A BIJECTION BETWEEN NONCROSSING AND NONNESTING PARTITIONS FOR CLASSICAL REFLECTION GROUPS

Senior Thesis

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Dedicated to the interested reader.
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Abstract

This document intends to be, on the one hand, an introduction to one of the very popular unsolved problems in the combinatorics of Coxeter groups: finding a bijection between noncrossing and nonnesting partitions for finite Weyl groups. On the other hand, it is also a place to present original results in this direction. It is an extension of the article with the same name at [9].

Several objects related to Coxeter groups have been found to be counted by the Coxeter-Catalan numbers [2] and there is a general willing to understand the connection between these objects. This is work is framed in that context, known as the Coxeter-Catalan Combinatorics.

First, we introduce the reader to the problem at hand, and this we do from the author's own point of view. Then, we present a type preserving bijection between noncrossing and nonnesting partitions for all classical reflection groups, answering a question of Athanasiadis and Reiner. The bijections for the abstract Coxeter types $B$, $C$ and $D$ are new in the literature. To find them we define, for every type, sets of statistics that are in bijection with noncrossing and nonnesting partitions, and this correspondence is established by means of elementary methods in all cases. The statistics can be then seen to be counted by the generalized Catalan numbers $\text{Cat}(W)$ when $W$ is a classical reflection group. In particular, the statistics of type $A$ appear as a new explicit example of objects that are counted by the classical Catalan numbers.
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This work is also due to my advisor Federico Ardila. He introduced me to the Coxeter-Catalan Combinatorics and to several open problems on noncrossing and nonnesting partitions. He taught me the fundamentals of Coxeter groups from his own perspective, which helped this work develop until its very end. I specially thank Federico for his many useful comments and suggestions, but more than anything, for his motivation and encouragement.

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CHAPTER 1

Introduction and background

Suppose we have two lines $l_1$ and $l_2$ in the real $xy$-plane, each of which passes through the origin. Consider the operators that reflect points of the plane on these lines. So, for instance, denote by $r_1$ the operator reflecting points on $l_1$ and denote by $r_2$ the operator reflecting on $l_2$. If we are given a point $x$ of the plane, the point $r_1(x)$ would be the reflection of this point on the line $l_1$. We understand this action very well and we could say many things about it. For example, we could claim $(r_1(x) - x) \cdot y = 0$ for all $y \in l_1$.

We could ask as well about the operator $r_1 r_2$. How much do we know about this one? This is well known from linear algebra: $r_1 r_2$ is a rotation about the origin. And what about $r_2 r_1$? We know again the answer from linear algebra: $r_2 r_1$ is back again a reflection. We could consider a new line $l_3$ different from $l_1$ and $l_2$, and its associated reflection operator $r_3$. Can we say something of $r_1 r_2 r_3$? Suppose $r_2 r_3$ rotates points at an angle $\theta$ about the origin. If we rotated $l_1$ at an angle $-\theta/2$, we would get a new line $l'_1$: the operator $r_1 r_2 r_3$ would fix this line. Also, the points of the line $l_1^\perp$ would be sent to their negatives by $r_1 r_2 r_3$, allowing $r_1 r_2 r_3$ to be nothing else than a reflection, once again. In general, a product of $n$ reflection operators in the plane is a rotation if $n$ is even and a reflection if $n$ is odd. We understand both of these actions very well, so we should not be afraid of saying how much we understand products of reflections in the real plane.

We could probably raise the difficulty by raising the dimension. Suppose $r_1$ and $r_2$ are now reflection operators in the Euclidean space with corresponding reflecting planes $h_1$ and $h_2$, each of which contains the origin. What can we say of $r_1 r_2$ in this case? The answer is available to us, $r_1 r_2$ fixes the line $h_1 \cap h_2$ and rotates points in the plane $(h_1 \cap h_2)^\perp$ about the origin, so $r_1 r_2$ is exactly a rotation about the line $h_1 \cap h_2$. What about $r_2 r_1$? We know this one fixes again $h_1 \cap h_2$. It is also a reflection in $(h_1 \cap h_2)^\perp$, say with reflecting line $l'$. We could then see $r_2 r_1$ is precisely a reflection on the plane $l' \times (h_1 \cap h_2)$. We understand this very well. Introducing a new reflection operator $r_3$ with respect to a plane $h_3$ different from $h_1$ or $h_2$, could we say something that good about $r_1 r_2 r_3$? Being afraid to appear ignorant, we could only answer: not that we know. We know $r_1 r_2 r_3$ is orthogonal, we could calculate its determinant $-1$, its inverse operator $r_3 r_2 r_1$ and the like, we could even guarantee that it is neither a reflection nor a rotation, but it should not be so easy to picture in our head what this operator does.
In general, suppose we were given \( m \) reflection operators \( r_1, r_2, \ldots, r_m \) in \( \mathbb{R}^n \). What can we say about the operator \( r_1 r_2 \ldots r_m \)? Something of the sort \( r_2 r_1 r_2 r_1 \ldots r_1 \) is either a reflection or a rotation and maybe some specific examples like this one could be treated, but things as general as \( r_1 r_2 \ldots r_m \) are mysterious to people like us. There could be other interesting questions to ask. For example, could we find reflection operators \( r_1, r_2, \ldots, r_m \) in \( \mathbb{R}^n \) such that the algebraic structure generated by taking products of them is finite? However hard this question sounds, it has already been answered and we will discuss some details later. But we are still left with the bad feeling: what can we really say about \( r_1 r_2 \ldots r_m \)? For example, are there more ways to express this operator as product of other reflections? If so, how is this done precisely?

Each one of the operators \( r_1, r_2, \ldots, r_m \) can be defined completely only by knowing the normal vectors to the reflecting hyperplanes, \( v_1, v_2, \ldots, v_m \). We understand vector spaces and we know how to operate these vectors (by usual addition) and how to multiply them by reals. A very innocent question to pose would be, could this vector space algebra between \( v_1, v_2, \ldots, v_m \) tell us more about something like \( r_1 r_2 \ldots r_m \)? If we forget for one moment how innocent that question is, we should realize how great it would be to actually know something like it.

We do not pretend to answer this question, clearly. But for us, the problem of finding a bijection between noncrossing and nonnesting partitions of classical reflection groups is, in a very concrete scenario, deeply related to that question. We hope this becomes clear later.

1.1. Reflection Groups

Let \( V \) denote a real vector space with dimension \( n \) and inner product \( \langle \cdot, \cdot \rangle \). We say a linear operator in \( V \) is a reflection if it takes some vector \( \alpha \in V \) to \(-\alpha\) and fixes pointwise the hyperplane \( \alpha^\perp \). A basis for \( V \) can be constructed from \( \alpha \) and \( n - 1 \) vectors from \( \alpha^\perp \), so this is indeed a good definition for a linear operator in \( V \). We denote this reflection by \( t_\alpha \).

**Proposition 1.1.1.** The operator \( t_\alpha \) with \( \alpha \in V \) is given by the equation,

\[
t_\alpha(\lambda) = \lambda - 2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha
\]

**Proof.** We have \( t_\alpha(\alpha) = \alpha - 2\alpha = -\alpha \) and for \( \lambda \in \alpha^\perp \), we have \( t_\alpha(\lambda) = \lambda - 0 = \lambda \). \( \square \)

Reflections have useful properties as linear operators. For instance, notice that \( (t_\alpha)^2 \) fixes the vector \( \alpha \) and fixes pointwise \( \alpha^\perp \), so it fixes a basis of \( V \). Therefore, \( (t_\alpha)^2 \) is the identity operator in \( V \). Also,

**Proposition 1.1.2.** The reflection operator \( t_\alpha \) is an orthogonal operator in \( V \).
Proof. We use the representation of Theorem 1.1.1. For \( \beta, \lambda \in V \) we have
\[
\langle t_{\alpha}(\beta), t_{\alpha}(\lambda) \rangle = \langle \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha, \lambda - 2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \rangle = \langle \beta, \lambda \rangle - 4\frac{\langle \beta, \alpha \rangle \langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} + 4\frac{\langle \beta, \alpha \rangle \langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \langle \beta, \lambda \rangle
\]
\[
\square
\]

In this setting, a finite reflection group is a finite group of operators in \( V \) generated by reflections. As a consequence of the last theorem, finite reflection groups are subgroups of the orthogonal group \( O(V) \).

Example 1.1.3. \( (A_{n-1}, n \geq 2) \) The symmetric group. With \( V = \mathbb{R}^n \), let permutations act on vectors by permuting coordinates. For example, with \( \sigma \in S_n \) and \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \), let \( \sigma(a) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) \). Exactly the same action can be defined if we say the following: let permutations act on \( \mathbb{R}^n \) by permuting the vectors \( e_1, e_2, \ldots, e_n \) of the canonical basis (permuting subscripts). The key is to realize that the transposition \( (ij) \) with \( i \neq j \) acts on \( \mathbb{R}^n \) precisely as a reflection: it takes \( e_j - e_i \) to its negative and fixes pointwise the hyperplane \( x_j = x_i \), which is \( (e_j - e_i)^\perp \) with the canonical inner product of \( \mathbb{R}^n \). Following our convention, we would represent this reflection as \( t_{e_j - e_i} \). Now, it is a well known fact that transpositions generate the symmetric groups. Therefore, this is an example of a reflection group of order \( n! \). More can be said about this particular action of the symmetric group. Notice that it fixes pointwise the line \( x_1 = x_2 = \cdots = x_n \) (generated by \( e_1 + e_2 + \cdots + e_n \) and stabilizes its orthogonal complement \( x_1 + x_2 + \cdots + x_n = 0 \). No nonzero vector in the hyperplane \( x_1 + x_2 + \cdots + x_n = 0 \) is left fixed by all permutations: it cannot have all its components equal, so we can swap the greatest and least components. Thus, the symmetric group \( S_n \) is essential with respect to a space of dimension \( n - 1 \). We denote this reflection group by \( A_{n-1} \).

What we did in this example can be generalized in the following result.

Proposition 1.1.4. Let \( W \) be a subgroup of \( O(V) \). Then \( W \) stabilizes the orthogonal complement of its set of fixed points \( \text{Fix}(W) \) and is essential with respect to it.

Proof. That \( W \) is essential when acting on \( \text{Fix}(W)^\perp \) is clear. We show that \( W \) stabilizes this set. Take \( w \in W, v \in \text{Fix}(W)^\perp \) and let \( u \) be any vector in \( \text{Fix}(W) \). Then \( \langle w(v), u \rangle = \langle w(v), w(u) \rangle = \langle v, u \rangle = 0 \), so \( w(v) \in \text{Fix}(W)^\perp \).

Example 1.1.5. \( (B_n, n \geq 2) \) Group of signed permutations. Again with \( V = \mathbb{R}^n \), consider the reflections that send \( e_i \) to \( -e_i \) for all \( 1 \leq i \leq n \): they form a group \( S \) of order \( 2^n \) isomorphic to \( \mathbb{Z}_2^n \) and such that \( S \cap A_{n-1} = \emptyset \). If \( a \in A_{n-1} \) and \( s \in S \), then \( asa^{-1} \in S \) clearly, so we are free to speak of the semidirect product of \( S \) and \( A_{n-1} \). Following the standard abstract algebra theory, this is a reflection group of order \( |S||A_{n-1}| = 2^n n! \) which contains a normal subgroup \( S \). This group is
denoted by \( B_n \) and is known as the group of signed permutations. Now, if we let \( S \) be the group generated by the reflections taking vectors \( e_i + e_j \) \((i \neq j)\) to their negatives, the same construction would give us the reflection group \( D_n, n \geq 4 \). The groups \( B_n \) and \( D_n \) are clearly essential with respect to \( R_n \).

1.2. Root Systems

Let \( W \) be a finite reflection group with set of reflections \( T_W = \{t_{\alpha_1}, t_{\alpha_2}, \ldots, t_{\alpha_m}\} \). The following result holds.

**Proposition 1.2.1.** If \( w \in W \) and \( 1 \leq i \leq m \), we have \( wt_{\alpha_i}w^{-1} = t_{w(\alpha_i)} \).

**Proof.** Let \( \lambda \in V \). Our operator \( w \) is orthogonal, so \( \langle w^{-1}(\lambda), \alpha_i \rangle = \langle \lambda, w(\alpha_i) \rangle \) and \( \langle \alpha, \alpha \rangle = \langle w(\alpha), w(\alpha) \rangle \). Using the representation theorem,

\[
wt_{\alpha_i}w^{-1}(\lambda) = w\left(w^{-1}(\lambda) - 2\frac{\langle w^{-1}(\lambda), \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i \right) = \lambda - 2\frac{\langle \lambda, w(\alpha_i) \rangle}{\langle \alpha_i, \alpha_i \rangle} w(\alpha_i) = t_{w(\alpha_i)}(\lambda).
\]

Consider the set of lines \( \mathcal{L} = \{\alpha_1 \mathbb{R}, \alpha_2 \mathbb{R}, \ldots, \alpha_m \mathbb{R}\} \). For \( 1 \leq i \leq m \) and \( w \in W \), we have \( w(\alpha_i \mathbb{R}) = w(\alpha_i) \mathbb{R} \). But \( t_{w(\alpha_i)} = wt_{\alpha_i}w^{-1} \in W \) so \( t_{w(\alpha_i)} = t_{\alpha_j} \) for some \( 1 \leq j \leq m \). This implies \( w(\alpha_i) \mathbb{R} = \alpha_j \mathbb{R} \). Now, \( w \) is a bijection in \( V \), so the operator \( w \) permutes the elements of \( \mathcal{L} \). More generally, \( W \) acts on \( V \) by permuting the set of lines \( \mathcal{L} \). Suppose from each line \( \alpha_i \mathbb{R} \) with \( 1 \leq i \leq m \) we choose two vectors \( \alpha'_i \) and \( -\alpha'_i \). Picking these pairs of vectors strategically to form a set \( \Phi \), we could make \( W \) stabilize \( \Phi \). For example we could let \( \alpha_1 \) and \( -\alpha_1 \) be our first choice of vectors. Then \( W \) would stabilize the set of vectors \( \pm \alpha'_i \) with \( \alpha'_i \in \alpha_i \mathbb{R} \) and \( |\alpha'_i| = \alpha_1 \) for all \( i \). The set \( \Phi \) so constructed would satisfy certain key properties.

**Definition 1.2.2.** Let \( \Phi \) be a nonempty finite set of nonzero vectors in \( V \) such that for all \( \alpha \in \Phi \)

\[
(R1) \ \Phi \cap \alpha \mathbb{R} = \{\alpha, -\alpha\} \\
(R2) \ t_\alpha \Phi = \Phi
\]

Then \( \Phi \) is called a root system and its elements are called roots.

From a finite reflection group \( W \) we can obtain a specific root system \( \Phi \) as explained. Recalling the construction of \( \Phi \) presented, it could be useful to ask whether all root systems should have its
vectors of equal length. Suppose we could manage to find nonempty disjoint sets of lines \( A \) and \( B \) stabilized by \( W \). Picking pairs of vectors of length \( l_A > 0 \) from each line in \( A \) and of length \( l_B > 0 \) from each line in \( B \) with \( l_A \neq l_B \), we could form root systems \( \Phi_A \) and \( \Phi_B \), respectively. But then \( \Phi = \Phi_A \cup \Phi_B \) would also be a root system whose vectors are not of equal length. The important thing to keep in mind here is that we can produce root systems from finite reflection groups.

It is natural to wonder what happens in the opposite direction. Suppose we are only given a root system \( \Phi \) and let \( W \) be the group generated by the reflections \( t_\alpha \) with \( \alpha \in \Phi \). Clearly, \( W \) stabilizes \( \Phi \). By definition, \( W \) fixes pointwise the orthogonal complement of the span of \( \Phi \). If for some \( w, w' \in W \) we had \( w(\alpha) = w'(\alpha) \) for all \( \alpha \in \Phi \), then \( w \) and \( w' \) would coincide in a basis for \( V \) and so \( w = w' \). But there is only a finite number of ways to permute the elements of the finite set \( \Phi \). Therefore, we conclude that the group \( W \) should be a finite reflection group. The important fact: from root systems we can produce finite reflection groups. We call \( \Phi \) a root system associated to the group \( W \). As we suggested, this set \( \Phi \) is not uniquely associated to \( W \), there can be infinitely many choices of roots systems for a finite reflection group \( W \).

Now, suppose we are given a total ordering of \( V \) and we want to understand how this order sorts out the elements of a root system \( \Phi \). For some \( \alpha \in \Phi \) we have either \( \alpha > 0 \) or \( \alpha < 0 \). We would then have \( -\alpha < 0 \) or \( -\alpha > 0 \), respectively. This shows that \( \Phi \) gets partitioned into two sets \( \Pi \) and \( -\Pi \) of roots, where every element of \( \Pi \) is strictly greater than 0 in the total ordering. This set \( \Pi \) is called positive system and its elements positive roots. Obviously, positive systems exist and they do so as much as total orderings of \( V \) exist. One such order could be the canonical lexicographic order. There is one other important subset of \( \Phi \) that we will often encounter.

**Definition 1.2.3.** A subset \( \Delta \) of \( \Phi \) is called a simple system if \( \Delta \) is a vector space basis for the span of \( \Phi \) in \( V \) and if, additionally, every root \( \alpha \in \Phi \) can be written as a positive or negative combination of elements from \( \Delta \). The elements of \( \Delta \) are called simple roots.

The existence of simple systems inside root systems is by no means obvious.

**Theorem 1.2.4.** Fix a choice of root system \( \Phi \) in our real vector space \( V \). Then, for each simple system there exists a unique positive system which contains it and, for each positive system there exists a unique simple system contained inside it. In particular, simple systems exist.

**Proof.** That simple systems have uniquely determined positive systems containing them is somewhat clear. Let \( \Pi \) be the set of roots which are positive combinations of elements of \( \Delta \). Their negatives, which are negative combinations of elements of \( \Delta \), will be \( -\Pi \). Extending \( \Delta \) to an ordered basis for \( V \), this positive system would be obtained by imposing the corresponding lexicographic order on \( V \), ie. the lexicographic order on the coefficient (coordinate) vectors with respect to our basis.
For the other direction we refer the reader to Humphreys [10].

Two additional results will be proven of great value later.

**Theorem 1.2.5.** Let $\Phi$ be a root system with generated group $W$ and let $\Delta$ be a simple system of $\Phi$. Then $W$ is generated by the set of reflections $T_\Delta = \{t_\alpha | \alpha \in \Delta\}$.

**Proof.** We need only prove that the set of reflections $t_\beta$ with $\beta \in \Phi$ is generated by our reflections in $T_\Delta$. Moreover, this requires that we only deal with reflections $t_\beta$ where $\beta \in \Pi$ and $\Pi$ is the positive system containing $\Delta$.

So let $W'$ be the group generated by $T_\Delta$ and let $\beta$ be a positive root, $\beta \in \Pi$. For each $\gamma \in \Pi$, define its height to be the sum $\sum_{\alpha \in \Delta} c_\alpha$ where $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$. Consider the $W'$-orbit $W'\beta$. We have $W'\beta \cap \Pi \neq \emptyset$ because $\beta$ is itself positive. Choose $\gamma \in W'\beta \cap \Pi$ of minimal height and assume $\gamma$ is not a simple root so that $\gamma$ is a positive combination of at least two different simple roots.

As $\gamma$ is a positive combination of elements of $\Delta$ and $\langle \gamma, \gamma \rangle > 0$, there must exist some $\alpha \in \Delta$ for which $\langle \gamma, \alpha \rangle > 0$. We know $t_\alpha(\gamma) = \gamma - A\alpha$ for some $A > 0$. But $t_\alpha(\gamma)$ is a root and $\gamma$ is a positive combination of at least two simple roots, so $\gamma - A\alpha$ must still be a positive combination of simple roots and, moreover, the height of $t_\alpha(\gamma)$ should be less than the height of $\gamma$, a contradiction. Thus $\gamma$ must be a simple root, $\gamma \in \Delta$.

We just showed how there must exist $w' \in W'$ such that $w'(\beta) = \alpha \in \Delta$. The inverse $w'^{-1}$ can be expressed by inverting the expression for $w'$ and we have $\beta = w'^{-1}(\alpha)$. But using Proposition 1.2.1 we obtain $t_\beta = t_{w'^{-1}(\alpha)} = w'^{-1}t_\alpha w' \in W'$, so indeed $t_\beta \in W'$.

Interesting enough is the technique employed in the last proof, which we present in a separate result.

**Corollary 1.2.6.** Given a simple system $\Delta$ of a root system $\Phi$ with group $W$, for every $\beta \in \Pi$ there exists $w \in W$ such that $w(\beta) \in \Delta$.

We may discern the relevance of understanding reflection operators in the study of finite reflection groups, which motivates the following definition. We presented this earlier and we now formalize it.

**Definition 1.2.7.** Let $W$ be a finite reflection group in the vector space $V$. We denote by $T_W$ the set of reflection operators of the group $W$. 
We have explained now how a finite reflection group \( W \) can be obtained from a root system \( \Phi \), but we have not answered the question of whether the set of reflections \( T_W \) is exactly \( T_\Phi = \{ t_\alpha | \alpha \in \Phi \} \) or some proper superset of it.

**Proposition 1.2.8.** Let \( W \) be a finite reflection group generated by the root system \( \Phi \). Then \( T_W = T_\Phi \).

**Remark 1.2.9.** This is a deceitfully hard theorem to prove. We do not include a full proof here because it involves a much more detailed development of the classical theory that is not relevant for our study.

**Proof Sketch:** Fix a choice of simple system \( \Delta \) and let \( \Pi \) the positive system containing it. Let \( t_\nu \) be a reflection in \( W \) and suppose \( \lambda \in \nu^\perp \). Introduce the partial ordering of \( V \) in which, for \( \mu, \gamma \in V \), we have \( \mu \leq \gamma \) if and only if \( \gamma - \mu \) is in the nonnegative span of \( \Delta \). Then choose a maximal element \( w(\lambda) \) with \( w \in W \) from the nonempty finite set \( \{ \mu | \mu = w(\lambda) \text{ for some } w \in W \text{ and } \lambda \leq \mu \} \). We can check that the maximality condition implies \( \langle w(\lambda), \alpha \rangle \geq 0 \) for all \( \alpha \in \Delta \). Now, \( t_{w(\nu)} = wt_\nu w^{-1} \in W \), so let \( t_{w(\nu)} = t_{\alpha_1}t_{\alpha_2} \ldots t_{\alpha_m} \) be an expression for \( t_{w(\nu)} \) of minimal length with \( \alpha_1, \alpha_2, \ldots, \alpha_m \in \Delta \). It can be proved that \( t_{w(\nu)}(\alpha_m) \in (-\Pi) \). By orthogonality we have \( t_{w(\nu)}(w(\lambda)) = w(\lambda) \), so \( \langle w(\lambda), \alpha_m \rangle = \langle t_{w(\nu)}(w(\lambda)), t_{w(\nu)}(\alpha_m) \rangle = \langle w(\lambda), t_{w(\nu)}(\alpha_m) \rangle \). But this is true for all \( \alpha \in \Delta \). Therefore, \( \langle w(\lambda), \alpha_m \rangle \geq 0 \) and \( \langle w(\lambda), t_{w(\nu)}(\alpha_m) \rangle \leq 0 \) because \( t_{w(\nu)}(\alpha_m) \) is a negative combination of elements of \( \Delta \). This forces \( \langle w(\lambda), \alpha_m \rangle = 0 \) and thus \( \alpha_m = w(\nu) \) for some \( r \in \mathbb{R}/\{0\} \). Inverting we get \( rv = w^{-1}(\alpha_m) \in \Phi \) and \( t_v = t_{rv} = t_{w^{-1}(\alpha_m)} \in T_\Phi \).

Few has been said so far about the relation between finite reflection groups and finite Coxeter groups. This is provided by the following important theorem that we present without proof.

**Theorem 1.2.10.** For \( \alpha, \beta \in \Delta \) let \( m_{\alpha, \beta} \) be the order of \( t_\alpha t_\beta \) in \( W \), that is, let \( m_{\alpha, \beta} \) be the least positive integer such that \( (t_\alpha t_\beta)^{m_{\alpha, \beta}} = 1 \in W \). Then \( W \) is generated as the free group of \( T_\Delta \)-words modulo the relations

\[
(t_\alpha t_\beta)^{m_{\alpha, \beta}} = 1 \text{ with } \alpha, \beta \in \Delta
\]

The simple roots \( \alpha \) and \( \beta \) in the previous statement need not be different and we actually have \( m_{\alpha, \alpha} = 1 \) for all \( \alpha \in \Delta \). This realization provides the remaining ingredients for the following corollary.

**Corollary 1.2.11.** All finite reflection groups are finite Coxeter groups.

The relations of Theorem 1.2.10 characterizing the algebra of \( T_\Delta \)-words are called Coxeter relations. This information is usually encoded in certain graphs. Let \( \Gamma \) be an undirected graph with vertex set \( T_\Delta \). For \( \alpha, \beta \in \Delta \), join vertices \( t_\alpha \) and \( t_\beta \) of our graph by an edge labelled \( m_{\alpha, \beta} \).
whenever this number is at least 3. If distinct vertices \( t_\alpha \) and \( t_\beta \) are not joined, it is then understood that \( m_{\alpha,\beta} = 2 \), so that \( t_\alpha \) and \( t_\beta \) commute. As a simplifying convention, the label \( m_{\alpha,\beta} = 3 \) is usually omitted. Then, \( \Gamma \) is called the Coxeter graph of \( W \) with respect to \( T_\Delta \).

Finite irreducible (with connected Coxeter graphs) Coxeter groups have all been classified, see for instance [6]. Their respective Coxeter graphs are all trees. From now on, we assume \( W \) to be irreducible but we will mention this occasionally when convenient.

Example 1.2.12. We present the standard choice of simple systems and positive systems that give rise to the so called classical reflection groups, which are the central object of study in this writing.

\[
\Delta_W = \begin{cases} 
\{ e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1} \} & \text{ for } W = A_{n-1} \\
\{ e_1, e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1} \} & \text{ for } W = B_n \\
\{ 2e_1, e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1} \} & \text{ for } W = C_n \\
\{ e_1 + e_2, e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1} \} & \text{ for } W = D_n
\end{cases}
\]

\[
\Pi_W = \begin{cases} 
\{ e_j - e_i | 1 \leq i < j \leq n \} & \text{ for } W = A_{n-1} \\
\{ e_i | 1 \leq i \leq n \} \cup \{ e_j \pm e_i | 1 \leq i < j \leq n \} & \text{ for } W = B_n \\
\{ 2e_i | 1 \leq i \leq n \} \cup \{ e_j \pm e_i | 1 \leq i < j \leq n \} & \text{ for } W = C_n \\
\{ e_j \pm e_i | 1 \leq i < j \leq n \} & \text{ for } W = D_n
\end{cases}
\]

1.3. Invariant Theory

Following our convention, let \( V \) be a real vector space with inner product and dimension \( n \). Fix a choice of basis for \( V \). Considering the coefficient (coordinate) representations of vectors in \( V \) corresponding to our chosen basis, there is no loss of generality if we allow ourselves assume that \( V = \mathbb{R}^n \). We define an action of the group of invertible real \( n \times n \) matrices \( GL_n(\mathbb{R}) \) on the ring of polynomials \( \mathbb{R}[x_1, x_2, \ldots, x_n] = R[x] \). Elements of \( R[x] \) may be seen as real functions on \( \mathbb{R}^n \) in the natural way.

Consider an invertible matrix \( M \in GL_n(\mathbb{R}) \). For \( v \in \mathbb{R}^n \) and \( f \in R[x] \), let \( Mf(v) = f(M^{-1}(v)) \). This action may be seen to preserve the grading of \( R[x] \) by degree. We are interested in the cases where \( M \) belongs to the group of orthogonal matrices \( O_n(\mathbb{R}) \). In that case, we would have \( Mf(v) = f(M^tv) \) for all \( v \in \mathbb{R}^n \) and \( f \in R[x] \), which is a very intuitive action as the following example shows.

Example 1.3.1. Suppose

\[
M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]
and \( f(x) = x_1x_3 + x_2^5 + x_3 - x_1 \). Then we have the equality of functions \( Mf(x) = f(M^t x) = x_2x_1 + x_3^5 + x_1 - x_2 \).

We can understand this particular case very well. Moreover, we are interested in the concrete case where \( M \) is the matrix representation for some operator \( w \in W \), \( W \) a finite reflection group.

So let \( W \) be a finite reflection group consisting of matrices. Consider the set of polynomials \( f \in R[x] \) such that \( wf = f \) for all \( w \in W \). This set is called the ring of \( W \)-invariant polynomials and we denote it by \( \text{Inv}(W) \). It is easily seen that \( \text{Inv}(W) \) has the structure of a ring, as suggested by its name, so it is a \( R \)-subalgebra of \( R[x] \). But we should ask here how interesting is this subalgebra. Are all its elements constants?

**Example 1.3.2.** Consider \( R^n \) with the canonical inner product and let \( W \) be a finite reflection group of matrices. Consider the polynomial \( f(x) = x_1^2 + x_2^3 + \cdots + x_n^2 \). For all \( v \in R^n \) we have \( f(v) = \langle v, v \rangle \) so with \( w \in W \), \( wf(v) = f(w^tv) = \langle w^tv, w^tv \rangle = \langle v, v \rangle = f(v) \), which implies \( wf = f \) for all \( w \in W \). Therefore, \( f \in \text{Inv}(W) \) for all finite reflection groups \( W \).

It can be proved as well that, in our specific setting, the field of fractions of \( \text{Inv}(W) \) is of transcendence degree \( n \) over \( R \). The following theorem by Chevalley is the keystone of our discussion.

**Theorem 1.3.3 (Chevalley’s Theorem).** Let \( \text{Inv}(W) \) be the subalgebra of \( R[x] \) of \( W \)-invariant polynomials. Then \( \text{Inv}(W) \) is generated as an \( R \)-algebra by \( n \) homogeneous, algebraically independent polynomials of positive degree and by 1.

There may exist more than one generating set of this kind for \( \text{Inv}(W) \), but the degrees of its \( n \) elements is unique modulo a reordering.

**Proposition 1.3.4.** Suppose that \( f_1, f_2, \ldots, f_n \) and \( g_1, g_2, \ldots, g_n \) are two sets of homogeneous, algebraically independent generators of the ring \( \text{Inv}(W) \) of \( W \)-invariant polynomials. Denote the respective degrees of these polynomials by \( d_i \) and \( e_i \), \( 1 \leq i \leq n \). Then, after renumbering one of the sets if necessary, we have \( d_i = e_i \) for all \( i \).

**Remark 1.3.5.** Within our dimension \( n \) space, to any finite reflection group \( W \) we therefore associate its unique and well defined set of degrees \( d_1, d_2, \ldots, d_n \).

For a complete proof of these results refer to the book of Humphreys [10]. We say a few words about the idea.

Consider the ideal \( I_{\text{Inv}(W)}^0 \) generated in \( R[x] \) by the elements of \( \text{Inv}(W) \) with constant term equal to 0. By Hilbert Basis Theorem, \( I_{\text{Inv}(W)}^0 \) can be generated by only a finite number of polynomials in \( \text{Inv}(W) \) with constant term 0, say \( f_1, f_2, \ldots, f_r \). It turns out that these polynomials (along with 1) also generate \( \text{Inv}(W) \) as a \( R \)-algebra. This is refined further until we get to Theorem 1.3.3.
1.4. Introduction to Nonnesting Partitions

This section is a natural continuation of our discussion in 1.2 and we keep the conventions adopted there. Assume from now on that $V = \mathbb{R}^n$. To present the notion of nonnesting partition we need first introduce that of crystallographic groups and systems.

Definition 1.4.1. A finite reflection group $W$ is said to be crystallographic if $W$ stabilizes a lattice in $V$. More explicitly, $W$ is crystallographic if for some $\mathbb{Z}$-span $L$ of a basis for $V$ we have $w(L) \subseteq L$ for all $w \in W$.

Example 1.4.2. Consider the group $A_{n-1}$. In the standard representation, it has a simple system $\Delta = \{ e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1} \}$. By definition, the elements of $\Delta$ are linearly independent. The vector $e_1 + e_2 + \cdots + e_n$ is orthogonal to $\Delta$, so the set

$$\{ e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}, e_1 + e_2 + \cdots + e_n \}$$

is a basis for $\mathbb{R}^n$. Without loss of generality, consider the action of $t_{e_2-e_1}$ on $\Delta$. We have

$$t_{e_2-e_1}(e_2 - e_1) = -(e_2 - e_1),$$
$$t_{e_2-e_1}(e_3 - e_2) = e_3 - e_1 = (e_2 - e_1) + (e_3 - e_2),$$
$$t_{e_2-e_1}(e_i - e_{i-1}) = e_i - e_{i-1} \text{ for } 3 < i \leq n \text{ and}$$
$$t_{e_2-e_1}(e_1 + e_2 + \cdots + e_n) = e_1 + e_2 + \cdots + e_n.$$  

All the images are in the $\mathbb{Z}$-span of our basis and therefore, $W$ stabilizes that lattice.

Crystallographic groups are characterized by the specific Coxeter relations they satisfy, cf. recall Theorem 1.2.10. This is an impressive result from our point of view. A proof may be found in Humphreys [10] or Armstrong [1].

Theorem 1.4.3. Let $W$ be a finite reflection group with simple system $\Delta$. If $W$ is crystallographic, then each integer $m_{\alpha,\beta}$ must be 2, 3, 4 or 6 whenever $\alpha \neq \beta$ and $\alpha, \beta \in \Delta$. On the contrary, if $W$ satisfies this necessary condition for some simple system $\Delta$, then the lengths of the simple roots may be modified so that $W$ stabilizes a lattice in $V$.

Proof. We prove only one direction and refer the reader to the references for the remaining one. Suppose $W$ is a finite crystallographic group with simple system $\Delta$ and let $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$. The orthogonal operator $t_\alpha t_\beta$ can be seen geometrically as follows. Consider the plane of $V$ spanned by $\alpha$ and $\beta$. Our operator stabilizes this plane and fixes pointwise its orthogonal complement $\alpha^\perp \cap \beta^\perp$. The restriction of the operator $t_\alpha t_\beta$ to this bidimensional space is still orthogonal and a product of two reflections. Therefore, it is a rotation in that plane. Suppose the rotation angle is $\theta$ and choose an orthonormal basis for our plane. Completing this basis with $n-2$
appropriate vectors from $\alpha^\perp \cap \beta^\perp$ we get a basis for $V$. The representation matrix for $t_\alpha t_\beta$ with respect to this basis is

$$M_{t_\alpha t_\beta} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & I \end{pmatrix}$$

where $I$ is the $(n-2) \times (n-2)$ identity matrix. The trace of this operator is thus $n - 2 + 2\cos(\theta)$. If we now consider the basis whose $\mathbb{Z}$-span is stabilized by $W$, then the trace of the representation matrix for our operator with respect to this basis would be integer. But similar matrices have equal trace, so $n - 2 + 2\cos(\theta) \in \mathbb{Z}$. That is, we require $\cos(\theta) = 0$, $\pm 1/2$ or $\pm 1$ and we know $\theta$ is of the form $\theta = \frac{2\pi k}{m_{\alpha,\beta}}$ for some $k \in \mathbb{Z}^+$. Getting our hands dirty and working out the possible cases we may see that this is only possible when $m_{\alpha,\beta}$ is 2, 3, 4 or 6. \qed

There is also a notion of crystallographic root systems.

**Definition 1.4.4.** Let $\Phi$ be a root system. We say that $\Phi$ is **crystallographic** if in addition to the two conditions of root systems (R1) and (R2) it satisfies a third condition

$$(R3) \quad \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi.$$ 

A reflection group $W$ arising from a crystallographic root system $\Phi$ is called a **Weyl Group of $\Phi$**.

This is the point where we should ask ourselves whether these two concepts are compatible or related to each other in some degree. The answer is the best one we can expect, but we need a lemma first.

**Lemma 1.4.5.** Let $\Phi$ be a crystallographic root system with simple system $\Delta$ and suppose that $\Phi$ generates a reflection group $W$. Then, the positive roots of $\Phi$ can be expressed as positive integer combinations of simple roots.

**Proof.** Let $\beta \in \Pi$, where $\Pi$ is the positive system. By corollary to Theorem 1.2.5 there exists $w \in W$ such that $w(\beta) = \alpha \in \Delta$. Inverting, we get $\beta = w^{-1}(\alpha)$. We know $w^{-1}$ is a product of simple reflections from $T_\Delta$. So, expanding $w^{-1}(\alpha)$ using the reflection representation equation successively and using the crystallography of $\Phi$ we express $\beta$ as an integer combination of simple roots, which must be a positive integer combination. \qed

**Proposition 1.4.6.** Let $W$ be a Weyl group of $\Phi$, then $W$ is a crystallographic group. If $W$ is a crystallographic group with simple system $\Delta$, then the length of the simple roots may be modified to obtain a crystallographic root system $\Phi$ associated to $W$.

**Proof.** We only prove one direction. Suppose $\Phi$ is a crystallographic root system generating the group $W$. By the previous lemma, the set $\{\sum_{\alpha \in \Delta} n_\alpha \alpha | n_\alpha \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ is stabilized by
W. But $W$ actually fixes pointwise any basis $\xi$ of the orthogonal complement of the span of $\Phi$, so it would stabilize as well the $\mathbb{Z}$-span of the basis $\Delta \cup \xi$ for $V$. \hfill \square

**Example 1.4.7.** The finite irreducible Weyl groups have all been classified into **combinatorial types**. They are,

I. The infinite family of groups $A_{n-1}, n \geq 2$, called the type $A$.
II. The infinite family of groups $B_n, n \geq 2$, called the type $B$.
III. The infinite family of groups $C_n, n \geq 2$, called the type $C$.
IV. The infinite family of groups $D_n, n \geq 4$, called the type $D$.
V. The group $E_6$, called the type $E_6$.
VI. The group $E_7$, called the type $E_7$.
VII. The group $E_8$, called the type $E_8$.
VIII. The group $F_4$, called the type $F_4$.
IX. The group $G_2$, called the type $G_2$.

The types $A$, $B$, $C$ and $D$ constitute the so called classical reflection groups. The remaining groups (types) $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$ are called **exceptional Weyl groups**.

To define nonnesting partitions of finite reflection groups we require the condition of crystallographicity. Assume $W$ is crystallographic.

**Definition 1.4.8.** The **root poset** of $W$ is its set of positive roots $\Pi$ with the partial order $\leq$ under which, for $\beta, \gamma \in \Pi$, $\beta \leq \gamma$ just if $\gamma - \beta$ lies in the nonnegative integer span of the simple roots.

**Remark 1.4.9.** This definition of the root poset is distinct from, and more suited for connections to nonnesting partitions than, the one given in Björner & Brenti [6], which does not require crystallographicity (and which in fact has a strictly weaker order than Definition 1.4.8).

**Example 1.4.10.** Figure 1.1 exhibits the root posets of the classical reflection groups with simple systems $\Delta_W$ and positive systems $\Pi_W$ as given in Example 1.2.12. We annotate the lower verges of the root posets with a line of integers, which for reasons of space we bend around the left side. Given a dot in Figure 1.1, if $i$ and $j$ are the integers in line with it on downward rays of slope 1 and $-1$ respectively, then it represents the root $\alpha = e_j - e_i$, where $e_k = -e_{-k}$ for $k < 0$ and $e_0 = 0$. Define the system of coordinates $ud$ in which this point has coordinates $u = i$ and $d = j$. By doing this, we have uniformized to some degree the root posets of classical reflection groups. The system $ud$ will be of help later.

We invite the reader to figure out what the lowest elements of each poset are.
1.5. Introduction to Noncrossing Partitions

Figure 1.1. The root posets for groups (left to right) $A_4$, $B_4$, $C_4$, and $D_4$.

Definition 1.4.11. A (uniform) nonnesting partition for $W$ is an antichain in the root poset of $W$. We denote the set of nonnesting partitions of $W$ by $NN(W)$.

Here we should naturally raise the question: why do we require crystallographicity to define nonnesting partitions? After all, we could be tempted to redefine the root poset in a slightly different way, requiring only that $\beta \leq \gamma$ just if $\gamma - \beta$ lies in the nonnegative real span of the simple roots and therefore including the non-crystallographic cases as well. It turns out that the poset so defined has completely the wrong combinatorial properties, refer to Section 5.4 of Armstrong [1]. We mention one of these key properties which are lost in the non-crystallographic case. This property is not only relevant for us, but merely central in this work: the set of nonnesting partitions $NN(W)$ is counted by the $W$-Catalan number $Cat(W)$. But this discussion is one of the most important open problems in the area, how to define the root poset for all finite reflection groups in such way that the nice combinatorial properties are preserved for Weyl groups and so that the same properties are satisfied within the remaining cases.

The following theorem is fundamental in the study of nonnesting partitions. A proof may be found in [14].

Theorem 1.4.12. Let $W$ be a Weyl Group. Then, the elements of an antichain of its root poset are linearly independent.

1.5. Introduction to Noncrossing Partitions

Let $W$ be a finite reflection group with set of reflections $T_W$, root system $\Phi$ and simple system $\Delta$. We have already discussed how every element of $W$ may be written as a $T_\Delta$-word modulo the Coxeter relations. We are going to do something similar here.

The orthogonal operator $w \in W$ can be written as a product of simple reflections so it follows that it can also be expressed as a product of reflections, ie. as a $T_W$-word (or equivalently a $T_\Phi$-word). One thing is important to realize here: by doing this we have just stepped away from the usual Coxeter groups setting. Several theorems have to be proved again to understand this new...
algebra of $T_W$-words and they do not necessarily coincide with the standard theorems of Coxeter groups. However, $W$ remains a Coxeter group anyway, we now just consider a different way to express its elements.

A well known fact in the theory of finite reflection groups (and accordingly also standard in the theory of Coxeter groups) is that the set of reflections $T_W$ of $W$ is exactly the set of elements of $W$ which can be expressed as palindromic $T_\Delta$-words. Actually, we have already gathered the necessary background to prove this. Every reflection of $W$ is of the form $t_\beta$ with $\beta \in \Pi$. The $W$-orbit of $\beta$ reaches $\Delta$ so $\beta = w^{-1}(\alpha)$ for some $\alpha \in \Delta$, but then $t_\beta = w^{-1}t_\alpha w$, which is palindromic. The pair $(W, T_W)$ is an example of a dual Coxeter system, denoting the Coxeter group $W$ with the algebra of $T_W$-words.

**Example 1.5.1.** The letter $T$ to denote reflections is a bit unfortunate, but the reason for this is not simple arbitrariness: the set $T_W$ coincides with the set of transpositions in the case $W = A_{n-1}$, $n \geq 2$. The symmetric group is the group that probably motivated many of the studies in the area and it is quite common to find objects named because of how they present themselves in $A_{n-1}$.

As in the case of $T_\Delta$-words, we are interested in the representation of the elements of $W$ as $T_W$-words of minimal length. For some fixed $w \in W$, let the set of $T_W$-reduced words for $w$ be the set of $T_W$-words with minimal length expressing $w$. This set is almost connected by shifts, see Lemma 2.5.1 of [1], but more important for us is that each $T_W$-reduced word for $w$ has the same length, which we denote by $l_{T_W}(w)$. The following is a breakthrough in the study of $T_W$-reduced words.

**Lemma 1.5.2 (Carter’s Lemma).** Let $W$ be a finite reflection group and $w \in W$. Then, the $T_W$-word

$$w = t_{\alpha_1}t_{\alpha_2}\ldots t_{\alpha_m}$$

is reduced if and only the set of roots $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is linearly independent.

Several things can be done with Lemma 1.5.2, we will get to them in time. For now we define the fundamental poset in the study of noncrossing partitions.

**Definition 1.5.3.** Let $W$ be a finite reflection group. The absolute order $\text{Abs}(W)$ of $W$ is the partial order on $W$ such that for $w, x \in W$, $w \leq x$ if and only if

$$l_{T_W}(x) = l_{T_W}(w) + l_{T_W}(w^{-1}x).$$

More explicitly, we say $w \leq x$ if there exist $T_W$-reduced words $w', x'$ and $y'$ such that $w'$ is a reduced expression for $w$, $x'$ is a reduced expression for $x$, and

$$x' = w'y'.$$
The absolute order of $W$ is a poset graded by $l_{T_W}$ and has a unique minimal element, the identity $1 \in W$. On the contrary, it can have several different maximal elements. Actually, if we go into the exceptional Weyl groups we may find maximal elements of the absolute order which do not even have equal lengths as $T_W$-words.

Some maximal elements of the absolute order of $W$ are of importance to us. Consider a simple system $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ associated to $W$. Let $\sigma$ be a permutation of the set $[m]$ and let $c = \prod_{i=1}^{m} t_{\alpha_{\sigma(i)}}$. By definition, $\Delta$ is linearly independent. Using Lemma 1.5.2 we conclude that $\prod_{i=1}^{m} t_{\alpha_{\sigma(i)}}$ is a $T_W$-reduced word for $c$. But moreover, as the dimension of the span of $\Phi$ is precisely $m$, we see that for all $w \in W$ we must have $l_{T_W}(w) \leq m$, so indeed $c$ is a maximal element of the absolute order $\text{Abs}(W)$ of $W$. Some, though usually not all of the maximal elements of the absolute order are of this form, but we will exclusively be concerned with these.

**Definition 1.5.4.** Let $W$ be a finite reflection group with associated simple system $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. A **standard Coxeter element** of $W$ is any element of the form $c = t_{\alpha_{\sigma(1)}} t_{\alpha_{\sigma(2)}} \cdots t_{\alpha_{\sigma(m)}}$, where $\sigma$ is a permutation of the set $[m]$.

The length of a standard Coxeter element in $\text{Abs}(W)$ is also a very important number in the combinatorics of Coxeter groups.

**Definition 1.5.5.** Let $W$ be a finite reflection group. Let $h$ denote the length of any standard Coxeter element of $W$ as a $T_W$-word. Then $h$ is the **Coxeter number** of $W$.

We are now ready to define the other central object of this work. The set $\text{NC}(W)$ of (uniform) noncrossing partitions of $W$ is defined as an interval of the absolute order.

**Definition 1.5.6.** Relative to any standard Coxeter element $c$, the poset of (uniform) noncrossing partitions is the interval $\text{NC}(W, c) = [1, c]$.

Although this definition appears to depend on the choice of standard Coxeter element $c$, the intervals $[1, c]$ are isomorphic as posets for all $c$. We briefly discuss the reason here. Assume $W$ is irreducible and consider standard coxeter elements $c$ and $c'$. Let $c = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_m}$ with $\alpha_1, \alpha_2, \ldots, \alpha_m \in \Delta$. Consider the following two types of movements:

1. $t_{\alpha_1} c t_{\alpha_1} = t_{\alpha_1} (t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_m}) t_{\alpha_1} = t_{\alpha_2} \cdots t_{\alpha_m} t_{\alpha_1}$ and
2. for $i < m$, swapping adjacent reflections $t_{\alpha_i}$ and $t_{\alpha_{i+1}}$ in $c$ whenever $m_{\alpha_i, \alpha_{i+1}} = 2$.

Using induction on the length $m$ of $c$ and the fact that the Coxeter graph of $W$ is a tree we can prove that $c$ and $c'$ are connected by these two types of movements. Therefore, $c$ and $c'$ are $W$-conjugates: there exists $w \in W$ such that $w^{-1} c w = c'$. But conjugation by a fixed element $w$ of $W$
is well known to be an automorphism of $\text{Abs}(W)$, so $\text{NC}(W,c) \simeq w^{-1}\text{NC}(W,c)w = \text{NC}(W,c')$ as posets.

This frees us to use the notation $\text{NC}(W)$ for the poset of noncrossing partitions of $W$ with respect to any $c$.

**Example 1.5.7.** For each classical reflection group $W$ there is a canonical choice of Coxeter element $c$ for the interval $\text{NC}(W)$, which is obtained from the root poset of $W$ by taking the product of the simple roots as they appear (from left to right) in the root poset of $W$. We refer here to the root poset of $W$ associated to the standard choices of simple systems $\Delta_W$ and positive systems $\Pi_W$ presented in Example 1.2.12, *cf.* Figure 1.1:

$$c = \begin{cases} t_{e_2-e_1}t_{e_3-e_2}\cdots t_{e_n-e_{n-1}} & \text{for } W = A_{n-1} \\ t_{e_1}t_{e_2-e_1}t_{e_3-e_2}\cdots t_{e_n-e_{n-1}} & \text{for } W = B_n = C_n \\ t_{e_1+e_2}t_{e_2-e_1}t_{e_3-e_2}\cdots t_{e_n-e_{n-1}} & \text{for } W = D_n \end{cases}$$

An example of the absolute order for $W = A_3$ is shown in Figure 1.2, but in the permutation representation to be presented later.

**Figure 1.2.** The absolute order on the symmetric group $A_3$. 
COMMENTARY 1.5.8. For noncrossing partitions we have followed Armstrong [1, §2.4–6]. The treatment of nonnesting partitions is due to Postnikov [12].

1.6. The Coxeter-Catalan combinatorics

The Coxeter-Catalan combinatorics is an active field of study in the theory of Coxeter groups. Let $W$ be an irreducible Weyl group.

Define the Coxeter-Catalan number $\text{Cat}(W)$ of $W$ by

$$\text{Cat}(W) = \prod_{i=1}^{n} \frac{h + d_i}{d_i} = \prod_{i=1}^{n} \frac{h + d_i}{|W|}$$

where $h$ is the Coxeter number of $W$ and $d_1, \ldots, d_n$ its degrees.

We should here be wondering about the nature of this strange number we have just defined. To begin unveiling the mystery, consider the case $W = A_{n-1}$. We have $\text{Cat}(A_{n-1}) = \frac{1}{n+1} \binom{2n}{n} = C_n$, the classical Catalan number. The coincidence is surprising and interesting: we defined the Coxeter-Catalan number in terms of degrees and the Coxeter number, but we have arrived to a very well known and classical combinatorial number. In fact, it is nowadays known that the classical Catalan number $C_n$ counts more than 160 essentially different objects and this Coxeter-Catalan number $\text{Cat}(W)$ of $W$ presents itself as a generalization of $C_n$, so: what can be said about $\text{Cat}(W)$ in general? This is again an example of how the case of the symmetric group motivates the names of certain objects in the area.

Several independently motivated sets of objects to do with $W$ are known already to have cardinality $\text{Cat}(W)$. Two of these sets of objects are

- the noncrossing partitions $\text{NC}(W)$, which in their classical (type $A$) avatar are a long-studied combinatorial object harking back at least to Kreweras [11], and in their generalization to arbitrary Coxeter groups are due to Bessis and Brady and Watt [5], [7]; and

- the nonnesting partitions $\text{NN}(W)$, introduced by Postnikov [12] for all the classical reflection groups simultaneously.

Athanasiadis in [3] proved in a case-by-case fashion that $|\text{NN}(W)| = |\text{NC}(W)|$ for the classical reflection groups $W$, proving the equality for each type one at a time. He asked for a bijective proof. This was later improved by Athanasiadis and Reiner [4] to a proof for all Weyl groups which also proved equidistribution by type, cited as Theorem 2.1.8 below. Our contribution has been to provide a uniform bijection for the classical reflection groups, given the standard choice of coordinates in the representation. Our proof also addresses equidistribution by type. The cases of
our bijection for types $B$, $C$, and $D$ have not before appeared in the literature. The ultimate goal in connecting $NN(W)$ and $NC(W)$, a case-free bijective proof for all Weyl groups, remains open.

Independently of and essentially simultaneously with this work, Stump [16] has obtained bijections for types $A$ and $B$, but these do not preserve type.
CHAPTER 2

Classical partitions and Fixed Spaces

2.1. Fixed Spaces

From now on, we fix the choice of Coxeter element given in Example 1.5.7 to define $NC(W)$, and we let $W$ be a classical reflection group with standard positive system and standard simple system. See Example 1.2.12.

We have already encountered and made use of fixed spaces. These are key vector spaces to understand the combinatorics of $NN(W)$ and $NC(W)$. We now formalize them.

**Definition 2.1.1.** For all $w \in W$, let $Fix(w) = \ker(w - 1)$ be the fixed space of $w$, where $1 \in W$ is the identity operator.

We study how to obtain fixed spaces from $T_W$-reduced words. The next theorem is a consequence of Lemma 2 of [8]. We prove it here for explicitness.

**Theorem 2.1.2.** Let $W$ be a Weyl group and consider a $T_W$-reduced expression for some $w \in W$, 

$$w = t_{\alpha_1}t_{\alpha_2}t_{\alpha_3} \cdots t_{\alpha_m}$$

and $\alpha_1, \alpha_2, \ldots, \alpha_m \in \Phi$.

Then $Fix(w) = \bigcap_{i=1}^{m} \alpha_i^\perp$.

**Proof.** The $\supseteq$ containment is immediate. To prove $\subseteq$, suppose $v$ is a vector such that $w(v) = v$. From $t_\alpha(\lambda) = \lambda - 2\langle \alpha, \lambda \rangle / \langle \alpha, \alpha \rangle \alpha$ we obtain an expression of the form $v = v + k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + \cdots + k_m\alpha_m$ with $k_1, \ldots, k_m \in \mathbb{R}$. It follows that $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + \cdots + k_m\alpha_m = 0$ so $k_1 = k_2 = k_3 = \cdots = k_m = 0$ since we had a reduced expression for $w$ with a linearly independent set of reflections $\{\alpha_1, \ldots, \alpha_m\}$. We inductively obtain from here $v \in \alpha_1^\perp \cap \cdots \cap \alpha_m^\perp$, so $v \in \bigcap_{i=1}^{m} \alpha_i^\perp$. That is, $Fix(w) \subseteq \bigcap_{i=1}^{m} \alpha_i^\perp$. $\square$

**Corollary 2.1.3.** Let $\sigma$ be a permutation of the set $[m]$. Define

$$w_\sigma = t_{\alpha_{\sigma(1)}} t_{\alpha_{\sigma(2)}} t_{\alpha_{\sigma(3)}} \cdots t_{\alpha_{\sigma(m)}}.$$

Then $Fix(w) = Fix(w_\sigma)$.

\[\text{1} Thanks to Christian Stump.\]
Theorem 2.1.2 and Corollary 2.1.3 free us to define fixed spaces of antichains of the root poset in a very natural way.

**Definition 2.1.4.** Let $A$ be an antichain of the root poset of $W$, i.e., let $A \in NN(W)$. Then, the **fixed space** $Fix(A)$ of $A$ is the set $Fix(\pi_A)$, where $\pi_A = \prod_{\alpha \in A} t_\alpha$ with the product taken in any order.

Theorem 1.4.12 shows that antichains of the root poset are linearly independent. Therefore, $Fix(A)$ is well-defined.

Fixed spaces provide additional insight into the absolute order of $W$, as shown by the next proposition.

**Proposition 2.1.5.** Consider $W$ ordered with the absolute order $Abs(W)$ and let $w, w' \in W$. Suppose there exists some $x \in W$ such that $w \leq x$ and $w' \leq x$. Then, $w \leq w'$ if and only if $Fix(w') \subseteq Fix(w)$.

A proof of Proposition 2.1.5 may be found in Armstrong [1]. We do not include it here because it requires further background in the algebra of $T_W$-words which deviates from our purpose. More important for us is the fact that fixed spaces characterize noncrossing and nonnesting partitions. To understand this, we first need acquaintance with the partition lattice.

As we know, associated to each root $\alpha$ we have an orthogonal hyperplane $\alpha^\perp$. The set of all hyperplanes associated to roots defines a hyperplane arrangement and a poset of intersections.

**Definition 2.1.6.** The **partition lattice** $\mathcal{P}(W)$ of $W$ is the collection of intersections of reflecting hyperplanes

$$\left\{ \bigcap_{\alpha \in S} \alpha^\perp : S \subseteq \Pi \right\}$$

ordered by reverse-inclusion of subspaces.

The following important proposition holds:

**Proposition 2.1.7.** The maps $w \mapsto Fix(w)$ and $A \mapsto Fix(A)$ are injections from $NC(W)$ to $\mathcal{P}(W)$ and from $NN(W)$ to $\mathcal{P}(W)$, respectively.

**Proof Sketch:** Suppose $w, w' \in NC(W)$. This is equivalent to saying $w \leq c$ and $w' \leq c$, where $c$ is the standard Coxeter element. But then, Proposition 2.1.5 shows that $w$ and $w'$ can only have equal fixed spaces if $w = w'$.

The case for nonnesting partitions may be found in Athanasiadis and Reiner [4].
Now, consider the map
\[ P(W) \hookrightarrow P(W)/W \]
\[ U \mapsto \{w(U) | w \in W\} \]
that sends vector subspaces of the partition lattice to their \(W\)-orbits. Two subspaces with equal \(W\)-orbits are said to be \(W\)-conjugate subspaces. In 2003, Athanasiadis and Reiner [4] proved the following fundamental theorem in a case-by-case fashion, going through each combinatorial type, one at a time. See Example 1.4.7. The cases of exceptional Weyl groups were all proven with the aid of specialized software.

**Theorem 2.1.8.** Define the maps
\[ f : NC(W) \rightarrow P(W) \hookrightarrow P(W)/W \]
\[ g : NN(W) \rightarrow P(W) \hookrightarrow P(W)/W \]
Then, for each \(W\)-orbit \(O \in P(W)\), we have
\[ |f^{-1}(O)| = |g^{-1}(O)|. \]

Theorem 2.1.8 is conceptually probably the most significant theorem known to imply the equality \(|NC(W)| = |NN(W)|\) and it reveals a profound connection between the two independently defined objects, \(NN(C)\) and \(NC(W)\).

The following definition has a very intuitive combinatorial interpretation in the classical partitions setting, presented in next section.

**Definition 2.1.9.** Let \(w \in W\). The *orbital type* of \(w\) is the \(W\)-orbit of the vector subspace \(\text{Fix}(w)\). Respectively, let \(A\) be an antichain of the root poset of \(W\). The *orbital type* of \(A\) is the \(W\)-orbit of the vector subspace \(\text{Fix}(A)\).

Athanasiadis [3] proposed the problem of finding a bijection between \(NC(W)\) and \(NN(W)\) that preserved orbital type, therefore refining Theorem 2.1.8. More explicitly, he proposed to find a bijection
\[ \sigma : NC(W) \rightarrow NN(W) \]
satisfying \(f(w) = g(\sigma(w))\) for every \(w \in NC(W)\), where \(f\) and \(g\) are the maps of Theorem 2.1.8. We do this in Section 3.

### 2.2. Classical partitions and permutation groups

To begin, we introduce the notion of classical partition for \(W\). We are not restricted here to noncrossing or nonnesting cases, as the following definition shows.
Definition 2.2.1. Consider some \( U \in \mathcal{P}(W) \) and let \( \rho_U \) be the orthogonal projection of \( \mathbb{R}^n \) on \( U \). The operator \( \rho_U \) induces a partition of the set

\[
E = \{\pm e_i : i = 1, \ldots, n\}
\]

into fibers, by which \( a, b \in E \) belong to the same block if and only if \( \rho_U(a) = \rho_U(b) \). We denote this partition by \( \text{Part}(U) \). Any partition of \( E \) induced in this way is called a classical partition of \( W \). Denote the set of classical partitions of \( W \) by \( \text{Part}(W) \).

We will streamline the notation of classical partitions by writing \( \pm i \) for \( \pm e_i \). Thus, a classical partition for \( W \) can be also realized as a partition of \( \pm [n] = \{1, \ldots, n, -1, \ldots, -n\} \) for some \( n \).

Classical partitions are symmetric under negation: if \( a, b \in E \) satisfy \( \rho_U(a) = \rho_U(b) \), then \( \rho_U(-a) = \rho_U(-b) \). A zero block is a block fixed by negation. A classical partition never has more than one zero block: if \( \pm e_i \) are projected to the same point of \( U \) this point is 0, the fiber over which contains every such pair. Thus a typical classical partition might look like

\[
\{1, 2\}, \{-1, -2\}, \{3, -7, -8\}, \{-3, 7, 8\}, \{5\}, \{-5\}, \{4, 6, -4, -6\}
\]

in which \( \{4, 6, -4, -6\} \) is the zero block: this is the partition depicted in Figure 2.2.

Given a minimal set of equations for \( U \), each of which must be of the form

\[
s_1 x_{i_1} = \cdots = s_k x_{i_k} (= 0)
\]

where the \( s_i \in \{+1, -1\} \) are signs, the classical partition can be read off, one block from each equation. To the above corresponds \( \{s_1 i_1, \ldots, s_k i_k\} \) if the \( = 0 \) is not included, and \( \{\pm i_1, \ldots, \pm i_k\} \) if it is.

Example 2.2.2. Let \( n = 4 \). Consider the root \( e_3 - e_1 \). We can check that \( U = \text{Fix}(t_{e_3 - e_1}) \) is the set of points \( x = (x_1, x_2, x_3, x_4) \) such that \( x_1 = x_3 \), so \( \text{Part}(U) \) is

\[
\pm \{e_1, e_3\}, \pm \{e_2\}, \pm \{e_4\}.
\]

To this corresponds

\[
\pm \{1, 3\}, \pm \{2\}, \pm \{4\}.
\]

Example 2.2.3. Let \( n = 6 \). Take \( U = \text{Fix}(t_{e_1} t_{e_2 - e_1} t_{e_4 + e_3} t_{e_6 - e_4}) = \text{Fix}(t_{e_1}) \cap \text{Fix}(t_{e_2 - e_1}) \cap \text{Fix}(t_{e_4 + e_3}) \cap \text{Fix}(t_{e_6 - e_4}) \), which is exactly the set of points \( x = (x_1, x_2, x_3, x_4, x_5, x_6) \) satisfying \( x_1 = -x_1, x_1 = x_2, x_3 = -x_4 \) and \( x_4 = x_6 \). So actually \( U = \{x | x_1 = x_2 = 0 \text{ and } x_3 = -x_4 = -x_6\} \). We conclude that \( \text{Part}(U) \) is

\[
\pm \{e_3, -e_4, -e_6\}, \pm \{e_5\}, \{e_1, e_2, -e_2, -e_1\}.
\]

To this corresponds

\[
\pm \{3, -4, -6\}, \pm \{5\}, \{1, 2, -2, -1\}.
\]
Remark 2.2.4. We have long ago convened to deal with standard positive systems and standard simple systems. The form of the roots in these systems is intentionally included in our examples, so that a quick thought suffices to realize the validity of the previous discussion.

In case $W = A_{n-1}$, all roots are of the form $e_j - e_i$ with $i \neq j$, so they have orthogonal complement $x_i = x_j$. Therefore, minimal equations for subspaces $U \in \mathcal{P}(W)$ are of the form

$$x_{i_1} = \cdots = x_{i_k} \text{ with } 1 \leq i_1 < \cdots < i_k \leq n$$

and a classical partition for $W$ will be the union of a partition of $[n]$ and its negative, a partition of $-[n]$. Here, and in everything we do henceforth with type $A$, we will omit the redundant negative blocks and treat type $A$ partitions as partitions of $[n]$.

Example 2.2.5. We present here the standard permutation representation of $W$. We have had glimpses of this in previous examples, where we suggested particular colloquial names (e.g. the symmetric group, the group of signed permutations, and the like). Let $S_{\pm n}$ be the group of signed permutations of $[n]$ (or the group of permutations of the set $\pm [n]$ already defined). Consider the map $\zeta^*: T_{\Delta W} \mapsto S_{\pm n}$ for which

$$t_{e_1} \mapsto (1,-1)$$

$$t_{e_{i+1} - e_i} \mapsto (i, i+1)(-i, -(i+1)) \text{ for all } 1 \leq i < n$$

$$t_{e_2 + e_1} \mapsto (1,-2)(-1,2)$$

This map extends uniquely to a monomorphism of groups $\zeta: W \mapsto S_{\pm n}$. Refer to the classical literature in Coxeter groups for a proof, for example, the book of Björner and Brenti [6], or simply remember from the first examples of this writing how reflections in $T_W$ act on vectors of $\mathbb{R}^n$. In the case $W = A_{n-1}$, by symmetry we may simply consider $S_n$ instead of $S_{\pm n}$ in the definition of $\zeta^*$, in which case $\zeta: A_{n-1} \mapsto S_n$ is an isomorphism of groups.

Using the permutation representation, the standard Coxeter elements of $W$ presented in Example 1.5.7 are

$$c = \begin{cases} (1 \ 2 \ \ldots \ n) & \text{for } W = A_{n-1} \\ (1 \ \ldots \ n \ (-1) \ \ldots \ (-n)) & \text{for } W = B_n = C_n \\ (1 \ (-1))(2 \ \ldots \ n \ (-2) \ \ldots \ (-n)) & \text{for } W = D_n \end{cases}$$

Remark 2.2.6. We invite the reader to check the following nice fact: classical partitions $\text{Part}(W)$ of $W$ are exactly the orbits of elements of $W$ in the permutation representation. As a hint, recall again how reflections in $T_W$ act on vectors of $\mathbb{R}^n$.

We have introduced classical partitions $\text{Part}(W)$ of $W$, but why should we care?
To partially answer the question, remember we are studying noncrossing and nonnesting partitions. Specifically, we would like to find a bijection between these objects. The least we should expect is the maps taking \( w \in NC(W) \) to \( \text{Part}(\text{Fix}(w)) \) and \( A \in NN(W) \) to \( \text{Part}(\text{Fix}(A)) \) to be injective. It does not take much effort to realize this is indeed the case. Suppose we are given a classical partition for \( W \), i.e. \( \text{Part}(U) \) with \( U \in \mathcal{P}(W) \). We know \( \text{Part}(U) \) is a partition of the set \( \pm[n] \) symmetric under negation and containing at most one zero block. Its blocks define uniquely a set of equations of the form

\[
s_1x_i = \cdots = s_kx_i (= 0)
\]

where the \( s_i \in \{+1, -1\} \) are signs. But these equations define minimally \( U \), so the map taking \( U \mapsto \text{Part}(U) \) is an injection of \( \mathcal{P}(W) \) into \( \text{Part}(W) \). By definition, this map is also surjective. Thus, it is a bijection between the two sets. We knew already the maps \( w \mapsto \text{Fix}(w) \) and \( A \mapsto \text{Fix}(A) \) to be injections of \( NC(W) \) and \( NN(W) \) into \( \mathcal{P}(W) \), respectively. Therefore, the maps

\[
\begin{align*}
  f^{NC} : w \mapsto \text{Fix}(w) & \longrightarrow \text{Part}(\text{Fix}(w)) \\
  f^{NN} : A \mapsto \text{Fix}(A) & \longrightarrow \text{Part}(\text{Fix}(A))
\end{align*}
\]

are injective maps of \( NC(W) \) and \( NN(W) \) into \( \text{Part}(W) \), as we wanted. We have probably managed to uniformize concepts with these injections: we are now able to directly compare noncrossing and nonnesting partitions via their respective images under \( f^{NC} \) or \( f^{NN} \) in \( \text{Part}(W) \). One might hope that these images are well behaved. As we will see, with some help, this is precisely the case.

Given our change of perspective then, let’s introduce the analogous concept of orbital type of a classical partition.

**Definition 2.2.7.** Let \( \text{Part}(U) \) be a classical partition for \( W \), so \( U \in \mathcal{P}(W) \). The **classical orbital type** of \( \text{Part}(U) \) is the \( W \)-orbit of the vector subspace \( U \).

Notice the following: for all \( w \in NC(W) \), the orbital type of \( w \) is equal to the classical orbital type of \( f^{NC}(w) \) and, for all \( A \in NN(W) \), the orbital type of \( A \) is equal to the classical orbital type of \( f^{NN}(A) \). This is as good as we might have expected, but there is even more to it. It turns out that understanding combinatorially these classical orbital types is quite simple: the type of a classical partition is essentially (completely characterized by) the multiset of its block sizes, adjusted to take care of the zero block.

**Example 2.2.8.** For example, the type of the partition (2.1) is the set of classical partitions with a zero block of size 4 and three pairs of nonzero blocks of sizes 3, 2, and 1.

Consequently, conjugacy of subspaces of the partition lattice \( \mathcal{P}(W) \) may be easily decided through the use of classical partitions, as the next proposition shows.
Proposition 2.2.9. Two subspaces $U, U' \in \Pi(W)$ are conjugate if and only if both of the following hold:

- the multisets of block sizes $\{|C| : C \in \text{Part}(U)\}$ and $\{|C| : C \in \text{Part}(U')\}$ are equal;
- if either $\text{Part}(U)$ or $\text{Part}(U')$ has a zero block, then both do, and these zero blocks have equal size.

For example, the type $A$ specialization of this result, where zero blocks are irrelevant and we drop the redundant negative elements, says that the conjugacy classes of the symmetric group $A_{n-1}$ on $n$ elements are enumerated by the partitions of the integer $n$.

2.3. Classical noncrossing and nonnesting partitions

Definitions of the classes of classical noncrossing and nonnesting partitions are perhaps most intuitively presented in terms of a diagrammatic representation that we will soon describe: after Armstrong [1, §5.1] we call these bump diagrams.

These bump diagrams were, in turn, intentionally defined for each combinatorial type so as to make them:

- agree with the uniform definitions of $NC(W)$ and $NN(W)$.
- nice combinatorial objects motivating the names “noncrossing” and “nonnesting”.

This should always be kept in mind during the following discussion. Except for type $A$, which is long known, all the subtleties and particularities of the bump diagrams of each combinatorial type should be thought of in this way. In even more colloquial terms, these particularities are there only to help Proposition 2.4.1 hold in some nice way.

Let $P$ be a partition of a totally ordered ground set $(S, \prec)$.

Definition 2.3.1. Let $G(P)$ be the graph with vertex set $S$ and edge set

$$\{(s, s') : s \prec_P s' \text{ and } \not\exists s'' \in S \text{ s.t. } s <_P s'' <_P s'\}$$

where $s <_P s'$ iff $s < s'$ and $s$ and $s'$ are in the same block of $P$.

A bump diagram of $P$ is a drawing of $G(P)$ in the $xy$ plane in which:

- the elements of $S$ are arrayed along the real $x$-axis (the real line) in their given order, so that $s < s'$ if and only if $s$ is to the left of $s'$
- all edges lie above the $x$-axis, and
- no two edges intersect more than once.
DEFINITION 2.3.2. $P$ is *noncrossing* if its bump diagram contains no two crossing edges, equivalently if $G(P)$ contains no two edges of form $(a,c), (b,d)$ with $a < b < c < d$.

DEFINITION 2.3.3. $P$ is *nonnesting* if its bump diagram contains no two nested edges, equivalently if $G(P)$ contains no two edges of form $(a,d), (b,c)$ with $a < b < c < d$.

We will abuse the terminology slightly and refer to the bump diagram of $P$ as noncrossing, resp. nonnesting, if $P$ is. We will denote the set of classical noncrossing and nonnesting partitions for $W$ by $NC^{\text{cl}}(W)$, resp. $NN^{\text{cl}}(W)$. To define these sets it remains only to specify the ordered ground set.

For $NN^{\text{cl}}(W)$, the ordering we use is read off the line of integers in Figure 1.1.

DEFINITION 2.3.4. A *classical nonnesting partition* for a classical reflection group $W$ not of type $D$ is a partition nonnesting with respect to the ground set

\[
\begin{align*}
1 < \cdots < n & \quad \text{if } W = A_{n-1}; \\
-n < \cdots < -1 < 0 < 1 < \cdots < n & \quad \text{if } W = B_n; \\
-n < \cdots < -1 < 1 < \cdots < n & \quad \text{if } W = C_n.
\end{align*}
\]

such that whenever $W = B_n$ or $W = C_n$, it is symmetric under negation and contains at most one zero block.

DEFINITION 2.3.5. A *classical nonnesting partition* $\pi$ of $D_n$ is a partition nonnesting with respect to the ground set

\[-n < \cdots < -1, 1 < \cdots < n\]

such that $\pi$ is symmetric under negation and contains at most one zero block. Furthermore, we require that if $\pi$ has a zero block, it contains properly the pair \{±1\}.

The ground set for classical nonnesting partitions for $D_n$ is not totally ordered but is merely a strict weak ordering, in which 1 and $-1$ are incomparable. Definitions 2.3.1 and 2.3.3 generalize cleanly to this situation, with no amendments to the text of the definitions themselves. That is, in a classical nonnesting partition for $D_n$, an edge with 1 as vertex and another with $-1$ as vertex are never considered to nest. We diverge in purely cosmetic fashion from Athanasiadis and reinforce this last point by aligning these two dots vertically when drawing a type $D$ nonnesting bump diagram.

Figure 2.1 exemplifies Definition 2.3.5, giving one nonnesting bump diagram for each classical type.

For $NC^{\text{cl}}(W)$, the ordering we use is read off of the standard Coxeter elements in (1.1).
2.3. CLASSICAL NONCROSSING AND NONNESTING PARTITIONS

Definition 2.3.6. A classical noncrossing partition for a classical reflection group \( W \) not of type \( D \) is a partition noncrossing with respect to the ground set

\[
1 < \cdots < n \quad \text{if } W = A_{n-1}; \\
-1 < \cdots < -n < 1 < \cdots < n \quad \text{if } W = B_n; \\
-1 < \cdots < -n < 1 < \cdots < n \quad \text{if } W = C_n.
\]

such that whenever \( W = B_n \) or \( W = C_n \), it is symmetric under negation and contains at most one zero block.

Observe that the order < in these ground sets differs from those for nonnesting partitions.

For example, Figure 2.2 presents an example of a noncrossing partition for types \( B \) and \( C \). Figure 2.3 is the same partition rendered circularly.

The subtleties that occur defining classical noncrossing partitions in type \( D \) are significant, and historically it proved troublesome to provide the correct notion for this case. Reiner’s first
definition [13] of classical noncrossing partitions for type $D$ was later superceded by that of Bessis and Brady and Watt [5, 7] and Athanasiadis and Reiner [4], which we use here, for its better agreement with the uniform definition of $NC(D_n)$. Indeed definitions 2.3.1 through 2.3.3 require tweaking to handle type $D$ adequately. (This said we’ll still use the name “bump diagram” for a diagram of a classical nonnesting partition for $D_n$.)

**Definition 2.3.7.** A classical noncrossing partition $\pi$ for $D_n$ is a partition symmetric under negation such that there exists $c \in \{2, \ldots, n\}$ for which $\pi$ is noncrossing with respect to both of the ordered ground sets

$$-2 < \cdots < -c < -1 < -(c + 1) < \cdots < -n < 2 < \cdots < c < 1 < c + 1 < \cdots < n$$

and

$$-2 < \cdots < -c < 1 < -(c + 1) < \cdots < -n < 2 < \cdots < c < -1 < c + 1 < \cdots < n$$

and such that $\pi$ contains at most one zero block. Furthermore, we require that if $\pi$ has a zero block, it contains properly the pair $\{\pm 1\}$.

Following [4] these are best drawn circularly. Arrange dots labelled $-2, \ldots, -n, 2, \ldots, n$ in a circle and place 1 and $-1$ in the middle. Then, a $D_n$ partition $\pi$ is classical noncrossing if and only if no two blocks in this circular bump diagram have intersecting convex hulls, except possibly two blocks $\pm B$ meeting only at the middle point. The single middle point is never considered to define a block by itself: the block $\{\pm 1\}$ never occurs in a classical noncrossing partition for $D_n$. The edges we will supply in these circular diagrams are those delimiting the convex hulls of the blocks. See Figure 2.4 for an example.

**Figure 2.4.** Circular bump diagram for the $D_{10}$ classical noncrossing partition $\pm\{1, 5, 6, -7, -8, -9\}, \pm\{2, -10\}, \pm\{3, 4\}$. 

![Diagram of a classical noncrossing partition for $D_{10}$](image-url)
2.4. Classical and Uniform partitions are equivalent

We state the relations between the classical objects and the uniform ones. As it might have been expected, the classical noncrossing and nonnesting partitions of \( W \) defined in the previous section coincide with the images under \( f^{NC} \) and \( f^{NN} \) of the corresponding uniform objects.

**Proposition 2.4.1.** The map \( f^{NC} : w \mapsto \text{Part}(\text{Fix}(w)) \) is a bijection between \( NC(W) \) and \( NC_{cl}(W) \).

To prove Proposition 2.4.1 we will use the permutation representation of \( W \) and the fact that classical partitions \( \text{Part}(W) \) of \( W \) are exactly the orbits of elements of \( W \) in this permutation representation. For the statement of \( f^{NC} \), we only treat the type \( A \) case in detail, it being the simplest case and the other cases being very similar in approach.

**Proof of Proposition 2.4.1.** To run the first bijection in the direction opposite to \( f^{NC} \), we need to restore the cycle structure of some \( w \in NC(W) \) given only its orbits, i.e. provide a cyclic ordering of the elements of each orbit. We choose the same order as is found in the standard Coxeter element \( c \): in case of \( A_{n-1} \) this is numerical order. So we claim that the elements of \( A_{n-1} \) in \([1, c]\) are exactly those each of whose cycles are in numerical order.

Given a partition \( w \in A_{n-1} \) in permutation representation, write it as a product of disjoint cycles, and let \( \rho = (i_1 \ldots i_k) \) be one of these. This \( \rho \) is minimally a product of \( k-1 \) transpositions, for instance as
\[
\rho = (i_1 \ i_2)(i_2 \ i_3) \cdots (i_{k-1} \ i_k)
\]
to which the corresponding roots are \( \{e_{i_2} - e_{i_1}, \ldots, e_{i_k} - e_{i_{k-1}}\} \), which span the subspace of \( (e_1 + e_2 + \cdots + e_n)^\perp \) which is nonzero only in coordinates occurring in the cycle. By Lemma 1.5.2, therefore, \( w(i \ j) \geq w \) in the absolute order if and only if \( i \) and \( j \) belong to different cycles of \( w \) in the disjoint cycle decomposition.

Now, given a product of two disjoint cycles
\[
(i_1 \ldots i_k)(j_1 \ldots j_l),
\]
the possible permutations that can be obtained by multiplying (on the left, say, though in fact the behavior is the same multiplying on the right) by a transposition of form \((i_r \ j_s)\) with \( 1 \leq r \leq k \) and \( 1 \leq s \leq l \) are just those that are obtained by, intuitively, cutting both cycles and patching them together: i.e. all permutations of form
\[
(i_r \ i_{r+1} \ldots j_{r-1} j_s j_{s+1} \ldots j_{s-1})
\]
(2.2)
So it remains to show that the products of disjoint cycles which can be patched together in this fashion to yield \( c = (1 \ldots n) \) are just the noncrossing partitions with the blocks cyclically ordered like \( c \).
Firstly, if any cycle contains three elements $i, j, k$ in the wrong cyclic order compared to $c$, then after any patching operation (2.2) they remain in the wrong order. So such cycles cannot be contained in any noncrossing permutation.

Next, if we have two cycles, one containing $i, k \in \lceil n \rceil$ and the other containing $j, l \in \lceil n + 1 \rceil$ with $i < j < k < l$, then one can exhaustively check all patchings of these two cycles and see that none of them have $i, j, k, l$ in the correct cyclic order; the relevant cyclic orders that do arise are $ikjl, iklj, kijl$, and $kilj$. So no noncrossing permutation can contain two such crossing cycles, since at some point in the patching process they will have to be patched, and after that it is impossible to get to $c$ as per the last paragraph. This is one direction of the equivalence we seek.

For the other direction let $w$ be a classical noncrossing partition. We claim that $w$ has some block $B$ of form $\{i, i + 1, \ldots, j\}$ consisting of a range of consecutive elements of $\lceil n \rceil$. Indeed, any block $B$ of $w$ which minimizes $b_{\text{max}} - b_{\text{min}}$, where $b_{\text{max}} = \max\{b : b \in B\}$ and $b_{\text{min}} = \min\{b : b \in B\}$, suffices. If $B$ did not contain all the elements between its two extreme elements, let $A$ be a subset that it misses. By the noncrossing condition every element of $A$ falls strictly between $b_{\text{min}}$ and $b_{\text{max}}$; but then the extreme elements of $A$ are closer together, contradiction.

So, having obtained such a $B$, we can patch it together with the cycle containing $b_{\text{min}} - 1$ (mod $n$) by inserting $b_{\text{min}} \in B$ immediately after $b_{\text{min}} - 1$. This gives us a noncrossing permutation (as is easily checked) with one cycle fewer. So, by induction on the number of cycles, we can continue patching to eventually obtain $c$, showing that $w \in [1, c] = \text{NC}(W)$. This completes the proof. $\square$

The only significant thing that needs changing to allow this to work for types $B$ and $C$ is that, instead of defining patching to work on single blocks, we must define it on pairs of blocks $B, -B$, except for zero blocks, which are unpaired. Given that, the generalization is straightforward. For noncrossing partitions in type $D$ a similar modification suffices also under which, in particular, patching a block to its negative yields a single zero block properly containing $\{\pm 1\}$ and the other elements (even though the cycle structure will differ).

**Proposition 2.4.2.** The map $f_{\text{NN}} : A \mapsto \text{Part}(\text{Fix}(A))$ is a bijection between $\text{NN}(W)$ and $\text{NN}^\text{cl}(W)$.

We deal only with types $A, B$ and $C$. Type $D$ is simply a further generalization.

**Proof Sketch:** We had convened to present the root posets in a coordinate system $ud$. Consider the map which draws, from a positive root with coordinates $u$ and $d$ in the root poset, an edge connecting vertices $u$ and $d$ in the corresponding bump diagram. Under this map, the image of two different positive roots is a nesting bump diagram if and only if the two roots are comparable in the root poset. Therefore, the image of a nonnesting partition is a nonnesting bump diagram.
and, by invariance under negation, we can complete this diagram down to a nonnesting bump diagram symmetric under negation. Using the permutation representation, it is easily seen that this is precisely the image of our nonnesting partition under $f_{NN}$. We can invert this process bijectively.

\[\gamma\]

**Figure 2.5.** How to relate antichains of the root poset with orbits in the permutation representation for type $C$.

We are now ready find a bijection between $NC(W)$ and $NN(W)$ that preserves the orbital type. By Proposition 2.4.1, this is tantamount to finding a bijection between $NC^{\text{cl}}(W)$ and $NN^{\text{cl}}(W)$ that preserves the classical orbital type, which is exactly what we will do.
CHAPTER 3

A type-preserving bijection for classical groups

Partitions will be drawn and spoken of with the greatest elements of their ground sets to the left.

Given any partition, define the order \(<_{lp}\) on those of its blocks containing positive elements so that \(B <_{lp} B'\) if and only if the least positive element of \(B\) is less than the least positive element of \(B'\).

3.1. Statement of the central theorem

We establish some notation.

Definition 3.1.1. Let \(\Psi^n\) be the set of \(n\)-tuples with entries in \(\{1, 0, -1\}\). For any \(u \in \Psi^n\) define \(#(u, 1)\) to be the number of entries equal to 1 in \(u\) and define \(#(u, -1)\) analogously. Let \(<_{lex}\) be the lexicographic order on \(n\)-tuples. For any two vectors \(a, b \in \mathbb{Z}^n\), let \(a\) be the set of elements of \(\mathbb{Z}^n\) \(<_{lex}\)-less than or equal to \(a\) and let \(\|a - b\| = (|a_1 - b_1|, \ldots, |a_n - b_n|)\).

To any nonnesting or noncrossing partition \(x\) of \(W\) we associate a set \(\Omega_x\) which is constructed inductively with \(i\) increasing from 1 to \(n\) stepwise. Initially, we begin with \(\Omega_x = \emptyset\). In step \(i\), let \(u_i\) be the element of \(\Psi^n \cap \text{Fix}(x)\) with \(\|e_i - u_i\| \leq_{lex}\)-minimal (actually \(\|e_i - u_i\| \in \Psi^n\)). Whenever \(u_i\) is linearly independent with the elements of \(\Omega_x\), let \(\Omega_x = \Omega_x \cup \{-u_i\}\) if \(u_i\) has some entry \(-1\) and let \(\Omega_x = \Omega_x \cup \{u_i\}\) if not. Let \(\Gamma_x\) be the number of canonical coordinate projections of \(\text{Fix}(x)\) with trivial image \(\{0\}\).

Lastly, let \(E\) be the canonical basis of \(\mathbb{R}^n\).

Theorem 3.1.2. Let \(x \in NN(W)\) [resp. \(x \in NC(W)\)]. Then, there is a unique \(y \in NC(W)\) [resp. \(y \in NN(W)\)] for which \(\Gamma_x = \Gamma_y\) and such that the sets \(\Omega_x\) and \(\Omega_y\) are related to each other in the following way:

There is a bijection \(\sigma\) between \(\Omega_x\) and \(\Omega_y\) such that for each \(u \in \Omega_x\) we have \(\sigma(u) \in \Omega_y\) satisfying

- \(#(u, 1) = #(\sigma(u), 1)\) and \(#(u, -1) = #(\sigma(u), -1)\)
- \(\text{card}(u \cap E) = \text{card}(\sigma(u) \cap E)\)
• $\text{card}(u \cap \Omega_x) = \text{card}(\sigma(u) \cap \Omega_y)$
• the product of the first two nonzero components of $u$ and $\sigma(u)$ is not equal whenever $\#(u, -1) > 1$ and $\#(u, 1) > 0$

Consequently, the induced mapping establishes a bijection between noncrossing and nonnesting partitions preserving orbital type.

In the remainder of this writing we will prove Theorem 3.1.2. The four sections that follow will give the individual type-preserving bijections for each of the classical types that arise from it in a case by case fashion; then in §3.6 we tie these together and complete the proof.

### 3.2. Type $A$

The bijection in type $A$, which forms the foundation of the ones for the other types, is due to Athanasiadis [3, §3]. We include it here to make this foundation explicit and to have bijections for all the classical groups in one place.

Let $\pi$ be a classical partition for $A_{n-1}$. Let $M_1 <_{lp} \cdots <_{lp} M_m$ be the blocks of $\pi$, and $a_i$ the least element of $M_i$, so that $a_1 < \cdots < a_m$. Let $\mu_i$ be the cardinality of $M_i$. Define the two statistics $a(\pi) = (a_1, \ldots, a_m)$ and $\mu(\pi) = (\mu_1, \ldots, \mu_m)$.

It turns out that classical nonnesting and noncrossing partitions are equidistributed with respect to these partition statistics, and that they uniquely determine one partition of either kind. This will be the mode in which we present all of our bijections, which will differ from this one in the introduction of more statistics.

We will say that a list of partition statistics $S$ establishes a bijection for a classical reflection group $W$ if, given either a classical noncrossing partition $\pi^{\text{NC}}$ or a classical nonnesting partition $\pi^{\text{NN}}$ for $W$, the other one exists uniquely such that $s(\pi^{\text{NC}}) = s(\pi^{\text{NN}})$ for all $s \in S$. We will say it establishes a type-preserving bijection if furthermore $\pi^{\text{NC}}$ and $\pi^{\text{NN}}$ always have the same classical orbital type.

**Convention 3.2.1.** When we define partition statistics, we shall observe the convention that Roman letters (like $a$) denote ground set elements or tuples thereof, and Greek letters (like $\mu$) denote cardinalities or tuples thereof.

**Theorem 3.2.2.** The statistics $(a, \mu)$ establish a type-preserving bijection for $A_{n-1}$.

Figure 3.1 illustrates this bijection.
The type-preserving assertion in Theorem 3.2.2 is easy: by Proposition 2.2.9 the tuple \( \mu \) determines the type of any partition that yields it.

As for the bijection itself, we will sketch two different descriptions of the process for converting back and forth between classical noncrossing and nonnesting partitions with the same tuples \( a, \mu \), with the intent that they will provide the reader with complementary suites of intuition. Routine verifications are left out of each.

**Proof of Theorem 3.2.2: chain by chain.** View \( M_1, \ldots, M_m \) as chains, i.e. connected components, in the bump diagram for \( \pi \), each corresponding to a block. By virtue of \( \mu \) we know the length of each chain, so we can view the chains as abstract unlabelled graphs in the plane and our task as that of interposing the vertices (or endpoints) of these chains in such a way that the result is nonnesting or noncrossing, as desired.

Suppose we start with \( \pi^{\mathrm{NN}} \). To build the noncrossing diagram of \( \pi^{\mathrm{NC}} \), we will inductively place the chains \( M_1, \ldots, M_m \), in that order. Suppose that, for some \( j \leq n \), we have placed \( M_i \) for all \( i < j \). To place \( M_j \), we insert its rightmost vertex so as to become the \( a_j \)th vertex counting from right to left, relative to the chains \( M_{j-1}, \ldots, M_1 \) already placed. We then insert the remaining vertices of \( M_j \) in the unique possible way so that no pair of crossing edges are formed: inserting them close enough to each other so that no vertex of \( M_i \) for \( i < j \) is left between the rightmost and leftmost vertices of \( M_j \).

Now, suppose we start with \( \pi^{\mathrm{NC}} \) and want \( \pi^{\mathrm{NN}} \). We again build the bump diagram chain by chain, at each step placing the rightmost vertex in exactly the same way and placing the remaining vertices in the unique way so that no pair of nesting edges are formed: this can be achieved if adjacent vertices are drawn at constant Euclidean distance 1 over the real line containing the vertices. Suppose we are in a situation where, for \( p, q \geq 1 \):

* we have drawn a number \( p - 1 \) of vertices and \( q - 1 \) edges following our algorithm, and
* we want to draw a new vertex in the graph, which would be the \( p \)-th vertex drawn.
Consider the \( p-1 \) vertices drawn and call the open intervals between consecutive vertices or between extremal vertices and \( \pm \infty \) places. If we choose one of these places to draw our vertex, it is irrelevant which exact point in this place we choose to draw it. So really, we just want to pick a place to draw. If our vertex is the rightmost vertex of a chain, the uniqueness of the right place to draw it is immediate. If not, per the description presented, there is at least one appropriate place (possibly between consecutive vertices already drawn) to draw the vertex and remain nonnesting: if we draw it to right of this place, either a nesting will be formed in which the edge of our vertex (the \( q \)-th edge) is nested within an edge already drawn, or the numbers of \( a \) will be unrespected; if we draw it to the left of this place, a nesting will be formed in which an edge already drawn is nested within the edge of our vertex (the \( q \)-th edge). Therefore, there is a unique way to draw vertices in places at each step and the final nonnesting diagram is uniquely determined.

\[ \square \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3_2.png}
\caption{The bijection of type \( A \) running (from left to right, top to bottom) in the case nonnesting-noncrossing. As a convention, dotted vertices in our diagrams will represent only labelled vertices. The classical nonnesting (top) and noncrossing (bottom) partitions correspond to \( a = (1, 2, 4), \mu = (2, 3, 3) \).}
\end{figure}

Viewing each block of \( \pi_{\text{NN}} \) as a chain with a fixed spacing is a particularly useful picture in terms of the connection between nonnesting partitions and chambers of the Shi arrangement [3, §5].

**Proof of Theorem 3.2.2: dot by dot.** Let \( M_1, \ldots, M_m \) be the blocks of a classical nonnesting partition \( \pi_{\text{NN}} \), such that the least vertex of \( M_i \) is \( a_i \). We describe an algorithm to build up a classical noncrossing partition \( \pi_{\text{NC}} \) with the same tuples \( a \) and \( \mu \) by assigning the elements 1, \ldots, \( n \), in that order, to blocks.

The algorithm maintains a set \( \mathcal{O} \) of open blocks: an open block is a pair \((M, \kappa)\) where \( M \) is a subset of the ground set of \( \pi_{\text{NN}} \) and \( \kappa \) a nonnegative integer. We think of \( M \) as a partially
completed block of $\pi^{\text{NC}}$ and $\kappa$ as the number of elements which must be added to $M$ to complete it. If $O$ ever comes to contain an open block of form $(M, 0)$, we immediately drop this, for it represents a complete block. When we begin constructing $\pi^{\text{NC}}$, the set $O$ will be empty.

Suppose we’ve assigned the elements $1, \ldots, j - 1$ to blocks of $\pi^{\text{NC}}$ already, and want to assign $j$. If $j$ occurs as one of the $a_i$, then we add a new singleton block $\{j\}$ to $\pi^{\text{NC}}$ and add $(\{a_i\}, \mu_i - 1)$ to $O$. Otherwise, we choose an open block from $O$ according to the Noncrossing open block policy. Given $O$, choose from it the open block $(M, \kappa)$ such that the maximum element of $M$ is maximal.

We add $j$ to this open block, i.e. we replace the block $M$ of $\pi^{\text{NC}}$ by $M' := M \cup \{j\}$ and replace $(M, \kappa)$ by $(M', \kappa - 1)$ within $O$. The desired partition $\pi^{\text{NC}}$ is obtained after assigning all dots.

The central observation to make is that this policy indeed makes $\pi^{\text{NC}}$ noncrossing, and there’s a unique way to follow it. Making a crossing of two edges $(a, c)$ and $(b, d)$, where $a < b < c < d$, requires assigning $c$ to an open block whose greatest element is then $a$, when there also exists one with greatest element $b > a$, which is witnessed to have been open at the time by its later acquisition of $d$; this is in contravention of the policy.

To recover $\pi^{\text{NN}}$ uniquely from $\pi^{\text{NC}}$, the same algorithm works, with one modification: instead of the noncrossing open block policy we use the Nonnesting open block policy. Given $O$, choose from it the open block $(M, \kappa)$ such that the maximum element of $M$ is minimal.
This policy makes $\pi^{NN}$ nonnesting and unique for a similar reason. If there are nested edges $(a, d)$ and $(b, c)$, where $a < b < c < d$, then $c$ was added to the block containing $b$ when by policy it should have gone with $a$, which was in an open block. □

A careful study of either of these proofs provides a useful characterisation of the pairs of tuples $a, \mu$ that are the statistics of a classical nonnesting or noncrossing partition of type $A$. We should think of it exactly in this way: the only necessary data to draw the bump diagrams came from the statistics. For example, using the first entry of $a$ with the first entry of $\mu$, we can directly draw a chain whose place in the bump diagram is uniquely determined. Figure 3.2 shows the presented bijection running in the case nonnesting-noncrossing. Figure 3.3 shows it running in the case noncrossing-nonnesting.

**Corollary 3.2.3.** Suppose we are given a pair of tuples of positive integers $a = (a_1, \ldots, a_m)$, $\mu = (\mu_1, \ldots, \mu_m)$ and let $n > 0$. Define $a_0 = 0$ and $\mu_0 = 1$. Then, $a$ and $\mu$ represent a classical noncrossing or nonnesting partition for $A_{n-1}$ if and only if

1. $m_1 = m_2 = m;
2. n = \sum_{k=1}^{m} \mu_k$; and
3. $a_{i-1} < a_i \leq \sum_{k=0}^{i-1} \mu_k$ for $i = 1, 2, \ldots, m$.

### 3.3. Type C

In the classical reflection groups other than $A_n$, the negative elements of the ground set must be treated, and so it will be useful to have some terminology to deal with these.

**Definition 3.3.1.** A positive block of a classical partition $\pi$ is a block of $\pi$ that contains some positive integer; similarly a negative block contains a negative integer. A switching block of $\pi$ is a block of $\pi$ that contains both positive and nonpositive elements, and a nonswitching block is one that contains only nonpositive elements or only nonnegative elements.

A single edge of the bump diagram is positive or negative or switching or nonswitching if it would have those properties as a block of size 2.

**Definition 3.3.2.** Suppose $\pi$ is a classical partition for $W$ and $W$ is not of type $A$. Let the positive side of $\pi$ be the partition of $\lceil n \rceil$ induced by taking the intersection of each block of $\pi$ with $\lceil n \rceil$. Define the negative side of $\pi$ analogously with $-\lceil n \rceil$.

Let $\pi$ be a classical partition for $C_n$. Given $\pi$, let $M_1 <_{lp} \cdots <_{lp} M_m$ be the positive nonswitching blocks of $\pi$, and $a_i$ the least element of $M_i$. Let $\mu_i$ be the cardinality of $M_i$. These two tuples are reminiscent of type $A$. Let $P_1 <_{lp} \cdots <_{lp} P_k$ be the switching blocks of $\pi$, let $p_i$ be the
least positive element of $P_i$, and let $\nu_i$ be the number of positive elements of $P_i$. Define the three statistics $a(\pi) = (a_1, \ldots, a_m)$, $\mu(\pi) = (\mu_1, \ldots, \mu_m)$, $\nu(\pi) = (\nu_1, \ldots, \nu_k)$. We have

$$n = \sum_{i=1}^{m} \mu_i + \sum_{j=1}^{k} \nu_j.$$

**Theorem 3.3.3.** The statistics $(a, \mu, \nu)$ establish a type-preserving bijection for $C_n$.

Figure 3.4 illustrates the bijection.

![Bump Diagram](image)

**Figure 3.4.** The type C nonnesting (top) and noncrossing (bottom) partitions corresponding to $a = (3, 4)$, $\mu = (2, 1)$, $\nu = (2, 3)$.

**Proof.** We state a procedure for converting back and forth between classical noncrossing and nonnesting partitions that preserve the values $a$, $\mu$, and $\nu$. Suppose we start with a partition $\pi$, be it noncrossing $\pi^{\text{NN}}$ or nonnesting $\pi^{\text{NC}}$, so that we want to find the partition $\pi'$, being $\pi^{\text{NC}}$ or $\pi^{\text{NN}}$ respectively. From $a$, $\mu$, and $\nu$ we inductively construct the positive side of $\pi'$, that is the partition it induces on the set of positive indices $[n]$, which will determine $\pi'$ by invariance under negation.

First we describe it from the chain-by-chain viewpoint. In the bump diagram of $\pi$, consider the labelled connected component representing $P_i$, which we call the *chain* $P_i$. Let the *(unlabelled)* partial chain $P'_i$ be the abstract unlabelled connected graph obtained from the chain $P_i$ by removing its nonpositive nonswitching edges and nonpositive vertices, leaving the unique switching edge incomplete, *i.e.* drawn as a half-edge with just one incident vertex, and by dropping the labels. Notice how the tuple $\nu$ allows us to draw these partial chains. The procedure we followed for type $A$ will generalize to this case, treating the positive elements of the switching edges first.

We want to obtain the bump diagram for $\pi'$, so we begin by using $\nu$ to partially draw the chains representing its switching blocks: we draw only the positive edges (switching and nonswitching) of every chain, leaving the unique switching edge incomplete. This is done by reading $\nu$ from back to front and inserting, each partial switching chain $P'_i$ in turn with its rightmost vertex placed to the right of all existing chains, analogously to type $A$. In the $\pi'$-is-noncrossing case, we end up with every vertex of $P'_i$ being strictly to the right of every vertex of $P'_j$ for $i < j$. In the $\pi'$-is-nonnesting case, the vertices of the switching edges will be exactly the $k$ first positions from right to left among
all the vertices of $P'_1, \ldots, P'_k$. It remains to place the nonswitching chains $M_1, M_2, \ldots, M_m$, and this we do also as in the type $A$ bijection, except that at each step, we place the rightmost vertex of $M_j$ so as to become the $a_j$th vertex, counting from right to left, relative to the chains $M_{j-1}, \ldots, M_1$ and the partial chains $P'_1, P'_2, \ldots, P'_k$ already placed. The construction in the case nonnesting-noncrossing should be straightforward. In the nontrivial case of the opposite direction where we have switching chains, it could be achieved like this: consider the switching chain corresponding to the last entry of $\nu$ and let $p$ be the rightmost vertex of this chain; draw all adjacent vertices at constant Euclidean distance $1$ and all vertices to the right of $p$ at Euclidean distance $<1$ from $p$. The proof of uniqueness is then exactly as in type $A$.

To take the dot-by-dot viewpoint, the type $A$ algorithm can be used with only one modification, namely that $O$ begins nonempty. It is initialised from $\nu$, as

$$O = \{(\{P_i^\prime\}, u_i) : i = 1, \ldots, k\},$$

where $P_i^\prime$ is a fictive element that represents the negative elements of $P_i$ which are yet to be added. We must also specify how these fictive elements compare, for use in the open block policies. A fictive element is always less than a real element. In the noncrossing case $P_i^\prime > P_j^\prime$ iff $i < j$, whereas in the nonnesting case $P_i^\prime > P_j^\prime$ iff $i > j$; the variation assures that $P_1^\prime$ is chosen first in either case.

We now possess a uniquely determined (by the statistics) and partially completed bump diagram. Consider the partition on the set of drawn vertices induced by connectedness in this diagram. This is a partition of $[n]$ and we denote its blocks by $P_1^*, \ldots, P_k^*$. We are being faithful to the statistics, ie. we want to preserve each one of them, so this partition is the only possible positive side for $\pi'$. We know $\pi'$ is invariant under negation, so we copy these blocks down again with all elements negated, and end up with a set of incomplete switching blocks $P_1^*, \ldots, P_k^*$ on the positive side and another equinumerous set $-P_1^*, \ldots, -P_k^*$ on the negative side that we need to pair up and connect with edges in the bump diagram.

There is a unique way to connect these incomplete blocks to get the partition $\pi'$, be it $\pi^{NC}$ or $\pi^{NN}$. This is clear if $\pi'$ is noncrossing. If $\pi'$ is nonnesting, the unique way to connect them would be to so as if $\pi'$ were noncrossing, and then inverting the connection in one of the sides, positive or negative. In every case $P_i^*$ gets connected with $-P_{k+1-i}^*$, and in particular symmetry under negation is attained. If there is a zero block it arises from $P_{(k+1)/2}^*$.

Finally, $\pi$ and $\pi'$ have the same type. Since the $P_i^*$ are paired up the same way in each, including any zero block, $\mu$ and $\nu$ determine the multiset of block sizes of $\pi$ and $\pi'$ and the size of any zero block, in identical fashion in either case. Then this is Proposition 2.2.9. □
3.3. TYPE C

Figure 3.5. The bijection of type C running (from left to right, top to bottom) in the case nonnesting-noncrossing. The classical nonnesting (top) and noncrossing (bottom) partitions correspond to $a = (3, 4), \mu = (2, 1), \nu = (2, 3)$.

Figure 3.5 shows the bijection presented running in the case nonnesting-noncrossing whereas Figure 3.6 shows in the opposite direction. Again, a careful look at the preceding proof gives the characterization of the tuples that describe classical noncrossing and nonnesting partitions for type $C$.

**Corollary 3.3.4.** Suppose we are given some tuples of positive integers $a = (a_1, \ldots, a_{m_1})$, $\mu = (\mu_1, \ldots, \mu_{m_2})$, $\nu = (\nu_1, \ldots, \nu_k)$ and let $n > 0$. Define $a_0 = 0$ and $\mu_0 = 1$. Then, $a$, $\mu$ and $\nu$ represent a classical noncrossing or nonnesting partition for $C_n$ if and only if

(1) $m_1 = m_2 = m$;

(2) $n = \sum_{i=1}^{m} \mu_i + \sum_{j=1}^{k} \nu_j$;
Figure 3.6. The bijection of type $C$ running (from left to right, top to bottom) in the case noncrossing-nonnesting. The classical noncrossing (top) and nonnesting (bottom) partitions correspond to $a = (1, 5), \mu = (2, 1), \nu = (1, 2, 2)$.

(3) $a_{i-1} < a_i \leq \sum_{k=0}^{i-1} \mu_k + \sum_{j=1}^k \nu_j$ for $i = 1, 2, \ldots, m$. 
3.4. Type $B$

We will readily be able to modify our type $C$ bijection to handle type $B$. Indeed, if it weren’t for our concern about type in the sense of Definition 2.2.7, we would already possess a bijection for type $B$, differing from the type $C$ bijection only in pairing up the incomplete switching blocks in a way respecting the presence of the element 0. Our task is thus to adjust that bijection to recover the type-preservation.

If $\pi$ is a classical partition for $B_n$, we define the tuples $a(\pi), \mu(\pi)$ and $\nu(\pi)$ as in type $C$.

Notice that classical noncrossing partitions for $B_n$ and for $C_n$ are identical, and that the strictly positive part of any classical nonnesting partition for $B_n$ is also the strictly positive part of some nonnesting $C_n$-partition, though not necessarily one of the same type. Thus Corollary 3.3.4 characterises the classical noncrossing or nonnesting partitions for $B_n$ just as well as for $C_n$.

Suppose $\pi$ is a classical nonnesting partition for $B_n$. In two circumstances its tuples $a(\pi), \mu(\pi), \nu(\pi)$ also describe a unique nonnesting partition for $C_n$ of the same type: to be explicit, this is when $\pi$ does not contain a zero block, and when the unique switching chain in $\pi$ is the one representing the zero block. If $P_1 <_{1p} \cdots <_{1p} P_k$ are the switching blocks of $\pi$, then $\pi$ contains a zero block and more than one switching chain if and only if $k$ is odd and $k > 1$. We notice that $P_k$ must be the zero block. On the other hand, if $\pi^C$ is a classical nonnesting partition for $C_n$, the zero block must be $P_{(k+1)/2}$. Reflecting this, our bijection will be forced to reorder $\nu$ to achieve type preservation.

Generalizing our prior language, we will say that two lists $S^{NC} = \{s_1^{NC}, \ldots, s_a^{NC}\}$ and $S^{NN} = \{s_1^{NN}, \ldots, s_a^{NN}\}$ of partition statistics, in that order, and a list $\Sigma = \{\sigma_1, \ldots, \sigma_a\}$ of bijections establish a (type-preserving) bijection for a classical reflection group $W$ if, given either a classical noncrossing partition $\pi^{NC}$ or a classical nonnesting partition $\pi^{NN}$ for $W$, the other one exists uniquely such that $\sigma_i(s_i^{NC}(\pi^{NC})) = s_i^{NN}(\pi^{NN})$ for all $1 \leq i \leq a$ (and furthermore $\pi^{NC}$ and $\pi^{NN}$ have the same type).

Suppose we have a tuple $\nu = (\nu_1, \ldots, \nu_k)$ with $k$ odd. Define the reordering

$$\sigma_B(\nu) = (\nu_1, \ldots, \nu_{(k-1)/2}, \nu_{(k+1)/2}, \ldots, \nu_k, \nu_{(k+1)/2}).$$

If $k$ is not odd then let $\sigma_B(\nu) = \nu$. Clearly $\sigma_B$ is bijective. For explicitness, the inverse for $k$ odd is given by

$$\sigma_B^{-1}(\nu) = (\nu_1, \ldots, \nu_{(k-1)/2}, \nu_k, \nu_{(k+1)/2}, \ldots, \nu_{k-1})$$

and for $k$ even $\sigma_B^{-1}(\nu) = \nu$.

Theorem 3.4.1. The lists of statistics $(a, \mu, \nu)$ and $(a, \mu, \nu)$ establish a type-preserving bijection for $B_n$ via the bijections $(\text{id}, \text{id}, \sigma_B)$.
Proof. We use the same procedures as in type $C$ to convert back and forth between classical nonnesting and noncrossing partitions, except that we must rearrange $\nu$ and handle the zero block appropriately, if it is present. When constructing a nonnesting partition we connect the incomplete switching blocks differently: in the notation of the dot-by-dot description, $P_k^*$ must be connected to $-P_k^*$ and the dot 0, so that we connect $P_i^*$ to $-P_{k-i}^*$ for $1 \leq i < k$. The conditions of Corollary 3.3.4, which as we noted above characterise type $B$ classical noncrossing and nonnesting partitions, don’t depend on the order of $\nu$. So if tuples $a$, $\mu$, and $\nu$ satisfy them then so do $a$, $\mu$ and $\sigma_B(\nu)$ (or $\sigma_B^{-1}(\nu)$). Thus our statistics establish a bijection between $NC^{cl}(B_n)$ and $NN^{cl}(B_n)$.

Type is preserved, by the definition of $\sigma$ and the preceding discussion. □

Figure 3.7 illustrates the resulting bijection and Figure 3.8 shows the handling of the zero block in the particular case noncrossing-nonnesting.

3.5. Type $D$

The handling of type $D$ partitions is a further modification of our treatment of the foregoing types, especially type $B$.

In classical partitions for $D_n$, the elements $\pm 1$ will play much the same role as the element 0 of classical nonnesting partitions for $B_n$. So when applying the order $<_lp$ and the terminology of Definition 3.3.1 in type $D$ we will regard $\pm 1$ as being neither positive nor negative.

Given $\pi \in NN^{cl}(D_n)$, define the statistics $a(\pi)$, $\mu(\pi)$ and $\nu(\pi)$ as in type $B$. In this case we must have

$$n - 1 = \sum_{i=1}^{m} \mu_i + \sum_{j=1}^{k} \nu_j.$$
Figure 3.8. The bijection of type $B$ running (from left to right, top to bottom) in the case noncrossing-nonnesting. The classical noncrossing (top) and nonnesting (bottom) partitions correspond respectively to $a = (2), \mu = (2), \nu = (1, 5, 1)$ and $a = (2), \mu = (2), \nu = (1, 1, 5)$.

Let $R_1 <_l \cdots <_l R_l$ be the blocks of $\pi$ which contain both a positive element and either 1 and $-1$. It is clear that $l \leq 2$. Define the statistic $c(\pi) = (c_1, \ldots, c_l)$ by $c_i = R_i \cap \{1, -1\}$. To streamline the notation we’ll usually write $c_i$ as one of the symbols $+, -, \pm$. Observe that $\pi$ contains a zero block if and only if $c(\pi) = (\pm)$.

To get a handle on type $D$ classical noncrossing partitions, we will transform them into type $B$ ones. Let $NC^cl_t(B_{n-1})$ be a relabelled set of classical noncrossing partitions for $B_{n-1}$, in which the ground elements $1, \ldots, (n-1)$ and $-1, \ldots, -(n-1)$ are changed respectively to $2, \ldots, n$ and $-2, \ldots, -n$. Define a map $CM : NC^cl_t(D_n) \rightarrow NC^cl_t(B_{n-1})$, which we will call central merging,
such that for $\pi \in NC^{cl}(D_n)$, $CM(\pi)$ is the classical noncrossing $B_{n-1}$-partition obtained by first merging the blocks containing 1 or $-1$ (which we’ve drawn at the center of the circular diagram) into a single block, and then discarding the elements $\pm 1$. Define the statistics $a$, $\mu$ and $\nu$ for $\pi$ to be equal to those for $CM(\pi)$, where the entries of $a$ should acknowledge the relabelling and thus be chosen from $\{2,\ldots,n\}$.

These statistics do not uniquely characterise $\pi$, so we define additional statistics $c(\pi)$ and $\xi(\pi)$. The definition of $c(\pi)$ is analogous to the nonnesting case: let $R_1 <_{lp} \cdots <_{lp} R_l$ be the blocks of $\pi$ which intersect $\{1,-1\}$, and define $c(\pi) = (c_1,\ldots,c_l)$ where $c_i = R_i \cap \{1,-1\}$. Also define $\xi(\pi) = (\xi_1,\ldots,\xi_l)$ where $\xi_l = \#(R_l \cap \{2,\ldots,n\})$ is the number of positive elements of $R_l$.

Observe that $CM(\pi)$ lacks a zero block if and only if $c(\pi) = ()$, the case that 1 and $-1$ both belong to singleton blocks of $\pi$. In this case $CM(\pi)$ is just $\pi$ with the blocks $\{1\}$ and $\{1\}$ removed, so that $\pi$ is uniquely recoverable given $CM(\pi)$. Otherwise, $CM(\pi)$ has a zero block. If $c(\pi) = (\pm)$ this zero block came from a zero block of $\pi$, and $\pi$ is restored by resupplying the elements $\pm 1$ to this zero block. Otherwise two blocks of $\pi$ are merged in the zero block of $CM(\pi)$. Suppose the zero block of $CM(\pi)$ is $\{c_1,\ldots,c_j,-c_1,\ldots,-c_j\}$, with $0 < c_1 < \cdots < c_j$, so that $j = \sum_{i=1}^l \xi_i$. By the noncrossing and symmetry properties of $\pi$, one of the blocks of $\pi$ which was merged into this block has the form $\{-c_{i+1},\ldots,-c_j,c_1,\ldots,c_i,s\}$ where $1 \leq i \leq j$ and $s \in \{1,-1\}$. Then, by definition, $c(\pi) = (s,-s)$ and $\xi(\pi) = (i,j-i)$, except that if $j-i = 0$ the latter component of each of these must be dropped. In this case the merged blocks of $\pi$ can be reconstructed since $c$ and $\xi$ specify $s$ and $i$.

Let a tagged noncrossing partition for $B_{n-1}$ be an element $\pi \in NC^{cl}_t(B_{n-1})$ together with tuples $c(\pi)$ of nonempty subsets of $\{1,-1\}$ and $\xi(\pi)$ of positive integers such that:

1. the entries of $c(\pi)$ are pairwise disjoint;
2. $c(\pi)$ and $\xi(\pi)$ have equal length;
3. the sum of all entries of $\xi(\pi)$ is the number of positive elements in the zero block of $\pi$.

**Lemma 3.5.1.** Central merging gives a bijection between classical noncrossing partitions for $D_n$ and tagged noncrossing partitions for $B_{n-1}$.

**Proof.** The foregoing discussion establishes that $CM$ is bijective. In view of this we need only check that the noncrossing property is preserved when moving between $\pi$ and $CM(\pi)$. In terms of bump diagrams, if $CM(\pi)$ is noncrossing $\pi$ is easily seen to be. For the converse, suppose $CM(\pi)$ has a crossing. This must be between the zero block $O$ and some other block $B$ of $CM(\pi)$, so that it is possible to choose $i,j \in B$ and $k \in O$ such that the segments $(i,j)$ and $(k,-k)$ within the bump diagram of $CM(\pi)$ cross. But these segments also cross in the bump diagram for $\pi$ and are contained within different blocks. \qed
We show next that partitions are uniquely determined by the data we have associated with them.

**Lemma 3.5.2.** A classical nonnesting partition \( \pi \) for \( D_n \) is uniquely determined by the values of \( a(\pi), \mu(\pi), \nu(\pi), \) and \( c(\pi) \).

**Proof.** We reduce to the analogous facts for classical nonnesting partitions of types \( B \) and \( C \). There are slight variations in the behavior depending on \( c(\pi) \), so we break the argument into cases.

If \( c(\pi) = () \), then dropping the elements \( \pm 1 \) from \( \pi \) and relabelling \( 2, \ldots, n, -2, \ldots, -n \) to \( 1, \ldots, n-1, -1, \ldots, -(n-1) \) yields a nonnesting partition \( \pi' \) for \( C_{n-1} \), and this is uniquely characterised by \( a(\pi'), \mu(\pi'), \) and \( \nu(\pi') \), which only differ from the statistics of \( \pi \) by the relabelling in \( a \).

If \( c(\pi) = (\pm) \), then merging the elements \( \pm 1 \) into a single element 0 and relabelling \( 2, \ldots, n, -2, \ldots, -n \) to \( 1, \ldots, n-1, -1, \ldots, -(n-1) \) yields a nonnesting \( B_{n-1} \)-partition, and this is again uniquely characterised by \( a(\pi'), \mu(\pi'), \) and \( \nu(\pi') \), which only differ from the statistics of \( \pi \) by the relabelling in \( a \).

The cases \( c(\pi) = (-) \) and \( c(\pi) = (+, -) \) are carried under the exchange of \( +1 \) and \( -1 \) respectively to \( c(\pi) = (+) \) and \( c(\pi) = (-, +) \), so it suffices to handle only the latter two.

We claim that, in these latter two cases, \( \pi \) is itself a classical nonnesting partition for \( C_n \). We will write \( \pi' \) for \( \pi \) when we mean to conceive of it as an element \( NN_{cl}(C_n) \); in particular \( \pi \) and \( \pi' \) will have different statistics. Since the ground set order for \( NN_{cl}(C_n) \) is a refinement of the order for \( NN_{cl}(D_n) \) in which only the formerly incomparable elements 1 and \( -1 \) in \( \pi \) have become comparable in \( \pi' \), \( \pi' \) will be in \( NN_{cl}(C_n) \) so long as no nestings involving edges of \( G(\pi) \) terminating at 1 and \( -1 \) are introduced. By symmetry, if there is such a nesting, there will be one involving the edges \((i, 1)\) and \((j, -1)\) of \( G(\pi') \) for some \( i, j > 1 \). But the fact that \( c(\pi) \) ends with \( + \) implies either \( i > j \) or the edge \((j, -1)\) does not exist, so there is no nesting of this form.

When we readmit 1 and \( -1 \) as positive and negative elements, respectively, every nonswitching block of \( \pi \) remains nonswitching in \( \pi' \), and every switching block of \( \pi \) remains switching unless its only nonpositive element was 1; in this latter case \( -1 \) is likewise the only nonnegative element of its block, which happens iff \( c(\pi) = (+) \).
Let \( a(\pi) = (a_1, \ldots, a_m) \), \( \mu(\pi) = (\mu_1, \ldots, \mu_m) \), \( \nu(\pi) = (\nu_1, \ldots, \nu_m) \). In the case \( c(\pi) = (-, +) \), we have
\[
(3.1) \quad \begin{align*}
 a(\pi') &= a(\pi) \\
 \mu(\pi') &= \mu(\pi) \\
 \nu(\pi') &= (\nu_k + 1, \nu_1, \ldots, \nu_{k-1}).
\end{align*}
\]
That is, the block containing 1 is the greatest switching block of \( \pi \) under \( \prec_{lp} \) by assumption, but in \( \pi' \) where 1 is positive it becomes the first switching block. The other switching blocks are unchanged in number of positive elements and order, and nothing changes about the switching blocks. In case \( c(\pi) = (+) \), the block containing 1 contains no other nonpositive element, so it becomes a nonswitching block, and in this case we get
\[
(3.2) \quad \begin{align*}
 a(\pi') &= (1, a_1, \ldots, a_m) \\
 \mu(\pi') &= (\nu_k + 1, \mu_1, \ldots, \mu_m) \\
 \nu(\pi') &= (\nu_1, \ldots, \nu_{k-1}).
\end{align*}
\]
In either case \( \pi' \) is a classical nonnesting partition for \( C_n \), and as such is determined by its statistics, but the translations (3.1) and (3.2) are injective so that \( \pi \) is determined by its statistics as well. \( \square \)

Note that, when \( c(\pi) \) is \( (+) \) or \( (-, +) \), \( \pi' \) is an arbitrary noncrossing partition for \( C_n \) subject to the condition that 1 is not the only positive element of its block. The cases \( (+) \) and \( (-, +) \) can be distinguished by whether \( a(\pi') \) starts with 1. Note also that the blocks of \( \pi \) which contain one of the elements 1 or \(-1\) are exactly those described by the last \( l \) components of \( \nu(\pi) \), where \( l \) is the length of \( c(\pi) \).

All that remains to obtain a bijection is to describe the modifications to \( \nu \) that are needed for correct handling of the zero block and its components (rather as in type \( B \)). For a classical nonnesting partition \( \pi \) for \( D_n \), find the tuples \( a(\pi), \mu(\pi), \nu(\pi) = (\nu_1, \ldots, \nu_k) \), and \( c(\pi) \). Let \( \xi(\pi) \) be the tuple of the last \( l \) entries of \( \nu(\pi) \), where \( l \) is the length of \( c(\pi) \). Define
\[
(\hat{\nu}(\pi), \xi_{inv}(\pi), c_{inv}(\pi))
\]
\[
= \begin{cases} 
 (\nu_1, \ldots, \nu_{k-1}, \nu_k - (\xi_2, \xi_1), (c_2, c_1)) & \text{if } l = 2 \\
 (\nu_1, \ldots, \nu_{k-1}/2, \nu_k, \nu_{k+1}/2, \ldots, \nu_{k-1}), \xi(\pi), c(\pi)) & \text{if } l = 1 \\
 (\nu(\pi), \xi(\pi), c(\pi)) & \text{if } l = 0
\end{cases}
\]
Define a bijection \( \sigma_D \) by \( \sigma_D (\nu(\pi), c(\pi)) = (\hat{\nu}(\pi), \xi_{inv}(\pi), c_{inv}(\pi)) \). This gives us all the data for a tagged noncrossing partition \( CM(\pi') \) for \( B_{n-1} \), which corresponds via central merging with a noncrossing partition \( \pi' \) for \( D_n \). Going backwards, from a noncrossing partition \( \pi' \) we recover a nonnesting partition \( \pi \) by applying central merging, finding the list of statistics \( (a(\pi), \mu(\pi), (\nu(\pi), c(\pi))) \)
via the equality 

\[(\nu(\pi), c_\pi) = \sigma_{D}^{-1} (\nu(\pi'), \xi(\pi'), c(\pi'))\]

(the other statistics remain equal) and using these statistics to make a nonnesting partition as usual. Type preservation is implied within these modifications of the statistics. When a zero block exists, the number of positive elements it contains is preserved because of the equality \(\xi(\pi') = \nu(\pi)_k\) which holds in that case. Our handling of \(\nu\) leaves the components corresponding to switching blocks not containing 1 or \(-1\) unchanged, so the number of positive elements in these blocks is also preserved. The number of positive elements in the blocks containing 1 or \(-1\) is preserved because \(\xi(\pi')\) corresponds to \(c(\pi')\) as the last entries of \(\nu(\pi)\) correspond to \(c(\pi)\) in all cases. The size of each nonswitching block is preserved in the statistic \(\mu\), as in previous cases.

All in all, we have just proved the following theorem.

**Theorem 3.5.3.** The lists of statistics \((a, \mu, (\nu, \xi, c))\) and \((a, \mu, (\nu, c))\) establish a type-preserving bijection for \(D_n\) via the bijections \((\text{id}, \text{id}, (\sigma_D)^{-1})\).

Figures 3.9 and 3.10 illustrate this bijection.

**Figure 3.9.** The \(D_{10}\) nonnesting partition corresponding to \(a = (3), \mu = (2), \nu = (1, 1, 2, 3), c = (+, -)\) (so \(\hat{\nu} = (1, 5, 1)\)).

**Figure 3.10.** (left) The \(D_{10}\) noncrossing partition corresponding to \(a = (3), \mu = (2), \nu = (1, 5, 1), \xi = (3, 2), c = (-, +)\). (right) The relabelled type \(B\) noncrossing partition obtained via central merging.

Finally we present a characterization of the values of \(a, \mu, \nu\) and \(c\) that describe type \(D\) classical nonnesting partitions. As for noncrossing partitions, between our discussion of type \(B\) and the definition of tagged partitions and Lemma 3.5.1, we have already presented all parts of the analogous result.
Corollary 3.5.4. Suppose we are given the tuples of positive integers \(a = (a_1, \ldots, a_m), \mu = (\mu_1, \ldots, \mu_m)\) and \(\nu = (\nu_1, \ldots, \nu_k)\), and a tuple \(c = (c_1, \ldots, c_l)\) of nonempty subsets of \(\{1, -1\}\). Let \(n > 0\). Define \(a_0 = 1\) and \(\mu_0 = 2\). Then, \(a, \mu, \nu\) and \(c\) represent a classical nonnesting partition for \(D_n\) if and only if

1. \(m_1 = m_2 = m\);
2. \(n - 1 = \sum_{i=1}^{m} \mu_i + \sum_{j=1}^{k} \nu_j\);
3. \(a_{i-1} < a_i \leq \sum_{k=0}^{i-1} \mu_k + \sum_{j=1}^{k} \nu_j\) for \(i = 1, 2, ..., m\);
4. the entries of \(c(\pi)\) are pairwise disjoint, so in particular \(l \leq 2\);
5. \(k - l\) is even.

3.6. Proof of the central theorem

Using the preceding bijections we are now ready to establish our central result.

Proof of Theorem 3.1.2. When defining statistics and using the terminology of Definition 3.3.1 we consider positive integers as positive elements of blocks and negative integers as negative ones, without exception. Tag these new statistics with \(\ast\) to distinguish them from the old statistics defined in Sections 3.2 through 3.5. Let \(x^{cl}\) be the classical partition representing \(x\). Let \(\eta^\ast(x^{cl})\) be the number of positive elements in the zero block of \(x^{cl}\). For any nonzero switching block \(P\) of \(x^{cl}\), define the joint block

\[S = \min_{<_{lp}} (P, -P)\]

and let \(S_1 <_{lp} \cdots <_{lp} S_{k'}\) be the joint blocks of \(x^{cl}\). The number of joint blocks \(k'\) is half the number of nonzero switching blocks. Let \(\vartheta^\ast_{+,i}\) be the number of positive elements in \(S_i\) and let \(\vartheta^\ast_{-,i}\) be the number of negative elements in \(S_i\) and define the statistic \(\vartheta^\ast(x^{cl}) = ((\vartheta^\ast_{+,1}, \vartheta^\ast_{+,1}, \ldots, (\vartheta^\ast_{+,k'}, \vartheta^\ast_{-,k'}))\). Finally, define as usual the statistics \(a^\ast(x^{cl}) = (a^\ast_1, \ldots, a^\ast_{m'})\) and \(\mu^\ast(x^{cl}) = (\mu^\ast_1, \ldots, \mu^\ast_{m'})\).

Let \(y^{cl}\) be the image of \(x^{cl}\) under the bijections of Theorems 3.2.2–3.5.3. There is a simple way to find a basis for \(\text{Fix}(x)\). From \(x^{cl}\) define a function \(f : x^{cl} \rightarrow \Psi^n\) in the following way. For any block \(B\) of \(x^{cl}\), let

\[f(B) = \text{sgn}(B) \sum_{b \in B} b \sum_{c|B} e_{|b|}\]

where \(\text{sgn}(B)\) is +1 or −1 so that \(f(B) \geq_{\text{lex}} 0\) when \(B\) is nonswitching and \(-f(B) \geq_{\text{lex}} 0\) when \(B\) is switching. The set \(\beta := f(x^{cl}) \setminus \{0\}\) is the basis we are looking for, which we call the canonical basis of \(\text{Fix}(x)\).
3.6. PROOF OF THE CENTRAL THEOREM

For a (positive) nonswitching block $C_i$ of $x^\cl$, we have
\begin{align}
\text{card}(f(C_i) \cap E) &= n + 1 - a^*_i \\
\#(f(C_i), 1) &= \mu^*_i \\
\#(f(C_i), -1) &= 0 \\
\text{card}(f(C_i) \cap \beta) &= (m' + 1 - i) + (k')
\end{align}
(3.3)

For a joint block $S_j$ of $x^\cl$, we have
\begin{align}
\text{card}(f(S_j) \cap E) &= 0 \\
\#(f(S_j), 1) &= \vartheta^*_j \\
\#(f(S_j), -1) &= \vartheta^*_j \\
\text{card}(f(S_j) \cap \beta) &= j
\end{align}
(3.4)

In any case, we have the equality
\begin{equation}
\Gamma_x = \eta^*(x^\cl)
\end{equation}
(3.5)

Note that
\begin{equation*}
f(S_1) <_{\text{lex}} \cdots <_{\text{lex}} f(S_{k'}) <_{\text{lex}} 0 <_{\text{lex}} f(C_{m'}) <_{\text{lex}} \cdots <_{\text{lex}} f(C_1)
\end{equation*}
and that $m' + k'$ is the number of vectors in the ordered basis $\beta$. In fact
\begin{equation*}
\beta = \{ f(S_1), \ldots, f(S_{k'}), f(C_{m'}), \ldots, f(C_1) \}
\end{equation*}

Suppose $z$ is nonnesting or noncrossing partition of $W$ and suppose $\{v_1, \ldots, v_p\}$ is the canonical basis of $\text{Fix}(z)$, ordered so that $v_1 <_{\text{lex}} \cdots <_{\text{lex}} v_n$, which we don’t know. Let $z^\cl$ be the classical partition of $z$. Then, knowing the statistics $S^* := (a^*, \mu^*, \vartheta^*, \eta^*)$ associated to $z^\cl$ allows us to recover the data in (3.3) through (3.5) associated to each of the $v_i$, and vice versa. Thus, the first step to reach our goal would be to prove that the bijections in Theorems 3.2.2 through 3.5.3 actually preserve the statistics $S^*$. Any of the old statistics for $y^\cl$ that is not mentioned in the following lines is trivially recovered from $S^*$.

Assume without loss of generality that $x$ is a nonnesting partition, the other direction being completely analogous.

We begin with the case where $x$ is a nonnesting partition of $A_{n-1}$. The bijection of Theorem 3.2.2 clearly preserves $S^*$. We have $a(y^\cl) = a^*(x^\cl)$ and $\mu(y^\cl) = \mu^*(x^\cl)$ so the uniqueness of $y^\cl$ is established directly from the statistics $S^*$. 
Suppose \( x \) is an antichain for \( C_n \). The statistics \( a^*, \mu^* \) and \( \eta^* \) are clearly preserved in Theorem 3.3.3. Also \( \vartheta^*_+ = \nu_i \) and \( \vartheta^*_- = \nu_{k+1-i} \) so \( \vartheta^* \) is also preserved. When there is a zero block we have \( \nu_{(k+1)/2} = \eta^* \) and this happens if and only if \( \eta^* > 0 \). Therefore \( y^{cl} \) is characterized by \( S^* \).

Consider the case when \( x \) is an antichain for \( B_n \). Again, the statistics \( a^*, \mu^* \) and \( \eta^* \) are clearly preserved in Theorem 3.4.1. If there is no zero block we have \( \vartheta^*_+ = \nu_i \) and \( \vartheta^*_- = \nu_{k+1-i} \). When there is a zero block we have

\[
\vartheta^*_+(x^{cl}) = \nu_i(x^{cl}) \quad \text{and} \quad \vartheta^*_-(x^{cl}) = \nu_{k-i}(x^{cl})
\]

but we also know that

\[
\nu_i(x^{cl}) = \nu_i(y^{cl}) \quad \text{and} \quad \nu_{k-i}(x^{cl}) = \nu_{k+1-i}(y^{cl})
\]

so \( \vartheta^* \) is preserved. There is a zero block if and only if \( \eta^* > 0 \) and here we know in addition that \( \nu_k(x^{cl}) = \eta^* \) and \( \nu_{(k+1)/2}(y^{cl}) = \eta^* \). Therefore \( y^{cl} \) is again characterized by \( S^* \).

We now consider the case when \( x \) is an antichain for \( D_n \). This part is divided into several subcases. Consider first when \( c(x^{cl}) = () \). Here, \( x^{cl} \) is a classical nonnesting partition for \( B_n \) and its image \( y^{cl} \) under Theorem 3.5.3 is the unique classical noncrossing partition from Theorem 3.4.1 so the previous type suffices. We know \( c(x^{cl}) = () \) holds exactly when \( a^*_1 = 1, \mu^*_1 = 1 \) and \( \eta^* = 0 \).

Suppose we have \( c(x^{cl}) = (+) \). Here the element +1 belongs to a nonswitching block of size \( > 1 \). In the bijection of Theorem 3.5.3 the statistics \( S^* \) are preserved and this case is characterized by \( a^*_1 = 1, \mu^*_1 > 1 \) and \( \eta^* = 0 \). Furthermore, on the noncrossing side we have

\[
a(y^{cl}) = (a^*_1, a^*_2, \ldots, a^*_{m'})
\]
\[
\mu(y^{cl}) = (\mu^*_1, \mu^*_2, \ldots, \mu^*_{m'})
\]
\[
\nu(y^{cl}) = (\vartheta^*_+1, \ldots, \vartheta^*_{+k'}, \mu^*_1 - 1, \vartheta^*_{-k'}, \ldots, \vartheta^*_{-1})
\]
\[
\xi(y^{cl}) = (\mu^*_1 - 1)
\]
\[
c(y^{cl}) = (+)
\]

so the uniqueness of \( y^{cl} \) is established directly from \( S^* \).
Suppose \( c(x^{\cl}) = (-) \). The statistics \( a^*, \mu^* \) and \( \eta^* \) are preserved. To check that \( \vartheta^* \) is preserved we have

\[
\begin{align*}
\vartheta^*_{+1}(x^{\cl}) &= 1 & \vartheta^*_{+1}(y^{\cl}) &= 1 \\
\vartheta^*_{-1}(x^{\cl}) &= \nu_k(x^{\cl}) & \vartheta^*_{-1}(y^{\cl}) &= \nu_k(y^{\cl}) \\
\vartheta^*_{+i}(x^{\cl}) &= \nu_{i-1}(x^{\cl}) & \vartheta^*_{+i}(y^{\cl}) &= \nu_{i-1}(y^{\cl}) \quad \text{for } i > 1 \\
\vartheta^*_{-i}(x^{\cl}) &= \nu_{k-i}(x^{\cl}) & \vartheta^*_{-i}(y^{\cl}) &= \nu_{k+1-i}(y^{\cl}) \quad \text{for } i > 1
\end{align*}
\]

However, we know the following equalities hold.

\[
\nu_k(x^{\cl}) = \nu_{k+1}(y^{\cl}) \\
\nu_{i-1}(x^{\cl}) = \nu_{i-1}(y^{\cl}) \quad \text{and} \quad \nu_{k-i}(x^{\cl}) = \nu_{k+1-i}(y^{\cl}) \quad \text{for } i > 1
\]

Hence, \( \vartheta^* \) is indeed preserved. We also know that \( c(x^{\cl}) = (-) \) if and only if \( a^*_1 > 1 \), \( \vartheta^*_+ = 1 \) and \( \eta^* = 0 \). Using the previous equations we may see that \( \nu(y^{\cl}) \) is obtained uniquely from \( \vartheta^* \), therefore \( y^{\cl} \) is characterized by \( S^* \).

Suppose \( c(x^{\cl}) = (\pm) \). Here it is easily seen that \( S^* \) are preserved. The characterization for the case is \( \eta^* > 0 \) and the uniqueness of \( y^{\cl} \) is also easily established.

Finally, consider the case when \( l = 2 \) so either \( c(x^{\cl}) = (+, -) \) or \( c(x^{\cl}) = (-, +) \) holds. To start, suppose that \( c(x^{\cl}) = (+, -) \). The bijection of Theorem 3.5.3 preserves \( a^*, \mu^* \) and \( \nu^* \) clearly. To see that \( \vartheta^* \) is also preserved we need the more intricate equalities

\[
\begin{align*}
\vartheta^*_{+1}(x^{\cl}) &= \nu_{k-1}(x^{\cl}) + 1 & \vartheta^*_{+1}(y^{\cl}) &= \xi_2(y^{\cl}) + 1 \\
\vartheta^*_{-1}(x^{\cl}) &= \nu_k(x^{\cl}) & \vartheta^*_{-1}(y^{\cl}) &= \xi_1(y^{\cl}) \\
\vartheta^*_{+i}(x^{\cl}) &= \nu_i(x^{\cl}) & \vartheta^*_{+i}(y^{\cl}) &= \nu_i(y^{\cl}) \quad \text{for } i > 1 \\
\vartheta^*_{-i}(x^{\cl}) &= \nu_{k-1-i}(x^{\cl}) & \vartheta^*_{-i}(y^{\cl}) &= \nu_{k+1-i}(y^{\cl}) \quad \text{for } i > 1
\end{align*}
\]

But we know from the handling of the statistics for type \( D \) that

\[
\nu_{k-1}(x^{\cl}) = \xi_2(y^{\cl}),
\]

because of the function \( \sigma_D \);

\[
\nu_k(x^{\cl}) = \xi_1(y^{\cl}),
\]

also because of the function \( \sigma_D \); and

\[
\nu_i(x^{\cl}) = \nu_i(y^{\cl}) \quad \text{and} \quad \nu_{k-1-i}(x^{\cl}) = \nu_{k+1-i}(y^{\cl}) \quad \text{for } i > 1.
\]

This implies that \( \vartheta^* \) is preserved in Theorem 3.5.3. Note that \( c(x^{\cl}) = (+, -) \) or \( c(x^{\cl}) = (-, +) \) occurs whenever none of the previous cases holds or whenever \( a^*_1 > 1 \), \( \vartheta^*_+ > 1 \) and \( \eta^* = 0 \). Note also that we can obtain \( a(y^{\cl}), \mu(y^{\cl}), \nu(y^{\cl}) \) and the number of positive and negative elements.
in the block containing +1 directly from $S^*$, but we cannot characterize $y^{cl}$. This is because the information in $S^*$ does not tell apart two noncrossing partitions $y^{cl}_1$ and $y^{cl}_2$ with identical statistics $a, \mu$ and $\nu$ but such that $\xi(y^{cl}_1) = \xi_{inv}(y^{cl}_2)$ and $c(y^{cl}_1) = c_{inv}(y^{cl}_2)$, $y^{cl}_1$ and $y^{cl}_2$ have the same statistics $S^*$. In particular $\vartheta^*_+ (y^{cl}_1) = \vartheta^*_+ (y^{cl}_2)$ and $\vartheta^*_+ (y^{cl}_1) = \vartheta^*_+ (y^{cl}_2)$. If additionally we require that the element with smallest absolute value $> 1$ in the block containing +1 changes sign from $x^{cl}$ to $y^{cl}$, then this would be Theorem 3.5.3. This extends to all cases and types in the following way. For a joint block $S_i$ of $x^{cl}$ with more than one positive element, we require that the element with smallest nonminimal absolute value in $S_i$ and the equivalent element in its image block $S'_i$ of $y^{cl}$ have opposite signs. This new requirement is simply a necessary condition for noncrossing (or nonnesting) bump diagrams in all other cases and it was discussed in the proof of Theorem 3.3.3, so there is no loss or change in the previous analysis if we consider it as being part of the bijections. However this is tantamount to requiring that for any such block $S_i$, the product of the first two nonzero components in $f(S_j)$ and $f(S'_j)$ is not equal. Clearly $S_i$ satisfies #($f(S_j), -1) > 1$ and #($f(S_j), 1) > 0$ and these inequalities are equivalent to the condition imposed on $S_i$.

Therefore, if we can prove that $\Omega_x = \beta$ in the general case where $x$ is a nonnesting or noncrossing partition of $W$, we will be done.

Suppose $x$ is a noncrossing or nonnesting partition and we are on step $i$ in the construction of $\Omega_x$. We have $e_i$ and we want to obtain $u_i \in \Psi^n \cap \text{Fix}(x)$ with $\|u_i - e_i\| <_{\text{lex}}$-minimal.

Consider the case when $i$ belongs to a zero block of $x^{cl}$. This means that $\pi^i(\text{Fix}(x)) = \{0\}$ where $\pi^i$ is the canonical projection on the $i$-th coordinate. If $v$ belongs to $\Psi^n \cap \text{Fix}(x)$ then $\|v - e_i\| \geq_{\text{lex}} e_i$ because $\|v - e_i\|_i = 1$. Hence $u_i = 0$ and $u_i$ does not enter $\Omega_x$.

Now consider when $i$ does not belong to a zero block. We first prove the uniqueness of $u_i$. Suppose there exist two vectors $u_i$ and $u'_i$ such that $\|u_i - e_i\|$ and $\|u'_i - e_i\|$ are $<_{\text{lex}}$-minimal. This implies $\|u_i - e_i\| = \|u'_i - e_i\|$ but then $(u_i)_i = (u'_i)_i$ and $(u_i)_j = (u'_i)_j$ for $j \neq i$. The $<_{\text{lex}}$-minimality condition implies that $u_i = u'_i = 0$ or $(u_i)_i = (u'_i)_i = 1$. Suppose $(u_i)_i = (u'_i)_i = 1$ holds and suppose $(u_i)_j = -(u'_i)_j$ for some $j \neq i$. Then $(u_i + u'_i)/2$ belongs to $\Psi^n \cap \text{Fix}(x)$ and $\|(u_i + u'_i)/2 - e_i\| <_{\text{lex}} \|u_i - e_i\| = \|u'_i - e_i\|$, a contradiction. Thus $u_i$ is unique. Again, the $<_{\text{lex}}$-minimality condition implies $u_i = 0$ or $(u_i)_i = 1$. If $u_i = 0$ then it does not enter $\Omega_x$. Suppose there exists some $j < i$ such that $(u_i)_j \neq 0$. In this case $\|0 - e_i\| <_{\text{lex}} \|u_i - e_i\|$ and we obtain a contradiction. Therefore, if $u_i$ enters $\Omega_x$ then $(u_i)_j = 0$ for $j < i$ and $(u_i)_i = 1$. The $<_{\text{lex}}$-minimality condition shows that actually $u_i$ is the vector in $\Psi^n \cap \text{Fix}(x)$ with the least number of nonzero components such that $(u_i)_j = 0$ for $j < i$ and $(u_i)_i = 1$. Now $u_i$ satisfies these conditions if and only if $i$ is the least nonzero component of $f(B)$ for some nonzero block $B$ of $x^{cl}$ so that $u_i = f(B)$ or $u_i = -f(B)$ (according to whether $B$ is nonswitching or switching, respectively). But
the sets

\[ S_1, \ldots, S_{k'}, C_1, \ldots, C_{m'} \]

are pairwise disjoint and their minimal positive elements are all different, and \( u_i \) (or \(-u_i\)) always enters \( \Omega_x \), so we obtain a correspondence between the elements of \( \Omega_x \) and \( \beta \).

\[ \square \]
CHAPTER 4

Further Directions

We have obtained a bijection between $NC(W)$ and $NN(W)$ when $W$ is a classical reflection group. The bijection is uniform up to the choice of standard simple systems, root systems and Coxeter elements. We do not think it is too far from uniform.

For example, an initial foray into the exceptional groups has verified that the same theorem holds for $G_2$ with $(1, -1, 0)$ and $(-2, 1, 1)$ chosen as simple roots.

Some interesting questions are left open.

**Question 4.0.1.** Can a bijection along the lines of Theorem 3.1.2 be extended to all Weyl groups? Is there a coordinate-free form of that theorem?

We have used a form of lexicographical order on $\mathbb{Z}^n$ to study certain canonical bases for the fixed spaces of noncrossing and nonnesting partitions. If we are given $w \in W$, there must be in general a “nice” way to express $w$ as product of reflections which reveals the connection with $\text{Fix}(x)$ directly. This is true in the most basic types and we would like to ask about the remaining cases. A restatement of 3.1.2 in this way would possibly be more prone to generalization.

Nonnesting partitions come from the root poset, which consists of positive integer combinations of simple roots. Studying the root posets of Weyl groups we realize there is a relation between how positive roots are expressed as sums of simple roots and how they are expressed as product of simple roots. It would nice to understand this better.

Finally, we propose what we think is a very interesting problem,

**Question 4.0.2.** For each combinatorial type, what is the relation between the content of this writing and the polytopal realization of the corresponding generalized associahedron?
Bibliography