The character table for E_8

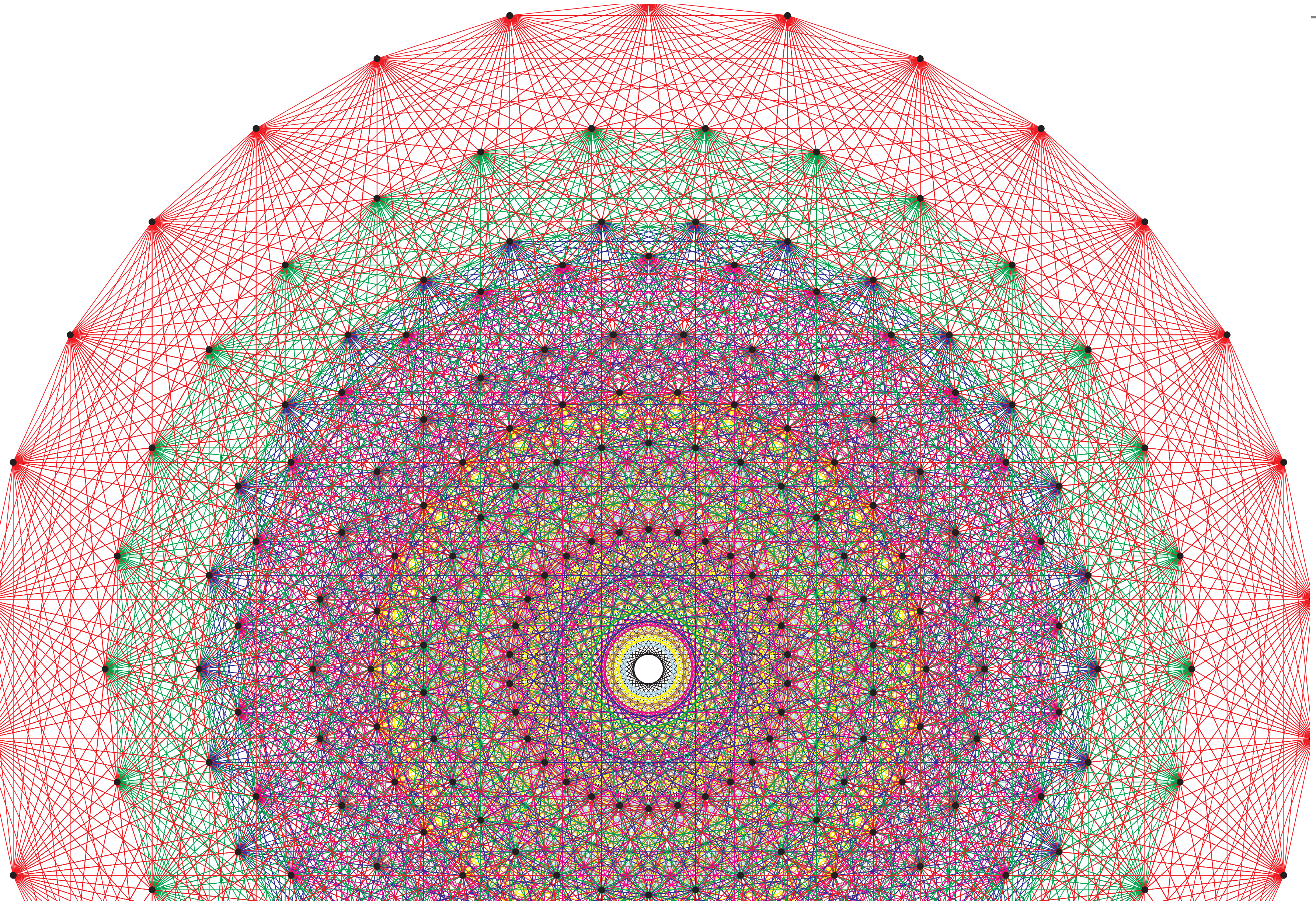
or

how we wrote down
a 453060×453060 ***matrix***
and found happiness

David Vogan

Department of Mathematics, MIT

Root system of E_8



The Atlas members:

Jeffrey Adams
Dan Barbasch
Birne Binegar
Bill Casselman
Dan Ciubotaru
Fokko du Cloux
Scott Crofts
Tatiana Howard
Marc van Leeuwen
Alfred Noel
Alessandra Pantano
Annegret Paul
Siddhartha Sahi
Susana Salamanca
John Stembridge
Peter Trapa
David Vogan
Wai-Ling Yee
Jiu-Kang Yu

American Institute of Mathematics `www.aimath.org`
National Science Foundation `www.nsf.gov`

`www.liegroups.org`

The Atlas members:

The story in code:

At 9 a.m. on January 8, 2007, a computer finished writing sixty gigabytes of files: Kazhdan-Lusztig polynomials for the split real group $G(\mathbb{R})$ of type E_8. Their values at 1 are coefficients in irreducible characters of $G(\mathbb{R})$. The biggest coefficient was 11,808,808, in

$$152q^{22} + 3472q^{21} + 38791q^{20} + 293021q^{19}$$
$$+ 1370892q^{18} + 4067059q^{17} + 7964012q^{16} + 11159003q^{15}$$
$$+ 11808808q^{14} + 9859915q^{13} + 6778956q^{12} + 3964369q^{11}$$
$$+ 2015441q^{10} + 906567q^{9} + 363611q^{8} + 129820q^{7}$$
$$+ 41239q^{6} + 11426q^{5} + 2677q^{4} + 492q^{3} + 61q^{2} + 3q$$

Its value at 1 is 60,779,787.

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Excellent questions. Since it's my talk, I get to rephrase them a little.

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- **How do you write a character table?**
  - RTFM (by Weyl, Harish-Chandra, Kazhdan/Lusztig).

Our Contribution

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Here are longer versions of those answers.

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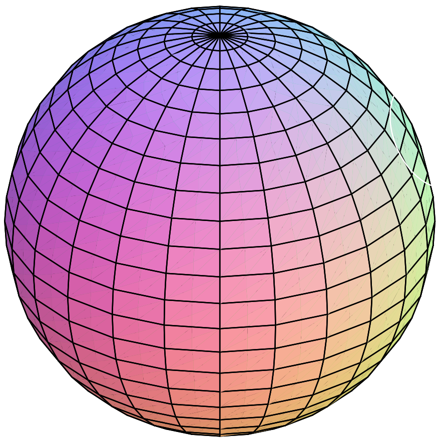

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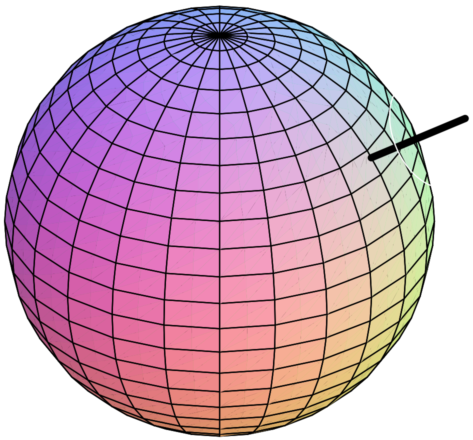

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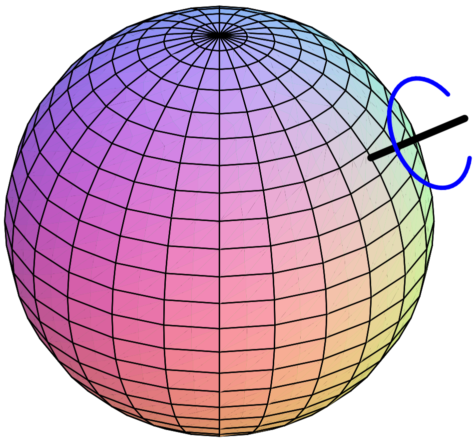

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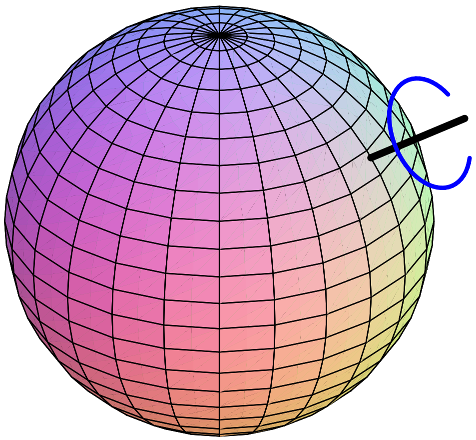

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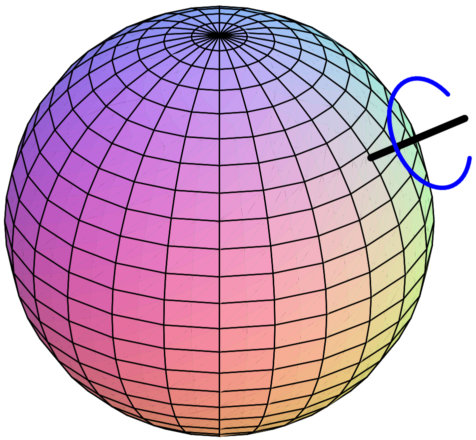

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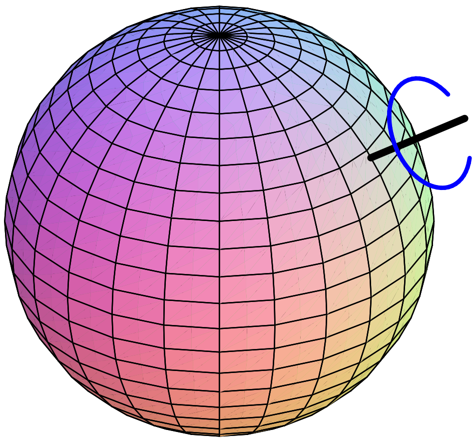

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Other groups $\leftrightsquigarrow$ other geometries, other physics. . .

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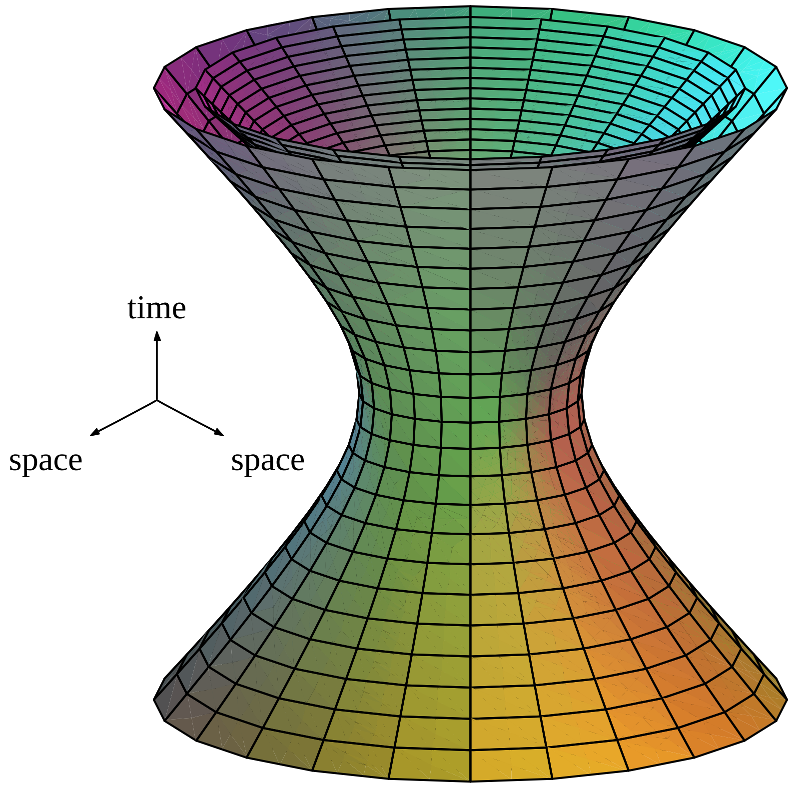

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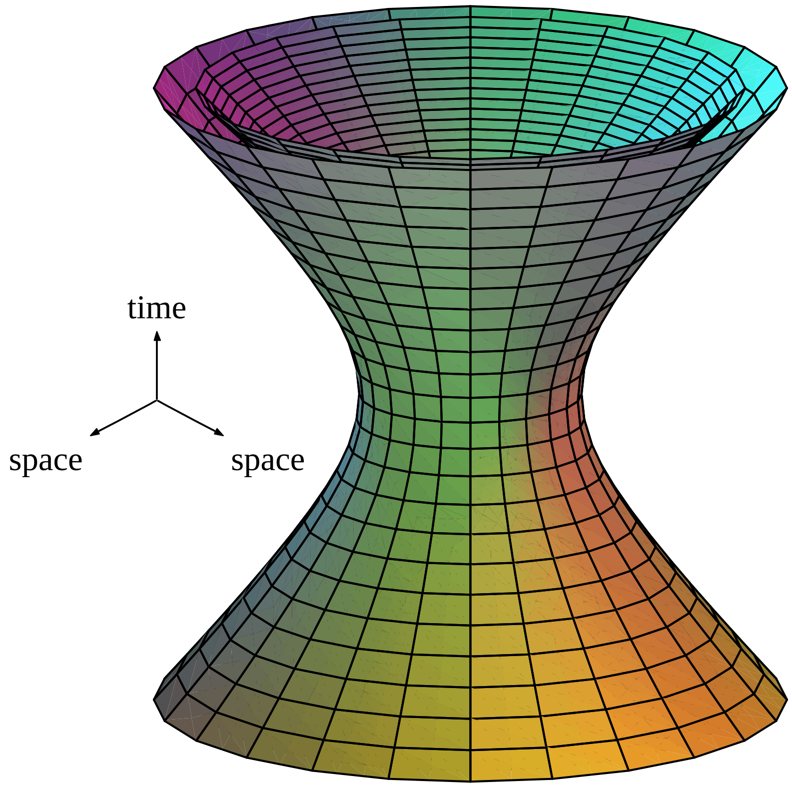

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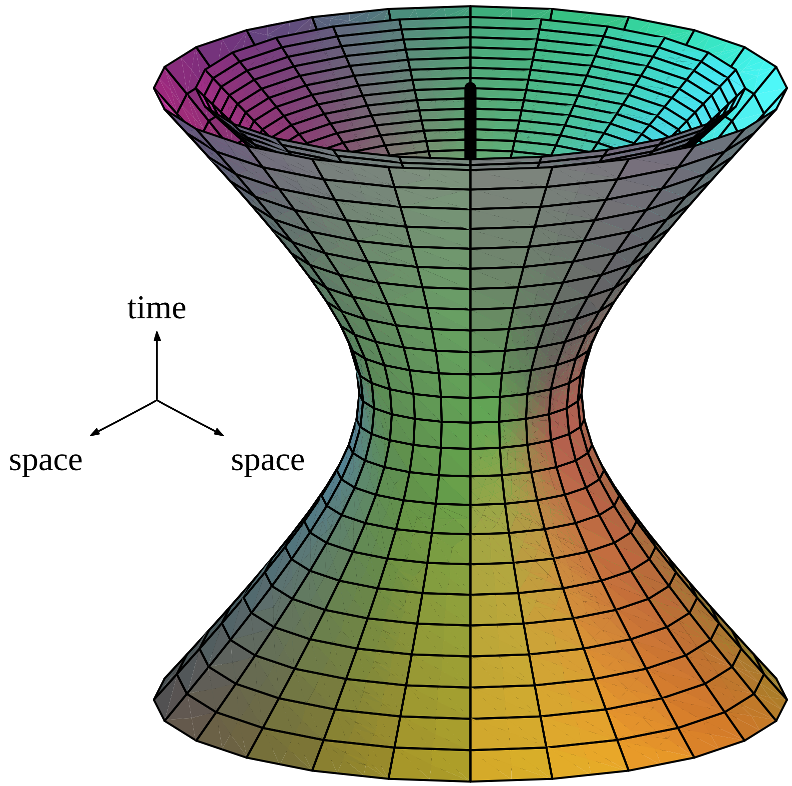

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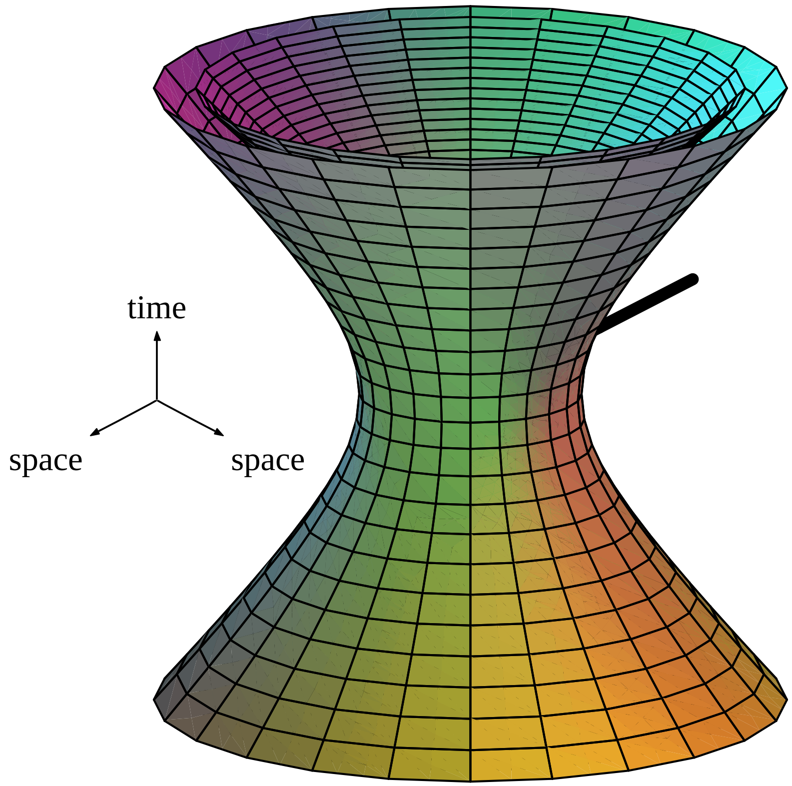

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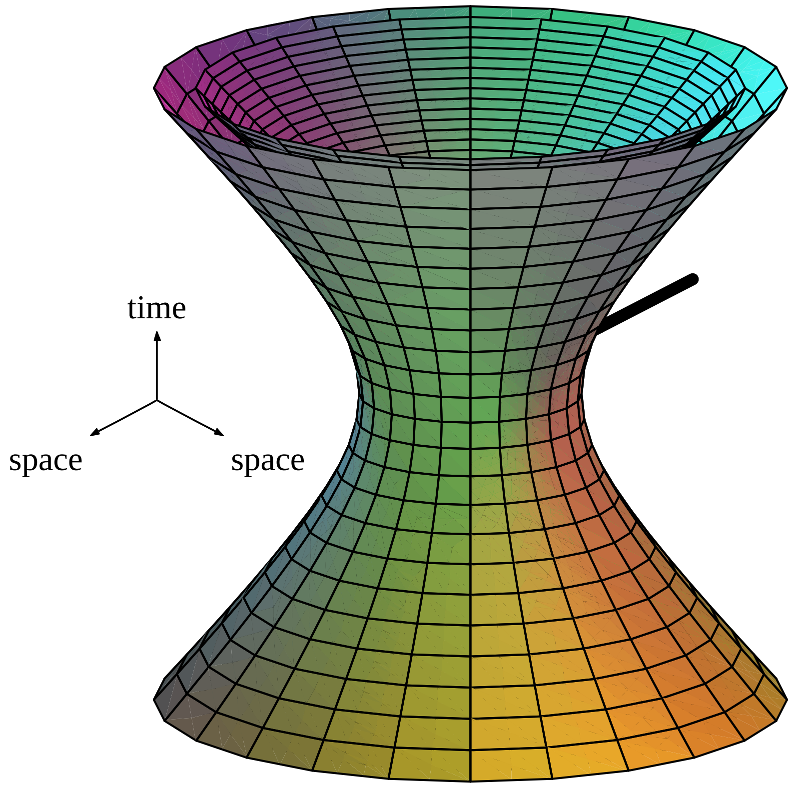

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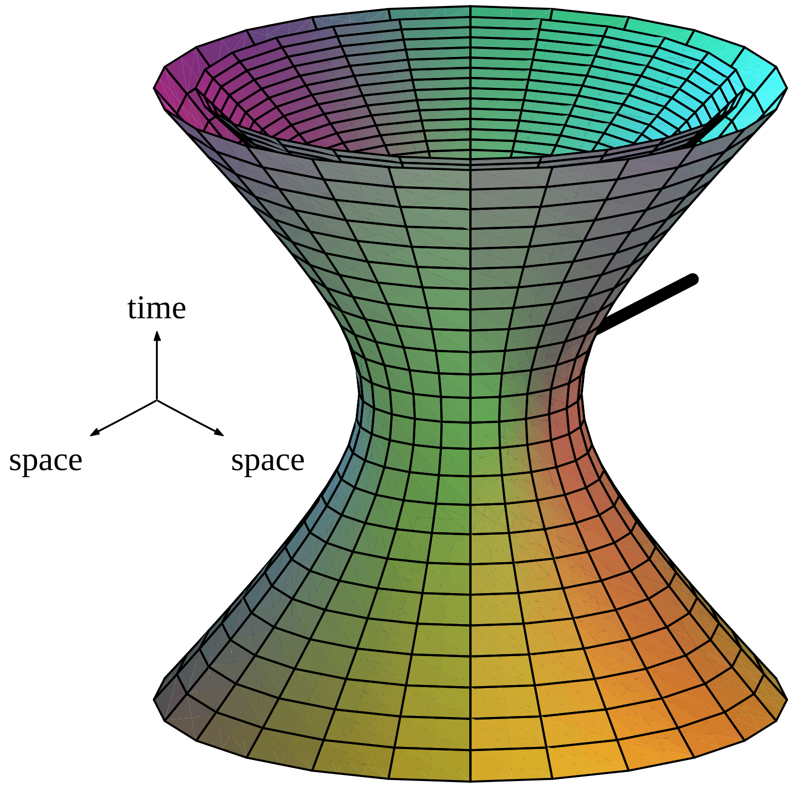

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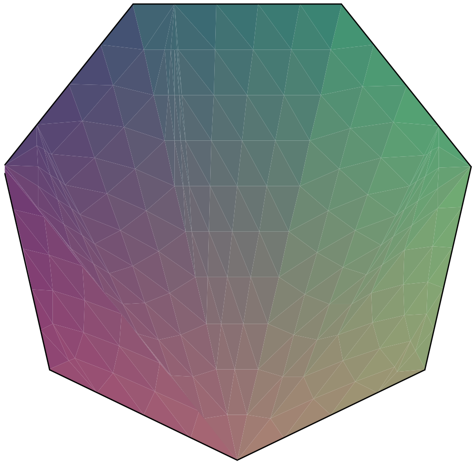

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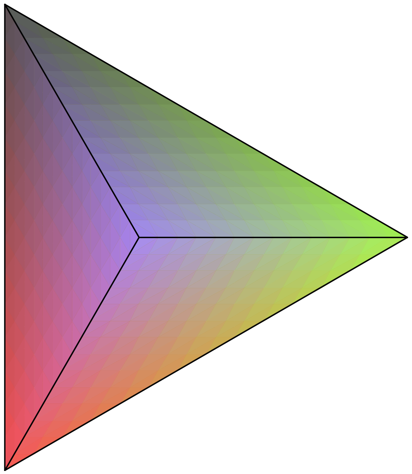

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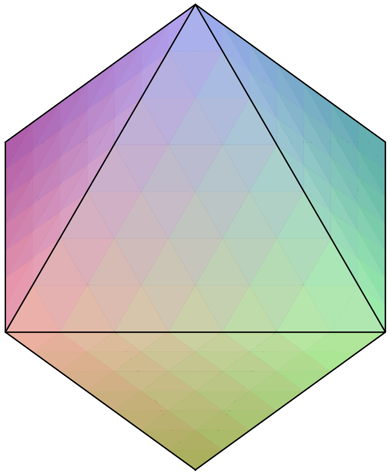

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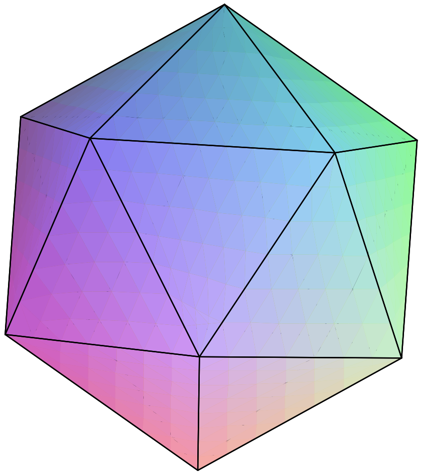

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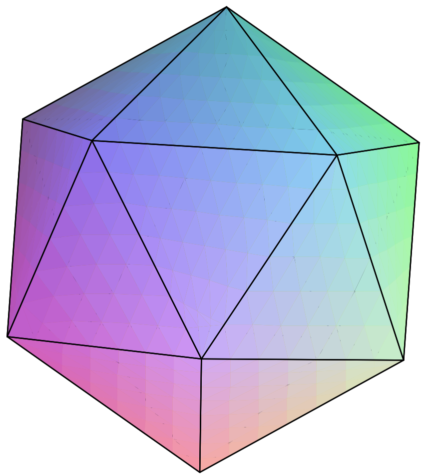

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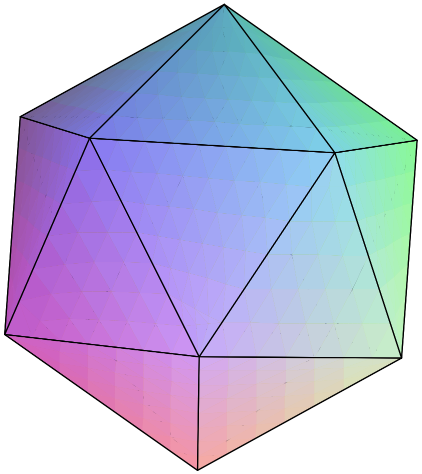

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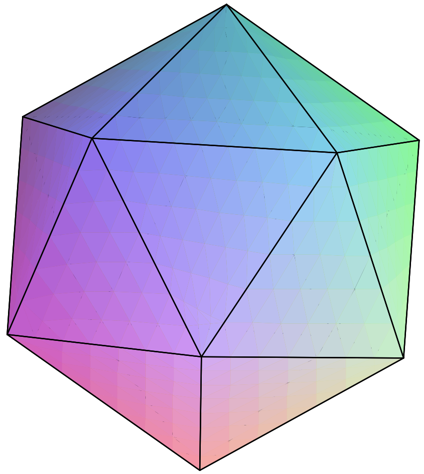

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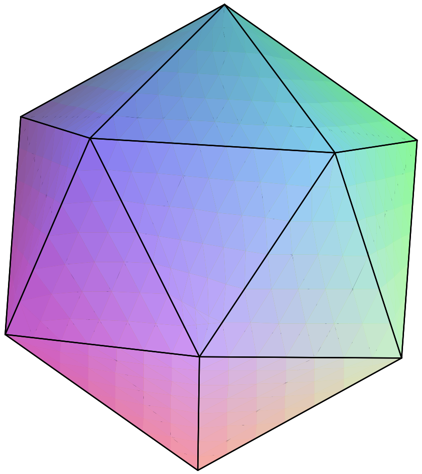

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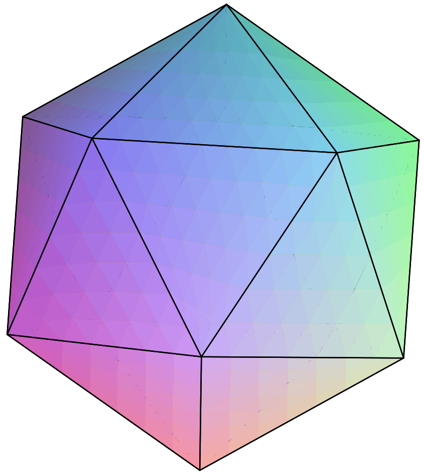

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- Building general Lie groups from simple is hard.

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- Split E_8. This is the tough one.

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First Lie group is 1-dimensional: symmetry in time.

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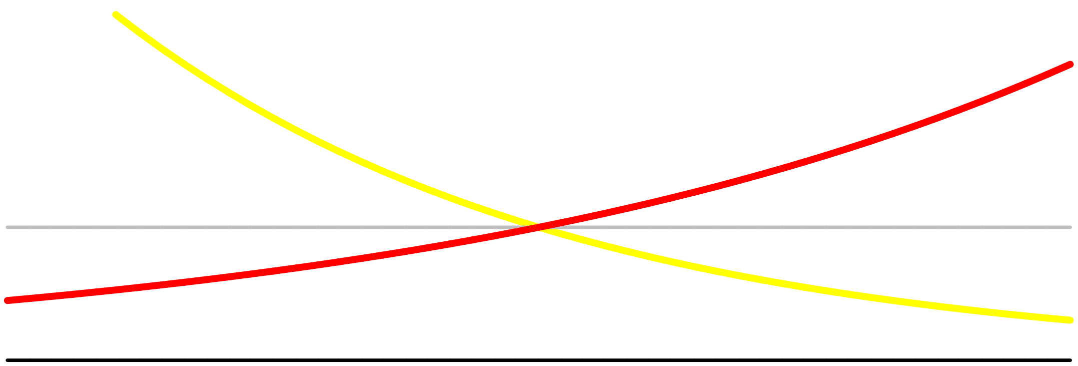

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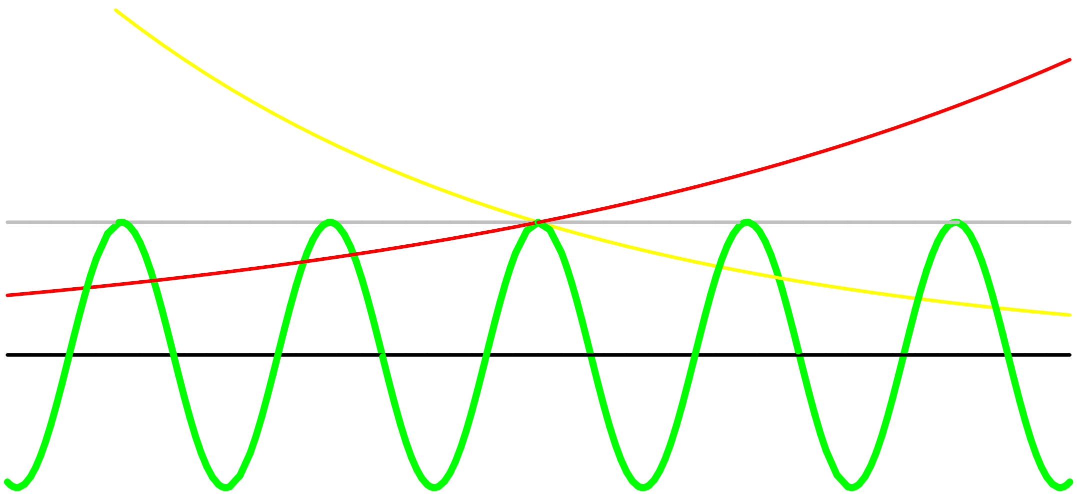

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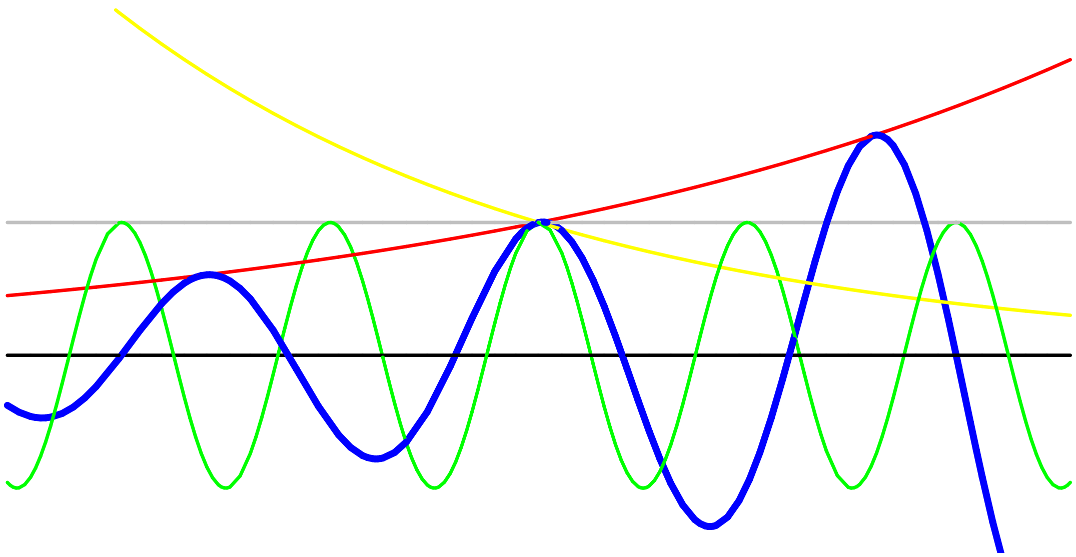

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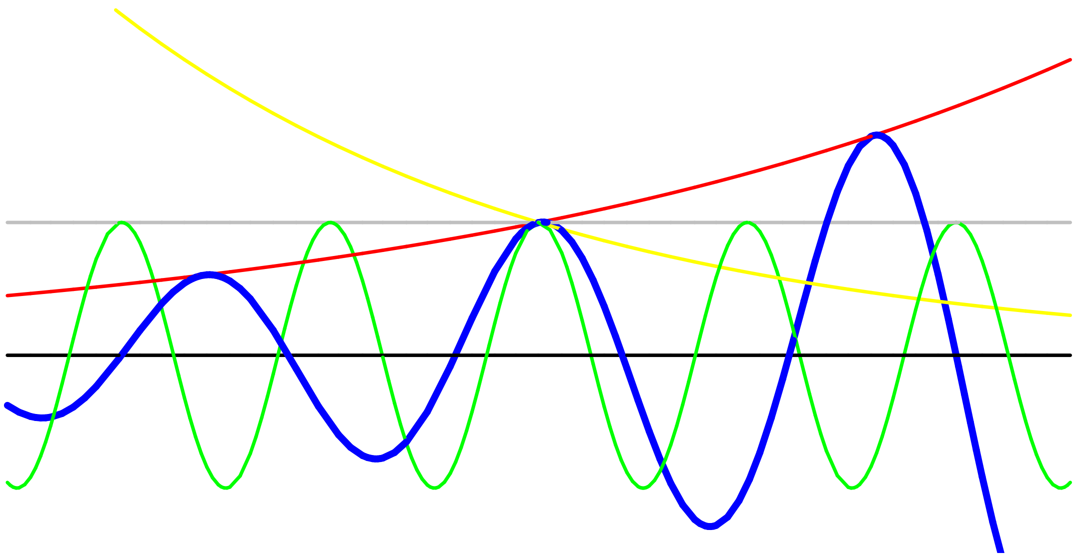

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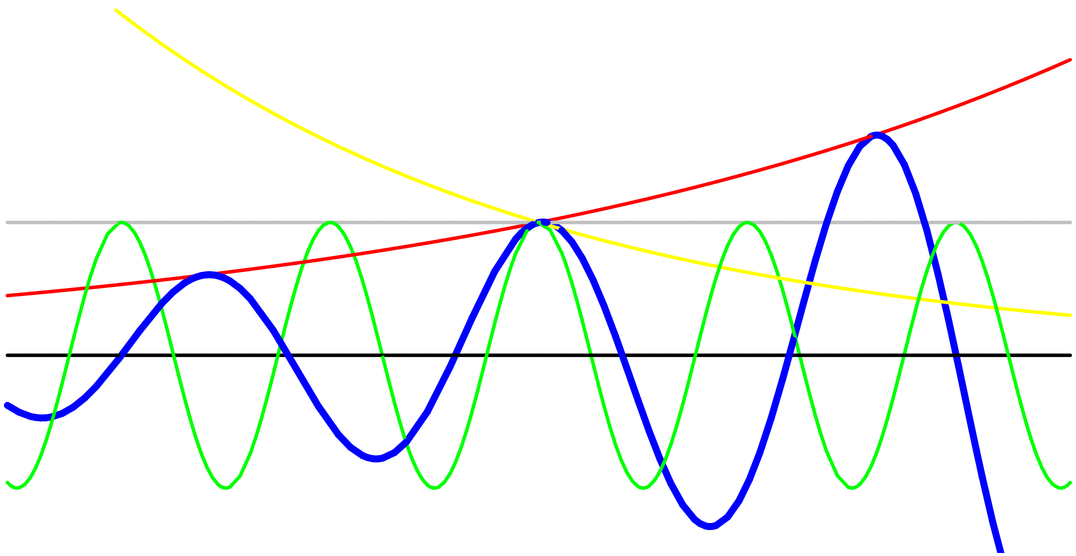

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$$\frac{df}{dt} = z \cdot f$$

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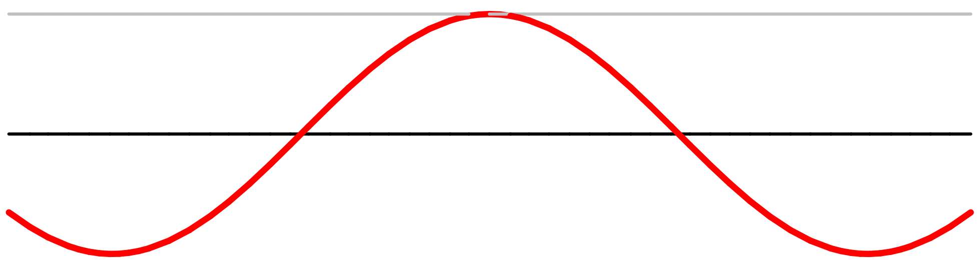

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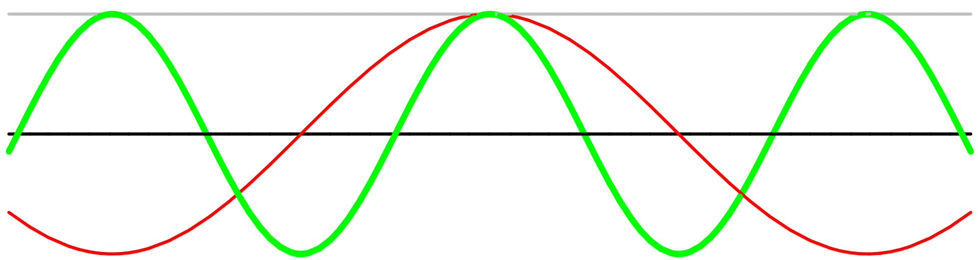

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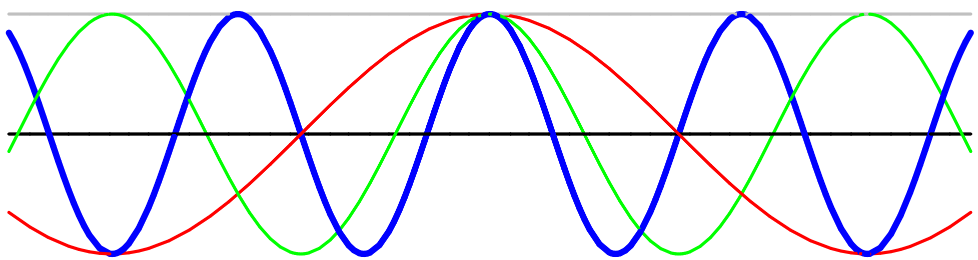

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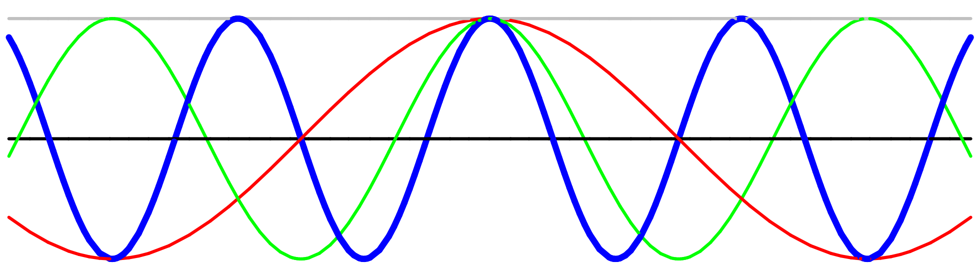

That's all the irreducible repns for compact time symmetry. Given by one integer: frequency.

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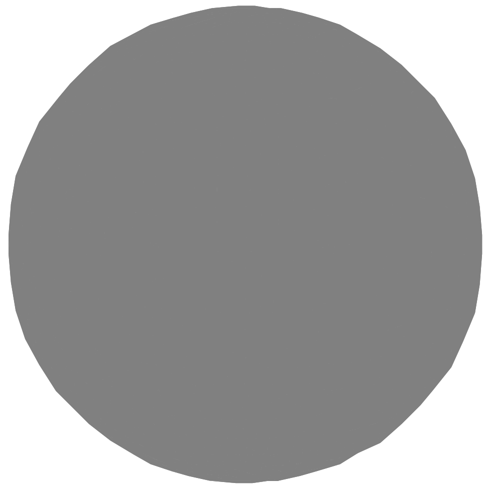

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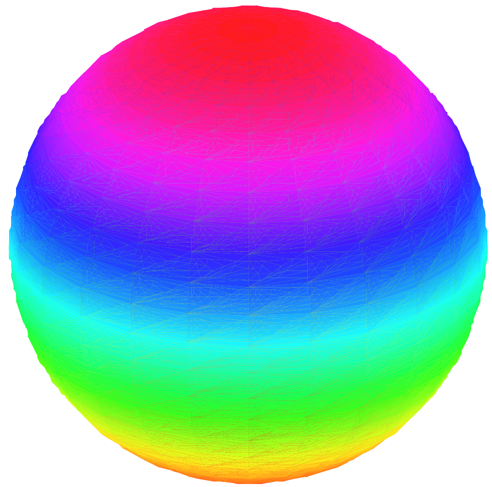

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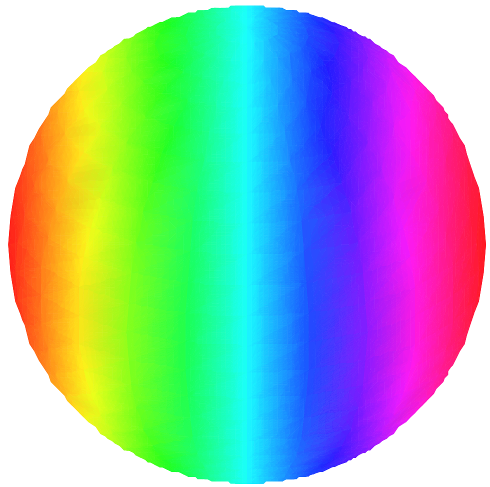

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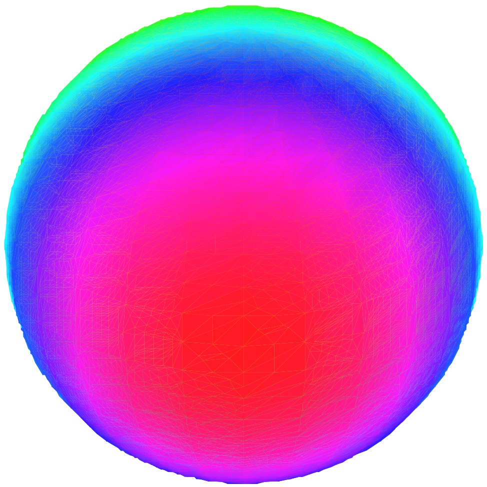

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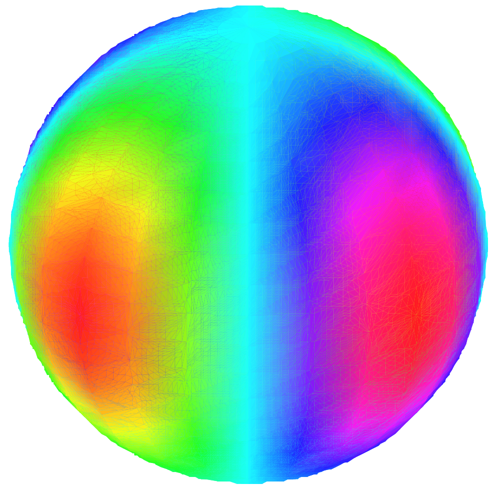

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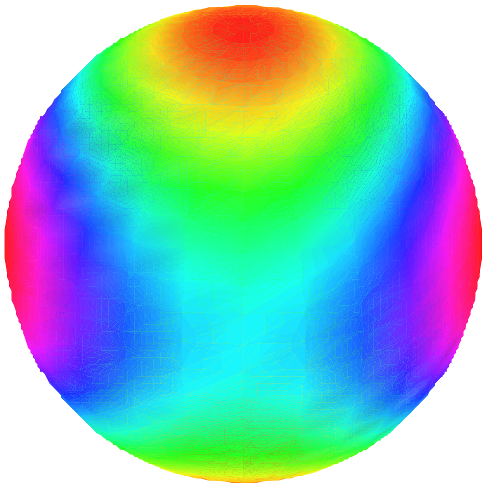

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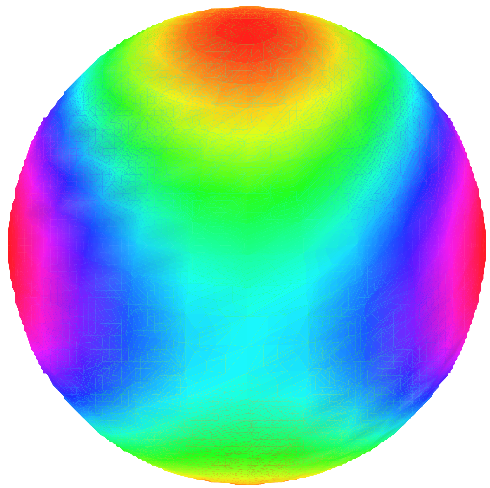

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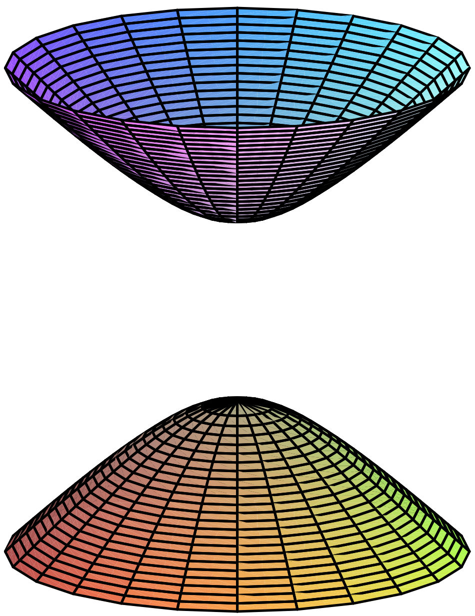

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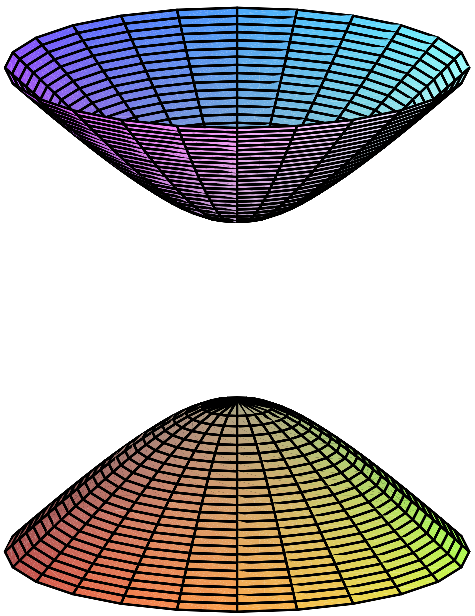

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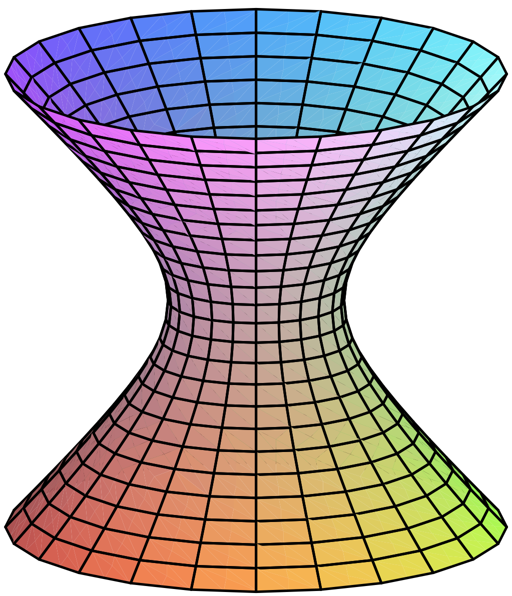

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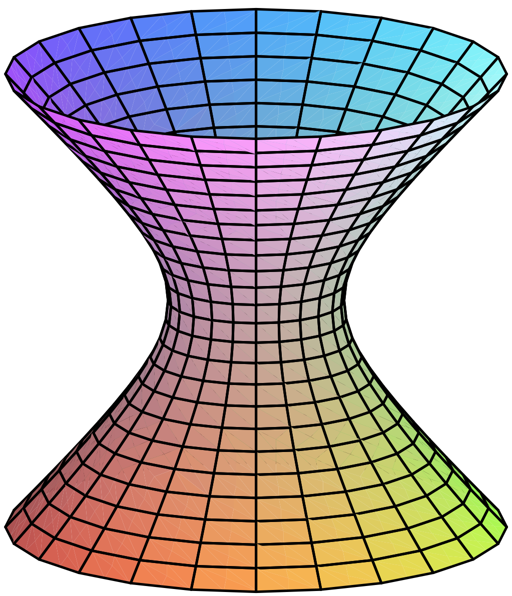

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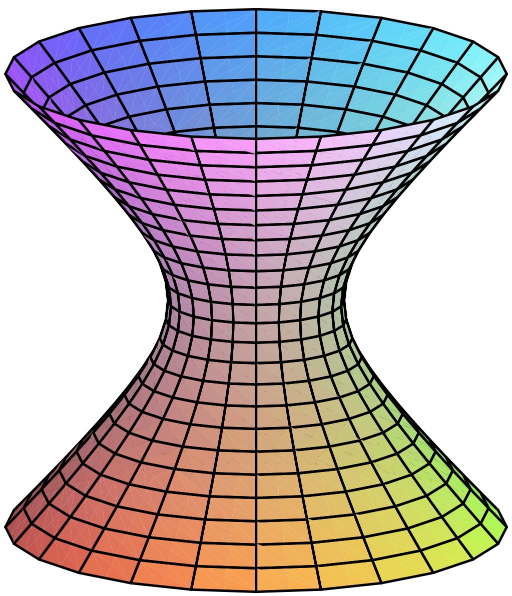

That's all irreducible representations for the Lorentz group: two families, indexed by integer F or complex number z. Representations are infinite-dimensional, except principal series $z = \pm 1, \pm 2, \ldots$.

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  Mathematical basis of integers in quantum physics.

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. . . but discrete series $f=-1/4,-3/4 \leftrightsquigarrow$ quantum harmonic oscillator.

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- Langlands (1970): Character matrix is upper triangular matrix of integers, ones on diagonal.

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Example: Lorentz group

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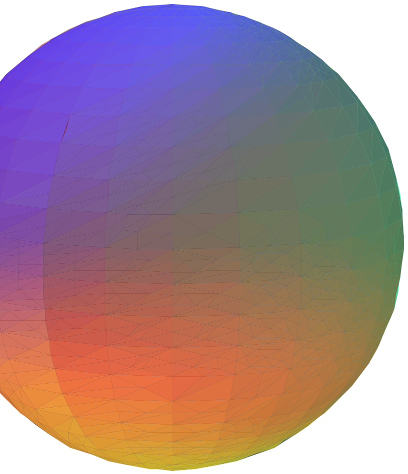
- Flag variety is sphere.



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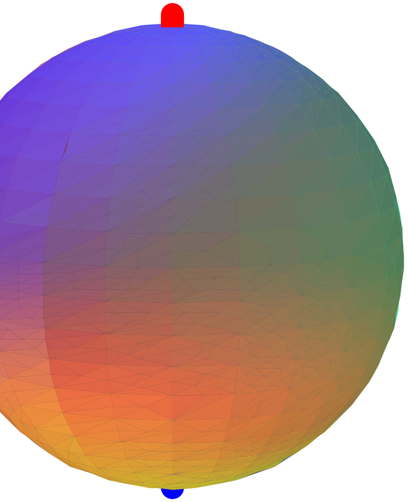

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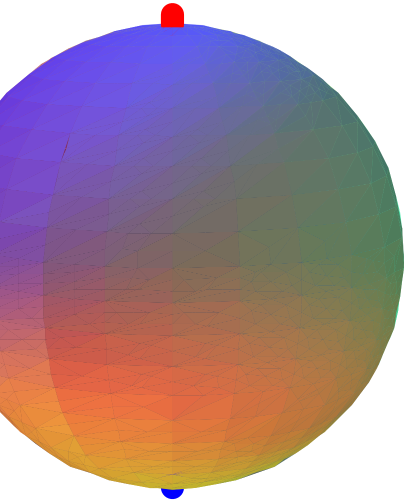

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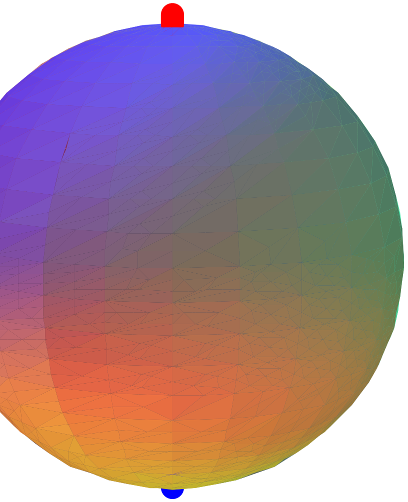

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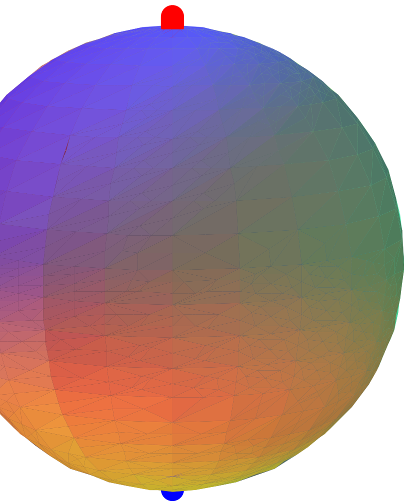

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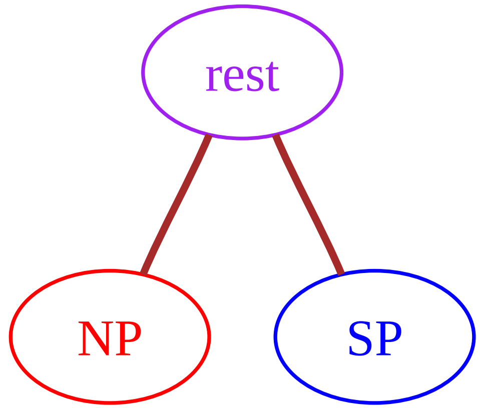
NP	1	0	1
SP	0	1	1
rest	0	0	1



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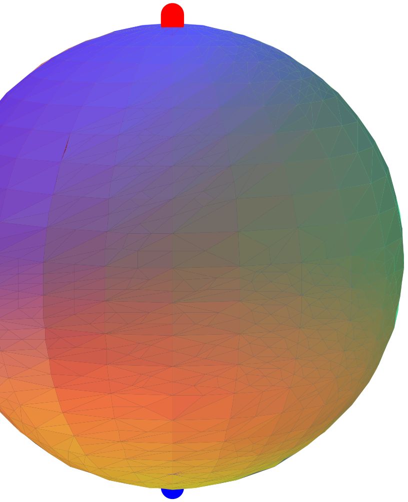

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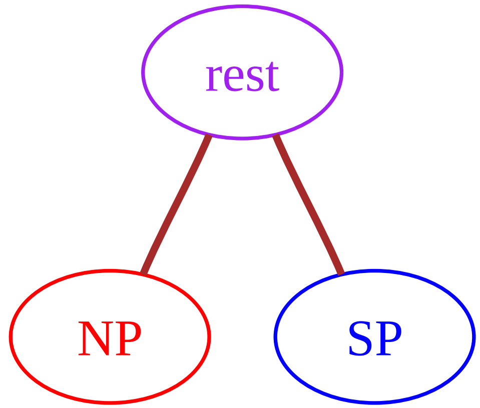

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  For big groups: let graph tell you what algebra to do.

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We read TFM.

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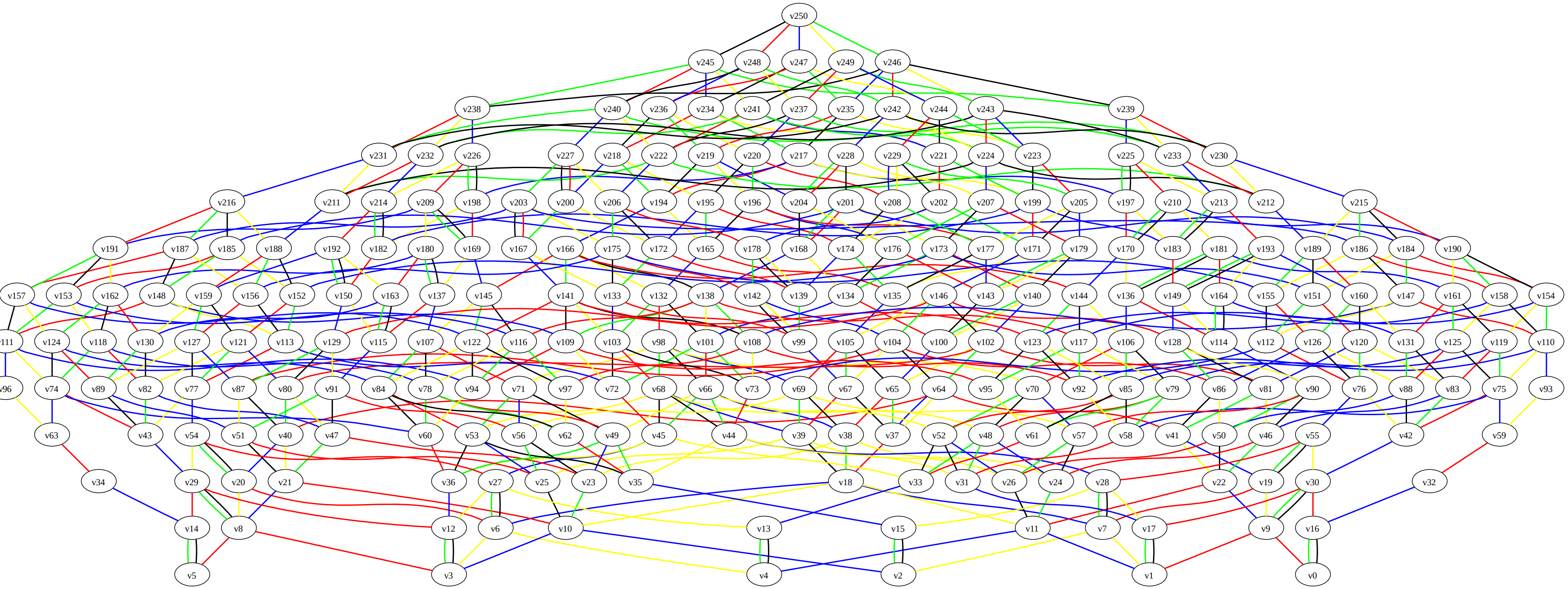

Graph for group $SO(5,5)$ (corresponding to equilateral $\triangle$).

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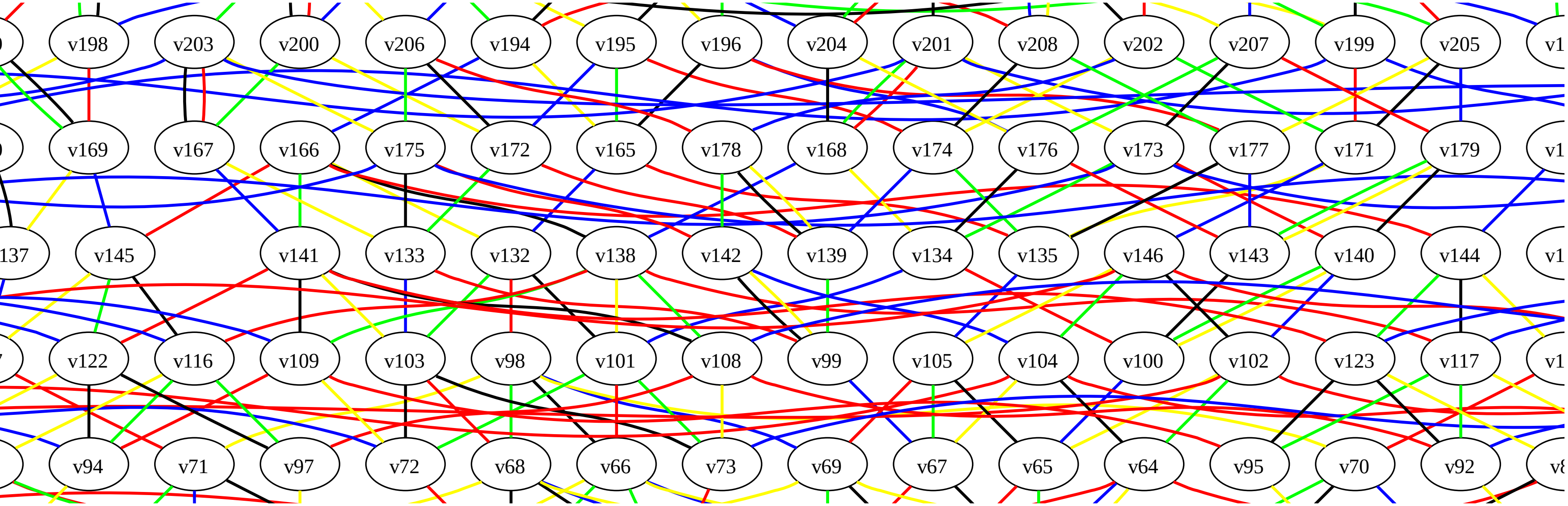

closeup view

Graph for group $SO(5,5)$ (corresponding to equilateral $\triangle$).

251 vertices $\rightsquigarrow$ 251 pieces of 40-dimensional flag variety.

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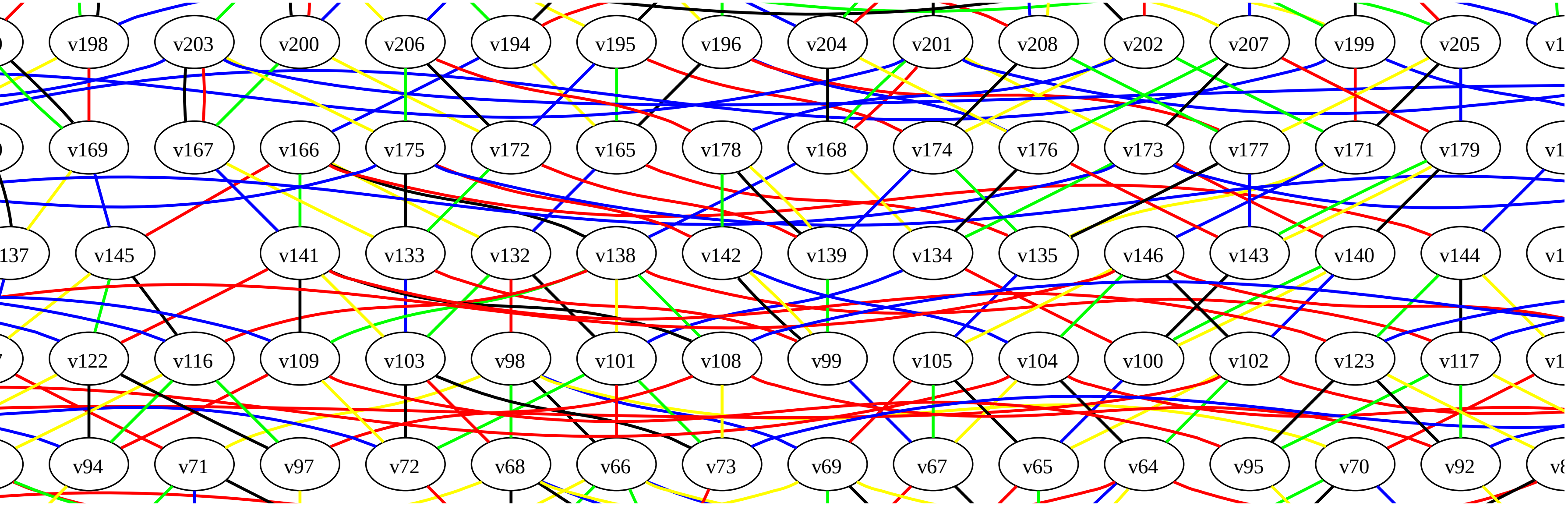

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E_8: $453,060$ vertices $\rightsquigarrow$ pieces of 240-dimensional flag variety.

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For E_8, the big sum averages about 150 nonzero terms.

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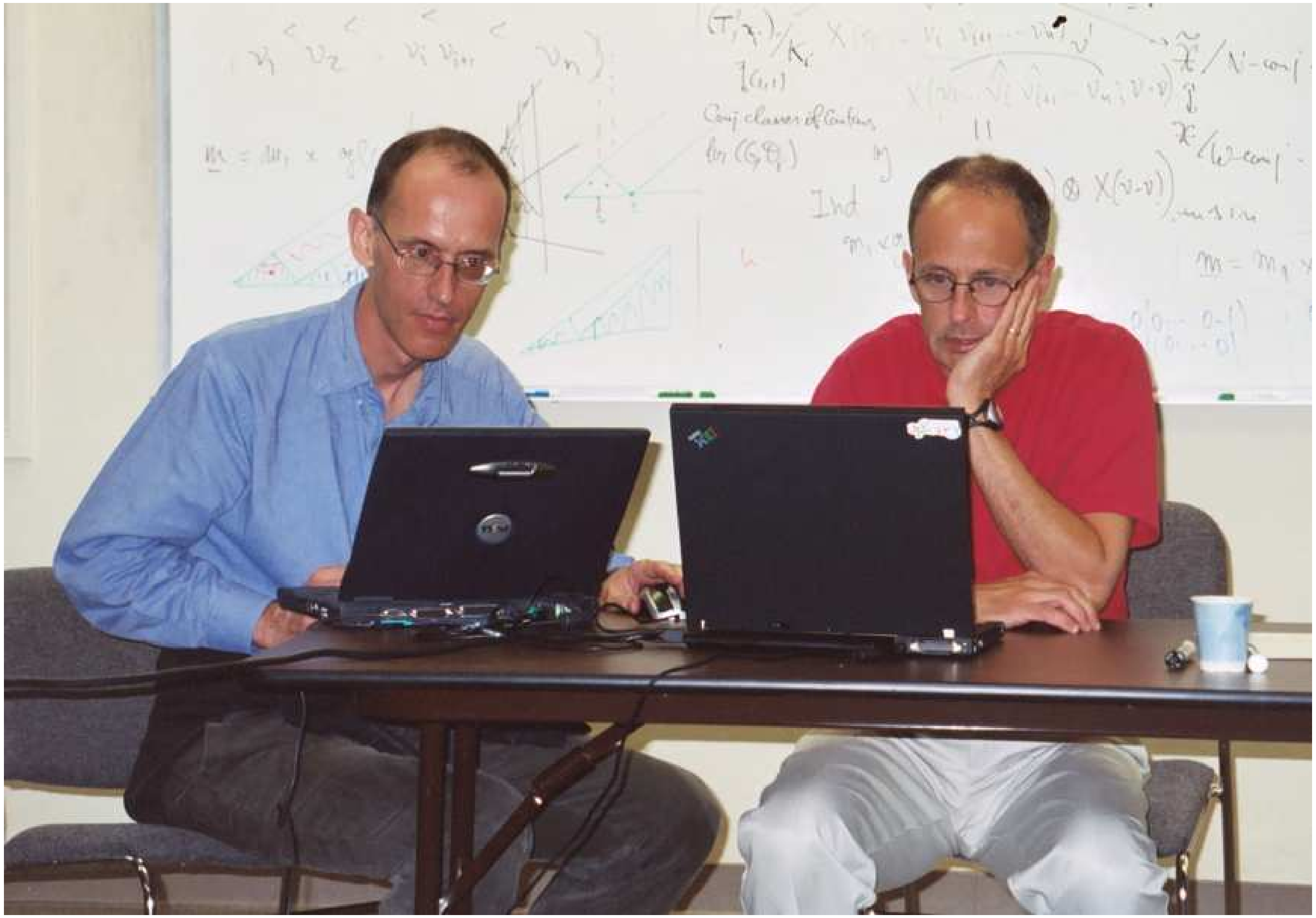

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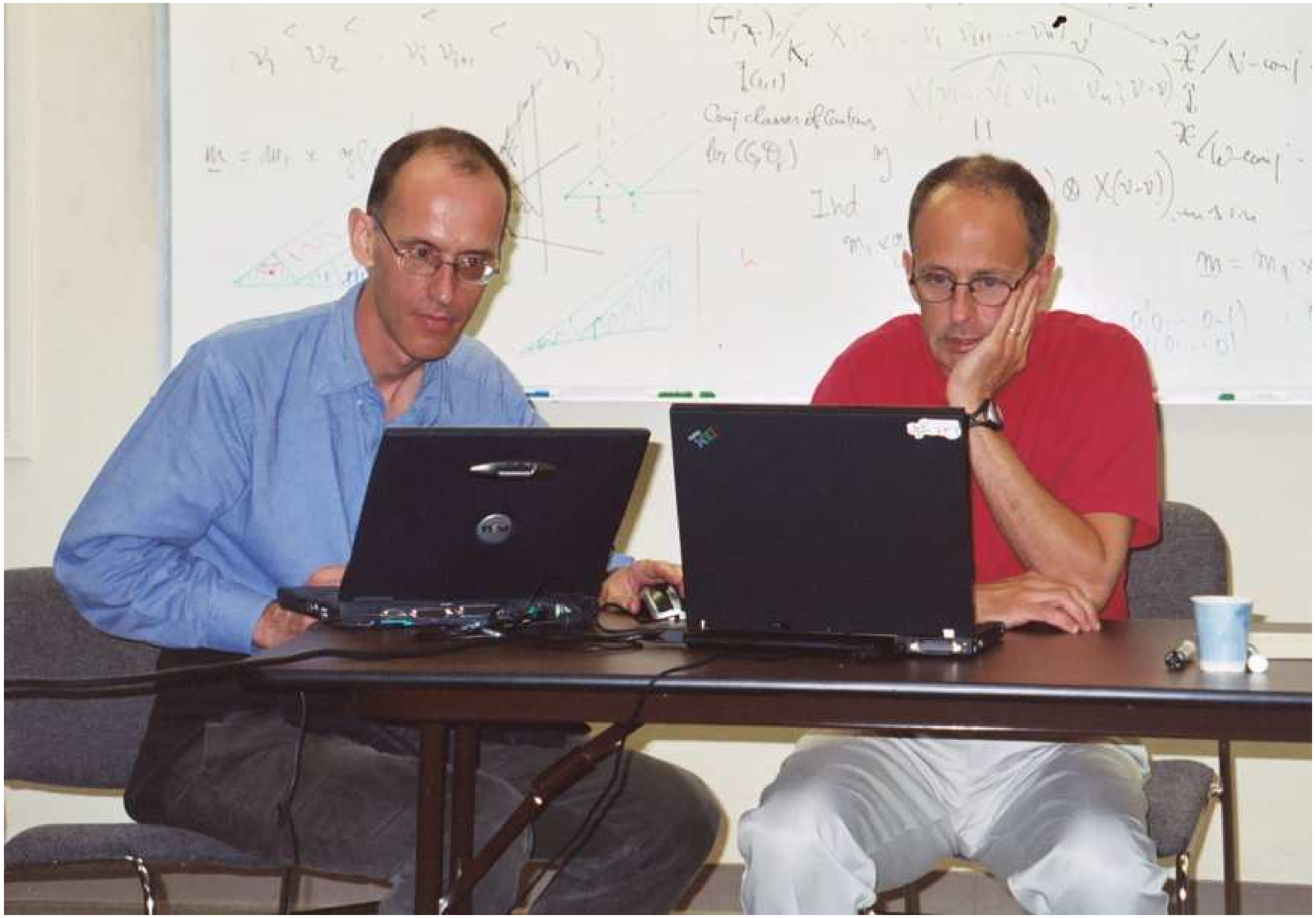

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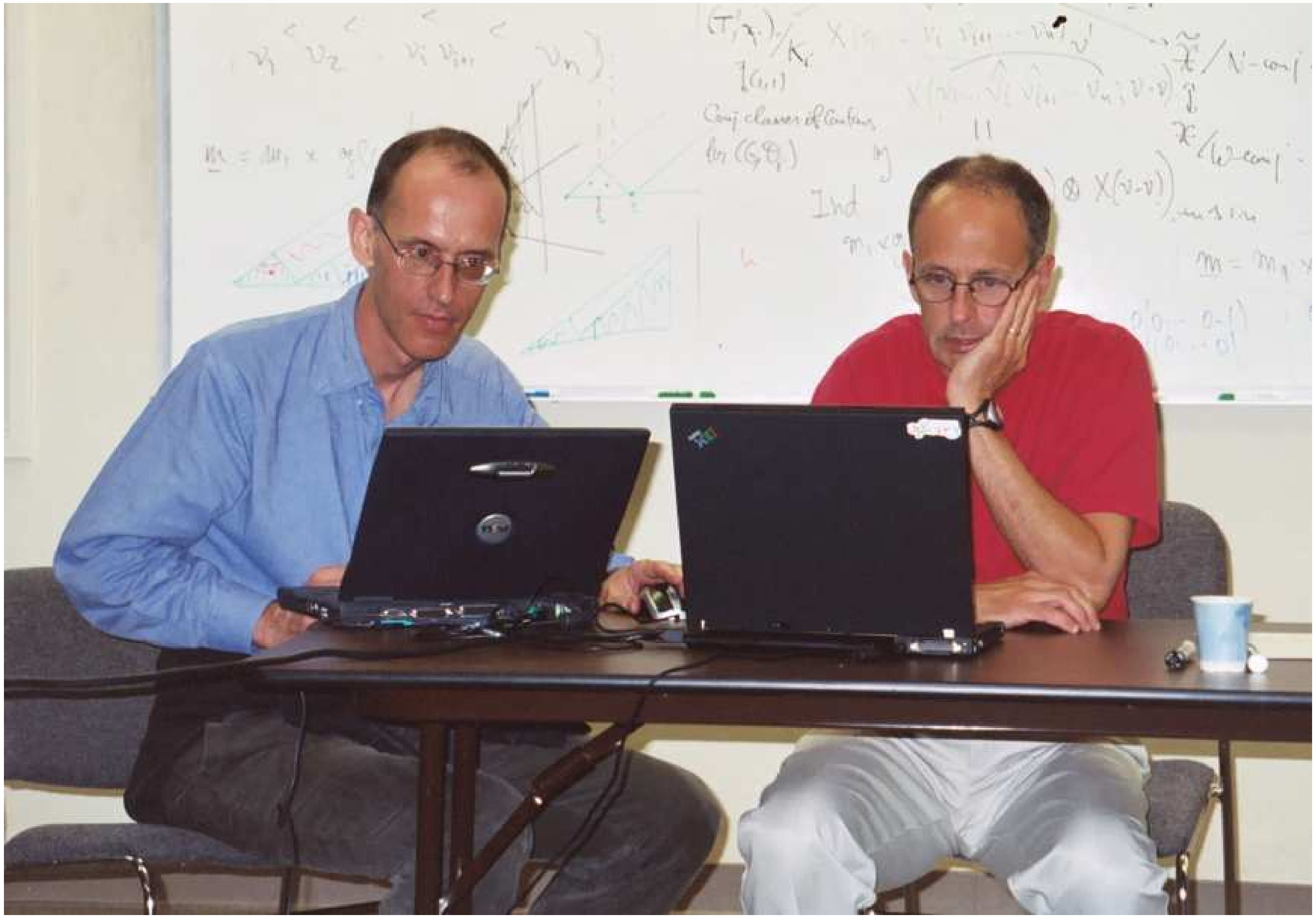

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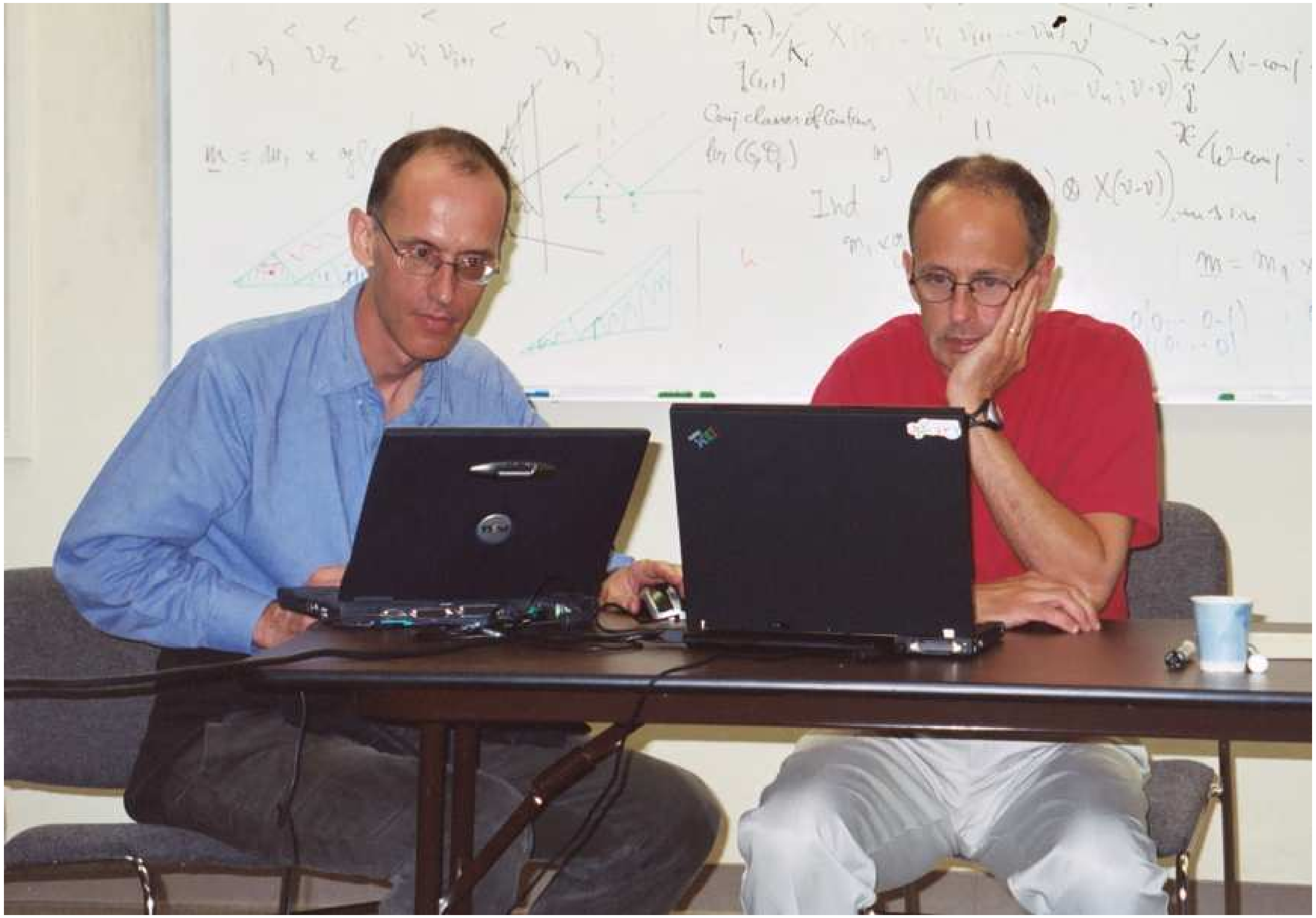

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Wasn't that easy?

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Feared 1 billion $\rightsquigarrow$ 150G total RAM.

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d_store.size() = 1181642979, prim_size = 3393819659

VmData:  64435824 kB
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Writing to disk took two days. Investigating why $\rightsquigarrow$ output bug, so mod 251 character table no good.

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One little computation for each of 13 billion coefficients.

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Fokko was startled by this remark, but not at a loss for words. "I don't know about you, but I'm having the time of my life!"

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Fokko du Cloux

December 20, 1954–November 10, 2006