Graph Orientations and Linear Extensions.

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Acyclic Orientation of a Graph.

Definition:

An acyclic orientation of a simple undirected graph $G = G(V, E)$ is an orientation of its edges with no induced directed cycles.

Acyclic orientations from labelings of $V$:

i. If $|V| = n$, label $V$ (bijectively) with elements of the totally ordered set $n$ and orient $E$ accordingly: All acyclic orientations of $G$ can be obtained in this way.

ii. Any given acyclic orientation of $G = G(V, E)$ induces a partial order on $V$, and a linear extension of one such poset recovers the corresponding orientation as in i.
Main Problem.

Problem:
Given an underlying simple graph \( G = G(V, E) \), each acyclic orientation of \( G \) induces a partial order on \( V \):

a. Which of these posets has the maximal number of linear extensions?

b. If \( |V| = n \), given a uniformly chosen random bijective labeling of \( V \) with \( n \), what is the most likely acyclic orientation of \( G \) so obtained?

We call this problem the **Main Problem for** \( G \), and the combinatorial statistic of interest is denoted by \( \varepsilon(G) \).

Complexity of Enumeration:
Counting the number of linear extensions of an arbitrary poset is \( \#P \)-complete (Brightwell and Winkler (1991)).
Bipartite Graphs and Odd Cycles.

A simple combinatorial algorithm provides the answer for bipartite graphs:

(B.I.) Optimal orientations of bipartite graphs have no directed 2-paths.

Proof Idea: Start with an arbitrary acyclic orientation + linear extension, and transform injectively the linear extension into a labeling with no induced directed 2-paths.
Bipartite Graphs and Odd Cycles.

... and similarly for odd cycles:

(B.I.) Optimal Orientations have exactly one directed 2-path.

Proof Idea: Start with an arbitrary acyclic orientation + linear extension, fix a directed 2-path, and transform injectively the linear extension into a labeling with one 2-dipath.
Connections to Theory.

- Stachowiak (1988) obtained the same result for bipartite graphs using poset theory (More on this later).
- Analogous combinatorial algorithms seem difficult to find for general graphs.
- Geometry of polytopes and poset theory provide better insight.

\[ \mathbb{R}^n \] with euclidean topology and standard basis \( \{e_j\}_{j \in [n]} \):

1. For \( J \subseteq [n] \), \( e_J := \sum_{j \in J} e_j \) and \( e_{\emptyset} := 0 \).
2. For \( x \in \mathbb{R}^n \), \( x_J := \sum_{j \in J} x_j \) and \( x_{\emptyset} := 0 \).
Definition: (Stanley (1986))

Given a partial order \( P \) on \([n]\), the order polytope of \( P \) is defined as:

\[
\mathcal{O}(P) := \{ x \in \mathbb{R}^n : 0 \leq x_i \leq 1 \text{ and } x_j \leq x_k \text{ whenever } j \leq_P k, \forall \ i, j, k \in [n] \}.
\]

The chain polytope of \( P \) is defined as:

\[
\mathcal{C}(P) := \{ x \in \mathbb{R}^n : x_i \geq 0, \forall \ i \in [n] \text{ and } x_C \leq 1 \text{ whenever } C \text{ is a chain in } P \}.
\]
Results About Polytopes.

Definition: (Stanley (1986))

Given a poset $P$ on $[n]$, **Stanley’s transfer map**

$\phi : O(P) \rightarrow C(P)$ is the function given by:

$$\phi(x)_i := \begin{cases} 
    x_i - \max_{j \leq_P i} x_j & \text{if } i \text{ is not minimal in } P, \\
    x_i & \text{if } i \text{ is minimal in } P.
\end{cases}$$

Theorem: (Stanley (1986))

For a poset $P$ on $[n]$:

1. Linear extensions of poset $P$ give a triangulation of $O(P)$ with simplices of equal volume $\frac{1}{n!}$, so $\text{Vol}(O(P)) = \frac{e(P)}{n!}$.

2. $\phi$ is a continuous bijective map, linear and unimodular on each simplex of the same triangulation of $O(P)$, so $\text{Vol}(C(P)) = \frac{e(P)}{n!}$.
Results About Polytopes.

Definition:

Given a simple undirected graph $G = G([n], E)$, the **stable polytope** $\text{STAB}(G)$ of $G$ is the full dimensional polytope in $\mathbb{R}^n$ obtained as the convex hull of all vectors $e_I$, where $I$ is a stable (a.k.a. independent) set of $G$.

Observations:

- **a.** Vertices of $\mathcal{O}(P)$ are given by the indicator vectors of the order filters of $P$.
- **b.** Vertices of $\mathcal{C}(P)$ are given by the indicator vectors of the antichains of $P$, hence the **Main Problem for** $G$ is equivalent to finding the chain polytope of maximal volume contained in $\text{STAB}(G)$.
- **c.** $\phi^{-1}$ maps every antichain $A$ of $P$ to its induced order filter $A^\vee$.
- **d.** Both $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are subpolytopes of the $n$-dimensional hypercube. As $P$ ranges over all acyclic orientations of a graph $G$, $\mathcal{O}(P)$'s give a subdivision of the hypercube.
Example of Stanley’s Theory on a 2-path Graph.
Comparability Graphs.

Definition:

A **comparability graph** is a simple undirected graph $G = G([n], E)$ for which there exists a partial order $P$ on $[n]$ under which two different vertices $i, j \in [n]$ are comparable in $P$ if and only if $\{i, j\} \in E$. All induced acyclic orientations of $G$ induced by such posets (at least two) are called **transitive orientations of $G$**.

Relation to other families of graphs:

i. Comparability graphs are perfectly orderable graphs, hence perfect.

ii. Complete graphs, bipartite graphs, complements of interval graphs, permutation graphs, cographs, and trivially perfect graphs, are all comparability graphs.

Motivations:

a. Poset theory, perfect graph theory.

b. Data-mining, correlation or causality analyses.
Comparability Graphs.

Key Observations:

For $G = G([n], E)$ a comparability graph:

i. Number of transitive orientations of $G$ depends on modular decomposition of $G$ (Gallai et al. (2001)).

ii. If poset $P$ on $[n]$ is obtained from an acyclic orientation of $G$, then $C(P) = \text{STAB}(G)$ precisely when $P$ is transitive.

A Comparability Invariant:

If two posets $P$ and $Q$ have isomorphic comparability graphs, then $e(P) = e(Q)$. 
Comparability Graphs and Geometry.

Theorem: (B.I.)

Let $G$ be a comparability graph. Then, the acyclic orientations of $G$ whose poset has the maximal number of linear extensions are exactly the transitive orientations of $G$.

Geometrical Proof:

Any other acyclic orientation induces a poset whose set of antichains is strictly contained in the set of independent sets of $G$, and this gives a strict containment of chain polytopes.
Comparability Graphs and Poset Theory.

Proposition: (Edelman et al. (1989), B.I.)

Let $P$ be a partial order on $[n]$. If $A$ is an antichain of $P$, then:

$$e(P) \geq \sum_{i \in A} e(P \setminus i),$$

where $P \setminus i$ denotes the induced subposet of $P$ on $[n] \setminus i$.

Similarly, if $S$ is a cutset of $P$, then:

$$e(P) \leq \sum_{i \in S} e(P \setminus i).$$

Moreover, if $J \subseteq [n]$ is either a cutset or an antichain of $P$, then

$$e(P) = \sum_{i \in J} e(P \setminus i)$$

if and only if $J$ is both an antichain and a cutset of $P$. 
Comparability Graphs and Poset Theory.

Proof Example:

A recursive approach resemblant of the classic recursion to compute chromatic polynomials leads to network flows:

Hasse diagram of poset $P$ on $[5]$

Linear extensions, $e(P) = 8$

12345
21345
12354
21354
23145
23154
12534
21534
Theorem: (B.I.)

Let $G$ be a comparability graph. Then, the acyclic orientations of $G$ whose poset has the maximal number of linear extensions are exactly the transitive orientations of $G$.

Poset-Theoretical Proof:

Every induced subgraph of $G$ is a comparability graph and, moreover, the restriction of any transitive orientation of $G$ to any induced subgraph is also transitive. Hence, we can use induction and the Proposition.

Note to Proof: Since both the set of minimal and maximal elements of a poset are antichain cutsets, Stachowiak (1988) had resolved the case of bipartite graphs using an analogous inductive technique.

Odd cycles and other pathological cases:

The same idea allows us to re-obtain the result for odd cycles and for other (rather restrictive) families of graphs.
Proposition: (B.I.)

Let $G = G([n], E)$ be a comparability graph. For $J \subseteq [n]$, let $G \backslash J$ be the induced subgraph of $G$ on vertex set $[n] \setminus J$. Then:

$$\varepsilon(G) \geq \frac{1}{\chi(G)} \cdot \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{\prod_{i=1}^{n-1} \chi(G \backslash \sigma[i])},$$

where $\mathfrak{S}_n$ denotes the symmetric group on $[n]$ and $\chi$ denotes the chromatic number of the graph.

Corollary: (B.I.)

Let $G = G([n], E)$ be a simple graph with chromatic number $k := \chi(G)$. Then:

$$\varepsilon(G) \geq \frac{n!}{k^{n-k} k!}.$$
Further Enumerative Results.

Proposition: (B.I.)

Let $G = G([n], E)$ be a simple graph and $\overline{G}$ its complement graph. Then:

$$\varepsilon(G) \leq (-1)^n \chi(\overline{G}, -1),$$

where $\chi(\overline{G}, \lambda)$ denotes the chromatic polynomial of $\overline{G}$. Equality is attained if and only if $G$ is a complete $p$-partite graph, with $1 \leq p \leq n$.

Statistical behavior of $\varepsilon(G)$ for general simple graphs:

- In general, upper and lower bounds for $\varepsilon(G)$ are too far apart and $\varepsilon(G)$ seems unlikely to behave well (exponential approximation).
- **Idea:** Consider instead $\log_2 \varepsilon(G)$.
- Are there any tight concentration results for the statistic $\log_2 \varepsilon(G)$ when $G \sim G_{n,p}$, $0 < p < 1$?
Random Graphs.

Theorem: (B.I.)

Let $G \sim G_{n,p}$ with $0 < p < 1$, $b = \frac{1}{1-p}$. Then, almost surely:

$$\log_2 \varepsilon(G) \sim \mathbb{E}[\log_2 \varepsilon(G)] \sim n \log_2 (2 \log_b n - 2 \log_b \log_b n).$$

Proof Idea:

Consider a random graph $G \sim G_{n,p}$ and let $n \to \infty$. Color $G$ with a minimal number of colors $k := \chi(G)$. Then:

$$\log_2 \varepsilon(G) \geq k \log_2 \lfloor n/k \rfloor!.$$

From Kahn and Kim (1995) we obtain:

$$n(\log_2 n - H(G)) \geq \log_2 \varepsilon(G),$$

where $H(G)$ is the graph entropy.

Lastly, from Bollobás (1988) and McDiarmid (1990)'s theorem for $\chi(G)$, and applying both Azuma's and Jensen's inequalities, we obtain the desired result.
From here, we can obtain an exponential approximation to the volume of the stable polytope of “most” graphs:

**Corollary: (B.I.)**

Let $G \sim G_{n,p}$ with $0 < p < 1$, $b = \frac{1}{1-p}$ and $s = 2 \log_b n - 2 \log_b \log_b n$. Then, almost surely:

$$\frac{s^n}{n!} \cdot \left(\frac{1}{e}\right)^n \leq \text{Vol}(\text{STAB}(G)) \leq \frac{s^n}{n!} \cdot \kappa^{n/s},$$

where $\kappa = 2 \left(\frac{e}{2}\right)^{2/(\log_2 b)}$. 

**Stable Polytopes.**
Graphical Arrangements.

**Definition:**

For a simple undirected graph $G = G([n], E)$, the *graphical arrangement* of $G$ is the central hyperplane arrangement in $\mathbb{R}^n$ given by:

$$A_G := \{ x \in \mathbb{R}^n : x_i - x_j = 0 \text{ for all } \{i, j\} \in E \}.$$
Observations:

i. The complete fan in \( \mathbb{R}^n \) given by \( A_G \) is combinatorially dual to the **graphical zonotope of** \( G \):

\[
\mathcal{Z}(G) := \sum_{\{i,j\} \in E} [e_i - e_j, e_j - e_i].
\]

ii. Regions of the graphical arrangement \( A_G \) with \( G = G([n], E) \) are in bijection with acyclic orientations of \( G \), and correspond to vertices of \( \mathcal{Z}(G) \).
Fractional Volumes.

**Definition: (Klivans and Swartz (2011))**

Given a central hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$, the **fractional volume** of a region $\mathcal{R}$ of $\mathcal{H}$ is the quantity:

$$\text{Vol}^\circ (\mathcal{R}) = \frac{\text{Vol}(B^n \cap \mathcal{R})}{\text{Vol}(B^n)},$$

where $B^n$ is the unit $n$-dimensional ball in $\mathbb{R}^n$.

**Proposition: (B.I., but more folklore)**

Let $G = G([n], E)$ be a simple graph with graphical arrangement $\mathcal{A}_G$. If $\mathcal{R}$ is a region of $\mathcal{A}_G$ and $P$ is its corresponding partial order on $[n]$, then:

$$\text{Vol}^\circ (\mathcal{R}) = \frac{e(P)}{n!}.$$

**A possible connection:**

Since $\mathcal{A}_G$ and $\mathcal{Z}(G)$ are combinatorially dual, can we use $\mathcal{Z}(G)$ to obtain information about the region of $\mathcal{A}_G$ with largest fractional volume?
Lemma: (B.I.)

Given a simple graph $G = G([n], E)$ and an acyclic orientation $O$ of its edges, the vertex $x^O$ of $\mathcal{Z}(G)$ corresponding to $O$ is coordinate-wise described via:

$$x_i^O = \text{indeg}(i) - \text{outdeg}(i),$$

where $\text{indeg}(\cdot)$ and $\text{outdeg}(\cdot)$ are calculated in $O$. Furthermore:

$$\frac{1}{2} \|x^O\|_2^2 = \frac{1}{2} \sum_{i \in [n]} \left(\text{indeg}(i) - \text{outdeg}(i)\right)^2 = \text{(expression)}.$$
Two optimization problems.

Proposition: (B.I.)

- Let $G = G([n], E)$ be a simple graph with $E = \{E_1, \ldots, E_m\}$. Choose an orientation of $G$ and let $Q$ be the corresponding $n \times m$ incidence matrix. Consider the two quadratic programs:

$$\max_{x \in [-1,1]^m} x^T Q^T Qx, \quad (\text{Problem } P_1)$$

and

$$\max_{x \in [-1,1]^n} x^T QQ^T x = x^T Lx, \quad (\text{Problem } P_2)$$

where $L$ is the combinatorial Laplacian of $G$.

- Then, Problem $P_1$ solves maximal-norm vertex of $Z(G)$, and Problem $P_2$ solves max-cut for $G$.

- Furthermore, if $G$ is a comparability graph or an odd cycle, then Problem $P_1$ solves the Main Problem for $G$. 
Main Problem.
Introduction.
Results.

Largest eigenvalue of the combinatorial Laplacian.

An interesting relaxation:

- For a general graph $G$, Problem $P_1$ and Problem $P_2$ are difficult problems. Consider instead the mutual relaxation:

$$\max_{x \in \mathbb{R}^n, ||x||_2 = 1} x^T L x,$$

(Problem $Q$)

where $L$ is the combinatorial Laplacian of $G$.

- Then, Problem $Q$ asks to describe the eigenspace of $L$ corresponding to the largest eigenvalue.

Theorem: (B.I.)

1. Let $G = G([n], E)$ be a comparability graph with combinatorial Laplacian $L$, largest eigenvalue $\lambda_{\text{max}}$ and associated eigenspace $E_{\lambda_{\text{max}}}$. Then, labeling vertices of $G$ with eigenvectors in $E_{\lambda_{\text{max}}}$ and orienting edges accordingly, we obtain precisely Gallai’s modular decomposition theory for $G$.

2. For a general simple graph $G$, a similar result holds.
Thank you.

- Richard Stanley, Carly Klivans, Tanya Khovanova, Federico Ardila, Diego Cifuentes.
- Full details in arxiv.org/abs/1405.4880