0.1. **Proof of Localization Theorem via differential operators on G/U.** This proof mimics the standard proof of the fact that projective space is D-affine (see Bernstein's notes). For background see [BBP].[1]

Let M be a twisted D-module on G/B and $\tilde{M}$ its pull back to G/U. Let $\Gamma_\lambda(M)$ be the weight component of weight λ in global sections of $\tilde{M}$ (here weights are taken with respect to the action of Cartan coming from the right action on G/U).

We need to show that for a regular dominant λ we have

1) If $M \neq 0$ then $\Gamma_\lambda(M) \neq 0$.

2) (also true for dominant not necessarily regular) The functor $M \mapsto \Gamma_\lambda(M)$ is exact.

Let $\widetilde{D}$ be the ring of global differential operators on G/U, thus the space $\Gamma(\tilde{M})$ is a $\widetilde{D}$ module.

The ring $\mathcal{O}(G/U) = \oplus_\mu V_\mu$ (where V_μ is representation with highest weight μ) is a subring in $\widetilde{D}$. Also $\widetilde{D}$ carries an action of the Weyl group W where the simple reflections act by relative Fourier transform for the $\mathbb{A}^2$ projection corresponding to the simple root α.

In particular, let C_μ be the image of the invariant element in $V_\mu \otimes V_{\mu'}$ (where μ' is the dual weight, i.e. $V_{\mu'} \cong V_\mu^*$) under the map $\phi \otimes \psi \mapsto \phi w_0(\psi)$ and C'_μ be the image of that element under the map $\phi \otimes \psi \mapsto w_0(\phi)\psi$; here $\phi \in V_\mu \subset \mathcal{O}(G/U)$, $\phi \in V'_\mu \subset \mathcal{O}(G/U)$. The elements C_μ, $C_{\mu'}$ lie in the enveloping of the abstract Cartan acting on the right on G/U and can be explicitly computed, see [BBP].

The formula shows that C_μ acts on $\Gamma_\lambda(M)$ by a nonzero constant provided that $\lambda - \mu$ is regular dominant. Thus (substituting $\nu = \lambda - \mu$) we see that if $\Gamma_\nu(M) = 0$ for a regular dominant ν then $\Gamma_{\nu+\mu}(M) = 0$ for any dominant μ, which implies that $M = 0$. This shows 1).

Likewise C'_μ acts by a nonzero constant on $\Gamma_\lambda(M)$ when λ is dominant. This implies 2): to see this consider a surjection $N \to M$. Given $a \in \Gamma_\lambda(M)$, there exists μ such that fa lies in the image of $\tilde{\Gamma}(N)$ for $f \in V_\mu \subset \mathcal{O}(G/U)$. Substituting the formula for C'_μ we see that a also lies in this image.

0.2. **Derived localization.** The derived global sections functor $R\Gamma_\lambda$ is an equivalence when the twisting λ is regular (not necessarily dominant).

First, we have the left adjoint derived localization functor L_λ such that $R\Gamma_\lambda \circ L_\lambda \cong Id$. The latter isomorphism is clear since $R\Gamma(D_\lambda) = U_\lambda$ in self-explanatory notation, which follows from the fact that $R\Gamma^i(T^*(G/B), \mathcal{O}) = 0$ for $i > 0$ and $\Gamma^i(T^*(G/B), \mathcal{O}) \cong \mathcal{O}(\mathcal{N})$. But one needs to check that $L_\lambda : M \mapsto D_\lambda \overset{\mathrm{L}}{\otimes}_{U_\lambda} M$ lands in the bounded derived category. This can be checked using that λ is regular so the map from $\mathfrak{t}$ to $\mathfrak{t}/W$ is etale at λ, which implies that $D_\lambda \overset{\mathrm{L}}{\otimes}_{U_\lambda} M \cong \tilde{D}_{\hat{\lambda}} \overset{\mathrm{L}}{\otimes}_U M$.

It remains to show that the right orthogonal to $Im(L_\lambda)$ (which is also the kernel of $R\Gamma_\lambda$) vanishes.

0.2.1. *First proof, after [MN].* [2] We check that for *any* λ and $M \in D_\lambda - mod$, the functor $R\Gamma_\lambda$ induces an isomorphism

$$RHom(M, D_\lambda) \to RHom(R\Gamma_\lambda(M), R\Gamma_\lambda(D_\lambda)). \tag{1}$$

[1]Bezrukavnikov, Braverman, Positselskii, "Gluing of abelian categories and differential operators on the basic affine space"

[2]McGerty, Nevins "DERIVED EQUIVALENCE FOR QUANTUM SYMPLECTIC RESOLUTIONS

In view of (1), the right orthogonal to $Im(L_\lambda)$ coincides with the left orthogonal, so if it was nonzero then the category $D^b(D_\lambda - mod)$ would split as a direct sum of two nonzero subcategories. This is easily seen to be impossible.

The idea of proof of (1) is that it holds since the similar statement holds for the associated graded sheaves:

$$RHom_{Coh(T^*(G/B)}(\mathcal{F}, \mathcal{O}) \cong RHom_{Coh(\mathcal{N})}(R\Gamma(\mathcal{F}), \mathcal{O}_{\mathcal{N}}),$$

which follows from Serre duality and the fact that canonical class of $T^*(G/B)$ is trivial.

Remark 1. This proof is close to the proof of the similar fact in positive characteristic [BMR],[3] where instead of passing to the associated graded we have used the Azumaya property of differential operators in positive characteristic.

0.2.2. *Second proof, close to the original argument of Beilinson-Bernstein.* There exists $w \in W$ such that for every integral dominant μ the weight $\lambda + w(\mu)$ is regular. Then considering the action of Casimir on $D_\lambda \otimes V_\mu$ we see that $D_\lambda \otimes \mathcal{O}(w(\mu))$ is a summand of $D_\lambda \otimes V_\mu$, hence lies in the image of L_λ. But coherent sheaves $\mathcal{O}(w(\mu))$ generate $D^b(Coh(G/B))$ in the sense that their right orthogonal is zero[4], hence the objects $D_\lambda \otimes \mathcal{O}(w(\mu))$ (or, in fact, a finite subset of this set) generate the derived category of D_λ-modules.

0.3. **Another proof of exactness for dominant weights.** [Gaitsgory, unpublished] Let π denote the projection $G \to G/B$.

For a D_λ module M we have

$$\Gamma(M) = Hom_{\mathfrak{g}}(M_\lambda, \Gamma(\pi^* M))$$

where M_λ denotes the Verma module with highest weight λ and $\Gamma(\pi^* M)$ is considered as a $\mathfrak{g}$ module via the right action of G on itself.

It is easy to see that with respect to that action $\Gamma(\pi^* M)$ is an (ind) object in category O, where M_λ is a projective object. Exactness follows.

0.4. **Localization and translation functors.** Let λ and μ be regular dominant, where μ is integral. Then we have an equivalence $D_\lambda - mod \cong D_{\lambda+\mu} - mod$,

$$M \mapsto M \otimes \mathcal{O}(\lambda). \tag{2}$$

Applying the localization equivalence one gets a functor between the corresponding categories of $\mathfrak{g}$ modules. This turns out to be the *translation functor* defined as a direct summand of the functor of tensoring with a finite dimensional module with an extremal weight μ [BG].[5]

Moreover, the isomorphism

$$T_{\lambda \to \lambda+\mu}(\Gamma_\lambda(M)) \cong \Gamma_{\lambda+\mu}(M \otimes \mathcal{O}(\mu)$$

holds when λ is dominant regular and $\lambda + \mu$ is dominant but not necessarily regular, where $T_{\lambda \to \lambda+\mu}$ is the translation functor.

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[3] Bezrukavnikov, Mirkovic, Rumynin, "Localization of modules for a semisimple Lie algebra in prime characteristic"

[4] In fact, for an ample line bundle $\mathcal{L}$ on a projective variety X there exists n such that $\mathcal{O}$, $\mathcal{L}$, ..., $\mathcal{L}^{\otimes n}$ generate $D^b(Coh(X))$ in this sense. Clearly, the set $\mathcal{O}(w(\mu))$ where μ is strictly dominant contains a twist of such a set by a line bundle.

[5] Bernstein, Gelfand, "Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras"

0.5. **Singular blocks.** (see [BeGi][6]) In fact, translation functor of the type considered in the last formula can be enhanced slightly. Let $\tilde{U} = U(\mathfrak{g}) \otimes_{\mathcal{O}(\mathfrak{t}/W)} \mathcal{O}(\mathfrak{t})$. Consider the category of $\tilde{U}$ modules where the center acts locally finitely, it splits as a direct summand indexed by points of $\mathfrak{t}$. The summands corresponding to regular weights are equivalent to the corresponding summand in the category of U-modules locally finite over the center. The *extended translation functor* is a functor from such a regular summand corresponding to λ to the summand corresponding to the singular weight $\lambda + \mu$. It realizes the latter category as a *Serre quotient* of the former. This allows one to prove things about singular central characters via D-modules, although localization theorem only applies to regular ones.

0.6. **Braid action.** Let μ be integral, regular and dominant. For every $w \in W$ we can use (2) with $\lambda = w \cdot \mu - \mu$ to identify the categories of D_λ and $D_{w\cdot\lambda}$ modules. On the other hand, both $R\Gamma_\lambda$ and $R\Gamma_{w\cdot\lambda}$ land in the same derived category of $\mathfrak{g}$-modules. Thus we obtain an auto-equivalence I_w of the derived category $D^b(U_\lambda - mod)$. These auto-equivalences generate an action of Artin braid group on $D^b(U_\lambda - mod)$.

It turns out to coincide with the action generated by the functors $M \mapsto \pi'_{w*}\pi_w^*(M)$ where π_w, π'_w are the two projections from the G-orbit $(G/B)^2_w \subset (G/B)^2$ to G/B. The original reference is Beilinson-Bernstein "On Casselman submodule theorem".

[6] Beilinson, Ginzburg, "D-modules and wall-crossing functors"