Perverse Sheaves and Quiver Representations

UROP+ Final Paper, Summer 2020

Peter Rowley

Mentor Timothy Ngotiaoco

Project Suggested by Roman Bezrukavnikov

August 31, 2020

Abstract

The complexification of a real hyperplane arrangement $\mathcal{H}$ determines a stratification $S^{(0)}$ of the complex numbers. A perverse sheaf is a cochain complex of sheaves with cohomology locally constant on these strata, satisfying two other conditions. We state an equivalence of categories proven by Kapranov and Schechtman between perverse sheaves and a certain set of quiver representations, and explicitly lay out one direction of the correspondence. We categorize the simple objects of the category of quiver representations for the hyperplane arrangement A_1, and find some of these simple objects for A_2.

1 Introduction

Let $\mathcal{H}$ be a real hyperplane arrangement on $\mathbb{R}^n$. This gives us stratification of $\mathbb{R}^n$, which we can define based on the intersections of various sets of hyperplanes. These strata also have a naturally induced partial order based on inclusion of closures.

In the mid-1970's, Goresky and MacPherson introduced intersection homology to study properties of surfaces with singularities. Given a stratification of the surface, one is able to relate the homology of the entire surface with the intersection homology of the various strata. Goresky, MacPherson, and Deligne continued to develop this theory and developed what are now known as perverse sheaves. See [4] for a more detailed history.

A perverse sheaf on $\mathbb{C}^n$ is a cochain complex of sheaves on $\mathbb{C}^n$ which satisfies a few additional properties, especially regarding its cohomology. Perverse sheaves are defined with respect to the complexification of our real hyperplane arrangement $\mathcal{H}$.

Although perverse sheaves are normally very complicated objects, the following theorem of [3] allows us to describe them using linear algebra.

Theorem 1. *There exists an equivalence of categories between perverse sheaves and quiver representations that satisfy the properties of monotonicity, transitivity, and invertibility.*

In this paper, we detail the construction in [3] on how to produce a perverse sheaf from a double representation. We look at A_1 and classify the irreducible representations, using the equivalence of categories to describe the simple perverse sheaves. We also look at A_2 and describe the irreducible representations involving small vector spaces.

1.1 Outline

Section 2 defines quivers, quiver representations, and related concepts. Sections 3-4 discuss hyperplane arrangements and their complexifications, and define the stratifications of $\mathbb{R}^n$ and $\mathbb{C}^n$ based on these arrangements. Section 5 discusses sheaves and other necessary related concepts, and section 6 defines perverse sheaves.

At this point the most important theorem of [3] is described in section 7. This regards the equivalence of categories described above. In section 8 we describe in more detail the functor bringing quiver representations to perverse sheaves in this equivalence. In section 9, we classify the irreducible representations of A_1 and calculate the corresponding simple objects in the category of perverse sheaves, and in section 10 we calculate the irreducible representations of A_2 with the limitation of two-dimensional vector spaces.

1.2 Acknowledgements

I would like to thank my mentor, Timothy Ngotiaoco, for guiding me throughout the course of this project. I would also like to thank Professor Roman Bezrukavnikov for suggesting the project. Finally, I would like to thank the UROP+ program for making this possible.

2 Quiver Representations

A **quiver** is a directed graph that allows self-loops and multiple edges. A **representation** of a quiver Q is a map that associates a vector space V_i with each vertex i of the quiver, and a linear map $A_h : V_i \to V_j$ with each arrow $h : i \to j$ of the quiver.

A **subrepresentation** of a representation of Q is a representation that assigns subspaces V_i' of V_i to each vertex i, and assigns linear maps A_h' by restricting the domain of A_h to V_i' for each h, such that $A_h' V_i' \subset V_j'$ for each $h : i \to j$. A subrepresentation is called **proper** if it is not equal to the original representation.

A representation is **irreducible** if it has no nonzero proper subrepresentations.

A **homomorphism** ϕ between two representations (V_i, A_h) and (V_i', A_h') of Q is a collection of linear maps $\phi_i : V_i \to V_i'$ such that if $h : i \to j$ is an edge of Q, then $\phi_j A_h = A_h' \phi_i$. A homomorphism is an **isomorphism** if all the ϕ_i are isomorphisms. Two representations are **isomorphic** if there exists an isomorphism between them.

3 Hyperplane Arrangements

A **hyperplane arrangement** is a set of hyperplanes passing through the origin in a finite-dimensional vector space. It is a real hyperplane arrangement if the space is $\mathbb{R}^n$, and a complex hyperplane arrangement if the space is $\mathbb{C}^n$. Each hyperplane is determined by a fixed linear equation.

Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^n$. A **flat** of $\mathcal{H}$ is something of the form

$$L = \bigcap_{H_i \in S} H_i, S \subset \mathcal{H}.$$

Now consider the open part of L, which is

$$L \setminus \bigcup_{H_i \notin S} H_i.$$

The connected components of this are called **faces** of $\mathcal{H}$. Note that faces of $\mathcal{H}$ can be of any dimension.

The faces of $\mathcal{H}$ create a stratification $\mathcal{C}$ of $\mathbb{R}^n$. We give $\mathcal{C}$ a poset structure by saying that if $A, B \in \mathcal{C}$, then $A \leq B$ if $A \subset \overline{B}$. This means that the intersection of all the hyperplanes in $\mathcal{H}$ is the smallest element of the poset, and faces of codimension zero are maximal elements.

Each face $A \in \mathcal{C}$ is assigned a unique sign vector $(a_H)_{H\in\mathcal{H}}$, where $a_H \in \{+,-,0\}$ is the sign of A determined by the equation of H.

Given any poset Σ, we can define an operation $\star$ for any $a, b \in \Sigma$

$$a \star b = \begin{cases} b & \text{if } a < b, \\ a & \text{otherwise.} \end{cases}$$

We consider $\{+,-,0\}$ as a poset with $0 < +, 0 < -$, and $+$ and $-$ incomparable. Then if $C, D \in \mathcal{C}$ with sign vectors $(C_H)_{H\in\mathcal{H}}$ and $(D_H)_{H\in\mathcal{H}}$, we define the operation $\circ$ as follows:

$$C \circ D = (C_H \star D_H)_{H\in\mathcal{H}}.$$

An ordered triple (A, B, C) of faces in $\mathcal{C}$ is called **collinear** if there exists a straight line that intersects all three in the given order. In other words, there must exist $a \in A, b \in B, c \in C$ such that b lies on the line segment $[a, c]$.

A **double representation** of a stratification $\mathcal{C}$ assigns a finite-dimensional vector space E_C to each face $C \in \mathcal{C}$, and two maps

$$\gamma_{C'C} : E'_C \to E_C, \delta_{CC'} : E_C \to E'_C$$

for each $C, C' \in \mathcal{C}$ such that $C' \leq C$, such that the $\gamma_{C'C}$ form a representation of the poset $(C, \leq)$ and the $\delta_{CC'}$ form a representation of the poset $(C, \geq)$. Throughout this paper, the base field will be $\mathbb{C}$. The category of complex double representations of $\mathcal{C}$ will be denoted $\mathrm{Rep}^{(2)}(\mathcal{C})$.

4 Complexification

Given a real hyperplane H, its **complexification** $H_{\mathbb{C}}$ is the complex hyperplane with the same equation.

Given a hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$, the **complexified arrangement** $\mathcal{H}_{\mathbb{C}}$ is the arrangement in $\mathbb{C}^n$ made up of the complexifications of the hyperplanes in $\mathcal{H}$.

The complexification $L_{\mathbb{C}}$ of a real flat L of $\mathcal{H}$ is the intersection of the same hyperplanes in the complexified hyperplane arrangement:

$$L_{\mathbb{C}} = \bigcap_{H \supset L} H_{\mathbb{C}}.$$

In the **complex stratification** $S^{(0)}$, as defined in [3], the strata are the open parts of the complexified flats:

$$L^{\circ}_{\mathbb{C}} = L_{\mathbb{C}} \setminus \bigcup_{H \not\supset L} H_{\mathbb{C}}.$$

Although we will mostly use $S^{(0)}$ in this paper, we will also define the $S^{(2)}$ stratification. Here, the strata are of the form $iC_1 + C_2$, for $C_1, C_2 \in \mathcal{C}$.

There exists an $S^{(1)}$ stratification, but we will not need it for this paper.

5 Sheaves

Let X be a topological space. A **presheaf** $\mathcal{F}$ of abelian groups on X is a structure that assigns the following:

1. To each open set $U \subset X$, an abelian group $\mathcal{F}(U)$,
2. To each inclusion $V \subset U$ of open sets, a restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,

satisfying the following conditions:

- $\mathcal{F}(\emptyset) = 0$,
- $\rho_{UU} = \mathrm{id}_U$,
- For $W \subset V \subset U$ open sets, $\rho_{UW} = \rho_{VW}\rho_{UV}$.

For any U, the elements of $\mathcal{F}(U)$ are called **sections**. If $V \subset U$ and $s \in \mathcal{F}(U)$, $\rho_{UV}(s)$ is sometimes denoted as $s|_V$.

A presheaf is called a **sheaf** if it satisfies the following conditions:

3. If U is an open set, $s \in \mathcal{F}(U)$, and V_i is an open covering of U such that $s|_{V_i} = 0 \forall i$, then $s = 0$.
4. If U is an open set, $\{V_i\}$ is an open cover of U, and if $s_i \in \mathcal{F}(V_i)$ are such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i and j, then there exists an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for all i.

Condition 3 implies that s in condition 4 is unique, since if s, s' both satisfy 4, then $s - s'$ restricts to 0 on all V_i, so it is zero.

A **morphism** $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves consists of a morphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each U, such that the $\phi(U)$ commute with restriction maps in $\mathcal{F}$ and $\mathcal{G}$. This means that for each $V \subset U$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \rho_{UV}\downarrow & & \downarrow\rho'_{UV} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

Here ρ and ρ' are the restriction maps in $\mathcal{F}$ and $\mathcal{G}$, respectively.

A morphism is an **isomorphism** if it has a two-sided inverse.

The following proposition is proven in [2].

Proposition 1. *Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathscr{F}^+$ such that for any sheaf $\mathcal{G}$ and morphism $\phi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\phi = \psi \circ \theta$. The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism.*

Here, $\mathcal{F}^+$ is called the **sheaf associated** to the presheaf $\mathscr{F}$.

A sheaf $\mathcal{F}'$ is called a **subsheaf** of a sheaf $\mathscr{F}$ if the following two conditions hold:

- For each open set U, $\mathcal{F}'(U) \subset \mathcal{F}(U)$,
- For all pairs of nested open subsets $V \subset U$, $\rho'_{UV} = \rho_{UV}|_{\mathcal{F}'(U)}$.

If $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, then the **quotient sheaf** $\mathcal{F}/\mathcal{F}'$ is the sheaf associated to the presheaf that assigns $\mathcal{F}(U)/\mathcal{F}'(U)$ to each open set U, with restriction maps induced by restriction maps in $\mathcal{F}$.

Let $U \subset X$ be an open set with the subspace topology, and let $\mathcal{F}$ be a sheaf on X. We define the **restriction** $\mathcal{F}|_U$ of $\mathcal{F}$ to U by letting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for each open $V \subset U$. The restriction maps in $\mathcal{F}|_U$ are the same as in $\mathcal{F}$.

Given a group A, the **constant sheaf** $\mathcal{A}$ determined by A is the sheaf defined as follows. Giving A the discrete topology, we let $\mathcal{A}(U)$ be the group of continuous maps from U into A. If $f \in \mathcal{A}(U)$, and $V \subset U$ is open, then the restriction map is defined as $\rho_{UV} f = f|_V$. If U is connected, then we can see based on this definition that $\mathcal{A}(U)$ is isomorphic to A.

A sheaf $\mathcal{F}$ on X is called **locally constant** if for each $p \in X$, there exists a neighborhood U of p such that $\mathcal{F}|_U$ is a constant sheaf on U.

The **stalk** $\mathcal{F}_p$ of $\mathcal{F}$ at $p \in X$ is defined as the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing p, under the restriction maps ρ. We can see $\mathcal{F}_p$ as a collection of elements of the form $\langle U, s \rangle$, where U is an open set containing p and $s \in \mathcal{F}(U)$. Two such elements $\langle U_1, s_1 \rangle$ and $\langle U_2, s_2 \rangle$ are equal if and only if there exists an open set V containing p with $V \subset U_1 \cap U_2$ such that $s_1|_V = s_2|_V$.

Let $f : X \to Y$ be a continuous map of topological spaces. If $\mathcal{G}$ is a sheaf on Y, then we define the **inverse image** or **pullback** sheaf $f^{-1}\mathcal{G}$ by assigning $f^{-1}\mathcal{G}(U)$ the direct limit

$$\lim_{V \supset f(U)} \mathcal{G}(V),$$

where the direct limit is taken over all open sets V containing $f(U)$. Note that if X is an open subset of Y and f is the inclusion map, then $f^{-1}\mathcal{G} = \mathcal{G}|_X$.

Proposition 2. *The pullback functor f^{-1} is exact.*

Proof. Sheaf functors are exact if and only if they are exact on the stalks ([2], page 65). It is known that if $p \in X, q \in Y$ such that $f(p) = q$, then $(f^{-1}\mathcal{G})_p = \mathcal{G}_q$. Clearly this implies that f^{-1} is exact on stalks, so it is exact. □

6 Perverse Sheaves

A **cochain complex** of sheaves, or just a **complex**, is a collection $\mathcal{F} : \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ with morphisms $d^i : \mathcal{F}_i \to \mathcal{F}_{i+1}$, such that $d^{i+1} \circ d^i = 0$ for all i. Equivalently, $\mathrm{im}(d^i) \subset \ker(d^{i+1})$.

We call $\ker(d^{i+1})/\mathrm{im}(d^i)$ the ith **cohomology sheaf** $\underline{H}^i(\mathcal{F})$ of a complex of sheaves.

Let $\mathcal{F}$ be a sheaf. If for every stratum $C \in S^{(n)}$ and inclusion map $i : C \to \mathbb{C}^n$ the sheaf $i^{-1}\mathcal{F}$ is locally constant, $\mathcal{F}$ is called $S^{(n)}$**-smooth**. It is called $S^{(n)}$**-constructible** if it is $S^{(n)}$-smooth and it has finite-dimensional stalks. A complex of sheaves is $S^{(n)}$-smooth or $S^{(n)}$-constructible if all of its cohomology sheaves are $S^{(n)}$-smooth or $S^{(n)}$-constructible, respectively.

The category $D^b_{S^{(0)}}\mathrm{Sh}\mathbb{C}^n$ is a full subcategory of the bounded derived category, the definition of which is beyond the scope of this paper. The reader can consult [6] for more on this. What's important for us is that an element of this category is represented by a $S^{(0)}$-constructible cochain complex of sheaves.

The category of perverse sheaves, $\mathrm{Perv}(\mathbb{C}^n)$, is defined in [3] as the full subcategory of $D^b_{S^{(0)}}\mathrm{Sh}_{\mathbb{C}^n}$ with the following two conditions:

(P^-) For each p the sheaf $\underline{H}^p(\mathcal{F})$ is supported on a closed complex subspace of codimension $\geq p$.

(P^+) If $l : Z \to X$ is a locally closed embedding of a smooth analytic submanifold of codimension p, then the sheaf $\underline{H}^q(l^!\mathcal{F}) = \mathbb{H}^q_Z(\mathcal{F})|_Z$ is zero for $q < p$.

Although we haven't defined the exceptional inverse image functor $l^!$, what's important for us is that if Z is open in X, then $l^!$ is exactly our inverse image functor l^{-1}. These two properties give rise to the following fact about perverse sheaves.

Proposition 3. *A perverse sheaf $\mathcal{F}$ can have nonzero cohomology on the open part of the stratification only in degree zero.*

Proof. Property (P^-) implies that for any $p > 0$, $\underline{H}^p(F)$ is supported on a closed complex subspace of positive codimension, which means it is zero on the open part U. Therefore, $\underline{H}^p(F)|_U \neq 0$ only if $p \leq 0$.

If we let Z be the open part of the stratification, then (P^+) states that $\underline{H}^q(l^!\mathcal{F}) = 0$ for $q < 0$. However, since Z is an open set, $l^!\mathcal{F}$ is simply the pullback of the inclusion map from Z to X applied to $\mathcal{F}$. Since this is an exact functor, it commutes with $\underline{H}^q$: $l^{-1}\underline{H}^q(\mathcal{F}) = \underline{H}^q(l^{-1}\mathcal{F})$. This implies that $\mathcal{F}$ has zero cohomology on the open part in degrees less than zero.

Putting these two facts together, we obtain our desired result. □

7 Main Theorem

Theorem 8.1 of [3] states that the category of perverse sheaves on $\mathcal{H}$ is equivalent to the full subcategory of $\mathrm{Rep}^{(2)}(\mathcal{C})$ of representations with the following properties:

- **Monotonicity:** For any $C' \leq C$, $\gamma_{C'C}\delta_{CC'} = \mathrm{Id}_{E_C}$.
  - Because of this, we can define $\phi_{AB} : E_A \to E_B$ by $\phi_{AB} = \gamma_{CB}\delta_{AC}$, for any $C \in \mathcal{C}$ with $C \leq A, C \leq B$. This is well-defined because for all C,

    $$\gamma_{CB}\delta_{AC} = \gamma_{CB}\gamma_{0C}\delta_{C0}\delta_{AC} = \gamma_{0B}\delta_{A0}.$$

    Here, 0 denotes the minimal element of $\mathcal{C}$.

- **Transitivity:** If (A, B, C) is a collinear triple of faces, then $\phi_{BC}\phi_{AB} = \phi_{AC}$.

- **Invertibility:** If C_1, C_2 are two d-dimensional faces on opposite sides of the same $d-1$-dimensional face, then $\phi_{C_1C_2}$ is an isomorphism.

8 Quiver Representation to Perverse Sheaf

Our goal for this section is to describe the above correspondence in one direction, from quiver representations to perverse sheaves.

8.1 Linear Algebra Data

Let $\mathcal{F}$ be a sheaf on $\mathbb{R}^n$ or $\mathbb{C}^n$ that is constant on the strata given by the hyperplane arrangement $\mathcal{H}$. (For $\mathbb{C}^n$ we use the $S^{(2)}$ stratification here.) Then Proposition 1.8 of [3] states that $\mathcal{F}$ is uniquely determined by its **linear algebra data**, which is defined as follows.

First, for each stratum σ we associate the vector space $\mathcal{F}_\sigma$ of the stalk on any point of that vector space. (This is well-defined because $\mathcal{F}$ is constant on the strata.) For strata σ', σ with $\sigma' \leq \sigma$, we define a **generalization map** as follows.

Take some $x' \in \sigma'$, and take a neighborhood U' of x' small enough that $\mathcal{F}(U') = \mathcal{F}_{\sigma'}$. Then take $x \in \sigma \cap U'$, and let $U \subset U'$ be a neighborhood of X that is small enough that $\mathcal{F}(U) = \mathcal{F}_\sigma$. Then the generalization map $\gamma_{\sigma'\sigma}$ is simply the restriction map $\rho_{U'U}$.

We will also define the inverse map, to reconstruct the sheaf based on its linear algebra data. Given an open set $U \subset X$, sections of $\mathcal{F}(U)$ are uniquely determined by their stalks at every point of U. Therefore, in order to determine the sections of $\mathcal{F}(U)$, it is sufficient to determine which stalks are compatible with each other.

We know as above that there for each point $x \in U$, there should be some neighborhood $U' \subset U$ of x such that $\mathcal{F}_x = \mathcal{F}(U')$. Therefore, each stalk corresponds to a section of a small neighborhood of that point. In order for stalks to be compatible, these sections have to restrict to the same value on their intersections.

Trivially, stalks at two points $x, x' \in U$ are compatible if their corresponding neighborhoods do not intersect. If they do intersect, say that $x \in \sigma$ and $x' \in \sigma'$, where $\sigma' \leq \sigma$. Then based on the above calculation, they agree if and only if $x = \gamma_{\sigma'\sigma} x'$. This condition is sufficient to construct all sections of $\mathcal{F}(U)$, so we are done.

8.2 The Correspondence

We will now describe how to obtain a perverse sheaf given a quiver representation.

Let Q be a double quiver satisfying the properties in section 7. For each $C \in \mathcal{C}$ we will define a sheaf $\mathcal{E}_C(Q)$ on $\mathbb{C}^n$ that is $S^{(2)}$-smooth. Since the $S^{(2)}$ strata are contractible, $\mathcal{E}_C(Q)$ is in fact constant on them, rather than just locally constant. Therefore, we can define $\mathcal{E}_C(Q)$ based on its linear algebra data:

$$\mathcal{E}_C(Q)|_{iC_1+D} = \begin{cases} E_{C\circ D} & \text{if } C_1 \leq C, \\ 0 & \text{otherwise.} \end{cases}$$

We now define the generalization maps:

$$\gamma^{\mathcal{E}_C(Q)}_{iC_1'+D', iC_1+D} = \gamma_{C\circ D', C\circ D}, C_1' \leq C_1 \leq C, D' \leq D.$$

We require $C_1' \leq C_1 \leq C, D' \leq D$ here because we need $C_1' \leq C_1, D' \leq D$ in order for $iC_1' + D' \leq iC_1 + D$, and if $C_1 \not\leq C$, then $\mathcal{E}_C(Q)|_{iC_1+D}$ is zero and the generalization map is trivial.

Having defined $\mathcal{E}_C(Q)$, we now define for any $C', C \in \mathcal{C}$ with $C' <_1 C$ a sheaf morphism

$$\underline{\delta}_{CC'} : \mathcal{E}_C(Q) \to \mathcal{E}_{C'}(Q).$$

We start by defining the action of $\underline{\delta}_{CC'}$ on stalks of the form $iC' + D$ for any $D \in \mathcal{C}$ as

$$\underline{\delta}_{CC'}|_{iC'+D} = \gamma_{C'K'}\delta_{KC'} : E_{C\circ D} \to E_{C'\circ D}.$$

Here, $K = C \circ D$ and $K' = C' \circ D$, for convenience.

For any $iC'' + D$ where $C'' \not\leq C'$, $\underline{\delta}_{CC'}$ must be zero, since $\mathcal{E}'_C(Q)|_{iC''+D} = 0$. If $C'' < C'$, then $\mathcal{E}'_C(Q)|_{iC''+D} = \mathcal{E}'_C(Q)|_{iC'+D}$ by definition. The same is true in $\mathcal{E}_C(Q)$, so the above definition is sufficient to define $\underline{\delta}_{CC'}$. This is a sheaf morphism because the maps of stalks commute with generalization maps ([3], page 661).

The definition of $\underline{\delta}_{CC'}$ allows us to assemble the following complex:

$$\Theta(Q) = \left\{ \bigoplus_{\text{codim}(C)=0} \mathcal{E}_C(Q) \otimes \text{or}(C) \xrightarrow{\delta} \bigoplus_{\text{codim}(C)=1} \mathcal{E}_C(Q) \otimes \text{or}(C) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{E}_0(Q) \right\}.$$

Here, $\text{or}(C)$ is a one-dimensional orientation vector space. An orientation of C is defined by an open cover (U_i) and diffeomorphisms $\phi_i : U_i \cong (0,1)^n$ or

$U_i \cong (0,1)^{n-1} \times (0,1]$ such that $\phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ has a positive Jacobian. By picking an orientation, we get an isomorphism $\mathrm{or}(C) \cong \mathbb{C}$. There are only two possible isomorphisms based on the orientation, which send $1 \in \mathbb{C}$ to additive inverse elements of $\mathrm{or}(C)$.

The orientation of C induces an orientation on its boundary, since it should correspond to the $\{1\} \times (0,1)^{n-1}$ of the $U_i \cong (0,1)^{n-1} \times (0,1]$ maps.

For $C' <_1 C$, the map from $\mathrm{or}(C)$ to $\mathrm{or}(C')$ is defined by a map $\mathbb{C} \to \mathbb{C}$, through the isomorphisms of each space with $\mathbb{C}$. This map is $\pm\mathrm{Id}$. It is positive if the orientation on C induces the same orientation on C' as we have chosen; it is negative if it is the opposite. The reader can consult [5] for more detail on this.

This definition makes $\Theta(Q)$ into a complex. It is shown in [3] that this is a perverse sheaf.

9 The Arrangement A_1

Given the above equivalence of categories, we wish to find the simple objects of the category of quiver representations, so that we can classify the simple objects of the category of perverse sheaves. In quiver representations, the simple objects are the irreducible representations. We will focus on the simplest case, that of the hyperplane arrangement A_1, and mention some of the irreducible representations of A_2 later.

9.1 Irreducible Representations of A_1

The hyperplane arrangement A_n is an arrangement in n-dimensional space. It is defined as the intersection in $n+1$-dimensional space of the hyperplane $x_1 + \cdots + x_{n+1} = 0$ with the hyperplane arrangement consisting of the hyperplanes $x_i - x_j = 0$ for every $1 \le i < j \le n+1$.

We wish to find the irreducible elements of the full subcategory described above for the hyperplane arrangement A_1. It is clear that A_1 is the one-hyperplane arrangement consisting of the origin in $\mathbb{R}^1$. Therefore we have three vector spaces as defined above, corresponding to the positive reals, the negative reals, and zero. We will call them E_+, E_-, and E_0, respectively. The corresponding quiver is represented by the diagram below.

$$E_- \underset{\gamma_{0-}}{\overset{\delta_{-0}}{\rightleftarrows}} E_0 \underset{\delta_{+0}}{\overset{\gamma_{0+}}{\rightleftarrows}} E_+$$

Monotonicity implies that when $C' \le C$, E'_C has at least as many dimensions as E_C, since $\delta_{CC'}$ is a section of $\gamma_{C'C}$. For A_1, this implies that E_0 has at least as many dimensions as each of E_+ and E_-. Additionally, invertibility implies that E_+ and E_- have the same number of dimensions.

We denote by 0—1—0 the representation where $E_+, E_- = 0$ and E_0 is one-dimensional. It is clear that this is the only nonzero irreducible representation where $E_+, E_- = 0$. It is represented more completely by the following diagram.

$$0 \rightleftarrows \mathbb{C} \rightleftarrows 0$$

Assume that $E_+, E_- \neq 0$. Then $\phi_{+-}\phi_{-+}$ is an automorphism of E_+, since ϕ_{+-} and ϕ_{-+} are isomorphisms by invertibility. Since E_+ is a complex vector space, this automorphism has an eigenvector v_+. Then v_+ induces a subrepresentation with E'_+, E'_- one-dimensional and E'_0 either one- or two-dimensional. Therefore, the only irreducible representations of A_1 with nonnzero E_+, E_- have dimensions 1—1—1 or 1—2—1.

Assume E_+, E_- are one-dimensional. If E_0 is one-dimensional, then monotonicity implies that the representation is isomorphic to assigning every map to the identity map on $\mathbb{C}$. This implies that there is only one class of irreducible representations in the 1—1—1 case, which is shown below.

$$\mathbb{C} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathbb{C} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathbb{C}$$

If E_0 is two-dimensional, the automorphism $\phi_{-+}\phi_{+-} : E_+ \to E_+$ need not be the identity map on $\mathbb{C}$. We will show that the image of 1 under this automorphism uniquely identifies an isomorphism class of irreducible representations.

Clearly, any two representations with different values of $\phi_{-+}\phi_{+-}(1)$ cannot be isomorphic, because an isomorphism of representations holds under compositions of maps. Therefore, we are left with showing that any two representations with the same value are isomorphic.

Let (E_+, E_0, E_-) be an irreducible representation of A_1 with E_0 two-dimensional. Fix a basis element 1_+ of E_+, and let $\delta_{+0}1_+ = v_1$, the first basis element of E_0. Define $1_- \in E_-$ as $\gamma_{0-}v_1$.

We know that $\delta_{-0}1_-$ is not a multiple of v_1, because then E_0 would be one-dimensional. Therefore, let $v_2 = \delta_{-0}1_-$. Now we can define $c = \gamma_{0+}v_2$.

Clearly whenever c is the same between two representations, the above bases provide an isomorphism. Therefore, we have proven that c uniquely determines the isomorphism class. Additionally, this construction makes it clear that c can be any nonzero complex number.

We can see by simple calculation that any subrepresentation of such a representation which contains a nonzero element of E_+ or E_- cannot be a proper representation. Therefore, the only way for there to be a proper subrepresentation is if $\ker \gamma_{0+} = \ker \gamma_{0-}$, so we have a subrepresentation of 0—1—0. Using the above bases, $\ker \gamma_{0+} = \mathrm{span}(-c, 1)$, and $\ker \gamma_{0+} = \mathrm{span}(-1, 1)$. Therefore, the representation is irreducible if and only if $c \neq 1$.

We now know that the 1—2—1 case has one isomorphism class per element of $\mathbb{C}^\times$ other than 1, which is shown below.

$$\mathbb{C} \underset{c \mapsfrom (0,1)}{\overset{1 \mapsto (1,0)}{\rightleftarrows}} \mathbb{C}^2 \underset{(0,1) \mapsfrom 1}{\overset{(1,0) \mapsto 1}{\rightleftarrows}} \mathbb{C}$$

9.2 Fundamental Group Representations

Theorem 4.3.5 of [1] states that the simple objects in the category of perverse sheaves, or equivalently quiver representations, are parametrized by pairs of the form (S, V), where S is one of the $S^{(0)}$ strata and V is an irreducible representation of the fundamental group of S.

The $S^{(0)}$ stratification of $\mathbb{C}$ under A_1 has two strata, 0 and $\mathbb{C}^\times$. The fundamental group of 0 is the trivial group, which we will also denote by 0, the only irreducible representation of which is the trivial representation. This gives the pair that we will denote $(0, 1)$ for simplicity.

The fundamental group of $\mathbb{C}^\times$ is $\mathbb{Z}$. The irreducible complex representations of $\mathbb{Z}$ biject with $\mathbb{C}^\times$, based on the image of 1, since $\mathrm{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$. Therefore, this gives us the pair $(\mathbb{C}^\times, c)$ for any $c \in \mathbb{C}^\times$.

Given a simple object in the category of perverse sheaves, we can identify which pair it corresponds to as follows. We know from section 6 that a perverse sheaf only has nonzero cohomology on the open part in one place. We identify this cohomology sheaf. If there is no such cohomology sheaf, then the perverse sheaf corresponds to $(0, 0)$.

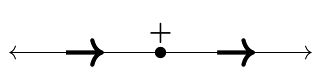
If there is such a cohomology sheaf, it will have $\mathbb{C}$ in the open part. We consider the operator on $\mathbb{C}$ given by traversing a loop around the origin. This operator will correspond to an element of $\mathbb{C}^\times$, which will determine our representation of $\mathbb{Z}$.

9.3 Corresponding Perverse Sheaves

We will use the functor Θ as described in section 8.2 to construct the perverse sheaves that correspond to the irreducible representations calculated in section 9.1. We will also calculate the corresponding $S^{(0)}$ strata and fundamental group representations, as discussed in section 9.2.

We will continue to denote the strata using the notation $iC+D$; for example, the top left is $i(+) + (-)$, and the positive x axis is $i(0) + (+)$. Here, the real strata are always in parentheses, to distinguish from operations.

In order to construct the corresponding perverse sheaves, we need to assign an orientation to each of the strata of A_1. We will do this as follows:



The orientation of the positive reals and the negative reals is represented by an arrow in the direction from 0 to 1 under the map described in section 8.2. The orientation of 0 is the orientation of a point, because it's zero-dimensional. Here, the orientation of the negative reals induces the + orientation on 0, while the orientation of the positive reals induces the opposite. Therefore, the map from the negative reals to zero is $\underline{\delta}_{-0} \otimes \mathrm{Id}$, while the map from the positive reals is $\underline{\delta}_{+0} \otimes$ -Id.

We will begin with the most simple case, 0—1—0. It is clear that $\mathcal{E}_+(A_1) = \mathcal{E}_-(A_1) = 0$, because $+ \circ D = +, - \circ D = -$ for all D. Additionally, we can see that $\mathcal{E}_0(A_1)$ is zero only at $i(0) + (0)$, because $0 \circ D = D$ for all D. Therefore, the corresponding perverse sheaf is a complex of all zeros, except for one sheaf that is $\mathbb{C}$ at the origin, as below.

$$
\begin{array}{ccccccccc}
 & & & & \uparrow & & & & \\
 & & & 0 & 0 & 0 & & & \\
 & & & & | & & & & \\
\cdots\ 0 & \longrightarrow & \longleftarrow & 0 & -\mathbb{C}- & 0 \longrightarrow & \longrightarrow & 0\ \cdots & \\
 & & & & | & & & & \\
 & & & 0 & 0 & 0 & & & \\
 & & & & \downarrow & & & &
\end{array}
$$

The only cohomology sheaf which is nonzero anywhere is in degree 1. There, the kernel is the whole sheaf, and the image of the previous map is zero, so the cohomology is the whole sheaf as well. Since this sheaf is zero on the open part, the above perverse sheaf corresponds to $(0, 1)$.

We now move on to the representation 1—1—1. As always, $\mathcal{E}_+(A_1)$ is zero at any stratum of the form $i(-) + D$. In any other case, $\mathcal{E}_+(A_1)|_{iC+D} = \mathbb{C}$, because $E_+ = E_0 = E_- = \mathbb{C}$. Symmetrically, $\mathcal{E}_-(A_1)|_{iC+D} = \mathbb{C}$ everywhere except at $C = +$, where it is 0. For both of these, it is simple to calculate the generalization maps, which are just Id when from $\mathbb{C}$ to $\mathbb{C}$ and 0 otherwise.

Simple calculation shows that in this case $\mathcal{E}_0(A_1)|_{i(0)+D} = \mathbb{C}$ as always, and $\mathcal{E}_0(A_1)|_{iC+D} = 0$ if $C \neq 0$.

Therefore, we can see that the perverse sheaf here is nonzero in two places. It is therefore just a map $\underline{\delta} : \mathcal{E}_+ \oplus \mathcal{E}_- \to \mathcal{E}_0$. The domain is a sheaf with stalks of $\mathbb{C}$ off of the reals, and stalks of $\mathbb{C}^2$ on the reals. The range is as described above.

It is clear that $\underline{\delta}$ is zero on the stalks at points off of the real line. On the real line, based on the above orientations, it is the map $(c_1, c_2) \mapsto -c_1 + c_2$. This completes the description for the quiver representation 1—1—1, represented by the following diagram.

$$
\begin{array}{ccccccccccc}
 & & & \uparrow & & & & & \uparrow & & \\
 & & \mathbb{C} & \mathbb{C} & \mathbb{C} & & & 0 & 0 & 0 & \\
 & & & | & & & & & | & & \\
\cdots & \longleftarrow & \mathbb{C}^2 - & \mathbb{C}^2 & - \mathbb{C}^2 \longrightarrow & \xrightarrow{d^0} & \longleftarrow & \mathbb{C} - & \mathbb{C} & - \mathbb{C} \longrightarrow & \cdots \\
 & & & | & & & & & | & & \\
 & & \mathbb{C} & \mathbb{C} & \mathbb{C} & & & 0 & 0 & 0 & \\
 & & & \downarrow & & & & & \downarrow & &
\end{array}
$$

By simple calculation, we can see that the only nonzero cohomology sheaf is in degree zero. The image of the previous map is zero, so the cohomology is equal to the kernel of the displayed map. We can see that this is the following sheaf.

$$\begin{array}{ccccccc} & & & \uparrow & & & \\ & & & | & & & \\ & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & \\ & & & | & & & \\ \longleftarrow & \mathbb{C}(1,1) & - & \mathbb{C}(1,1) & - & \mathbb{C}(1,1) & \longrightarrow \\ & & & | & & & \\ & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & \\ & & & | & & & \\ & & & \downarrow & & & \end{array}$$

We wish to calculate the image of 1 in $\mathbb{C}$ when we start (arbitrarily) in the top left and go counterclockwise around the origin.

We know that $\gamma^{-1}_{i(0)+(-),i(+)+(-)}(1) = \{(1,1)\}$ in the above cohomology sheaf, because $\gamma_{i(0)+(-),i(+)+(-)}$ is a restriction of the same map in $\mathcal{E}_+(A_1) \oplus \mathcal{E}_-(A_1)$. By similar reasoning, we know that $\gamma_{i(0)+(-),i(-)+(-)}$ sends $(1,1)$ to 1. Repeatedly using these steps yields that the image of 1 is after going around the origin is 1, so we have the trivial representation of $\mathbb{Z}$. Therefore, this perverse sheaf corresponds to $(\mathbb{C}^\times, 1)$.

We will now consider 1—2—1. The analysis for $\mathcal{E}_+$ and $\mathcal{E}_-$ is the same here as in the previous case, and we obtain the same results. For $\mathcal{E}_0$, it is again only nonzero on the reals. Here, since $0 \circ D = D$, we get $\mathcal{E}_0|_{i(0)+D} = E_D$, and the generalization maps are the same maps as in the original representation.

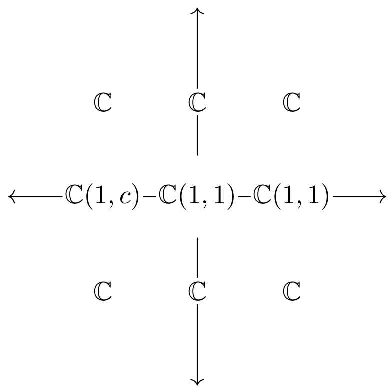
Therefore, we get a similar complex as in the previous case. The dimensions of the stalks are displayed below.

$$\cdots \quad \begin{array}{ccccccc} & & & \uparrow & & & \\ & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & \\ & & & | & & & \\ \longleftarrow & \mathbb{C}^2 & - & \mathbb{C}^2 & - & \mathbb{C}^2 & \longrightarrow \\ & & & | & & & \\ & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & \\ & & & \downarrow & & & \end{array} \quad \xrightarrow{d^0} \quad \begin{array}{ccccccc} & & & \uparrow & & & \\ & 0 & & 0 & & 0 & \\ & & & | & & & \\ \longleftarrow & \mathbb{C} & - & \mathbb{C}^2 & - & \mathbb{C} & \longrightarrow \\ & & & | & & & \\ & 0 & & 0 & & 0 & \\ & & & \downarrow & & & \end{array} \quad \cdots$$

The sheaf in degree 0 has along the x-axis three two-dimensional vector spaces, each equal to $E_+ \oplus E_-$. At $i(0) + (-)$, the sheaf in degree 1 has E_-. We can therefore see that $d^0|_{i(0)+(-)} = -\phi_{+-} \oplus \mathrm{Id}$, with the minus sign coming from orientation. Using the basis from section 9.1, we can see that this is the map $(a_1, a_2) \mapsto -ca_1 + a_2$.

Using similar reasoning, we can see that $d^0|_{i(0)+(0)}$ is the map $(a_1, a_2) \mapsto -(a_1, a_2)$, and $d^0|_{i(0)+(+)}$ is the map $(a_1, a_2) \mapsto -a_1 + a_2$.

Therefore, the cohomology in degree zero is as below.



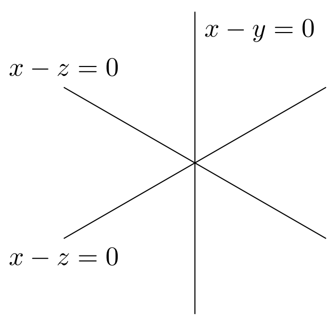
Again, we wish to calculate the image of 1 in $i(+) + (-)$ after going counterclockwise around the origin. By similar reasoning as in the previous case, $1 \in i(+) + (-)$ goes to $(1, c) \in i(0) + (-)$, and $\gamma_{i(0)+(-),i(-)+(-)}$ brings $(1, c)$ to $c \in i(-) + (-)$. The rest of the maps are the same as in 1—1—1, so the image of 1 is c. Therefore, this perverse sheaf coresponds to the pair $(\mathbb{C}^{\times}, c)$.

This completes the description of the perverse sheaf in this case, so we are done.

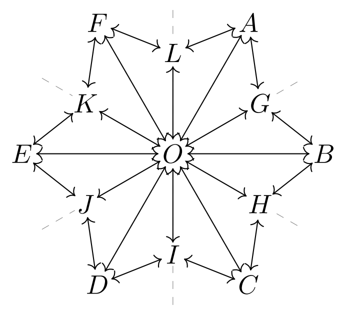
10 Irreducible Representations of A_2

We will now discuss the irreducible representations of A_2 that use vector spaces of dimension of most 2.

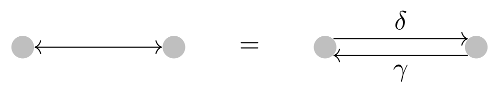
As a subset of the hyperplane $x+y+z = 0$ in $\mathbb{R}^3$, the hyperplane arrangement A_2 looks like the diagram below.



We will represent the poset of A_2 using the following quiver. Throughout this section, we will refer to the faces of A_2 using these labels.



In the above diagram, the double-sided arrows represent an arrow in each direction, as below.



We know by invertibility that the codimension zero faces, A through F, all have the same dimension in the representation. Also by invertibility, we know each codimension one face has the same dimension as its opposite (K/H, G/J, and L/I) . By monotonicity, O has at least as many dimensions as any of these.

We will organize by the dimension of the representation on the open part of A_2. We will only consider dimension zero and one below. This is because if the open part has dimension two, then everything has dimension two, because of our restriction. However, monotonicity implies that such a representation consists entirely of isomorphisms for every map. Therefore, there is a subrepresentation of all $\mathbb{C}$s, so it is not irreducible.

10.1 Open Part Dimension Zero

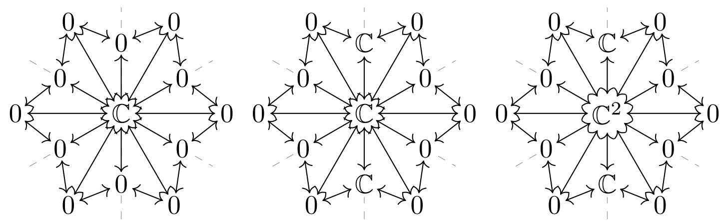
We will first consider the case that the representation is zero on the open part. If no codimension 1 faces have nonzero representations, then the only possible nonzero irreducible representation has $\mathbb{C}$ for the origin, and zero everywhere else.

Now assume that at least one codimension 1 face has a nonzero representation, say L. We wish to show that if an irreducible representation is zero on the open part, then it can only be nonzero along one line.

Assume for contradiction that L and I are nonzero, and G and J are also nonzero. By transitivity, $\phi_{LG} = \phi_{LA}\phi_{AG} = 0$, by the same reasoning $\phi_{LJ} = 0$, $\phi_{IG} = 0$, and $\phi_{IJ} = 0$. By definition, $\phi_{LG} = \delta_{LO}\gamma_{OG}$, so $\mathrm{im}(\delta_{LO}) \subset \ker(\gamma_{OG})$. Again, we can similarly conclude that $\mathrm{im}(\delta_{LO}) \subset \ker(\gamma_{OJ})$, $\mathrm{im}(\delta_{IO}) \subset \ker(\gamma_{OG})$, and $\mathrm{im}(\delta_{IO}) \subset \ker(\gamma_{OJ})$.

Because of the above result, the representation induced by E_L is zero on G, contradicting that the original representation was irreducible.

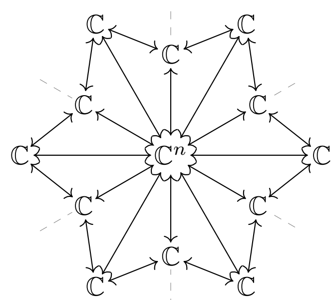
Since the representation can only be nonzero on one hyperplane, this case is reduced to the A_1 case. Thus, we have the following three cases:



Recall that we arbitrarily chose L and I to be nonzero. Therefore, we also have the same three cases on either of the other two hyperplanes. As with A_1, the third of these cases has one isomorphism class per element of $\mathbb{C}^\times$ not equal to 1.

10.2 Open Part Dimension One

We will begin by considering the case that an irreducible representation is $\mathbb{C}$ at every vertex other than the origin, as in the diagram below. We will show that it is isomorphic to the representation with $\mathbb{C}$ everywhere and the identity for every map.



Despite our restriction, n here can be any positive integer; we will show that it must be 1 if the representation is irreducible.

First, we wish to show that on the above conditions, we can choose a basis of each one-dimensional space such that all the maps not involving the origin are the identity map. Choosing a basis for a one-dimensional vector space is just choosing what value to call 1, so we will use this terminology.

Lemma 1. *Let C, C' be two faces with the same dimension representation such that $C' <_1 C$. Then $\gamma_{C'C}$ and $\delta_{CC'}$ are inverses of each other.*

Proof. Monotonicity implies that $\delta_{CC'}$ is a section of $\gamma_{C'C}$. Since the two spaces are the same dimension, $\gamma_{C'C} = \delta_{CC'}^{-1}$. $\square$

Fix some value 1_A in A. We will define $B \ni 1_B = \phi_{AB} 1_A$, and 1_C through 1_L likewise. We will show that the image in O of each of these is the same, which will imply by monotonicity that we can take all the maps around the outside to be the identity.

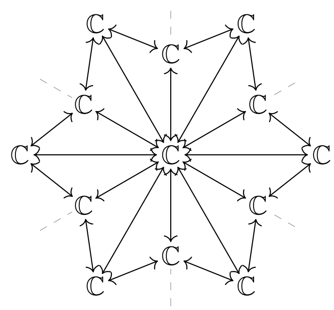
By transitivity, we know that $\phi_{AB} = \phi_{GB}\phi_{AG} = \gamma_{GB}\delta_{AG}$. Therefore,

$$\delta_{BG}1_B = \delta_{BG}\gamma_{GB}\delta_{AG}1_A = \delta_{AG}1_A.$$

This implies that

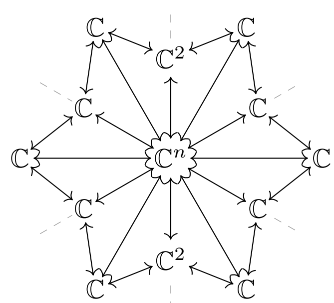
$$\delta_{AO}1_A = \delta_{GO}\delta_{AG}1_A = \delta_{GO}\delta_{BG}1_B = \delta_{BO}1_B.$$

This implies that the image in O of 1_A through 1_L is the same, as desired. If the representation is irreducible, then it must be one-dimensional at the origin, because otherwise there is a proper subrepresentation that is one-dimensional everywhere. As above, this implies that every map is the identity on $\mathbb{C}$.



We will now consider the case that some of the codimension 1 spaces have two-dimensional representations. We will split this case up by how many of the hyperplanes have two-dimensional representations.

If only one hyperplane has a two-dimensional representation, then we have the following diagram.



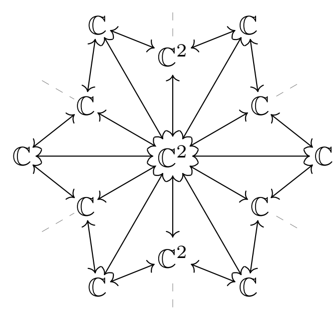
Again, despite our restriction, n here can be anything; we will show that it must be 2.

Note that there are two sets of five one-dimensional spaces that are separated by the central hyperplane. By Lemma 1, in each set, all maps between these five spaces are isomorphisms. Therefore, we can assume all these maps are the identity map on $\mathbb{C}$.

If the images in E_L of 1_A and 1_F are constant multiples of each other, then all of 1_A through 1_F have the same image in E_O up to a constant factor. Since

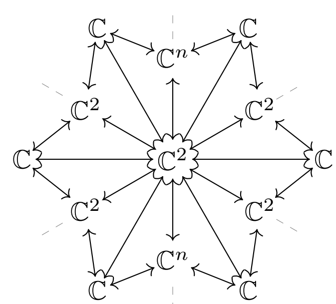
δ_{IO} is injective, this implies that the images in E_I of 1_C and 1_D are consant multiples of each other. Thus, we can see that 1_A induces a subrepresentation isomorphic to the representation with $\mathbb{C}$ at every vertex. Therefore, for the representation to be irreducible, 1_A and 1_F have to have different images in E_L, and by the same reasoning 1_C and 1_D have to have different images in E_I.

This implies that the images of E_L and E_I in E_O are the sum of the images in E_O of 1_A and 1_F. This means that if $n > 2$, then there is a proper subrepresentaion with E_O two-dimensional. Therefore, $n = 2$ is the only option, so we have the following representation.



By identifying each side of the hyperplane as above, and the three $\mathbb{C}^2$s with each other, we can see that this reduces to the A_1 case. Therefore, there is exactly one isomorphism class here per element of $\mathbb{C}^\times$ not equal to 1.

We will now consider the case that more than one of the hyperplanes have two-dimensional representations. This case looks like the diagram below, for $n = 1, 2$.



We wish to show that no such representation exists.

Consider the map ϕ_{GH}. By definition,

$$\phi_{GH} = \gamma_{OH}\delta_{GO}.$$

However, by transitivity,

$$\phi_{GH} = \delta_{BH}\gamma_{GB}.$$

By Lemma 1, the right side of the first equation above is an isomorphism. However, since E_G is two-dimensional and E_B is one-dimensional, γ_{GB} has a nonempty kernel. This is a contradiction, so no such representation exists.

References

[1] Pramod N. Achar. Perverse Sheaves and Applications to Representation Theory. Draft of book.

[2] Robin Hartshorne. *Algebraic Geometry*. Springer, New York, New York, 1977.

[3] M. Kapranov and V. Schechtman. Perverse sheaves over real hyperplane arrangements. *Annals of Mathematics*, 183(2):619–679, 2016.

[4] Steven L. Kleiman. The development of intersection homology theory. 2007.

[5] Claire Voisin. *Hodge Theory And Complex Algebraic Geometry, I.* Cambridge University Press, Cambridge, United Kingdom, 2002.

[6] Charles A. Weibel. *An Introduction to Homological Algebra.* Cambridge University Press, Cambridge, United Kingdom, 1994.