18.01A Practice Exam 1 50 mins.

Directions: 1. No books, notes, calculators, cell phones.
2. Exam has 100 pts, lasts 50 mins.: works out to 2 pts./minute, including checking.
3. Read it through before starting, and do first the problems which are easiest for you.
4. Problem 4. below is partly based on the next lecture (Tues.).

Problem 1. (10) For the function $f(x) = \dfrac{e^{-x}}{1+2x}$, find the best linear and the best quadratic approximations for values of x close to 0. (Suggestion: use algebraic methods.)

Problem 2. (10) Evaluate: a) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}$ b) $\displaystyle\int_0^1 2x\cos(3x^2)\,dx$

Problem 3. (10) By making the substitution $x = \sin u$, change $\displaystyle\int_0^{\sqrt{2}/2} \frac{dx}{(1-x^2)^{3/2}}$ to a definite integral in terms of u, **not evaluated**, but simplified as much as possible.

Problem 4. (10) A solid is formed by rotating about the vertical y-axis the region under the graph of $y = e^x$ and over the x-interval $0 \le x \le 1$. Find the volume of the solid.

Problem 5. (20: 10, 10) Let $F(x) = \displaystyle\int_0^x \frac{4-t^2}{4+t^2}dt.$ Without explicitly calculating the value of this integral, answer the following; show work or briefly indicate reasoning.

a) Find the domain of $F(x)$, (i.e., for what x-values the function is defined), the x-values where it is increasing, and the x-values of its local maximum and of its local minimum ("local" = "relative").

b) Express the value of $\displaystyle\int_0^x \frac{1-u^2}{1+u^2}du$ in terms of values of $F(x)$.

Problem 6. (10) Find the average distance to the origin of a point $P : (x, x^2)$ on the parabola $y = x^2$, if its x-coordinate is chosen at random on the interval $0 \le x \le 2$.

Problem 7. (10) A factory operates continuously, producing a drug which must be immediately chilled to 0^oC as soon as it is produced, and then stored at that temperature. It costs one cenIntegration by partial fractions
t to refrigerate one kg. of the drug for one hour: \$.01/kg/hr.

The rate of drug production over a 1000 hour time interval rises linearly from 10kg/hr to 60kg/hr. Set up but **do not evaluate** a definite integral which measures how much has been spent on refrigeration over the 1000 hours, as follows:

a) About how much is spent refrigerating the drug produced in a small time interval $[t_i, t_i + \Delta t]$?

b) Finish the problem.

Other material The following topics are not included in the questions above; use for review the relevant Part I problems on the three problem sets and examples worked in lecture.

L'Hospital for ∞/∞ Estimating integrals Partial fractions Trig int's: 341-3/exs.1-7
Geometric applications of integration: volumes by horizontal "washers", arclength
Physical applications of integration: work, mass with varying density

Required formulas Standard differentiation, including De^x, $D\ln x$, $D\sin^{-1} x$, $D\tan^{-1} x$
Trig identities for $\sin 2x$, $\cos 2x$, $\sin^2 x$, $\cos^2 x$; $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$

18.01A Practice Exam 1 Solutions

1 Using alg. methods:

$$\frac{e^{-x}}{1+2x} = \left(1 - x + \frac{x^2}{2} \ldots\right)\left(1 - 2x + 4x^2 \ldots\right)$$

$$= 1 - 3x + x^2\left(\tfrac{1}{2} + 2 + 4\right) + \ldots$$

linear: $1 - 3x$

quad: $1 - 3x + \frac{13}{2}x^2$

2 a) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} \underset{\text{L'Hosp}}{=} \lim_{x\to 0} \frac{2\sin x \cos x}{\sin x} = 2$

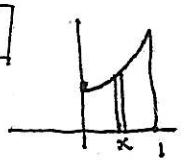
b) $\displaystyle\int_0^1 2x\cos(3x^2)\,dx = \frac{1}{3}\sin(3x^2)\Big]_0^1 = \frac{\sin 3}{3}$

3 $\displaystyle\int_0^{\sqrt{2}/2} \frac{dx}{(1-x^2)^{3/2}} = \int_0^{\pi/4} \frac{\cos u\,du}{\cos^3 u} = \int_0^{\pi/4} \sec^2 u\,du$ $\left(\text{or } \displaystyle\int_0^{\pi/4} \frac{du}{\cos^2 u}\right)$

$x = \sin u \quad dx = \cos u\,du$

$1 - x^2 = \cos^2 u \quad \sin \pi/4 = \frac{\sqrt{2}}{2}$

4



$$\text{vol.} = \int_0^1 2\pi x y\,dx = \int_0^1 2\pi x e^x\,dx$$

$$\int_0^1 x e^x\,dx \underset{\text{int by parts}}{=} x e^x\Big]_0^1 - \int_0^1 e^x\,dx = e - e^x\Big]_0^1 = 1$$

Ans: 2π

5 $\displaystyle F(x) = \int_0^x \frac{4-t^2}{4+t^2}\,dt$

a) $F(x)$ defined for all x, since integrand is continuous for all t: $(4+t^2 > 0)$

$F'(x) = \dfrac{4-x^2}{4+x^2} \geq 0 \iff 4 - x^2 \geq 0 \iff \boxed{-2 \leq x \leq 2}$

$F'(x) = 0$ when $\underset{\text{max}}{x=2}$ or $\underset{\text{min}}{x=-2}$

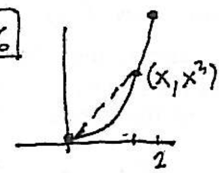
(max, min since $F(x)$ increasing on $[-2,2]$)

b) $\displaystyle\int_0^x \frac{1-u^2}{1+u^2}\,du = \int_0^x \frac{4-4u^2}{4+4u^2}\,du$

Set $t = 2u$, $dt = 2du$ $\displaystyle = \int_0^{2x} \frac{4-t^2}{4+t^2}\,\frac{dt}{2}$

$$= \frac{F(2x)}{2}$$

6



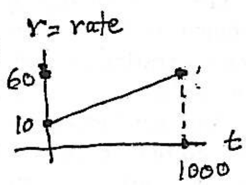
Average distance

$$= \frac{1}{2}\int_0^2 \sqrt{x^2 + x^4}\,dx$$

$$= \frac{1}{2}\int_0^2 x\sqrt{1+x^2}\,dx$$

$$= \frac{1}{6}(1+x^2)^{3/2}\Big]_0^2 = \frac{1}{6}\left[5\sqrt{5} - 1\right]$$

7 a)



$$r = \frac{50t}{1000} + 10 = \frac{t}{20} + 10$$

In $[t_i, t_i + \Delta t]$,

amt drugs produced $\approx \left(\frac{t_i}{20} + 10\right)\Delta t$ kg

refrigerate for $(1000 - t_i)$ hours @ .01/kg/hr

cost $\approx \underset{\text{kg}}{\left(\frac{t_i}{20} + 10\right)}\underset{\text{hours}}{(1000 - t_i)}(.01)$

b) Adding & passing to limit as $\Delta t \to 0$:

$$\int_0^{1000} \left(\frac{t}{20} + 10\right)(1000 - t)(.01)\,dt$$