1. Overview of SYZ approach to construction of mirrors
   - in CY case (mirror = CY)
   - in Fano case (mirror = LG model) \}
   \quad \text{ignoring instanton corrections}

2. Instanton corrections - examples
   \quad \text{mirror symmetry for pairs}

3. More examples: blowups of toric varieties along Gorenstein subvarieties
   \quad \rightarrow \text{mirror symmetry for general varieties} \quad [\text{conj. by Katzarkov}]
   \quad [\text{work in progress w/ Abramovich-Katzarkov}]

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**SYZ conjecture:** \( X \text{ Calabi-Yau & mirror } X' \text{ carry dual special Lefschetz fibration over a same affine manifold } B. \)

\[ \triangle \quad \rightarrow \text{existence of } \text{Slay } T^n \text{ fibrations is not clear unless } X \text{ is near } \quad \text{"large complex structure limit" degeneration} \]
\[ \rightarrow \text{geometry of mirror is modified by } \text{"instanton corrections"} \]

- **Def.:** \( (X^n, \omega, J) \text{ Kähler mfd is (almost) CY if } \Omega^{0,0} = O_x \),
  \[ \exists \exists R \in \mathbb{R}^{n,0} \text{ holomorphic volume form} \]
  We don't usually require \( |R|_g \text{ constant} \)

- **Def.:** \( L^n \subset X \text{ is special Lefschetz if } \omega_{1L} = 0 \text{ and } \text{Im } \Omega_{1L} = 0 \),
  \( \text{or more generally } \text{Im } (e^{-i\varphi} R)_{1L} = 0 \text{ for some fixed } \varphi \)
  Then \( \Omega_{1L} = \psi \cdot \omega \big|_{1L} = \psi \cdot \text{vol}_{1L} \), where \( \psi = |R|_g \in C^\infty(L, \mathbb{R}_+) \)

- **Deformation:** \( v \in C^\infty(NL) \text{ is a first-order SLay deform.} \) \[ \left\{ \begin{array}{l}
  L_v \omega = 0 \\
  L_v \text{Im } \Omega = 0
\end{array} \right. \]
  \[ \text{i.e.:} \quad \left( \begin{array}{c}
  \omega \\
  \text{Im } \Omega
\end{array} \right) \in L(L, \mathbb{R})^{\circ} \]
  \[ L_v \text{Im } \Omega = \psi \cdot \alpha \in \mathbb{R}^{n-1}(L, \mathbb{R}) \]
Then:

\[ \{ \text{Slay deformations} \} \rightarrow \{ \alpha \in \Omega^1(\mathcal{L}, \mathbb{R}) \mid \frac{\partial \alpha}{\partial x} = 0, \quad d^*(\psi \alpha) = 0 \} : = \mathcal{X}^s_{\psi}(\mathcal{L}) \]

"$\psi$-harmonic" 1-forms

\exists \psi$-harmonic representative in each cohomology class, so

\[ H^1_{\psi}(\mathcal{L}) \cong H^1(\mathcal{L}, \mathbb{R}) = H^{\text{\text{cyc}}}(\mathcal{L}, \mathbb{R}) \]

\[ \sim \quad [\alpha] \quad [\psi \alpha] \]

Prop (McLean, Joyce)

The moduli space of special Lagrangians is a smooth mild $\mathcal{B}$,

\[ T_L \mathcal{B} \cong H^1(\mathcal{L}, \mathbb{R}) \cong H^{\text{\text{cyc}}}(\mathcal{L}, \mathbb{R}) \]

For $\mathcal{L} = T^n$, can expect locally a fibration $T^n \rightarrow X \rightarrow \mathcal{B}$

Rank $\mathcal{B}$ carries two natural affine structures:

- "Syrphical": $T_L \mathcal{B} \cong H^1(\mathcal{L}, \mathbb{R})$
  
  local coords: syrphical areas swept by basis of $H^1(\mathcal{L})$

- "Cyclic": $T_L \mathcal{B} \cong H^{\text{\text{cyc}}}(\mathcal{L}, \mathbb{R})$
  
  local coords: $\int_{\Delta_i} \omega$ in $\omega$ swept by a basis of $H^{\text{\text{cyc}}}(\mathcal{L})$

Dual fibration: the dual form of $\mathcal{L}$ is $\text{Hom}(\pi_1(\mathcal{L}), \mathbb{U}(1))$

parameterizes flat $\mathbb{U}(1)$-conn. on $\mathcal{L}$ / gauge

Thus, given a Slay fibration $T^n \rightarrow X \rightarrow \mathcal{B}$,

\[ M = \{ (L, \nabla) \mid L \subset X \text{ Slay fiber of } \pi \]

\[ \nabla \text{ flat } \mathbb{U}(1) \text{ conn. on } \mathcal{L} / \text{gauge} \}

\[ T(L, \nabla) M = \{ (V, \alpha) \in C^\infty(NL) \oplus \Omega^1(\mathcal{L}, \mathbb{R}) \mid \nabla \omega + i \alpha \in H^1_{\psi}(\mathcal{L}) \otimes \mathbb{C} \}

\[ \downarrow \text{ connection 1-form} \quad \text{naturally a } \mathbb{C} \text{-vector space} \]
This defines an a.c.s. $J^u$ on $M$.

Proposition:

Set $J^u$ given by $T(c,\theta)M = H^1_X(L) \otimes C$

- $L^u((v_1,\alpha_1),..,(v_n,\alpha_n)) = \int L((v_1,\omega+i\xi_1)\wedge..\wedge(v_n,\omega+i\xi_n))$
- $\omega^u((v_1,\alpha_1),(v_2,\alpha_2)) = \frac{1}{[L,\theta][L]} \int L(\alpha_2 \wedge v_1 \text{Im}L - \alpha_1 \wedge v_2 \text{Im}L)$

Then $J^u$ is integrable, $\omega^u$ is a compatible Kähler form, $(M,J^u,L^u,\omega^u)$ is almost-CY, and $\pi^u: M \to B$ is a Slav holomorphic fibration.

* $M$ is the (uncorrected) mirror of $X$.

\[ X \xrightarrow{\pi} M \xrightarrow{\psi} B \]

**duality** = interchange of the two affine structures on $B$.

* In real life: most Slav fibrations have singular fibers, where dualization breaks down. Instanton corrections will deal with this.

- Example: $X = T^2 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, $L = dz$, $\int_{T^2} \omega = \lambda$

(rhr. because I don't want to mention B-fields)

\[ T^2 \quad \xrightarrow{\pi} \quad S^2 \]

$L = \{ \text{Im } z = \text{const.} \}$ are Slag $S^1$'s

For symplectic affine structure, $B$ has size $\lambda = \text{area}(T^2)$

complex \quad ___ \quad ___ \quad $\tau = \text{modular parameter}$

Dualizing: $M = T^2$ with

- complex shr: $\mathbb{C}/\mathbb{Z} + i\lambda \mathbb{Z}$
- symplectic area $\tau$
Motivation from HMS: expect $D^b \text{Coh}(M) \cong D^c \text{Fuk}(X)$

so: point $p \in M \iff O_p \in D^b \text{Coh}(M) \iff z_p \in D^c \text{Fuk}(X)$

$\text{Ext}^k(O_p, O_p) \cong H^k(T^n, \mathbb{C}) = $ Floer cohomology of $z_p$ is $H^k(\mathbb{T}^n)$

expect most likely $z_p$ is a Lagrangian tori + flat $U(1)$ conn.

(However: some pts of $M$ might not correspond to honest Lagr. in $X$)

If only want $M$ as a symplectic manifold, enough to work with

Lagrangian tori in $X$; "special" condition needed to define $\omega$ on $M$.

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Non-CY cases: from now on $(X,J,\omega)$ Kähler, $D = X$ hyperurface $D \equiv \text{Ker}(K_X)$

effective anticanonical divisor

reduced, at most normal crossings.

$\Rightarrow D = \sigma^* \in \Sigma^{\geq 0}(X, D) \text{ holom vol. form with poles along } D$

Can look for a SLag fibre fibration on the almost-CY manifold $X \times D$,

let $\mathcal{N} = \{(L, D) \mid L \subset X \times D \text{ SLag fibres} \}$ mirror of $X \times D$ as above.

The mirror of $X$ (mod. mirror equations) will be a

Landau-Ginzburg model $(M, W)$, $W: M \rightarrow \mathbb{C}$ holomorphic function

= "superpotential"

$W$ modifies geometric interpretation of mirror symmetry, esp. B-model =

singularity of $W$.

Construction: $(L, D) \in M$, $\beta \in \pi_2(X, L)$

$\Rightarrow \mathcal{M}(L, \beta) \text{ moduli space of holom maps } u: (S^2, \mathbb{S}^2) \rightarrow (X, L), \ [u] = \beta$

$\text{exp. dim}_{\mathbb{R}} \mathcal{M}(L, \beta) = n - 3 + \mu(\beta)$

Maslov index: for $L$ SLag,

$\mu(\beta) = 2(\beta \cdot D)$
There exists a compactification $\overline{M}(L, \beta)$ formed by adding budded configurations $L \cong T^n$ spin $\Rightarrow M$ orientable.

- Assume there are no discs $\mu(\beta) \leq 0$, and $\mu = 2$ discs are regular.

Then, for $\mu(\beta) = 2$, $\exists \eta_\beta(L) = \# \text{holo. disc in class } \beta$ passing through a generic point $p\in L$.

(\text{A signed count, need orientations})

(or: $\text{ev}: \overline{M}_2(L, \beta) \to L$, $\eta_\beta(L) = \deg(\text{ev}_* [\overline{M}_2(L, \beta)])$)

1 boundary marked pt

Define:

$$ W(L, \nu) = \sum_{\substack{\beta \in \pi_2(L, \nu) \\mu(\beta) = 2}} \eta_\beta(L) z_\beta(L, \nu) $$

where $z_\beta(L, \nu) = \exp(-\int_\beta \omega) \text{hol}_{\beta}(\nu) \in \mathbb{C}^*$

Note: $z_\beta$ are local holomorphic coordinates on $M$!

Indeed $\partial \log z_\beta(u, \nu) = \int_\beta -i\nu\omega + i\alpha$

($\to \log z_\beta =$ complexified version of affine coordinates on $B$)

Hence $W$ is a holomorphic function on $M$ as long as things go well.... but.... 2 major issues:

1. convergence of the sum is unknown in general
    (except specific cases, e.g. toric Fano)
    so $W$ might only be defined as a formal sum $\in$ Nishino ring,
    not as an actual complex number

2. in general, $\eta_\beta(L)$ might be ill-defined - may depend on
    construction of virtual fundamental chain for $\overline{M}(L, \beta)$, and on
    additional data. This makes $W$ multivalued/discontinuous.
    This will be remedied by instanton corrections.
Example: \( \mathbb{CP}^2 \) (or any other toric Fano) [see: Hori, Cho-Chu, FO3]

\[ X = \mathbb{CP}^2, \quad D = \{x_0x_1x_2 = 0\}, \quad X \smallsetminus D = (\mathbb{C}^*)^2, \quad S = \text{dendy, w tric (in gen: X toric Fano, D toric divisor)} \]

Then product tori \( L = S'(r_1) \times S'(r_2) \subset (\mathbb{C}^*)^2 \subset \mathbb{CP}^2 \) are special Lagrangian bases \( B = \text{orbit space for } T^2 \text{-action on } (\mathbb{C}^*)^2 \)

\[ \rightarrow \text{symp. affine structure: } B = \text{int}(\Delta); \text{ interior of moment polytope} \]

\[ \text{fibration = moment map} \]

\[ \rightarrow \text{complex affine structure: } B = \mathbb{R}^2, \text{ fibration = log map} \]


- **Duality:** \( M = \{ (z_1, z_2) / (-1/2\pi \log |z|) \in \text{int} \Delta \} \subset (\mathbb{C}^*)^2 \)

- There are no \( \mu \leq 0 \) discs in \( (X, L) \) (would have to be \( \subset (\mathbb{C}^*)^2 \))

- \( \mu = 2 \) discs hit \( D = D_0 \cup D_1 \cup D_2 \) exactly once transversely

Exactly one family for each component of \( D \):

\[ L = S'(r_1) \times S'(r_2) \text{ bounds} \]

- \( D^2(r_1) \times \{ \text{pt} \} \subset \mathbb{C}^2 \subset \mathbb{CP}^2 \)

- \( \{ \text{pt} \} \times D^2(r_2) \)

- 3rd family through line at infinity

in classes \( \beta_1, \beta_2, \) and \( \beta_3 = [\mathbb{CP}^1] - \beta_1 - \beta_2 \)

Pichaually:

\[ \Delta \]

\[ \beta_1 \quad \beta_2 \quad \beta_3 \]

\[ L \]

\[ \rightarrow \text{variables } \begin{pmatrix} \mathbb{Z}_{\beta_1} = z_1 \quad (|z_1| = e^{-2\pi \mu_1}) \end{pmatrix} \]

\[ \begin{pmatrix} \mathbb{Z}_{\beta_2} = z_2 \end{pmatrix} \]

\[ \mathbb{Z}_{\beta_3} = \frac{e^{-\text{Area}(\mathbb{CP}^1)}}{\mathbb{Z}_1 \mathbb{Z}_2} \]

Moreover \( \eta_{\beta_1} = \eta_{\beta_2} = \eta_{\beta_3} = 1 \) (one disc through each point of \( L \))

\[ \Rightarrow W = z_1 + z_2 + \frac{e^{-\text{Area}(\mathbb{CP}^1)}}{z_1 z_2} \quad \text{ (classical)} \]

For general toric Fano, \( W = \) Laurent polynomial with one monomial per facet of \( \Delta \), with exponent \( \leftrightarrow \) normal vector to the facet.