Higher Bruhat orders for any Weyl group

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Abstract

We describe a framework for constructing a hierarchy of partially ordered sets for an arbitrary Weyl group which is analogous to the higher Bruhat orders defined by Manin and Schechtman for the symmetric group. We give an explicit description of such an order for type B, extending prior work by Shelley-Abrahamson and Vijaykumar. Finally, we discuss possible interpretations and applications of these orders.

1 Introduction

One way to describe permutations π on n elements is by the set of *inversions* of π. Specifically, the inversions of π are the pairs (i, j) with $i < j$ but $\pi(i) > \pi(j)$. The (left) *weak Bruhat order* says that $\pi_1 < \pi_2$ when the set of inversions of π_1 is a subset of the set of inversions of π_2. This defines a partially ordered set (hereafter poset) on the set S_n of permutations on n elements. The *higher Bruhat order* introduced by Manin and Schechtman [MS89] and described in Section 2 generalizes this perspective to larger sets of inversions.

In [Eli16], Elias relies on certain nice properties of the higher Bruhat order, and conjectures that there ought to exist similar objects for other types (specifically type B). In [SV16], Shelley-Abrahamson and Vijaykumar construct a higher Bruhat order for type B up to the second level, and show that it has some of these desirable properties (in particular the existence of a unique minimal and maximal element). In Section 4 we construct a higher Bruhat order for type B which is well-defined at every level, but can only conjecture that it has a unique minimal and maximal element.

In generalizing the higher Bruhat order to other Weyl groups, we look at the root hyperplane arrangement corresponding to an irreducible root system Φ. From this perspective, elements of the Weyl group are the chambers of the arrangement, and the partial order given by the weak Bruhat order corresponds to crossing walls between chambers. Paths from the identity permutation to the longest element ω_0 can be described as walks from the fundamental chamber of the arrangement to the opposite chamber which cross each hyperplane once. Our motivation is that elements of the higher Bruhat order should be seen as higher-dimensional walks, defined by the order in which they cross subspaces of higher codimension. We describe this definition formally in Section 3.

A result of Ziegler [Zie93] combined with the Bohne-Dress theorem [RZ94] shows that another way to interpret elements of the higher Bruhat order is as fine zonotopal tilings of the cyclic zonotope. The fact that "zonotopal tiling flips" correspond to the covering relation in the higher Bruhat order was used in [BW20] to study Postnikov's plabic graphs [Pos06] through a result of Galashin [Gal18] relating plabic graphs to zonotopal tilings. In Section 5 we speculate as to how the higher Bruhat order in type B could lead to a similar perspective.

1.1 Notation

We use $[n]$ as shorthand for the set $\{1, \ldots, n\}$, and $\binom{[n]}{d}$ for the set of subsets of $[n]$ of size d. When discussing type B, we use $[\pm n]$ for the set $\{-n, \ldots, -1, 1, \ldots, n\}$, and $\binom{[\pm n]}{d}$ is the set of size d subsets of $[\pm n]$ with all distinct absolute values, considered modulo negation of all elements in the set. For example, $\binom{[\pm n]}{1}$ has n elements, and $\binom{[\pm n]}{2}$ has $n \cdot (n-1)$ elements. We will often refer to elements of $\binom{[\pm n]}{d}$ by one of their representative elements, allowing us to force the sign of any one element to be as desired.

When discussing a poset with order $<$, we say that a *covers* b if $a < b$ but there is no c with $a < c < b$. The *covering relation* describes which pairs of elements cover each other, and is denoted by $\lessdot$. The transitive closure of this operation can be used to completely define a poset.

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2 Higher Bruhat Order for the Symmetric Group

In this section we describe the *higher Bruhat order* as defined by Manin and Schechtman [MS89] and some results about it. The higher Bruhat order is a generalization of the weak Bruhat order on the symmetric group.

Definition 2.1. Let S_n be the set of permutations on $[n]$. The i-th *elementary transposition* $\tau_i \in S_n$, for $1 \leq i < n$, is the permutation which swaps i and $i+1$ and leaves everything else unchanged. The (left) *weak Bruhat order* on S_n is the partially ordered set on S_n generated by the covering relation $\pi \lessdot \sigma$ whenever $\sigma = \tau_i \circ \pi$ for some i such that $\pi(i) < \pi(i+1)$. The maximal element of the weak Bruhat order is called ω_0 and satisfies $\omega_0(i) = n - i + 1$.

The weak Bruhat order is a ranked poset with rank $\binom{n}{2}$, with rank function given by counting the number of pairs $i < j$ such that $\pi(i) > \pi(j)$. Maximal chains in the weak Bruhat order correspond to *reduced expressions* for the maximal element ω_0 as a product of $\binom{n}{2}$ elementary transpositions.

The weak Bruhat order will serve as the $d = 1$ level of the higher Bruhat order. Elements of the weak Bruhat order are permutations of the set $[n]$; to generalize, elements of the level d higher Bruhat order will be certain permutations of $\binom{[n]}{d}$. The motivation is that elements at level d should correspond to maximal chains in the poset for level $d-1$ by describing the order in which sets should be moved around, like how reduced expressions for ω_0 are maximal chains in the weak Bruhat order.

Definition 2.2. For a set $S \in \binom{[n]}{d}$, the *d-packet* of S is defined to be $P(S) \coloneqq \{S \setminus x : x \in S\} \subseteq \binom{[n]}{d-1}$. The *lexicographic order* $<_{\text{lex}}$ on $\binom{[n]}{d}$ is the order which compares two sets by first comparing smallest elements, then comparing the second smallest element only if the sets have the same smallest element, and so on. A total order ρ on $\binom{[n]}{d}$ is *admissible* if for each $S \in \binom{[n]}{d+1}$, the subsequence of ρ formed by restricting to $P(S)$ is in lexicographic order, or the opposite of lexicographic order. If ρ is admissible, let $\text{inv}(\rho)$ denote the subset of $S \in \binom{[n]}{d+1}$ for which this restriction is in the opposite of lexicographic order.

In the case $d = 1$, all permutations of $[n]$ are admissible, and in the case $d = 2$, the admissible orderings correspond exactly to reduced expressions for ω_0. In general, let $A(A_n, d)$ be the set of all admissible orders of $\binom{[n]}{d}$. Then say that two admissible orders are *elementarily equivalent* if they differ by a swap of two adjacent sets which are not in any $(d+1)$-packet together. Let $B(A_n, d)$ be the quotient of $A(A_n, d)$ under the equivalence relation formed by the transitive closure of elementary equivalence. These equivalence classes of admissible total orders will be the elements of the level-d higher Bruhat order. The following is the main theorem of Manin and Schechtman, and completes the definition of the higher Bruhat order while giving some basic properties of it.

Theorem 2.3 ([MS89, Theorem 3]). *(a) The covering relation given by $\rho_1 \lessdot \rho_2$ if and only if ρ_2 can be obtained from ρ_1 by reversing a packet which appears consecutively in lexicographic order in ρ_1 generates a partial order on $B(A_n, d)$.*

(b) Under this partial order, the unique minimal element is the lexicographic order on $\binom{[n]}{d}$ and the unique maximal element is the anti-lexicographic order. The function $|\mathrm{inv}|$ is a rank function for this poset.

(c) Admissible orderings of $\binom{[n]}{d+1}$ are in bijection with maximal chains in $B(A_n, d)$ via the map which sends orderings to the chain which performs packet flips in the order given.

(d) The function inv *is injective on $B(A_n, d)$.*

A useful geometric realization of the higher Bruhat order is given by Ziegler in [Zie93] using hyperplane arrangements, which we now describe. The *cyclic vector arrangement* is any collection of n vectors $v(z_i) = (1, z_i, z_i^2, \dots, z_i^{d-1})$ in $\mathbb{R}^d$. Dually, we consider an arrangement $\mathcal{H}$ of n hyperplanes H_i in $\mathbb{R}^d$ with equations $x_1 + z_i x_2 + z_i^2 x_3 + \cdots + z_i^{d-1} x_d = 0$. We add in the hyperplane $x_1 = 1$ in order to get an affine arrangement restricted to $d-1$ dimensions. In [Zie93, Theorem 4.1 (B)], it is shown that the higher Bruhat order is isomorphic to the set of extensions of this affine hyperplane arrangement by a pseudo-hyperplane, ordered by single-step inclusion of the set of vertices which lie on one side of the hyperplane. Moreover, if we list the vertices of the affine hyperplane arrangement in order of x_d-coordinate, then (for suitable choices of z_i) they appear in lexicographic order when each vertex is labeled with the indices i of the hyperplanes which include it. This allows us to see geometrically that the lexicographic order does provide a valid sequence of packet flips to perform in the higher Bruhat order one dimension lower.

Ziegler also interprets these pseudo-hyperplane extensions as extensions of the *alternating oriented matroid*. Using duality for oriented matroids, these extensions can also be thought of as one-element lifts of the same oriented matroid, and then it follows by the Bohne-Dress theorem [RZ94] that elements of the higher Bruhat order are in bijection with *fine zonotopal tilings* of the cyclic zonotope $Z(n, d)$ formed by the Minkowski sum of the vectors $v(z_i)$. In this setting, the covering relation in the higher Bruhat order corresponds to certain elementary flips of tiles. In two dimensions these are the well-studied hexagon flips in rhombus tilings, and in three dimensions the flips were shown in [Gal18] to correspond to certain moves in the *plabic graphs* of Postnikov [Pos06]. We will not work much with this picture, but it is natural to wonder whether a similar story can be told for types other than A.

3 Type-Independent Framework

Let Φ be an irreducible root system in n dimensions, with Weyl group W. Then let $\mathcal{H}$ be the hyperplane arrangement in $\mathbb{R}^n$ formed by the reflecting planes for the roots of Φ. We call the chamber of $\mathcal{H}$ bounded by the hyperplanes corresponding to simple roots of Φ the *fundamental chamber* of $\mathcal{H}$, and we can label all the chambers of $\mathcal{H}$ by the element of W which sends that chamber to the fundamental chamber. Note that we can choose a different chamber to be the fundamental chamber and proceed in the same manner.We say that $C_1 \leq C_2$ in the *weak Bruhat order* if every hyperplane separating C_1 from C_2 also separates the fundamental chamber from C_2. This matches the usual definition of the weak Bruhat order on W using reduced expressions.

In this section, we will extend this perspective analogously to how the higher Bruhat order extends the reduced expression perspective on the weak order in type A. In Section 2, we defined elements of the higher Bruhat order $B(A_n, d)$ to be certain equivalence classes of total orderings on $\binom{[n]}{d}$. Here, certain subspaces of codimension $d-1$ will take the role of elements of $\binom{[n]}{d}$, as we now describe.

Definition 3.1. A subspace $V \subset \mathbb{R}^n$ is an *inversion* for Φ if the following two conditions hold for the set $S \coloneqq \{H \in \mathcal{H} : V \subseteq H\}$ of hyperplanes containing V

1. $\bigcap_{H \in S} H = V$
2. The hyperplane arrangement in $V^\perp$ formed by the restriction of the elements of S to $V^\perp$ is the set of reflecting hyperplanes for some irreducible root system.

We use $C(\Phi, d)$ to denote the set of inversions for Φ with codimension $d-1$, for $d > 1$.

Remark 3.2. When Φ is the type A_n root system, one can check that an element of $C(\Phi, d)$ is specified by equations $x_{i_1} = x_{i_2} = \cdots = x_{i_d}$, and so corresponds to the set $\{i_j : j \in [d]\} \in \binom{[n]}{d}$.

Next, we need to describe the notions of packets and admissibility in this context.

Definition 3.3. For an inversion $V \in C(\Phi, d)$, the *packet* $P(V)$ of V is defined to be the set $\{U \in C(\Phi, d-1) : V \subset U\}$ of all codimension $d-2$ inversions which contain V. A total order ρ on $C(\Phi, d)$ is *admissible* with respect to another total order ρ^* if for all $V \in C(\Phi, d+1)$, the restriction $\rho|_{P(V)}$ is equal to either $\rho^*|_{P(V)}$ or its reverse. When ρ is admissible, the *inversion set* $\mathrm{inv}_{\rho^*}(\rho)$ is the set of V such that $\rho|_{P(V)}$ is equal to the reverse of $\rho^*|_{P(V)}$. Two admissible orders ρ_1, ρ_2 with $\mathrm{inv}_{\rho^*}(\rho_1) = \mathrm{inv}_{\rho^*}(\rho_2)$ are said to be *equivalent* or to *differ by commutation.*

When a packet $P(V)$ appears as a substring (i.e., a consecutive subsequence) of a total ordering ρ, we can perform a *packet flip* in ρ which reverses the order of that substring. The result will be an admissible ordering with only one inversion changed (proved below), and we say that it *differs by a packet flip* from ρ.

Lemma 3.4. Suppose ρ_1 and ρ_2 are admissible total orderings with respect to ρ^*. Then

(a) If $\mathrm{inv}_{\rho^*}(\rho_1) = \mathrm{inv}_{\rho^*}(\rho_2)$, then ρ_1 and ρ_2 can be related by a sequence of swaps of adjacent elements which do not share a packet.

(b) $\mathrm{inv}_{\rho^*}(\rho_1) = \mathrm{inv}_{\rho^*}(\rho_2) \sqcup \{V\}$ if and only if there exists representative elements of $[\rho_1]$ and $[\rho_2]$ which differ only by a packet flip of $P(V)$.

Proof. For part (a), if ρ_1, ρ_2 are distinct, then there must exist some pair of consecutive elements V_1, V_2 in ρ_1 which do not appear in that order in ρ_2. There must not be a packet containing V_1 and V_2, since that would mean that ρ_1 and ρ_2 had different inversion sets. Therefore we can swap V_1 and V_2, reducing the number of pairs (V_1, V_2) which appear in opposite orders in ρ_1 and ρ_2. Induction on the number of such pairs shows that repeating this process eventually relates ρ_1 and ρ_2.

For part (b), we start by finding a representative element of $[\rho_1]$ for which $P(V)$ appears as a substring by using these commutation moves. This will be possible as long as there do not exist V_1, V_2, V_3, appearing in that order in ρ_1, such that $V_1, V_3 \in P(V)$, but $V_1, V_2 \in P(U_1)$ and $V_2, V_3 \in P(U_2)$ for some $U_1, U_2 \neq V$. Now, $P(U_1)$ and $P(U_2)$ must appear in ρ_2 in the same order as in ρ_1, so V_1, V_2, V_3 must occur in the same order there. This contradicts the fact that $P(V)$, which contains V_1, V_3, appears in the opposite order in ρ_2, so in fact we must be able to bring $P(V)$ together as a substring.

Finally, having $\rho \in [\rho_1]$ where $P(V)$ appears as a substring, we can perform the packet flip of $P(V)$. It suffices to show that no pair of elements of $P(V)$ appear in any packet other than $P(V)$, since then we can conclude that the result of the flip will be admissible and in $[\rho_2]$. Well, any element of an intersection $P(U) \cap P(V)$ of two packets must contain $\mathrm{span}(U, V)$, which has codimension at most $d-1$ if U and V have codimension d. Then the only possible element of the intersection is $\mathrm{span}(U, V)$, so no pair of elements of $P(V)$ can also be in some $P(U)$, as required. $\square$

In Section 2 we described the higher Bruhat order in terms of commutation classes and packet flips. Lemma 3.4 shows that we can equivalently consider the *consistent* inversion sets to be the elements of the higher Bruhat order, and single-step inclusion to take the role of packet flips. We do not classify exactly which sets of inversions are consistent in the sense of being realized by some element of the higher Bruhat order, but such a classification in type A can be found in [Zie93, Lemma 2.4].

Definition 3.5. For an irreducible root system Φ and a total ordering ρ^* on $C(\Phi, d)$, let $B(\Phi, d, \rho^*)$ be the set of subsets of $C(\Phi, d+1)$ which arise as inversion sets of admissible total orderings of $C(\Phi, d)$. Equivalently, $B(\Phi, d, \rho^*)$ consists of the equivalence classes of orderings up to commutation, and we will use these interpretations interchangeably. For

$d = 1$, we let $B(\Phi, 1, \rho^*)$ be the usual weak Bruhat order where ρ^* denotes the choice of fundamental chamber. We still label the elements of $B(\Phi, 1, \rho^*)$ by their inversion set, which in this case is the subset of hyperplanes (elements of $C(\Phi, 2)$) which separate the chamber from the fundamental chamber.

The *higher Bruhat order* on $B(\Phi, d, \rho^*)$ is generated by the covering relation of single-step inclusion of inversion sets. A sequence $\rho_1^*, \ldots, \rho_d^*$ such that ρ_i^* is an admissible total ordering on $C(\Phi, i)$ $(i > 1)$ and each of the prefixes of ρ_{i+1}^* (considered as an inversion set) is an element of $B(\Phi, d, \rho_i^*)$ is said to define a higher Bruhat order up to level d on Φ. In other words, the sequence of total orders needs to be *compatible* in the sense that each ρ_{i+1}^* provides a chain from ρ_i^* to its opposite.

A good thing to check first is that this definition encapsulates the ordinary higher Bruhat order for type A_n as a special case.

Proposition 3.6. Suppose $\Phi = A_n$, and $C(\Phi, d)$ is naturally identified with $\binom{[n]}{d}$ as in Remark 3.2. Then if ρ_i^* is the lexicographic order on $\binom{[n]}{i}$ for all i, the sequence of $\rho_1^*, \ldots, \rho_d^*$ is a higher Bruhat order up to level d on $\Phi = A_n$, and the posets $B(\Phi, i, \rho_i^*)$ are isomorphic to the posets $B(A_n, i)$ defined in Section 2 for all $1 \leq i \leq d$.

Proof. The lexicographic orders ρ_i are the minimal elements of the posets $B(A_n, i)$ defined in Section 2. Therefore by Theorem 2.3 part (c), prefixes of ρ_i^* are inversion sets for elements of $B(A_n, i-1, \rho_{i-1}^*)$. The function inv on $B(A_n, i)$ is the same as the function $\mathrm{inv}_{\rho_i^*}$ on $B(A_n, i, \rho_i^*)$, and the partial order on $B(A_n, i)$ can also be described as single-step inclusion of inversion sets, so the posets are isomorphic. $\square$

We can now prove a partial analogue of the main theorem of Manin and Schechtman [MS89] (cf. Theorem 2.3).

Theorem 3.7. *Let $\rho_1^*, \ldots, \rho_d^*$ be a higher Bruhat order up to level d on Φ. We have the following characterization of the orders $B(\Phi, i, \rho_i^*)$.*

(a) $[\rho_1] \lessdot [\rho_2]$ *in* $B(\Phi, i, \rho_i^*)$ *if and only if there exists representative elements of* $[\rho_1]$ *and* $[\rho_2]$ *which differ by a packet flip for an inversion in* $[\rho_2]$.

(b) $[\rho_i^*]$ *is a minimal element of* $B(\Phi, i, \rho_i^*)$*, and its reverse is a maximal element. The function* $|\mathrm{inv}_{\rho_i^*}|$ *is a rank function for this poset.*

(c) Total orderings of $C(\Phi, i+1)$ *which are admissible with respect to* ρ_{i+1}^* *can be considered as distinct chains from* $\emptyset$ *to* $C(\Phi, i+1)$ *in the poset* $B(\Phi, i, \rho_i^*)$.

Proof. Part (a) follows from Lemma 3.4. The inversion set of $[\rho_i^*]$ is $\emptyset$, so it certainly minimal, and the inversion set of its reverse is $C(\Phi, i)$, which is all possible inversions. The covering relation implies that if $[\rho_1] \lessdot [\rho_2]$ then $|\mathrm{inv}_{\rho_i^*}(\rho_1)| + 1 = |\mathrm{inv}_{\rho_i^*}(\rho_2)|$, so the poset $B(\Phi, i, \rho_i^*)$ is ranked with rank function $|\mathrm{inv}_{\rho_i^*}|$. It now remains to show part (c).

We can think of each ρ_{i+1}^* as specifying a chain in $B(\Phi, i, \rho_i^*)$ from $\emptyset$ to $C(\Phi, i+1)$ by writing down the order in which the elements of $C(\Phi, i+1)$ are added; the condition on the prefixes of ρ_{i+1}^* ensures that this will give inversion sets that are actually in $B(\Phi, i, \rho_i^*)$. We need to show that the same process for *any* element ρ of $B(\Phi, i+1, \rho_{i+1}^*)$ also gives a valid chain. The only issue that can come up is for a move specified by ρ to be illegal in $B(\Phi, i, \rho_i^*)$, which happens when the packet $P(y)$ which needs to be flipped cannot be brought together due to some element $x \in C(\Phi, i)$ which does not commute with at least two elements of $P(y)$. Suppose we could find an element $z \in C(\Phi, i+2)$ such that $P(z)$ contains every y' such that $P(y')$ contains x and intersects $P(y)$, or $y = y'$. Then since ρ has $P(z)$ in the same order as ρ_{i+1}^* or backwards, if x caused a problem in ρ with the flip of $P(y)$, it would also cause a problem in ρ_{i+1}^*.

It therefore suffices to find such an element z. The exact method here depends on the root system Φ, so we do not have a type-independent proof in this case. For type A_n, this can be done by taking z to be the union of x and y as subsets of $[n]$. □

Part (d) of Theorem 2.3 is included as part of the definition of the higher Bruhat order in our case, while part (a) was a definition in [MS89, Theorem 3]. We do not show a bijection in part (c) because as we have defined things such a bijection does not always exist; some chains may not match ρ_{i+1}^* or its reverse in some packets, where two flips may be interchanged without changing the inversion set. This phenomenon does not occur in type A, but it can in type B with $d > 2$. The other only detail not generalized by Theorem 3.7 is the uniqueness of the minimal and maximal elements of $B(\Phi, i, \rho_i^*)$, and again this is because it is false in general. Even in type A, if an ordering other than the lexicographic one is used, there may not be a unique minimum or maximum. In [FW00, Section 3] certain *reorientations* of the usual level-2 higher Bruhat order are considered, and an example of an order ρ_2^* is given which would not be the unique minimal element in $B(A_6, 2, \rho_2^*)$.

The key to determining what the higher Bruhat order should look like for Φ is choosing the sequence of orders $\rho_1^*, \ldots, \rho_d^*$. So far we have not even shown that any such sequence of orders must exist. We now describe a general procedure for constructing at least some valid higher Bruhat order for any Φ.

Definition 3.8. Suppose $\mathcal{H}$ is the hyperplane arrangement in $\mathbb{R}^n$ for an irreducible root system Φ. Let $\mathcal{B} = (e_1, \ldots, e_n)$ be an ordered basis for $\mathbb{R}^n$ which is in a generic position relative to $\mathcal{H}$. Then the affine space $E_d \coloneqq e_d + \mathrm{span}(e_1, \ldots, e_{d-1})$ intersects each of the spaces $V \in C(\Phi, d)$ at a point, since V has codimension $d-1$ and $\mathcal{B}$ is generic. Let $\rho_1(\mathcal{B})$ denote the chamber of $\mathcal{H}$ which e_1 lies in, and for $d > 1$, let $\rho_d(\mathcal{B})$ be the total order which lists the inversions $V \in C(\Phi, d)$ in increasing order according to the e_d-coordinate of the intersection point $E_d \cap V$. The sequence $\rho_1(\mathcal{B}), \ldots, \rho_n(\mathcal{B})$ is called a *geometric* higher Bruhat order on Φ.

In order to show that geometric higher Bruhat orders satisfy Definition 3.5, we construct a diagram like the "light leaves" diagrams shown in [EW14, Definition 1.14] for $d = 3$, to demonstrate that $\rho_d(\mathcal{B})$ defines a path from $\rho_{d-1}(\mathcal{B})$ to its opposite. This idea is described for type A after the statement of Theorem 3 in [MS89].

Proposition 3.9. Suppose $\mathcal{B} = (e_1, \ldots, e_n)$ is a generic basis relative to a hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ for the irreducible root system Φ. Then the geometric higher Bruhat order $\rho_1(\mathcal{B}), \ldots, \rho_n(\mathcal{B})$ actually defines a higher Bruhat order on Φ.

Proof. It suffices to show that for each $1 \leq d < n$, the order $\rho_{d+1}(\mathcal{B})$ gives a valid sequence of packet flips from $\rho_d(\mathcal{B})$ to its reverse. In the affine subspace $E_{d+1} = e_{d+1} + \mathrm{span}(e_1, \ldots, e_d)$, each of the elements of $C(\Phi, d)$ is a line, and the elements of $C(\Phi, d+1)$ are the points of intersection of these lines. We can project this picture onto the affine plane $P \coloneqq e_{d+1} + \mathrm{span}(e_{d-1}, e_d)$ by forgetting the other coordinates, and lines still represent $C(\Phi, d)$, and elements of $C(\Phi, d+1)$ are still points of intersection (although keep in mind that other intersections may have been introduced by this projection).

All points of intersection of the lines in P have e_{d-1}-coordinate less than x_0, for some $x_0 \gg 0$, since there are finitely many of them. The line $v \cdot e_{d-1} = x_0$ in P crosses the lines labeled by $C(\Phi, d)$ in the order listed by $\rho_d(\mathcal{B})$, since it was defined to list these points of intersection by e_d-coordinate in E_d.

As we sweep the line $v \cdot e_{d-1} = x_0$ down by decreasing x_0, the order in which the lines are crossed changes exactly at the intersection points of the lines. When the intersection point corresponds to a subspace $V \in C(\Phi, d+1)$, it includes exactly the lines labeled by an element of $P(V)$, and so moving the line $v \cdot e_{d-1} = x_0$ across that intersection point performs the packet flip of V. If the intersection point is not a subspace in $C(\Phi, d+1)$, then the intersection of any pair of subspaces whose lines go through the point must have smaller dimension. Therefore the elements of $C(\Phi, d)$ which go through the point must all *commute* in the sense that they do not share a packet, and so may be freely interchanged without changing the inversion set.

In the process of decreasing x_0 until the line $v \cdot e_{d-1} = x_0$ has passed every intersection point in $C(\Phi, d+1)$, we perform each of the packet flips in the order specified by $\rho_{d+1}(\mathcal{B})$, and no other packet flips (since all other moves are commutation). We started at $\rho_d(\mathcal{B})$, so we must end at its reverse, as required. $\square$

In type A, a geometric higher Bruhat order can also be visualized as a hyperplane arrangement in the following sense. For each $i \in [n]$, let H_i be the hyperplane in $\mathbb{R}^d$ of points $v = (v_1, \ldots, v_d)$ such that v_1 is the x_i-coordinate of $v_1 e_1 + v_2 e_2 + \cdots + v_{d-1} e_{d-1} + e_d$ in the root hyperplane arrangement. Then places where these planes intersect represent places in E_d where $x_i = x_j$, and so packet flips again correspond to vertices. A similar phenomenon can occur in type B, as we will see in the following section.

4 Construction for Type B

In this section we explicitly describe a sequence $\rho_1^*, \rho_2^*, \ldots, \rho_n^*$ of total orders ρ_d^* on $C(B_n, d)$ which is a higher Bruhat order on the type B_n root system. First we have to describe the inversion sets $C(B_n, d)$. The same inversion sets were used in [SV16] for their take on the type B higher Bruhat order. Recall the definitions from Section 1.1 regarding signed subsets, which are crucial for type B.

Lemma 4.1. The subspaces $V \in C(B_n, d)$ can be described by linear equations in $x_1, \dots, x_n$ in one of two ways. The first way is by equations $x_{i_1} = x_{i_2} = \cdots = x_{i_{d-1}} = 0$, in which case we correspond V to the set $\{i_1, \dots, i_{d-1}\} \in \binom{[n]}{d-1} = C(A_n, d-1)$. The other possibility is that V is described by equations $x_{i_1} = \varepsilon_2 x_{i_2} = \cdots = \varepsilon_d x_{i_d}$ for $\varepsilon_j \in \{\pm 1\}$, in which case we correspond B to the set $\{i_1, \varepsilon_2 i_2, \dots, \varepsilon_d i_d\} \in \binom{[\pm n]}{d}$. In this way we bijectively correspond $C(B_n, d)$ with $\binom{[n]}{d-1} \cup \binom{[\pm n]}{d}$.

Proof. The hyperplanes in the hyperplane arrangement $\mathcal{H}$ corresponding to $\Phi = B_n$ are of one of three forms: $x_i = 0$, or $x_i = x_j$, or $x_i = -x_j$. If V satisfies condition 1 in Definition 3.1 then it is equal to the intersection of some collection of these hyperplanes. Any such intersection can be described by (possibly multiple) chains of equations of the form in the statement of the Lemma. If multiple chains are required, then they involve disjoint sets of variables. Therefore any root system producing hyperplane arrangement in $V^\perp$ described in condition 2 of Definition 3.1 is reducible by expressing the space as a sum of the spaces using just the variables corresponding to each chain of equalities. We conclude that if V is an inversion for B_n, then a single chain of equations can describe V, completing the proof. $\square$

Remark 4.2. In type D, a similar proof shows that for $d > 2$, we have $C(B_n, d) = C(D_n, d)$. When $d = 2$, we rather have $C(D_n, 2) \subset C(B_n, 2)$, and we can obtain a higher Bruhat order for type D from a higher Bruhat order for type B by simply removing all the of the inversions which don't exist in type D. The discussion in this section can therefore be seen as also applying to type D.

Now we need to define our total orders on $C(B_n, d) = \binom{[n]}{d-1} \cup \binom{[\pm n]}{d}$, for each n and d. First we assign to each element S of $\binom{[\pm n]}{d}$ an element $A(S)$ of $\binom{[n]}{d-1}$ by removing all of the signs and also the smallest element. Then we order the subsets S according to $A(S)$ first, then by looking at the signs. In the case when $d \leq 3$, Elias and Williamson used the same ordering this definition [EW14].

Definition 4.3. We represent inversions $V \in C(B_n, d) = \binom{[n]}{d-1} \cup \binom{[\pm n]}{d}$ by the set $S_V \coloneqq \{a, x_1, \varepsilon_2 x_2, \dots, \varepsilon_{d-1} x_{d-1}\}$ for $\varepsilon_2, \dots, \varepsilon_{d-1} \in \{\pm 1\}$ and $|a| < x_1 < \cdots < x_{d-1}$, setting $a = 0$ if $V \in \binom{[n]}{d-1}$. Let $A(S_V) \coloneqq \{x_i : i \in [d-1]\} \in \binom{[n]}{d-1}$, and let $f(S_V) \coloneqq 0$ if $a = 0$ and $f(S_V) = a + \frac{a}{|a|} \cdot \sum_{i=2}^{d-1} \frac{\varepsilon_i}{(-3)^i}$ otherwise. Then we say that $V <_B W$ in $C(B_n, d)$ if $A(S_V) <_{\text{lex}} A(S_W)$, or $A(S_V) = A(S_W)$ and $f(S_V) < f(S_W)$.

Example 4.4. When $n = 5$ and $d = 4$, the following is the total order $<_B$ on $C(B_n, d)$, where each inversion V is represented by the set S_V with elements concatenated in increasing order of absolute value, and negative signs in superscripts to save space.

$$\begin{array}{l} 0123,\ 0124,\ 0125,\ 0134,\ 0135,\ 0145, \\ {}^{-}12^{-}34,\ {}^{-}12^{-}3^{-}4,\ {}^{-}1234,\ {}^{-}123^{-}4,\ 0234,\ 123^{-}4,\ 1234,\ 12^{-}3^{-}4,\ 12^{-}34, \\ {}^{-}12^{-}35,\ {}^{-}12^{-}3^{-}5,\ {}^{-}1235,\ {}^{-}123^{-}5,\ 0235,\ 123^{-}5,\ 1235,\ 12^{-}3^{-}5,\ 12^{-}35, \\ {}^{-}12^{-}45,\ {}^{-}12^{-}4^{-}5,\ {}^{-}1245,\ {}^{-}124^{-}5,\ 0245,\ 124^{-}5,\ 1245,\ 12^{-}4^{-}5,\ 12^{-}45, \\ {}^{-}23^{-}45,\ {}^{-}23^{-}4^{-}5,\ {}^{-}2345,\ {}^{-}234^{-}5,\ {}^{-}13^{-}45,\ {}^{-}13^{-}4^{-}5,\ {}^{-}1345,\ {}^{-}134^{-}5, \\ 0345,\ 134^{-}5,\ 1345,\ 13^{-}4^{-}5,\ 13^{-}45,\ 234^{-}5,\ 2345,\ 23^{-}4^{-}5,\ 23^{-}45. \end{array}$$

In order to show that the orders $<_B$ define a higher Bruhat order, we will show that they in particular define a geometric higher Bruhat order (Definition 3.8), and so by Proposition 3.9 they are also a higher Bruhat order.

Take $0 < z_1 < \cdots < z_n$ so that the cyclic hyperplane arrangement corresponding to the z_i gives the lexicographic order as discussed in Section 2, and let $v_{\pm i} = (1, z_i, z_i^2, \ldots, z_i^{d-1}, \pm a_i)$, where the signs on i and a_i are equal and a_i is arbitrarily large relative to any polynomials in z_j (for all j) as well as $a_1, \ldots, a_{i-1}$. The motivation is that this can be a sort of type B analogue of the cyclic vector configuration. Let $\mathcal{H}$ be the corresponding arrangement of normal hyperplanes to these vectors, as well as the plane $x_1 = 1$. We will show that the vertices in the $x_1 = 1$ plane, when labeled by the subset of $[\pm n]$ of indices of hyperplanes involved, are sorted by $<_B$ in terms of x_{d-1}-coordinate. It follows from the discussion in Section 3 that this means that $<_B$ corresponds to a geometric higher Bruhat order.

Theorem 4.5. *Let $\mathcal{H}$ be the hyperplane arrangement of $2n$ planes $H_{\pm i}$ in $\mathbb{R}^{d+1}|_{\{x_0=1\}}$ with equations $x_0 + z_i x_1 + z_i^2 x_2 + \cdots + z_i^{d-1} x_{d-1} \pm a_i x_d = 0$ with z_i and a_i as described before. Then for any subset $S \subseteq [\pm n]$, the intersection $v_S := \bigcap_{s\in S} H_s$ is a vertex if and only if $S \in C(B_n, d)$. For any $V, W \in C(B_n, d)$, the x_{d-1}-coordinate of v_V is less than the x_{d-1}-coordinate of v_W if and only if $V <_B W$.*

Proof sketch. For $S \subseteq [\pm n]$, the intersection v_S is the solution to the system of equations $z_{i_s} x_1 + \cdots + z_{i_s}^{d-1} x_{d-1} + \varepsilon_s a_{i_s} x_d = -1$ for all $\varepsilon_s i_s = s \in S$, where $i_s = |s|$ and ε_s is the sign of s. There is a unique solution when there are d such equations and they are independent, which is captured by $\binom{[n]}{d-1} \cup \binom{[\pm n]}{d} = C(B_n, d)$ as desired. Next, by Cramer's rule the x_{d-1}-coordinate of the solution to this system is given by the ratio of determinants:

$$\det \begin{bmatrix} \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ z_{i_s} & z_{i_s}^2 & \cdots & z_{i_s}^{d-2} & -1 & \varepsilon_s a_{i_s} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \end{bmatrix} \cdot \left(\det \begin{bmatrix} \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ z_{i_s} & z_{i_s}^2 & \cdots & z_{i_s}^{d-2} & z_{i_s}^{d-1} & \varepsilon_s a_{i_s} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \end{bmatrix} \right)^{-1}.$$

We compute these determinants by expanding cofactors along the last column. When we do this, the resulting minors will be Vandermonde determinants which we could compute but will just denote by $\mathcal{V}_S := \det(z_{i_s}^{j-1})_{s\in S, j\in[d-1]}$. Letting s_k be the s in the k-th row of the matrix, the first determinant is equal to $\pm \sum_{k=1}^{d} (-1)^k \cdot \varepsilon_{s_k} a_{i_{s_k}} \cdot \mathcal{V}_{S\setminus\{s_k\}}$.

The second determinant is equal to $\pm \sum_{k=1}^{d} (-1)^k \cdot \varepsilon_{s_k} a_{i_{s_k}} \cdot \mathcal{V}_{S\setminus\{s_k\}} \cdot \prod_{s \in S\setminus\{s_k\}} z_{i_s}$. How do we compare this ratio for $S \in C(B_n, d)$ to the ratio for another $S' \in C(B_n, d)$? The key is that $a_1 \gg a_2 \gg \cdots \gg a_n \gg z_n$, allowing us to just look at one term in these sums at a time.

If $i_1 < \cdots < i_d$ are the indices i_s in increasing order, then the ratio is very close to $\pm \left(\prod_{k\in[d]\setminus\{1\}} z_{i_k}\right)^{-1}$, by just looking at the dominant terms in the numerator and denominator. It follows that sorting by x_{d-1}-coordinate, like sorting by $<_B$, first compares the sets $A(S)$ using the lexicographic order. If those are equal, then the next step is to look at the next largest terms in the numerator and denominator, whose ratio is $\pm\varepsilon_2 \left(\prod_{k\in[d]\setminus 2} z_{i_k}\right)^{-1}$. This means that the ratio is slightly shifted either towards or away from the first approximation, depending on ε_2, and the amount depends most upon the value of z_{i_1}. This means that this order, just like $<_B$, compares the (signed) values of the smallest index next after checking that the sets agree on the rest of the (unsigned values). Note that this corresponds to the fact that the a term in the expression for $f(S)$ in Definition 4.3 dominates. Finally, if the two sets have the same unsigned values completely, then it remains to compare each of the subsequent adjustments due to lower order terms, in increasing order of index. For these, the comparison is only of the signs ε_i, and when they differ, we look to the parity of i to determine whether the set with negative ε_i should have larger or smaller x_{d-1}-coordinate. This again coincides with the behavior of the order $<_B$, where the size of $f(S)$ depends in the same manner on the ε_i when the unsigned sets are the same. $\square$

We have now completed our construction of the type B higher Bruhat order to arbitrary levels, which we can now denote $B(B_n, d)$ with the ρ_i^* being implicit. We have not yet shown that it has any particularly desirable properties that other type B higher Bruhat orders might not have.

Proposition 4.6. For any $S \subsetneq [n]$ with $|S| = m$, the poset $B(B_S, d)$ obtained from $B(B_n, d)$ by throwing out all inversions including elements of $[n] \setminus S$ is isomorphic to $B(B_m, d)$.

Proof. In Definition 4.3, the order $<_B$ depends only on the order $<_{\text{lex}}$ on unsigned subsets, and the function f. The lexicographic order does not change when numbers are relabeled but magnitudes are preserved, and the function f only cares about the relative sizes of the smallest element a and the signs ε_i. Therefore relabeling the elements of S with the numbers 1 through m in order provides the needed poset isomorphism. $\square$

This proposition establishes the "parabolic-compatible" condition desired in [Eli16]. Do the other conditions discussed there hold for our order? We expect that at least one does.

Conjecture 4.7. The higher Bruhat orders $B(B_n, d)$ as defined in this paper have a unique minimal element (namely $<_B$, with inversion set $\emptyset$), and a unique maximal element (namely $>_B$, with inversion set $\binom{[n]}{d-1} \cup \binom{[\pm n]}{d}$).

5 Zonotopes?

Warning to the reader: this section is highly speculative, and not to be taken too seriously.

As discussed in Section 2, the cyclic vector arrangement (formed by choosing n points $v_1, \ldots, v_n$ on the *moment curve* $(1, x, x^2, \ldots, x^{d-1})$) produces a hyperplane arrangement corresponding to the lexicographic order on $C(A_n, d)$. The *cyclic zonotope* $Z(n, d)$ is defined to be the Minkowski sum of the segments $[0, v_i]$, and elements of the higher Bruhat order $B(A_n, d)$ can be realized as fine zonotopal tilings of this polytope. The picture is most interesting when $d = 3$, where cross-sections correspond to plabic graphs [Gal18]. In Theorem 4.5, we showed that our type B higher Bruhat order can similarly be realized by a vector configuration. It is natural to ask what happens when we look at the zonotopes generated by this configuration. When $d = 2$, we can even correspond elements $B(B_n, 2)$ to *symmetric* tilings of $Z(2n, 2)$ with rhombi. However, when $d \geq 3$, the shape formed by the Minkowski sum of the vectors is not the same as $Z(2n, d)$ anymore, so it will not be as simple as looking at symmetric tilings.

The order we chose for type B is not necessarily the best one, however. At least for $d \leq 3$, there is another order one can choose (for $d = 2$ it can be described by repeating the lexicographic order on $\binom{[n]}{2}$ twice for the different signings, with the unsigned sets $\binom{[n]}{1}$ interspersed) which matches up better in this context. In particular, the polytope formed for $d = 3$ is isomorphic to $Z(2n, 3)$, and we believe that elements of the order $B(B_n, 3, \rho^*)$ do correspond to *almost-fine* symmetric tilings, where there are tiles along the plane of symmetry which are copies of $Z(4, 3)$ instead of parallelepipeds. Unfortunately it is hopeless for $d \geq 4$, as the nature of type B requires any vector configuration to create linear dependencies which do not exist in type A. We also did not mention this order in the previous sections because we do not see a natural continuation of it for $d \geq 4$. Nevertheless, it would be interesting to look at the cross-sections of the symmetric tilings in three-dimensions. They ought to be some sort of symmetric plabic graph, although a different sort than that described in [KS18].

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