Recall: \((X,J,\omega)\) Kahler \(\mathbb{D}\) ankaeoidal divisor, \(\Sigma \in \mathcal{L}^{\omega}(\mathbb{D})\) holom vol form \(M = \left\{ (L, D) / L \leq X - D \text{ Slag form} \right\}
\)
\[
W(L, D) = \sum_{\beta \in \pi_2(X, L) \leftarrow} \eta_{\beta}(L) z_{\beta}(L, D)
\]
where \(\eta_{\beta}(L) = \# \text{ holom divs through } p \in L\) generic point
\[
z_{\beta}(L, D) = e^{-\int_{\beta} \omega}. h_{\beta}(D, \beta)
\]

\[\text{Ex: } X = \mathbb{C}P^2, D = \{x_0 x_1 x_2 = 0\}, \quad M = \text{(subset of ) } (\mathbb{C}^e)^2, \quad W = z_1 + z_2 + e^{-\int_{\text{area}} \omega} / z_1 z_2\]

Today: 1. Motivation/interpretation
2. Wall-crossing & instanton corrections
3. Mirror symmetry for \(X\) vs. mirror symm for \(D\).

Interpretation: take \(L = S^{1}(r_1) \times S^{1}(r_2) \subset \mathbb{C}^e \subset \mathbb{C}P^2\).

in \((\mathbb{C}^e)^2\) \(L\) doesn't bound any holom divs, so \(HF^*(L, L) \cong H^*(T^2)\)

\((L, D) \leftrightarrow \text{ a point in } \text{mirr } (\mathbb{C}^e)^2\)

\(E^2(0_0, O_0) \cong H^*(T^2)\)

But in \(\mathbb{C}P^2\) \(L\) bounds holom divs, \(\text{Floer theory is obstructed!}\)

\[\text{Ex: } \begin{array}{c}
\text{L} \\
\text{Floer diff on } CF^*(L,L') \text{ has } \exists^2 \neq 0!
\end{array}\]

Obstruction encoded by \(m_0 \in CF^*(L, L)\) - count of divs \(\mathbb{P}^3\)
on \(CF^*(L, L')\): \(\exists^2(x) = m_0(L) \cdot x - x \cdot m_0(L)\)

We're in "weakly unobstructed" case where \(m_0\) is a multiple of the unit - in fact: \(m_0 = W.1\).

Then can define \(HF^*((L, D), (L', D'))\) only if \(W(L, D) = W(L', D')\).
2) moreover: \( \text{HF}(L,L) \) well defined, but typically \( \text{zero} \) \((=0 \text{ in } \text{Def})\) \(\text{indeed}, \text{except for Clifford forms, all product tori in } \mathbb{CP}^2 \text{ are displacable.}\)

- To replicate this on the mirror, introduce exponential \( W \) & replace \( \mathcal{D}^b \text{Coh}(M) \) by \( \mathcal{D}^{\text{b, sing}}(M,W) = \bigoplus_{\lambda \in \mathbb{C}} \text{D}^b \text{Coh}(W^{-1}(\lambda))/\text{Perf } W^{-1}(\lambda) \) \(\text{(Orlov)}\)

\(\text{"measured" singularities of } W^{-1}(\lambda)\)

In particular:

- different level sets of \( W \) don't "see" each other

\[ \text{in terms of matrix factors: } \epsilon, \epsilon' \in \text{MF}(W^{-1}) \]

\[ \Rightarrow \text{diff. hom}(\epsilon, \epsilon') \text{ square to } (\lambda^{-1} - \lambda) \text{Id} \]

\[ \rho \in M \Rightarrow \mathcal{O}_\rho \simeq 0 \text{ in } \mathcal{D}^{\text{b, sing}} \text{ unless } \rho \in \text{Crit } W. \]

---

**Wall-crossing:** recall \( \exp \dim \mathcal{M}(L,\beta) = n-3 + \mu(\beta), \quad \mu(\beta) = 2(\beta \cdot D) \)

- in dim. \( n=2 \), expected dim. < 0 for \( \mu(\beta) = 0 \) ... generically \( L \) shouldn't bound any \( \mu=0 \) disc. However, in codim. 1 \( L \) bounds such a disc; this defines a "wall" in \( \mathcal{M} \), across which \( n_{\beta} \) (hence \( W \)) jumps.

\[ \beta \quad (n=2) \quad \rightarrow \quad \beta' \quad (n=2) \quad \rightarrow \quad \beta' \quad (n=0) \]

**Typically:**

\[ n_{\beta} = 1 \]

\[ n_{\beta'} = 0 \]

\[ n_{\beta'} = 1 \]

- in dim. \( n \geq 3 \), a generic \( L \) may bound \( \mu=0 \) disc, so \( \text{ev}_* \left[ \text{HF}(L,\beta) \right] \) is an \( n \)-chain on \( L \), not a cycle (depends on auxiliary data).

\( \text{So } n_{\beta}(L) \text{ depends on choice of point } \rho \in L \ldots \text{ W multivalued!} \)

Can lift ambiguity by choosing extra data, e.g:
In any case, \( W \) is now a function of \((L,D,\text{extra data})\) and we still have discontinuities when \( r_p \) jumps across walls.

Example: \( X = \mathbb{CP}^2 \), \( D = \{xy=\xi\} \cup \{\text{line not }\infty \} \), \( \omega = \frac{dx \wedge dy}{xy-\xi} \), \( S^1 \) acts by \( e^{i\theta} (xy) \mapsto (xe^{i\theta}, ye^{i\theta}) \), quotient map \( \mu \left( \frac{1}{2} \frac{|x|^2 - |y|^2}{1+|x|^2+|y|^2} \right) \)

\( S^1 \)-inv. family of special Laplacian bds:

\[
T_{r,\lambda} = \{ \text{ such that } |xy-\xi| = r, \mu = \lambda \}
\]

Check: \( \bigwedge_{T_{r,\lambda}} = \text{span}(ix,-iy), v_2 \)

- \( \omega(v_1,v_2) = d\mu(v_2) = 0 \) (\( \mu = \text{cot} \))
- \( v_1 \wedge v_2 = i d \log (xy-\xi) \), which is real on \( v_2 \) (\( |xy-\xi| = \text{cd} \))

- \( T_{1,0} \) singular
- \( T_{\ell,0} \) bounded \( \mu = 0 \) dsns

For \( r > |\ell| \): \( T_{r,\lambda} \) deforms to product \( \mathbb{T}^2 \) (shift circle \( \infty \to 0 \)) without wall-crossing

- 2 families of dsns as before
- two are sections of \( f \) over \( D(\delta,\Gamma) \), class \( \beta_1, \beta_2 \); 3rd goes through line at \( \infty \)

\[
W = z_1 + z_2 + \frac{e^{-A}}{z_1 z_2} \quad (A = \text{area}(\Gamma))
\]
For \( r < |\varepsilon| \): \( T_{r,\lambda} \) deforms to Chekanov-type torus \( T_{r,0} \) 

bound 4 families of discs (Chekanov-Schlenk, Polterovich ...) 

one is a section of \( f \) over \( D(\varepsilon, r) \), 3 others through line at \( \infty \) 

\[
\eta_\beta = 1, \quad \eta_{[\mathbb{CP}^1]-2\beta+\alpha} = 1, \quad \eta_{[\mathbb{CP}^1]-2\beta} = 2
\]

\[
\Rightarrow W = u + \frac{e^{-A(1+v)^2}}{u^2 v}
\]

As we cross \( r = |\varepsilon| \) for \( \lambda > 0 \), the disc in class \( \beta \) becomes class \( \beta_2 \), so geometry of \( M \) would give \( u \leftrightarrow z_2, \quad v \leftrightarrow \frac{z_1}{z_2} \), but \( W \) is discontinuous across wall. 

Instead, notice that 

(smooths to discs in class \( \beta + \alpha = \beta_1 \) on side \( r > |\varepsilon| \) (not on side \( r < |\varepsilon| \) 

(\& vice versa for discs through line at infinity)

So: want to correct: 

\[
\begin{cases}
u \leftrightarrow z_2(1+v) = z_2 + z_1 \\
v \leftrightarrow \frac{z_1}{z_2}
\end{cases}
\]

Then \( W \) is continuous & analytic across wall! 

(similarly for portion \( \lambda < 0 \) of wall: exceptional disc in class \( -\alpha \) 

correl gluing \( u \leftrightarrow z_1 \) to \( u \leftrightarrow z_1(1+v') = z_1 + z_2 \ v \))

---

General wall-crossing result: (FO00): 

across wall, \( \exists \) analytic change of variable on \( M \) that makes \( W \) continuous.
More precisely, when wall corresponds to a Nodal 0 disc in class $\alpha$, gluing is $\beta \mapsto \beta \cdot h(z_\alpha)$ in dim. 2

\[ h(z_\alpha) = 1 + O(z_\alpha) \in \mathbb{Q}[[z_\alpha]] \]

where $h(z_\alpha)$ = coeff of $d$-fold cover of exceptional disc; in our example $h(v) = 1 + v$.

Corrected mirror $\Rightarrow$ glue chambers of $M$ by these changes of coordinates.

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**Mirror symmetry for the pair $(X,D)$:**

**Folklore statement:** "the fiber of $W: M \to C$ is mirror to $D"$

*D CX carries a natural induced holomorphic volume form, $\Omega_D = R_{\text{deg}}(\Sigma)$

Residue of $\Omega$ along $D$, satisfies $\Omega = R_D \wedge d\log \varphi_D$ + bounded

\[ \forall \text{ defining sect of } D \]

**Natural conjecture (supported by example):**

the curves of the $SL_2$ fibration $X, D, \pi, B$ lie in a neighborhood of $D$, and are $SL_2$-bundle over $SL_2$-tori in $(D, \omega_D, \Sigma_D)$

(Ex: toric case; $T_{r,0}$ for $r \to 0$ in above example)

In that case, $\exists B = base of SL_2$ fibration on $D$...

- Fibers $L$ of $\pi$ near $D$ bound a family of $\mu = 2$ median discs; small normal disc $D_D$.

*Call $S$ the class of these discs.

Assume $D$ smooth for simplicity & fibration $\pi$ has the expected boundary structure. Then:

* Near the boundary, $W = z_3 + o(1)$ (all other discs have area $\gg$ area$(S)$)
\[ \mathcal{M} = \{ |z_5| = 1 \} \quad (\text{i.e. area of minimal disc } \to 0) \]

L totally collapsed onto \( \Lambda C D \)

Actually \( \mathcal{M} \) is a bundle on \( S^1 \) with fiber \( \mathcal{M}_D = \{ z_5 = 1 \} \)

The fibration is given by \( \arg(z_5) = \) holonomy of \( D \) along minimal

i.e. \( \mathcal{M}_D = \{ L \text{ collapsed onto } \Lambda C D \}
\begin{align*}
\{ D \text{ pulled back from flat cone on } \Lambda \} \\
\end{align*} \)

Thus: \( \mathcal{M}_D = \text{SYZ mirror to } D \) ! (uncertainted so far).

\[ \begin{align*}
\mathcal{M}_D \cong \{ W = 1 \} \\
\end{align*} \]

Equivalently (recall \( W \leftarrow w + t g(X) \) e[large mirror]):

\[ \{ W = e^t \} \] family of fibers, as \( t \to \infty \), approximate

family of SYZ mirrors to \( (D, \omega_D + t g(X))_{ID} \)

However this only holds asymptotically, \( W \) to higher order terms.

\[ \begin{align*}
2 \text{ reasons: } \ & \ 1) \ W \neq z_5 \text{ so } W''(1) \neq z_5''(1) \\
\end{align*} \]

2) instanton corrections are different!

The geometry of \( \mathcal{M}_D \subseteq \mathcal{M} \) gets corrected by wall-crossing in \( X \) —

even in the collapsing limit \( \to D \), large tori in \( X \) still cross walls when
they bound \( \mu = 0 \) disc in \( X \), vs. when the corept. tori in \( D \)

bound \( \mu = 0 \) disc in \( D \).

\[ \begin{align*}
\text{On the other hand, all small fibers of } W \text{ are symplectomorphic}
\end{align*} \]

to each other and also to \( \{ z_5 = 1 \} \)!

+ kähler class unaffected by instanton corrections

\[ \Rightarrow \text{ expect: } \text{ SYZ mirror to the claim } D \]
Hochschild Conjecture for pairs $\mathcal{M}$: [necessarily $B$-model on $(X, D)$]
$\mathcal{A}$-model on $(M, M_b)$

The other side doesn't quite match...

- **Relative Fukaya category** $\mathcal{F}(M, M_b)$: (recall $M_b = \{x_b = 1\} \subset \mathcal{M}$)
  - objects = admissible Lagrangians $L \subset M$, $\partial L \subset M_b$ (possibly empty), $\mathcal{E}_L \in \mathbb{R}^+$ near $\partial L$.
  - morphisms $\text{Hom}(L_1, L_2) = \text{CF}(L_1, L_2^+)$ [Kontsevich, Seidel]

- `perarb` to positive direction

($\Leftarrow$) Fukaya cat. of LG-model $M \xrightarrow{\mathcal{E}} \mathbb{D}^2 \subset C$, recall $W \sim \mathcal{E}$ near $\partial M$, so expect $\mathcal{F}(M, M_b) \sim \mathcal{F}(M, W)$.

- **Restriction (Ad-functor)** $\rho : \mathcal{F}(M, M_b) \to \mathcal{F}(M_b)$
  - on objects, $L \mapsto \mathcal{E}L$

**Conj. (HNS):**

\[
\begin{align*}
\text{D}^{\mathcal{A}}\text{coh} (X) & \xrightarrow{\text{hnr}} \text{D}^{\mathcal{A}}\text{coh} (\mathcal{D}) \\
\text{HNS} \downarrow & \Leftarrow \quad \downarrow \text{HNS}
\end{align*}
\]

\[
\text{D}^\mathcal{M} \mathcal{F}(M, M_b) \xrightarrow{\text{hnr}} \text{D}^\mathcal{M} \mathcal{F}(\mathcal{D})
\]

**Ex.:** $\mathbb{C}P^2$ & Del Pezzo surfaces [A.-Kapferer-Orlov]