# Forming the *Atlas* group

Between 1980 and 2000, increasingly sophisticated computer programs calculated special kinds of Kazhdan-Lusztig polynomials; none dealt with the complications attached to general real reductive groups.

In 2001, **Jeff Adams** at University of Maryland decided computers and mathematics had advanced far enough to begin interesting work in that direction.

Adams formed a research group *Atlas of Lie groups and representations*, aimed in part at producing software to make old mathematics widely accessible, and to find new mathematics.

A first goal was to calculate Kazhdan-Lusztig polynomials for real reductive groups.

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# How do you make a computer do that?

- ▶ In June 2002, **Jeff Adams** asked **Fokko du Cloux**.
- ▶ In November 2005, Fokko finished the program.

Wasn't that easy?



- ▶ In 2006, Jeff Adams proposed to calculate KL polynomials for the 248-dimensional *exceptional Lie group*  $E_8$ .

# What's the computer have to do?

TASK	COMPUTER RQMT
Make graph: 453,060 nodes, 8 edges at each	<b>250M RAM, 10 minutes</b> (latest software: thirty seconds)
List primitive pairs of vertices: 6,083,626,944	<b>450M RAM, few seconds</b>
Calculate polynomial for each primitive pair	Fetch few kB from memory, few thousand integer ops $\times$ <b>6 billion</b>
Look for polynomial in store, add if it's new	$\frac{4 \text{ bytes}}{\text{coef}} \times \frac{20 \text{ coefs}}{\text{poly}} \times \text{?? polys}$ <b>RAM</b>
Write number for poly in table	<b>25G RAM</b>

Big unknown: number of distinct polynomials.

Hoped **400 million polys**  $\rightsquigarrow$  **75G total RAM.**

Feared **1 billion**  $\rightsquigarrow$  **150G total RAM.**

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# Saga of the end times

11/06 Experiments by **Birne Binengar** on **William Stein's** computer **sage** showed we needed 150G.

11/28/06 Asked about pure math uses for 256G computer.

11/30/06 **Noam Elkies** told us we didn't need one. . .

one 150G computation  $\xrightarrow{\text{(modular arithmetic)}}$  four 50G computations

12/03/06 **Marc van Leeuwen** made Fokko's code modular.

12/19/06 mod 251 computation on **sage**. Took 17 hours:

```
Total elapsed time = 62575s. Finished at l = 64, y = 453059  
d_store.size() = 1181642979, prim_size = 3393819659
```

```
VmData: 64435824 kB
```

Writing to disk took two days. Investigating why  $\rightsquigarrow$  output bug, so mod 251 answers no good.

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# The Tribulation (continued)

- 12/21/06 9 P.M. Started mod 256 computation on **sage**.  
Computed 452,174 out of 453,060 rows of KL  
polynomials in 14 hours, then **sage** crashed.
- 12/22/06 EVENING Restarted **mod 256**. **Finished** in just 11 hours

```
( hip, hip, HURRAH!  
hip, hip, HURRAH!      pthread_join(cheer[k], NULL) );
```

```
Total elapsed time = 40229s.  Finished at l = 64, y = 453059  
d_store.size() = 1181642979, prim_size = 3393819659
```

```
VmData:  54995416 kB
```

- 12/23/06 Started mod 255 computation on **sage**, which  
crashed.

**sage** down til 12/26/06  
(regional holiday in Seattle).



# So we've got mod 256...

12/26/06 `sage` rebooted. Wrote KL polynomials mod 255.

12/27/06 Started computation mod 253. Halfway, `sage` crashed.

consult experts  $\rightsquigarrow$  probably not Sasquatch.

Did I mention `sage` is in Seattle?

Decided not to abuse `sage` further for a year.

1/3/07 Atlas members one year older  $\rightsquigarrow$  thirty years wiser  
as team  $\rightsquigarrow$  safe to go back to work.

Wrote KL polynomials mod 253 (12 hrs).

Now we had answers mod 253, 255, 256.

Chinese Remainder Theorem (CRT)

gives answer mod  $253 \cdot 255 \cdot 256 = 16,515,840$ .

One little computation for each of 13 billion coefficients.

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# The Chinese Remainder

- 1/4/07 **Marc van Leeuwen** started his CRT software.  
On-screen counter displayed polynomial number:  
0,1,2,3,...,1181642978. Turns out to be a bad idea.
- 1/5/07 MORNING Restarted CRT computation, with counter  
0,4096,8192,12288,16536,...,1181642752,1181642978.  
Worked fine until **sage** crashed.  
**William Stein** (our hero!) replaced hard drive with one  
with backups of our 100G of files mod 253, 255, 256.
- 1/5/07 AFTERNOON Re-restarted CRT computation.
- 1/6/07 7 A.M. Output file 7G too big: **BUG** in output routine.
- 1/7/07 2 A.M. Marc found output bug. Occurred only after  
polynomial 858,993,459; had tested to 100 million.
- 1/7/07 6 A.M. Re-re-restarted CRT computation.

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# In Which we Come to an Enchanted Place. . .

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1/8/07 9 A.M. Finished writing to disk the KL polynomials for  $E_8$ .

## So what was the point?

In the fall of 2004, Fokko du Cloux was at MIT, rooming with fellow Atlas member Dan Ciubotaru. Fokko was halfway through writing the software I've talked about: the point at which neither the end of the tunnel nor the beginning is visible any longer.

Walking home after a weekend of mathematics, Dan said,  
"Fokko, look at us. We're spending Sunday alone at work."

Fokko was startled by this remark, but not at a loss for words.  
"I don't know about you, but I'm having the time of my life!"

Fokko du Cloux  
December 20, 1954–November 10, 2006

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