Generalizing endoscopic transfer

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Outline

Slides at
http://www-math.mit.edu/~dav/paper.html/

Introduction

Joint work with Jeffrey Adams and Lucas Mason-Brown generalizing endoscopic transfer for reductive groups.

Our results concern real reductive groups.

Subject is a morass of technical difficulties, many of which are much worse for $\mathbb{R}$ than for p-adic fields.

Example: need to change def of Langlands parameter/$\mathbb{R}$.

I'll avoid some difficulties by discussing mostly non-archimedean local field k, and connected reductive algebraic G/k.

Avoid remaining difficulties by ignoring them.

What's the plan?

Study rep theory of reductive algebraic G.

Typically G defined over a local field k, but details later.

Endoscopic group: smaller reductive H, often $H \not\subset G$.

Examples:

$$G = Sp(2(p+q), \mathbb{R}), \quad H = SO(p,p) \times Sp(2q, \mathbb{R})$$
$$G = Sp(2(p+r), \mathbb{R}), \quad H = GL(p, \mathbb{R}) \times Sp(2r, \mathbb{R}).$$

Endoscopic transfer: (virtual H-reps) $\rightarrow$ (virt G-reps).

Will define slightly larger class of such $H \not\subset G$.

New examples:

$$G = Sp(2(p+q+r), \mathbb{R}) \quad H = U(p,q) \times Sp(2r, \mathbb{R})$$
$$G = GL(2p+q, \mathbb{R}) \quad H = GL(p, \mathbb{C}) \times GL(q, \mathbb{R})$$

What's this have to do with Harish-Chandra?

Harish-Chandra's work on discrete series was rooted in what Hermann Weyl did for compact groups:

(Weyl integration) + (Schur orthog) $\rightsquigarrow$ (Weyl char formula).

Harish-Chandra's work was the same, except that every step required radically new ideas.

One such idea was his method of descent. If $s \in G$ semisimple, then $H = G^s$ is again reductive.

Harish-Chandra descent describes any character Θ_G near s in terms of a new character Θ_H on H.

In formal language, he defined a linear map descent

$$(K(G) = \text{virtual reps of } G) \longrightarrow (K(H) = \text{virtual reps of } H.)$$

Endoscopic transfer is Harish-Chandra descent applied in the Langlands L-group.

How do you name a group? (case of $\overline{k}$)

To ask about a group G, you need first to give it a name.

Lie, Chevalley and Grothendieck solved this problem:

(reductive algebraic group G) / algebraically closed $\overline{k}$ $\leftrightsquigarrow$

based root datum $\quad \mathcal{R}(G) = (X^*, \Pi, X_*, \Pi^\vee)$.

X^* and X_* are dual lattices: chars/ cochars of max torus in G.

finite sets $\Pi \subset X^*$ and $\Pi^\vee \subset X_*$: simple roots/simple coroots.

Any lattice is isomorphic to $\mathbb{Z}^n$, so the name $\mathcal{R}(G)$ of G is two finite collections of n-tuples of integers.

Two names are the same iff first collections differ by invertible integer matrix M, and second collections differ by ${}^tM^{-1}$.

Example: $GL(2)$ is given by $\Pi = \{(1,-1)\}, \quad \Pi^\vee = \{(1,-1)\}$.

Example: the exceptional group G_2 is given by

$$\Pi = \{(1,0),(0,1)\}, \qquad \Pi^\vee = \{(2,-1),(-3,2)\}.$$

How do you name a group? (case of k)

A reductive $G/\overline{k}$ named by the (combinatorial) based root datum $\mathcal{R}(G)$: two finite sets of n-tuples of integers.

Defining G/k gives action of $\Gamma = \mathrm{Gal}(\overline{k}/k)$ on $\mathcal{R}(G)$.

Concretely: repn of Γ by $n \times n$ integer matrices $\mu(\sigma)$ so

$$\mu(\sigma) \cdot \Pi = \Pi, \qquad {}^t\mu(\sigma)^{-1} \cdot \Pi^\vee = \Pi^\vee,$$

respecting axioms for a based root datum.

Shorthand: action of Γ on the Dynkin diagram of G.

k-forms of G are inner if $\rightsquigarrow$ same action of Γ on $\mathcal{R}(G)$.

Example A rank two unitary group/k starts with a separable quadratic extension of k; that is, subgroup $\Gamma_0 \subset \Gamma$ of index two.

Representation of Γ on $\mathbb{Z}^2$ is

$$M(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (\sigma \in \Gamma_0), \quad M(\sigma) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ (\sigma \notin \Gamma_0)$$

All unitary grps w fixed quad ext form a single inner class.

L-group

Defining G / any $k \rightsquigarrow$ action of $\Gamma = \mathrm{Gal}(\overline{k}/k)$ on $\mathcal{R}(G)$.

Axioms for based root data are symmetric in $(X^*, \Pi) \leftrightarrow (X_*, \Pi^\vee)$.

Dual based root datum is $\mathcal{R}^\vee = (X_*, \Pi^\vee, X^*, \Pi)$.

Gives reductive algebraic dual group ${}^\vee G$ and

$$\text{L-group} \quad {}^L G = {}^\vee G \rtimes \Gamma, \qquad \text{(defined over } \mathbb{Z}\text{)}.$$

Langlands' insight (local Langlands conjecture):

$$\text{(analytic rep theory}/K \text{ of } G(k)) \leftrightsquigarrow \text{(alg geom of } {}^L G(K)).$$

Typically $K = \mathbb{C}$ and k is local.

Complex reps of $G(k) \leftrightsquigarrow$ complex alg geom of ${}^L G(\mathbb{C})$

Endoscopic (and generalized endoscopic) groups H correspond to subgroups ${}^E H \subset {}^L G$.

By local Langlands, relating $\widehat{H(k)}$ to $\widehat{G(k)}$ means relating alg geom of ${}^L G(\mathbb{C})$ to alg geom of subgroup ${}^E H(\mathbb{C})$.

Easy! But what the hell is $\leftrightsquigarrow$?

Weil-Deligne group

k p-adic field, $\Gamma = \mathrm{Gal}(\overline{k}/k)$, G conn reductive alg$/k$.

${}^L G$ complex L-group: $1 \to {}^\vee G \to {}^L G \to \Gamma \to 1$.

Local Langlands explains irr reps $\widehat{G(k)}$ using ${}^L G$.

Recall: finite residue field $\mathbb{F}_q$ of k $\rightsquigarrow$ natural surjection

$$1 \to I_k \to \Gamma \to \widehat{\mathbb{Z}} = \overline{\langle \mathrm{Frob} \rangle} \to 1.$$

Inertia subgroup I_k is profinite compact.

Weil group W_k = (dense) preimage in Γ of $\langle \mathrm{Frob} \rangle$:

$$1 \to I_k \to W_k \to \mathbb{Z} = \langle \mathrm{Frob} \rangle \to 1.$$

Weil-Deligne group $W_k' = W_k \ltimes \mathbb{C}$: here I_k acts trivially on $\mathbb{C}$, and Frob acts by multiplication by q.

Langlands parameters

Recall $\Gamma = \mathrm{Gal}(\overline{k}/k)$, and $W_k \subset \Gamma$ is a dense subgroup.

Have two short exact sequences

$$\begin{array}{ccccccccc} 1 & \rightarrow & {}^\vee G & \rightarrow & {}^L G & \rightarrow & \Gamma & \rightarrow & 1 \\ & & \uparrow & & \uparrow \phi' & & \uparrow & & \\ 1 & \rightarrow & \mathbb{C} & \rightarrow & W_k' & \rightarrow & W_k & \rightarrow & 1 \end{array}$$

Langlands parameter is a group homomorphism $\phi' \colon W_k' \rightarrow {}^L G$ compatible with exact sequences.

Means $\phi'|_{\mathbb{C}} \colon \mathbb{C} \rightarrow {}^\vee G$ (one-param nilp alg subgp), and ϕ' descends to inclusion $W_k \hookrightarrow \Gamma$.

Loc Langlands conj: $\phi' \rightsquigarrow$ finite L-pkt $\Pi(\phi') \subset \widehat{G(k)}$.

More conjecture:

1. L-packets partition $\widehat{G(k)}$;
2. $\Pi(\phi')$ depends only ${}^\vee G$-conj class of ϕ';
3. if $G(k)$ quasisplit, then $\Pi(\phi') \neq \emptyset$.

Repn theory and algebraic geometry

Want to translate problems about reps of $G(k)$ to alg geom problems about parameters in ${}^LG$.

Infl char of ϕ' is $\phi = \phi'|_{W_k}$. Each infl char $\phi\colon W_k \to {}^LG$ extends in finitely many ways to $\phi'\colon W_k' \to {}^LG$: the parameters of infl char ϕ.

Since I_k compact, ${}^\vee G^{\phi(I_k)}$ = centralizer in ${}^\vee G$ of $\phi(I_k)$ is reductive algebraic in ${}^\vee G$.

Preimage $\widetilde{\text{Frob}}$ in W_k defines $\phi(\widetilde{\text{Frob}}) \in {}^LG$, so semisimple alg aut (indep of $\widetilde{\text{Frob}}$) $\sigma_\phi = \text{Ad}(\phi(\widetilde{\text{Frob}})) \in \text{Aut}({}^\vee G^{\phi(I_k)})$.

σ_ϕ defines ${}^\vee G^\phi = ({}^\vee G^{\phi(I_k)})^{\sigma_\phi}$, twisted pseudolevi of ${}^\vee G^{\phi(I_k)}$.

$\mathfrak{n}(\phi) =_{\text{def}}$ q-eigenspace of σ_ϕ on ${}^\vee\mathfrak{g}^{\phi(I_k)}$, a vector space of nilpotent Lie algebra elements on which ${}^\vee G^\phi$ acts.

The algebraic geom we want is ${}^\vee G^\phi$ orbits on $\mathfrak{n}(\phi)$.

$\mathfrak{n}(\phi)$ is prehomogeneous for ${}^\vee G^\phi$: finitely many orbits.

Could you repeat that?

Start with a Langlands parameter $\phi' \colon W_k' \to {}^L G$.

Restriction ϕ_I of ϕ' to inertia $I_k \subset \mathrm{Gal}(\overline{k}/k)$ is arithmetic; image is profinite (compact) subgroup of ${}^L G$:

$$Z_{{}^\vee G}(\phi(I_k)) = {}^\vee G^{\phi(I_k)} \quad \text{reductive algebraic.}$$

An extension ϕ of ϕ_I to W_k (called infinitesimal character) is given by a single element $\phi(\widetilde{\mathrm{Frob}})$ of ${}^L G$.

$\phi(\widetilde{\mathrm{Frob}})$ defines aut σ_ϕ of ${}^\vee G^{\phi(I_k)}$, fixed points ${}^\vee G^\phi$.

q-eigspace of $d\sigma_\phi$ = nilp subspace $\mathfrak{n}(\phi) \subset \mathfrak{g}^{\phi(I_k)}$.

$\mathfrak{n}(\phi)$ is prehomogeneous for ${}^\vee G^\phi$.

Parameters ϕ' of infl char $\phi \leftrightsquigarrow {}^\vee G^\phi$ orbits $\mathcal{O}'$ on $\mathfrak{n}(\phi)$.

irreps of infl char $\phi \overset{\text{LLC}}{\leftrightsquigarrow} {}^\vee G^\phi$-eqvt perv sheaves on $\mathfrak{n}(\phi)$.

L-packet of $\phi' \overset{\text{LLC}}{\leftrightsquigarrow}$ sheaves with support $\mathcal{O}'$.

What's the plan?

L-group has short exact seq $1 \to {}^\vee G \to {}^L G \to \Gamma \to 1$.

L-subgroup is ${}^L G \supset {}^E H \twoheadrightarrow \Gamma$, kernel ${}^\vee H$ reductive:

$$\begin{array}{ccccccccc} 1 & \to & {}^\vee G & \to & {}^L G & \to & \Gamma & \to & 1 \\ & & \cup & & \cup & & \| & & \\ 1 & \to & {}^\vee H & \to & {}^E H & \to & \Gamma & \to & 1 \end{array}$$

In this setting param ϕ'_H for ${}^E H \rightsquigarrow$ param ϕ for ${}^L G$;

$$\mathfrak{n}(\phi_H) \subset \mathfrak{n}(\phi), \qquad {}^\vee H^\phi \subset {}^\vee G^\phi.$$

This is the geometric part of local Langlands functoriality.

So relating reps of G to reps of H amounts to relating perv sheaves on $\mathfrak{n}(\phi)$ to perv sheaves on $\mathfrak{n}(\phi_H)$.

To get strong theorems relating perverse sheaves to a subvariety, need strong hypotheses on the subvariety.

Example is Goresky-MacPherson Lefschetz formula.

Need subvariety = fixed points of an automorphism.

What's an endoscopic group?

Langlands params are ${}^\vee G$ orbits on (algebraic variety).

So action of $s \in {}^\vee G \rightsquigarrow$ automorphism of params.

Endoscopic datum is

1. $s \in {}^\vee G$ semisimple;
2. L-subgroup ${}^E H \subset ({}^L G)^s \subset {}^L G$, with
3. ${}^\vee H$ = identity component of ${}^\vee G^s$ reductive in ${}^\vee G$.

Root datum $\mathcal{R}({}^\vee H)$ has dual root datum $\rightsquigarrow H/\overline{k}$.

$${}^E H \rightsquigarrow \text{action of } \Gamma = \mathrm{Gal}(\overline{k}/k) \text{ on root data,}$$

$$\rightsquigarrow \text{inner class of } k\text{-forms of } H.$$

Endoscopic group for $G = H/k$, any form in inner class.

Where's the fixed point formula?

$s \in {}^\vee G$ semisimple, L-subgroup ${}^E H \subset ({}^L G)^s \subset {}^L G$, ${}^\vee H = {}^\vee (G^s)_0$.

Hypotheses imply ${}^E H$ open in $({}^L G)^s$.

Simplify by assuming ${}^E H = ({}^L G)^s$. Then

(fixed pts of $Ad(s)$ on params) = (params for ${}^E H$).

This equality allows application of a Lefschetz formula.

More precisely:

$$\mathrm{tr}(s \text{ action on perv cohom for } {}^L G)$$
$$= \mathrm{tr}(s \text{ action on perv cohom for } {}^E H.)$$

Since s central in ${}^E H$, right side is easy.

Equality seems to require s to centralize ${}^E H$.

Generalization seems impossible...

Here's how to generalize

Generalized endoscopic datum is

1. $s \in {}^\vee G$ semisimple;
2. L-subgroup ${}^E H \subset {}^L G$ normalized by s;
3. ${}^\vee H$ = identity component of ${}^\vee G^s$ reductive in ${}^\vee G$;
4. quotient action of $\mathrm{Ad}(s)$ on $\Gamma = {}^E H/{}^\vee H$ is trivial.

As for endoscopic groups,

$$ {}^E H \rightsquigarrow \text{Galois action on root datum for } {}^\vee H $$
$$ \rightsquigarrow \text{inner class of } k\text{-forms of } H. $$

These k forms are generalized endoscopic groups.

Define $\xi \colon {}^E H \to {}^\vee H$ by $\xi(m) = sms^{-1}m^{-1} \quad (m \in {}^E H)$.

Equivalently: $\mathrm{Ad}(s)(m) = \xi(m)m$.

ξ measures failure of s to commute with ${}^E H$, or equivalently failure of of ${}^E H$ to be endoscopic.

Then ξ factors to $\Gamma = {}^E H/{}^\vee H$, values in $Z({}^\vee H)$.

Precisely: ξ is 1-cocycle of Γ with values in $Z({}^\vee H)$.

How do you generalize endoscopic transfer?

Endoscopic transfer: should correspond to map sheaves on ${}^LG$ params $\rightsquigarrow$ sheaves on ${}^EH$ params.

Classical endoscopy: s acts by conjugation on ${}^LG$ params; fixed points are ${}^EH$ params.

Only ${}^LG$-params in image are ${}^\vee G$-conj to ${}^EH$-params.

Generalized endoscopy: s still acts on ${}^LG$-params, but does not fix ${}^EH$ params: $\mathrm{Ad}(s)(\phi_H(\gamma)) = \xi(\gamma)\phi_H(\gamma)$.

Try modify $\mathrm{Ad}(s)$ by ξ^{-1}: $(s \overset{?}{\star}_\xi \phi)(\gamma) = \xi^{-1}(\gamma)\,\mathrm{Ad}(s)(\phi(\gamma))$.
But this is not an action except on ${}^EH$ params.

Solution: look only at params conjugate to ${}^EH$ params:

$${}^\vee G \times_{{}^\vee H} ({}^EH \text{ params}) \rightarrow ({}^LG \text{ params}), \quad (g, \phi'_H) \mapsto \mathrm{Ad}(g)\phi'_H.$$

s acts on left space by $s \star_\xi (g, \phi'_H) = \mathrm{Ad}(g)(\xi^{-1}\phi'_H)$.

Fixed points of $\star_\xi$ are ${}^EH$ params.

What's that $\star$ action on parameters?

To make Langlands params H (${}^\vee H = {}^\vee G^s$) into fixed points, needed to compose action of $\mathrm{Ad}(s)$ with mult by a 1-cocycle.

Following very special case may shed some light. Result stated is Theorem for k archimedean, and in various p-adic cases where local Langlands conj is proven.

Desideratum (Langlands); see Borel, Corvallis volume 2.

$$\phi' \colon W_k' \to {}^L G \rightsquigarrow L\text{-packet } \Pi(\phi') = \{\pi_\tau\}.$$

π_τ is irrep of an inner k-form of G. Suppose that

$$\xi \colon W_k' \to Z(G^\vee)$$

is a 1-cocycle. Define

$$\xi \cdot \phi' \colon W_k' \to {}^L G, \qquad (\xi \cdot \phi')(w) = \xi(w)\phi'(w).$$

1. The 1-cocycle condition means $\xi \cdot \phi'$ is also a group homomorphism, new Langlands parameter.
2. $\overline{\xi} \in H^1(W_k', Z({}^\vee G)) \rightsquigarrow$ smooth char $\gamma_{\overline{\xi}}$ of $G(k)$.
3. $\Pi(\xi \cdot \phi') = \{\gamma_{\overline{\xi}} \otimes \pi_\tau\}$.

Mult param by $Z({}^\vee G)$ cocycle tensors G reps with 1-diml rep.

Classical endoscopic groups

Suppose G/k reductive and $P = MN$ parabolic over k.

Put $X^*(M) =$ ratl chars of M, a Γ-fixed sublattice of $X^*(G)$.

$\rightsquigarrow$ Γ-fixed sub $\subset X_*({}^\vee G)$ $\rightsquigarrow$ Γ-fixed torus ${}^\vee A \subset {}^\vee G$.

${}^\vee M =_{\text{def}} {}^\vee G^{{}^\vee A}$ is Γ-stable, dual to M: ${}^L M \simeq {}^\vee M \rtimes \Gamma$.

Generic $s \in {}^\vee A \rightsquigarrow ({}^L G)^s = {}^L M \rightsquigarrow$ endoscopic group M.

Endoscopic transfer (reps of M) $\rightsquigarrow$ (reps of G) is Ind_{MN}^G.

Endoscopy is more powerful than parabolic induction.

Allows $Z_{{}^L G}$(Γ-fixed element), not just Γ-fixed torus.

But endoscopy also misses a lot of interesting subgroups.

Rational Cartan subgrp of G is almost never endoscopic.

Generalized endoscopic groups

Suppose L any rational Levi subgroup of $G \rightsquigarrow \Gamma$ action on root datum of L.

If G simply connected, easy to find ${}^L L \subset {}^L G$.

In general, get extended group ${}^E L \subset {}^L G$.

$s \in Z({}^\vee L)$ generic $\rightsquigarrow (s, {}^E L)$ gen endoscopic datum $\rightsquigarrow$ L generalized endoscopic for G.

Endoscopic transfer from general ratl Levi L should be important generalization of parabolic induction.

Over $\mathbb{R}$, this is Zuckerman's cohomological induction.

Over a p-adic field, this is still a mystery.

Harish-Chandra would tell us to get to work.