The maximum number of k-gons in the zone of a pseudoline
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Abstract. We show that the sum of $\max\{0, \deg F - 6.5\}$ over faces F in a zone in a pseudoline arrangement of $n+1$ pseudolines is at most n. This bound is tight up to constant terms, and the value 6.5 is optimal. We also prove a similar bound for half-zones.

We introduce the quantities L_I and $\widetilde{L}_I$, defined for each $I \subseteq \mathbb{Z}_{>0}$ as limits involving the proportion of faces in the zone or half-zone of a pseudoline ℓ whose number of sides lie in I. We determine the values of $\widetilde{L}_I$ for all sets I of positive integers. We determine the values of L_I for infinitely many sets I; in particular, we show that for each fixed $k \geqslant 10$, the maximum possible number of k-gons in the zone of a pseudoline in an arrangement of n pseudolines, as a function of n, is $\frac{n}{k-6.5} + o(n)$.

We define a class of closed polygonal chains corresponding to circular permutations of $2n$ vectors, and give a criterion on whether they could be embedded in the 1-skeleton of a zonotopal tiling of a $2n$-gon.

Contents

§1 Introduction

Pseudoline arrangements are generalizations of line arrangements in which lines are replaced by *pseudolines*, which are not necessarily straight. With fewer geometrical constraints, pseudoline arrangements can be treated in a more combinatorial way. A pseudoline arrangement divides the plane into regions called faces. The number of pseudolines supporting a face is called its *degree*. The *complexity* of a finite set of faces, named due to its applications to analysis of certain algorithms in computational geometry, is defined to be the sum of degrees of the faces in the set.

The *zone* of a pseudoline ℓ in an arrangement $\mathcal{A}$, first defined in [CGL85], is the set of faces of $\mathcal{A}$ having a side on ℓ. The so-called Zone Theorem, proved in its strongest form by Pinchasi [Pin11] based on work in [CGL85] and [BEPY91], states that the maximum possible complexity of a zone in an affine arrangement of $n+1$ pseudolines is $19n/2-3$.

Our paper consists of three main components. The first component (Sections 3 and 4) is on inequalities related to the Zone Theorem. The Zone Theorem concerns the sum $\sum_F \deg F$ over faces F in a zone. We consider sums of the form $\sum_F \max\{0, \deg F - c\}$ for varying c. Our main result (Theorem 4.1) in this part is a strong upper bound on this sum for $c \geqslant 6.5$. Specifically, we show that in an affine arrangement of $n+1$ pseudolines,

$$\sum_F \max\{0, \deg F - 6.5\} \leqslant n,$$

where the sum is over faces in a zone. This inequality does not hold if n is replaced by $(1-\varepsilon)n$ for any $\varepsilon > 0$, and the constant 6.5 in $\deg F - 6.5$ is also the smallest possible constant. We also prove a related inequality, Theorem 3.1, in which zones are replaced by *half-zones*. A half-zone is defined to be the set of faces supported by one side of a pseudoline. (For projective arrangements, this is defined as the set of faces in the zone of a line ℓ in one of the two regions bounded by ℓ and another given line.)

The second component of our paper, Section 5, concerns the number of faces with various degrees in a zone or half-zone. In his monograph *Arrangements and Spreads* [Grü72], Grünbaum asks about the maximum number of k-sided faces in an arrangement of n pseudolines. This problem is deeply linked to incidence geometry and particularly the Szemerédi-Trotter theorem. We consider a question similar to Grünbaum's but for zones. Specifically, we ask for the maximum number of faces in a zone whose degree lie in I, where I is a given set of positive integers. We prove that for all sets I, this maximum number is asymptotically linear in n. Using in part the inequalities we proved earlier, we obtain in our second main result (Theorem 5.6) the leading coefficient L_I for various sets I. (For a formal definition of L_I, see Definition 5.1.) In particular, using Theorem 4.1, we show that for fixed $k \geqslant 10$, $L_k = \frac{1}{k-6.5}$, i.e. the maximum number of k-sided cells in a zone in an arrangement of n pseudolines is $\frac{n}{k-6.5} + o(n)$. We also reduce the problem of finding L_I for all sets I of positive integers to eight remaining unsolved sets.

We also consider quantities $\widetilde{L}_I$, defined in a similar way as L_I but for half-zones. The half-zone problem is substantially easier, and we obtain in another main result (Theorem 5.5) the value of $\widetilde{L}_I$ (see Definition 5.2) for all sets I of positive integers.

In the last component of our paper, Section 7, we study the local structure of pseudoline arrangements. We introduce the concept of closed polygonal chains corresponding to circular permutations of $2n$ vectors, and our main result in this part is a criterion (Theorem 7.8) on whether a closed polygonal chain can be realized as a curve in the 1-skeleton of a zonotopal tiling, which is the dual of a pseudoline arrangement.

Our paper is structured as follows. Section 2 is a preliminaries section in which we formally define various notions related to pseudoline arrangements. In Section 3, we prove a strong inequality on the degrees of faces in half-zones. In Section 4, we prove a strong inequality on the degrees of faces in zones. In Section 5, we introduce the quantities L_I and $\widetilde{L}_I$, then determine the exact values of $\widetilde{L}_I$ for all sets $I \subseteq \mathbb{Z}_{>0}$ and the values of L_I for various sets I, including all sets $I \subseteq \mathbb{Z}_{\geqslant 10}$. We also give lower and upper bounds on L_I for other sets I. In Section 6, we present explicit configurations attaining the bounds we give in previous sections. In Section 7, we define proper closed polygonal chains, explore various properties of them, and give a criterion on when a proper closed polygonal chain is extendable to a zonotopal tiling. Finally, in Section 8, we give various comments, including a discussion of the differences between affine and projective arrangements as it applies to our results.

§2 Preliminaries

We review standard terminology and results. In the projective plane $\mathbb{P}^2$, a *pseudoline* is a simple closed curve whose removal does not disconnect $\mathbb{P}^2$. A *projective pseudoline arrangement* is a family of pseudolines in $\mathbb{P}^2$ in which each pair of pseudolines intersect exactly once [FG18].

In the affine plane $\mathbb{R}^2$, a *pseudoline* is an infinite curve that can be sent to a line under a homeomorphism of $\mathbb{R}^2$,and an *affine pseudoline arrangement* is a family of pseudolines in $\mathbb{R}^2$ in which each pair of pseudolines intersect exactly once and cross at their intersection.

Projective pseudoline arrangements and affine pseudoline arrangements are distinct but closely related objects. In many arguments it will be useful to switch between the two types of pseudoline arrangements, and we will often do so. One way to relate projective and affine arrangements is to pair each affine arrangement $\mathcal{A}$ with the projective arrangement $\mathcal{A}^+$ created by reanalyzing the underlying plane as a projective plane and adding the line at infinity ℓ_∞. An explicit example is given in Figure 1. Further discussion of the difference between projective and affine pseudoline arrangements is given in Section 8.

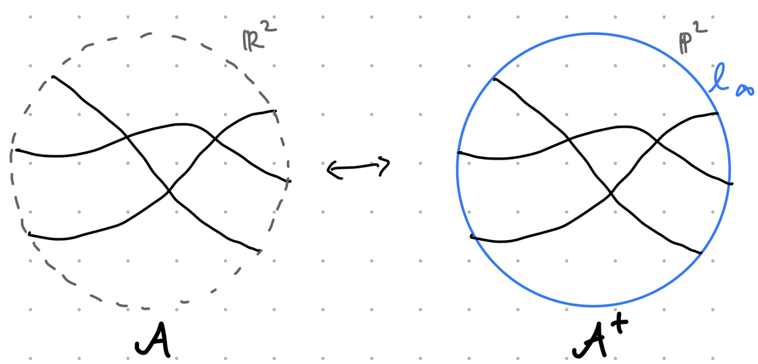

Figure 1: A relation between projective and affine pseudoline arrangements.

Unless specified otherwise, the following definitions apply to both projective and affine pseudoline arrangements. Two pseudoline arrangements are said to be *isomorphic* if there is a homeomorphism of the underlying plane which sends one pseudoline arrangement to the other. A pseudoline arrangement $\mathcal{A}$ is said to be *simple* if no three pseudolines pass through a single point, and *stretchable* if $\mathcal{A}$ is isomorphic to a line arrangement.

Pseudoline arrangements divide the underlying plane into *faces*. Given a pseudoline ℓ

in a pseudoline arrangement $\mathcal{A}$, the collection of faces which is supported by ℓ is called the *zone* of ℓ.

We now define the notion of *half-zones*. In affine arrangements it is possible to distinguish between two sides of a pseudoline; a half-zone is simply a collection of faces supported by one side of a pseudoline ℓ. In projective arrangements, defining half-zones requires an auxiliary pseudoline.

Definition 2.1. Let $\mathcal{A}$ be a projective pseudoline arrangement. Let ℓ and q be two distinct pseudolines of $\mathcal{A}$. The pseudolines ℓ and q divide the projective plane into two regions. A *half-zone* of ℓ with respect to q is the collection of faces in the zone of ℓ lying within the same region determined by ℓ and q.

The number of sides of a face F is often referred to as the *degree* of F. The sum of the degrees of a collection of faces is called the *complexity* of that collection. The Zone Theorem gives a tight upper bound on the complexity of a zone.

Theorem 2.2 (Zone Theorem; [BEPY91], [Pin11])**.** *The complexity of the zone of a pseudoline ℓ in an affine arrangement of $n+1$ pseudolines is at most $\frac{19n}{2}-3$, and this bound is tight.*

A strong upper bound on the complexity of a half-zone is also known, due to a theorem by Chazelle, Guibas, and Lee [CGL85] which we will refer to as the "Half-Zone Theorem."

Theorem 2.3 (Half-Zone Theorem; [CGL85])**.** *The maximum complexity of a half-zone in an affine arrangement of $n+1$ pseudolines is $5n+O(1)$.*

Both the Zone and the Half-Zone theorems are also known to hold for projective arrangements, up to constant terms. This can be shown by an argument similar to Corollary 3.3.

§3 Bounds for half-zones

In this section we derive an inequality related to a generalization of the notion of complexity of half-zones. Specifically, we consider the quantities $\widetilde{Q}(h,c) := \sum_{F\in h} \max\{0, \deg F - c\}$, where h is a half-zone in an affine arrangement and c is a constant. These quantities are generalizations of the complexity of half-zones. Theorem 3.1, the main result of this section, is an inequality giving an optimal bound on these quantities for $c \geqslant 5$, and is a counterpart of the Half-Zone Theorem, which gives an optimal bound for $c \leqslant 3$.

Theorem 3.1. *Let h be a half-zone in an affine arrangement of n pseudolines. Then*

$$\sum_{F\in h} \max\{0, \deg F - 5\} \leqslant n - 1,$$

where $\deg F$ *is the number of sides of F. This bound is tight up to constant terms.*

Remark 3.2. As all but two regions of a half-zone must have degree at least 3, the Half-Zone Theorem shows that

$$\max_{\mathcal{A},h} \widetilde{Q}(h,c) = (5-c)n + O(1)$$

holds for $c \leqslant 3$, where the maximum is over all half-zones h in affine arrangements of n pseudolines. In contrast, Theorem 3.1 shows that

$$\max_{\mathcal{A},h} \widetilde{Q}(h,c) = n + O(1)$$

for $c \geqslant 5$. The fact that this bound is optimal for any $c \geqslant 5$ comes from the arrangement where the half-zone h contains an n-gon, in which case it is clear that $\widetilde{Q}(h,c) \geqslant n - c$. Furthermore, Configuration 6.6 gives $\widetilde{Q}(h,c) \geqslant (7-c)n/2 + O(1)$ for $c \leqslant 7$, which shows that 5 cannot be replaced by a smaller constant.

Theorem 3.1 also applies to projective arrangements.

Corollary 3.3. Let h be a half-zone in a projective arrangement of n pseudolines. Then

$$\sum_{F\in h} \max\{0, \deg F - 5\} \leqslant n + O(1),$$

where $\deg F$ is the number of sides of F. This bound is tight up to constant terms.

Proof. Let $\mathcal{A}^+$ be a projective arrangement, and let h be the half-zone of ℓ with respect to q, where ℓ and q are two pseudolines in $\mathcal{A}$. Take a transformation sending q to the line at infinity ℓ_∞.

Consider the affine pseudoline arrangement $\mathcal{A}'$ formed as follows. First we remove $q = \ell_\infty$ from $\mathcal{A}^+$ and view the resulting arrangement as an affine arrangement $\mathcal{A}$. Draw a large disk which includes all intersections in $\mathcal{A}$, and then add two pseudolines q_1 and q_2 to $\mathcal{A}$, each covering around half of the large circle. (See Figure 2.) The resulting arrangement $\mathcal{A}'$ will contain four more faces in the zone of ℓ compared to $\mathcal{A}$, and at most 2 faces in the zone of ℓ will have their degrees increased by 1 or 2.

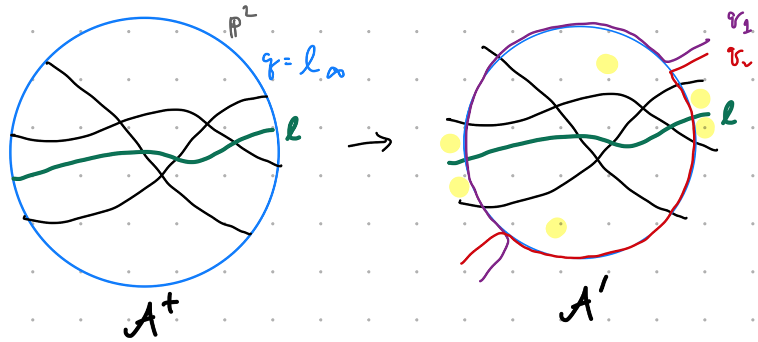

Figure 2: Creating an affine arrangement $\mathcal{A}'$ from a projective arrangement $\mathcal{A}$.

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Let h' be the half-zone of ℓ above ℓ in $\mathcal{A}'$. It follows that

$$\begin{aligned}\sum_{F \in h} \max\{0, \deg F - 5\} &\leqslant \sum_{F \in h'} \max\{0, \deg F - 5\} + O(1) \\ &\leqslant n + O(1).\end{aligned}$$

The arrangement in which h contains an n-gon gives $\sum_{F \in h} \max\{0, \deg F - 5\} = n - 5$, so this bound is tight up to constant terms. $\square$

The rest of this section is dedicated to proving Theorem 3.1. Throughout the rest of this section, let $\mathcal{A}$ be an affine arrangement of n pseudolines, one of which is a horizontal line ℓ. We can make this assumption because there is always a homeomorphism of $\mathbb{R}^2$ that sends a fixed pseudoline in an affine arrangement to a fixed straight line. We also assume that $\mathcal{A}$ is simple; this assumption is allowed because making a simple arrangement nonsimple always increases the degrees of faces. We will also assume that the half-zone h in question is the half-zone of ℓ above ℓ.

With this setting in mind, we now introduce the notions of left, right, roof, and normal edges. These notions are similar to the notions of left, right, top, and bottom edges in [Pin11], which were used to give a concise proof of the Half-Zone Theorem. Figure 3 shows examples of left, right, roof, and normal edges.

Definition 3.4. Let ℓ be a horizontal line in a simple affine pseudoline arrangement $\mathcal{A}$. For each face F, possibly unbounded, in the zone of ℓ, let $i(F)$ be the set of intersections of the pseudolines supporting F (except ℓ) with ℓ. We say an edge e of F is

- *left* if the pseudoline containing e intersects ℓ to the left of F's edge on e,
- *right* if the pseudoline containing e intersects ℓ to the right of F's edge on e,
- *roof* if the pseudoline containing e intersects ℓ at either the leftmost or rightmost points in $i(F)$, and
- *normal* if e does not have an endpoint on ℓ and e is not a roof edge.

The above definition covers unbounded edges. If F is unbounded then the roof edges of F are the two unbounded edges of F.

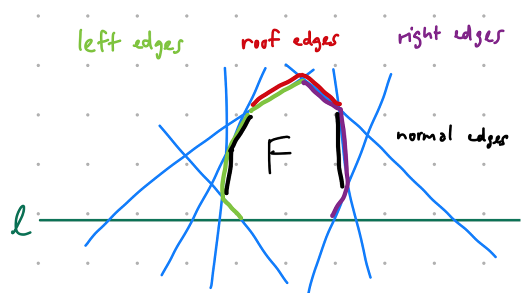

Figure 3: Examples of left, right, roof, and normal edges

The previous definition allows us to state succinctly the following lemma, which is our key observation. This lemma is in much of the same spirit as Pinchasi's [Pin11] phrasing of the main observation Chazelle, Guibas, and Lee [CGL85] used to prove the Half-Zone theorem.

Lemma 3.5. Each pseudoline $q \neq \ell$ contains at most one normal edge, in total, across all faces in a given half-zone of ℓ.

Proof. Suppose a pseudoline $q \neq \ell$ contains a normal edge of a face F in the half-zone h of ℓ above ℓ. Pick such a normal edge e that lies closest to the intersection $\ell \cap q$ on q. We show that e must in fact be the only normal edge on q above ℓ.

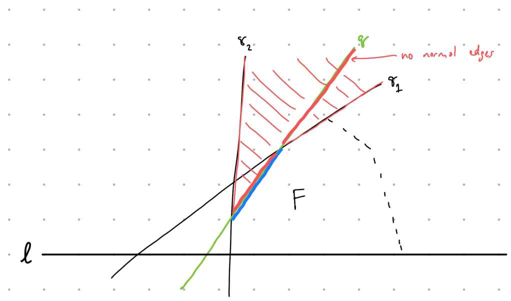

Figure 4: If a pseudoline $q \neq \ell$ contains normal edge (in blue) above ℓ, then it cannot contain another normal edge above ℓ.

Consider the two edges of F adjacent to e. As e is normal, the pseudolines q_1 and q_2 containing these two edges intersect ℓ at two points on ℓ which lie on the same side of the edge of F on ℓ. It follows that the section of q which is further from ℓ than e lies in the region bounded by q_1, q_2 which does not intersect ℓ. (See Figure 4.) Faces in this region cannot lie in the zone of ℓ and therefore cannot contain a normal edge. Consequently, q cannot contain a normal edge above ℓ other than e. □

With Lemma 3.5, we are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Each pseudoline $q \neq \ell$ can contain at most one normal edge in h, the half-zone of ℓ above ℓ. As there are $n-1$ pseudolines other than ℓ, there can be at most $n-1$ normal edges in h.

Each face F in h can contain at most five non-normal edges: three edges with at least one endpoint on ℓ, and two roof edges. Therefore, each face F must contain at least $\max\{0, \deg F - 5\}$ normal edges. Summing over all faces F gives the inequality

$$\sum_{F \in h} \max\{0, \deg F - 5\} \leqslant n - 1.$$

The affine arrangement in which all n pseudolines form an n-gon above ℓ shows that the sum given above can be at least $n-5$. Therefore the above inequality is tight up to constant terms. □

§4 Bounds for zones

In this section we derive an inequality comparable to the Zone Theorem. We define quantities $Q(\ell, c)$ for zones analogous to the quantities $\widetilde{Q}(h, c)$ for half-zones. Whereas the Zone Theorem gives an upper bound of $Q(\ell, c)$ for $c \leqslant 3$, our main result, Theorem 4.1, gives an upper bound of $Q(\ell, c)$ for $c \geqslant 6.5$.

Theorem 4.1. *Let ℓ be a pseudoline in an affine arrangement of n pseudolines. Then*

$$\sum_{F \in \mathrm{zone}(\ell)} \max\{0, \deg F - 6.5\} \leqslant n - 1,$$

where $\deg F$ is the number of sides of F. This bound is tight up to constant terms.

Remark 4.2. Theorem 4.1 is to the Zone Theorem what Theorem 3.1 is to the Half-Zone Theorem. Formally define the quantities $Q(\ell, c)$ by

$$Q(\ell, c) = \sum_{F \in \mathrm{zone}(\ell)} \max\{0, \deg F - c\},$$

where ℓ is a pseudoline in an affine arrangement and c is a constant. On one hand, for $c \leqslant 3$, the Zone Theorem gives the tight bound

$$\max_{\mathcal{A}, \ell} Q(\ell, c) = (9.5 - 2c)n + O(1),$$

where the maximum ranges over all zones in affine arrangements of n pseudolines. On the other hand, Theorem 4.1 gives the optimal bound

$$\max_{\mathcal{A}, \ell} Q(\ell, c) = n + O(1)$$

for $c \geqslant 6.5$. This bound does not hold for $c < 6.5$, as Configuration 6.13 gives $Q(\ell, c) \geqslant (10 - c)n/3.5 + O(1)$.

An argument identical to Corollary 3.3 shows that Theorem 4.1 also applies to projective arrangements.

Corollary 4.3. Let ℓ be a pseudoline in a projective arrangement of n pseudolines. Then

$$\sum_{F \in \mathrm{zone}(\ell)} \max\{0, \deg F - 6.5\} \leqslant n + O(1),$$

where $\deg F$ is the number of sides of F. This bound is tight up to constant terms.

The rest of this section is dedicated to proving Theorem 4.1. We make the same assumption as in the previous section: we assume that $\mathcal{A}$ is a simple affine arrangement of n pseudolines, one of which is a horizontal line ℓ. We again use the notions of left, right, roof, and normal edges, as defined in Definition 3.4. Our strategy is as follows.

We first introduce the notions of bases and eaves, which will simplify arguments involving relations between faces. We start by defining bases and eaves, which are segments on ℓ associated to faces. We then define a partial order on the set of segments of ℓ.

We construct auxiliary graphs $G, G_\uparrow, G_\downarrow$, corresponding to various pairs of normal edges lying on the same pseudoline. We then construct another auxiliary graph H based on G. By considering possible orders of segments on ℓ and faces in the zone of ℓ, we show that H has no cycles. This gives a strong upper bound on the average degree of G which in turn implies Theorem 4.1.

Definition 4.4. For each face F in the zone of ℓ, define the *base* $\overline{F}$ to be the edge, either bounded or unbounded, of F on ℓ.

Definition 4.5. For each face F in the zone of ℓ, define the *eave* $\widehat{F}$ of F to be the segment on ℓ joining the leftmost and rightmost points of $i(F)$. (Recall that $i(F)$ is the set of intersections of ℓ and the pseudolines supporting F.) Given faces F_1 and F_2 in the zone of ℓ, we say F_1 *dominates* F_2 if $\widehat{F_1} \supsetneq \widehat{F_2}$.

We also define a partial order on the closed intervals of ℓ, including points, which will be useful in arguments involving the relative order of points or pseudolines.

Definition 4.6. Identify ℓ with the real line, with $-\infty$ to the left and $+\infty$ to the right. We give a partial order on the closed intervals of ℓ as follows.

Let $a = [a_1, a_2]$ and $b = [b_1, b_2]$ be closed intervals of ℓ, possibly single points (when $a_1 = a_2$ or $b_1 = b_2$). We say $a \leqslant b$ if $a_2 \leqslant b_1$ as real numbers, and we say $a < b$ if $a \leqslant b$ and $a \neq b$.

When restricted to the set of bases of faces in a half-zone of ℓ, the partial ordering defined above becomes a total order. From now on, if faces F_1 and F_2 are on the same side of ℓ, we may use the interval notation such as $[F_1, F_2)$ to mean the set of faces F such that F lies on the same side of ℓ as F_1 and F_2, and $\overline{F_1} \leqslant \overline{F} < \overline{F_2}$.

Lemma 3.5 implies that a pseudoline $q \neq \ell$ can contain at most two normal edges: one in each half-zone of ℓ. In the next lemma, we show that the two normal edges cannot be both left or both right edges.

Lemma 4.7. If a pseudoline $q \neq \ell$ contains two normal edges, then one of them is a left edge, and the other is a right edge.

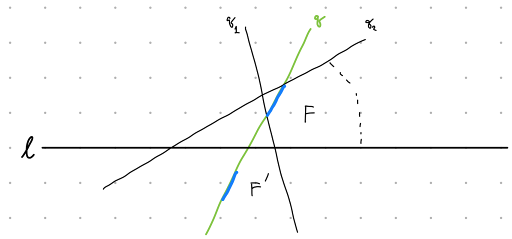

Figure 5: If a pseudoline $q \neq \ell$ contains a left normal edge of F and a left normal edge of F' then $\overline{F'} < \overline{F}$.

Proof. By symmetry, it suffices to show that a pseudoline $q \neq \ell$ cannot contain two left normal edges. Suppose for the sake of contradiction that a pseudoline $q \neq \ell$ contains left normal edges e_1 and e_2 of distinct faces F and F'. We show that $\overline{F}' < \overline{F}$.

See Figure 5. Let $P = q \cap \ell$. Consider the two edges of F adjacent to e_1. Let the pseudolines q_1 and q_2 containing them intersect ℓ at points P_1, P_2. One of these points, say P_1, must be closer to $\overline{F}$ than P. As F' has a left normal edge on q, it follows that F' is in the region below ℓ bounded by q and q_1, so $\overline{F}' < \overline{F}$.

Finally, we observe that we have not made any special distinction between F and F', so the same argument shows that $\overline{F} < \overline{F}'$ as well, and this is our desired contradiction. $\square$

We introduce notions of up-relatedness and down-relatedness for pairs of faces containing normal edges on the same pseudoline. We then introduce graphs $G_{\uparrow}, G_{\downarrow}$, and G which captures relations between faces. Our goal is to show that these graphs have low average degree. This would imply that few pseudolines can contain two normal edges, which in turn implies that the total number of normal edges is low. A strong enough bound on the average degree of G will imply Theorem 4.1.

Definition 4.8. We say faces A above ℓ and B below ℓ are *up-related* if there is a pseudoline $q \neq \ell$ that contains a left normal edge of A and a right normal edge of B. We say faces A above ℓ and B under ℓ are *down-related* if there is a pseudoline $q \neq \ell$ that contains a right normal edge of A and a left normal edge of B. We say two faces A and B are *related* if they are either up-related or down-related.

Define a graph $G_\uparrow$ with vertices v_F representing faces F with *at least two normal edges*, and edges joining pairs of up-related faces. Analogously define $G_\downarrow$ for down-related faces and G for related faces (either up-related or down-related). The edge set of G is the union of the edge sets of $G_\uparrow$ and $G_\downarrow$.

Lemma 4.9. Neither $G_\uparrow$ nor $G_\downarrow$ contains a cycle.

Proof. By symmetry, it suffices to prove that $G_\uparrow$ does not contain a cycle. Suppose for the sake of contradiction that $G_\uparrow$ contains a cycle C. Let F be the face above ℓ in this cycle whose base $\overline{F}$ lies rightmost on ℓ. Suppose the cycle C consists of faces $F - F_1 - F_2 - \cdots - F_r - F$ in this order. As each edge of $G_\uparrow$ connects two faces on opposite sides of ℓ, the graph $G_\uparrow$ is bipartite. Cycles in $G_\uparrow$ must therefore contain at least 4 edges, so $r \geqslant 3$.

Without loss of generality, assume that $\overline{F_1} < \overline{F_r}$. Let q be the pseudoline containing the normal edges of F and F_1. As $\overline{F_1} < \overline{F_r}$, it is clear that F_r and F must lie on the same side of q. See Figure 6.

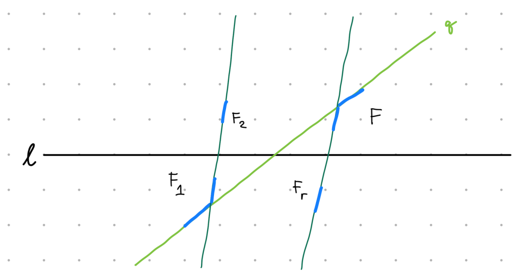

Figure 6: F_r and F must lie on the same side of q.

Consider $\overline{F_2}$. As $\overline{F}$ lies rightmost, we have $\overline{F_2} < \overline{F}$. In fact, F_2 and F_1 must lie on the same side of q (with F and F_r on the opposite side).

The two endpoints of the path $F_2 - \cdots - F_r$ in C are on opposite sides of q, so there must be a pair of consecutive faces $F_s - F_{s+1}$, distinct from F and F_1, such that F_s lies on the F_1 (and F_2)-side of q, but F_{s+1} lies on the F (and F_r)-side of q. Let q_s be the pseudoline containing normal edges of both F_s and F_{s+1}.

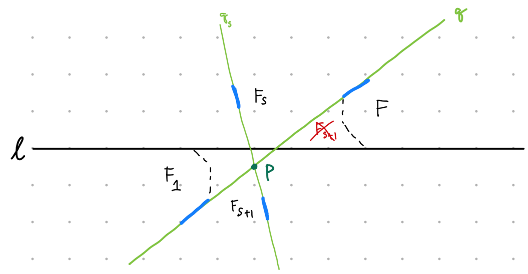

Figure 7: Depending on the location of P (above or below ℓ), either F_s or F_{s+1} cannot be in the zone of ℓ.

If F_{s+1} lies above ℓ then as $\overline{F_{s+1}} < \overline{F}$, the face F_{s+1} must lie in the region bounded by ℓ, q and F. It follows that q_s necessarily intersects the interior of F, which is impossible. Therefore F_s lies above ℓ and F_{s+1} lies below ℓ. Now suppose q_s intersects q at P, as in Figure 7. If P lies above ℓ then F_s cannot be in the zone of ℓ, and if P lies below ℓ then F_{s+1} cannot be in the zone of ℓ; in either case, there is a contradiction. Therefore $G_\uparrow$ does not contain a cycle. $\square$

We establish some basic properties of bases and eaves, which will be helpful in arguments related to the order of segments on ℓ.

Lemma 4.10. If faces F_1 and F_2 in the zone of ℓ are both above ℓ or both below ℓ and $\overline{F_1} \subseteq \widehat{F_2}$ then $\widehat{F_1} \subsetneq \widehat{F_2}$.

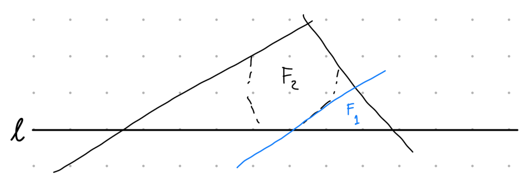

Figure 8: If $\overline{F_1} \subseteq \widehat{F_2}$ then F_1 is located in one of the regions bounded by F_2 and the lines containing its roof, and hence $\widehat{F_1} \subsetneq \widehat{F_2}$.

Proof. Without loss of generality, suppose that $\overline{F_1} > \overline{F_2}$. Then F_1 must be in the region bounded by ℓ, F_2, and the right roof of F_2. The left roof of F_1 cannot intersect F_2, so it must intersect ℓ to the right of $\overline{F_2}$. The right roof of F_1 also cannot intersect F_2, so it cannot intersect ℓ to the right of the right roof of F_2; it follows that $\widehat{F_1} \subsetneq \widehat{F_2}$. $\square$

Lemma 4.11. If faces A and B are related then $\overline{A} \subsetneq \widehat{B}$ and $\overline{B} \subsetneq \widehat{A}$.

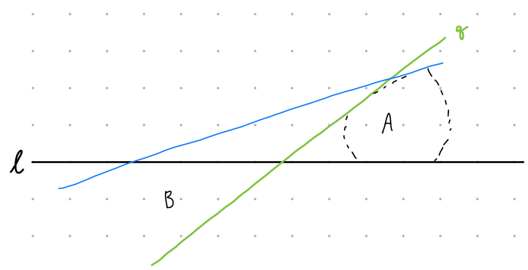

Figure 9: If faces A and B are related then $\overline{B} \subsetneq \widehat{A}$.

Proof. It suffices to show $\overline{B} \subsetneq \widehat{A}$. Without loss of generality, assume A lies above ℓ and B lies below ℓ. Also assume that A and B are up-related by a pseudoline q. The left roof of A cannot intersect B, so it must intersect ℓ to the left of $\overline{B}$, which implies $\overline{B} \subsetneq \widehat{A}$. □

Lemma 4.12. Let A, A_1, A_2 be faces in the zone of ℓ above ℓ, and let B, B_1, B_2 be faces in the zone of ℓ below ℓ. Assume that $\overline{A_1} < \overline{A_2}$ and $\overline{B_1} < \overline{B_2}$.

a) If A is up-related to B_1 and B_2 then $\widehat{B} \subsetneq \widehat{B_1}$ for all $B \in (B_1, B_2]$.

b) If A is down-related to B_1 and B_2 then $\widehat{B} \subsetneq \widehat{B_2}$ for all $B \in [B_1, B_2)$.

c) If B is up-related to A_1 and A_2 then $\widehat{A} \subsetneq \widehat{A_2}$ for all $A \in [A_1, A_2)$.

d) If B is down-related to A_1 and A_2 then $\widehat{A} \subsetneq \widehat{A_1}$ for all $A \in (A_1, A_2]$.

Proof. It suffices to prove a); all other parts are equivalent to a) by horizontal and/or vertical flipping.

Let q be the line containing a left normal edge of A and a right normal edge e_1 of B_1, and let $P = q \cap \ell$. Let q' be the right roof of B_1, and let q' intersect ℓ at P'. From Lemma 4.11, we have $\overline{A} \subsetneq \widehat{B_1}$. As P' is the rightmost point of $\widehat{B_1}$, it follows that $\overline{A} < P'$.

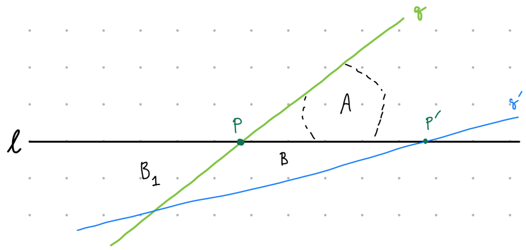

Figure 10: If faces A and B_1 are up-related then $\overline{A}$ lies to the left of the intersection of the right roof of B_1 and ℓ.

As A and B_2 are up-related, $\overline{B} \leqslant \overline{B_2} < \overline{A} < P'$. Therefore $\overline{B} \subseteq \widehat{B_1}$, so by Lemma 4.10 it follows that $\widehat{B} \subsetneq \widehat{B_1}$. $\square$

We now define a multigraph H whose vertices are the faces in the half-zone of ℓ below ℓ with at least two normal edges. The edges of H are colored either blue or green, and are defined according to the following procedure:

- For each face A above ℓ that is up-related to at least two faces $B_1, \ldots, B_r$ below ℓ such that $\overline{B_1} < \cdots < \overline{B_r}$, we add blue edges connecting B_i and B_{i+1} for each $i = 1, \ldots, r-1$.
- For each face A above ℓ that is down-related to at least two faces $B_1, \ldots, B_r$ below ℓ such that $\overline{B_1} < \cdots < \overline{B_r}$, we add green edges connecting B_i and B_{i+1} for each $i = 1, \ldots, r-1$.

Our end goal is to show that H has no cycles, which would give an upper bound on the number of edges in G strong enough to imply Theorem 4.1. Our strategy will be to eliminate various configurations of edges of H and show that a cycle in H must contain a configuration that has been eliminated.

Lemma 4.13. The graph H is in fact a simple graph.

Proof. By its definition, H has no loops. Suppose H has a double edge between B_1 and B_2. The two edges cannot be of the same color because that would lead to a cycle in either $G_\uparrow$ or $G_\downarrow$. The two edges cannot have different colors either because Lemma 4.12 would imply $\widehat{B_1} \subsetneq \widehat{B_2}$ and $\widehat{B_1} \supsetneq \widehat{B_2}$. $\square$

Lemma 4.14. Let B_1, B_2, B_3, B_4 be faces in the zone of ℓ below ℓ. The following two configurations are impossible:

a) $\overline{B_1} < \overline{B_2} \leqslant \overline{B_3} < \overline{B_4}$, B_1B_3 is a blue edge in H, and B_2B_4 is a green edge in H.

b) $\overline{B_1} \leqslant \overline{B_2} < \overline{B_3} \leqslant \overline{B_4}$, B_1B_3 is a green edge in H, and B_2B_4 is a blue edge in H.

Figure 11: The two configurations shown in this figure are impossible.

Proof. a) Let A_{13} be a face above ℓ up-related to B_1 and B_3, and let A_{24} be a face above ℓ down-related to B_2 and B_4. As A_{24} and B_2 are down-related, $\overline{A_{24}} < \overline{B_2}$ and $\overline{A_{24}} \subseteq \widehat{B_2}$. As A_{13} and B_3 are up-related, $B_{13} \subseteq B_3$. Lemma 4.12 gives $\widehat{B_1} \supset \widehat{B_2}$. It follows that $\overline{B_1} < \widehat{B_2}$, which implies $\overline{B_1} < \overline{A_{24}} < \overline{B_2} \leqslant \overline{B_3} < \overline{A_{13}}$.

As B_1 is up-related to A_{13}, $\overline{B_1} \subseteq \overline{A_{13}}$. Now $\overline{A_{24}}$ lies between $\overline{B_1}$ and $\overline{A_{13}}$, which are both segments of $\widehat{A_{13}}$. Therefore, $\overline{A_{24}} \subseteq \widehat{A_{13}}$. Lemma 4.10 then implies $\widehat{A_{24}} \subsetneq \widehat{A_{13}}$. However, a similar argument implies $\widehat{A_{13}} \subsetneq \widehat{A_{24}}$, which is a contradiction.

b) As B_2B_4 is blue, Lemma 4.12 implies $\widehat{B_3} \subsetneq \widehat{B_2}$. As B_1B_3 is green, Lemma 4.12 implies $\widehat{B_2} \subsetneq \widehat{B_3}$. These two relations clearly contradict.

$\square$

Lemma 4.14 implies the following corollary.

Corollary 4.15. a) If B_1B_2 is a blue edge in H then for each green edge e in H, either none of its endpoints lie in the interval $(B_1, B_2]$, or both of its endpoints lie in the interval $(B_1, B_2]$.

b) If B_1B_2 is a green edge in H then for each blue edge e in H, either none of its endpoints lie in the interval $[B_1, B_2)$, or both of its endpoints lie in the interval $[B_1, B_2)$.

Proof. The two parts are equivalent under a reflection over ℓ so it suffices to prove a). Suppose there is a green edge B_3B_4 (with $\overline{B_3} < \overline{B_4}$) with exactly one endpoint in $(B_1, B_2]$. Then either $\overline{B_3} \leqslant \overline{B_1} < \overline{B_4} \leqslant \overline{B_2}$ or $\overline{B_1} < \overline{B_3} \leqslant \overline{B_2} < \overline{B_4}$; the former violates Lemma 4.14(b) and the latter violates Lemma 4.14(a). $\square$

We show that one last configuration is forbidden in cycles.

Lemma 4.16. Let B_1, B_2, B_3, B_4 be faces in the zone of ℓ below ℓ. If $\overline{B_1} \leqslant \overline{B_2} < \overline{B_3} \leqslant \overline{B_4}$, and B_1B_4 and B_2B_3 are differently colored edges in H, then the edges B_1B_4 and B_2B_3 cannot be part of the same cycle in H.

Proof. Without loss of generality assume B_1B_4 is blue and B_2B_3 is green. (The case where B_1B_4 is green and B_2B_3 is blue is equivalent to this case under a reflection over a line perpendicular to ℓ.) From Lemma 4.14(b) we know that $B_1 \neq B_2$.

Recall that our partial ordering of bases gives a total ordering of faces below ℓ. Consider the union U of the half-open intervals $[L, R)$ where L, R are faces below ℓ such that $\overline{B_1} < \overline{L} < \overline{R} \leqslant \overline{B_4}$ and LR is a green edge of H. The set U can be written as the union $\bigcup_i [L_i, R_i)$ of disjoint half-open intervals such that L_i and R_i are distinct. (See the top part of Figure 12.) As $[B_2, B_3) \subseteq U$, there is an interval $[L_j, R_j)$ which contains $[B_2, B_3)$.

Call an edge of H *splitting* if exactly one endpoint of the edge is in the interval $[L_j, R_j)$. We claim that all splitting edges have R_j as an endpoint.

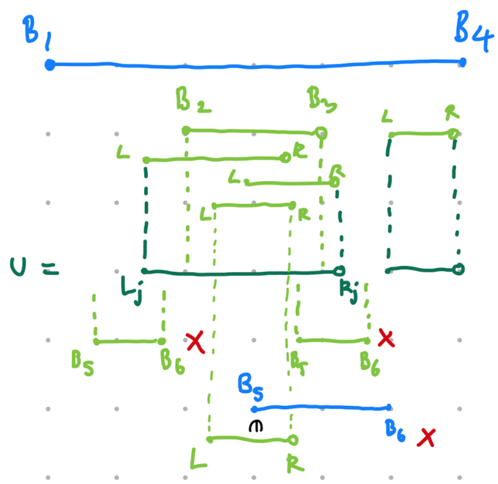

Figure 12: The top half of this figure shows the union U of the half-open intervals $[L, R)$ where L, R are faces below ℓ such that $\overline{B_1} < \overline{L} < \overline{R} \leqslant \overline{B_4}$ and LR is a green edge of H. The bottom half of this figure shows some invalid positions for a splitting edge B_5B_6; this shows that all splitting edges must have R_j as an endpoint.

Suppose there is a splitting edge B_5B_6 such that neither B_5 nor B_6 is R_j.

- If B_5B_6 is green then without loss of generality assume $\overline{B_5} < \overline{B_6}$. Since at least one of B_5 and B_6 $[L_j, R_j) \subseteq (B_1, B_4]$, by Corollary 4.15(a) the other must lie in $(B_1, B_4]$ as well. It follows that $[B_5, B_6)$ is an interval of the form $[L, R)$ in the union U, and therefore it must be contained within one of U's disjoint intervals; since one of B_5, B_6 lies in $[L_j, R_j)$, this interval must be $[L_j, R_j)$. It follows that $B_5, B_6 \in [L_j, R_j]$, so B_5B_6 cannot be a splitting edge unless $B_6 = R_j$.

- If B_5B_6 is blue then consider its endpoint within $[L_j, R_j)$. WLOG assume it is B_5. Then B_5 must be contained within one of the intervals $[L, R)$ in the union U, so by Corollary 4.15(b), B_6 must lie in $[L, R)$ as well, so in this case B_5B_6 cannot be splitting either.

Therefore all splitting edges must have R_j as an endpoint. Now assume for the sake of contradiction that B_1B_4 and B_2B_3 are in the same cycle in H. Consider the disjoint paths p_1 from B_2 to B_1 and p_2 from B_2 to B_4 in this cycle. As $B_2 \in [L_j, R_j)$ but $B_1, B_4 \notin [L_j, R_j)$, both p_1 and p_2 contains a splitting edge. It follows that both p_1 and p_2 passes through R_j, which is impossible because p_1 and p_2 does not have a vertex in common except for B_2, and $B_2 \neq R_j$. This is our desired contradiction. ☐

We have now eliminated enough configurations of edges in H.

Lemma 4.17. *H does not contain a cycle.*

Proof. Suppose for the sake of contradiction that H contains a cycle $c = B_1B_2 \cdots B_rB_1$.

If c is monochromatic, without loss of generality assume it is blue. For each $i = 1, \ldots, r$, let A_i be the face above ℓ up-related to B_i and B_{i+1} (where $B_{r+1} = B_1$). It follows that $B_1A_1B_2A_2 \cdots B_rA_rB_1$ is a closed walk in $G_\uparrow$, so $G_\uparrow$ must contain a cycle, and this contradicts Lemma 4.9.

Therefore c must contain both blue and green edges. In particular, there must be consecutive edges in c having distinct colors. Without loss of generality we may assume that these edges are B_1B_2 and B_2B_3, and that B_1B_2 is green and B_2B_3 is blue. As H is a simple graph, B_1, B_2, B_3 are all distinct. There are six orderings of B_1, B_2, B_3, and we show that none is possible.

Case 1: $\overline{B_1} < \overline{B_2} < \overline{B_3}$.

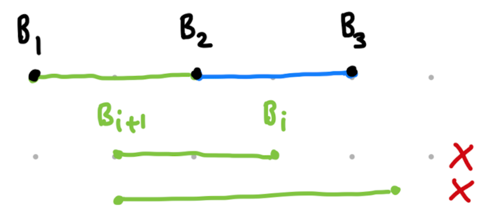

Figure 13: In the case where $\overline{B_1} < \overline{B_2} < \overline{B_3}$, there cannot be an edge B_iB_{i+1} such that $\overline{B_{i+1}} < \overline{B_2} < \overline{B_i}$.

In this case, since c contains the path $B_3B_4 \cdots B_rB_1$, there must be an edge B_iB_{i+1} in c such that $\overline{B_{i+1}} < \overline{B_2} < \overline{B_i}$. Without loss of generality assume this edge is green. If $\overline{B_i} \leqslant \overline{B_3}$ then $\overline{B_{i+1}} < \overline{B_2} < \overline{B_i} \leqslant \overline{B_3}$, but $B_{i+1}B_i$ is green and B_2B_3 is blue. This contradicts Lemma 4.14(b). On the other hand, if $\overline{B_i} > \overline{B_3}$ then $\overline{B_{i+1}} < \overline{B_2} < \overline{B_3} < \overline{B_i}$, and this contradicts Lemma 4.16.

Case 2: $\overline{B_3} < \overline{B_2} < \overline{B_1}$. In this case $\overline{B_3} < \overline{B_2} = \overline{B_2} < \overline{B_1}$ violates Lemma 4.14(a).

Case 3: $\overline{B_1} < \overline{B_3} < \overline{B_2}$, $\overline{B_2} < \overline{B_1} < \overline{B_3}$, $\overline{B_2} < \overline{B_3} < \overline{B_1}$, or $\overline{B_3} < \overline{B_1} < \overline{B_2}$. These four orderings all violate Lemma 4.16.

We have shown that none of the orderings of B_1, B_2, B_3 is possible; this is a contradiction, and therefore H cannot contain a cycle. $\square$

The fact that H cannot contain a cycle gives an upper bound on the average degree of H, which in turn gives an upper bound on the average degree of G.

Lemma 4.18. The average degree of G is less than 3.

Proof. Let $A_1, \ldots, A_r$ be the vertices of G above ℓ, i.e. faces with normal edges above ℓ. Let $B_1, \ldots, B_s$ be the vertices of G below ℓ. By the definition of H,

$$\begin{aligned}
\#\text{ edges in } H &= \sum_{i=1}^{r} \Big(\max\{0, \deg_{G_\uparrow} A_i - 1\} + \max\{0, \deg_{G_\downarrow} A_i - 1\} \Big) \\
&\geqslant \sum_{i=1}^{r} \Big(\{\deg_{G_\uparrow} A_i - 1\} + \{\deg_{G_\downarrow} A_i - 1\} \Big) \\
&= \left(\sum_{i=1}^{r} \deg_G A_i \right) - 2r
\end{aligned}$$

As H has no cycles, the number of edges in H is less than the number of vertices in H, which is s. Therefore

$$\sum_{i=1}^{r} \deg_G A_i < 2r + s.$$

Similarly we can show that $\sum_{i=1}^{s} \deg_G B_i < 2s + r$. It follows that

$$\sum_{v \in G} \deg_G v < 3(r+s) = 3|V|,$$

so the average degree of G is less than 3. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let B be the set of faces F in the zone of ℓ with at least two normal edges, and let $b = |B|$.

Call a normal edge *mundane* if it is on a face with at least two normal edges. By Lemma 4.7, each line other than ℓ contains at most two mundane edges, and each line with two mundane edges corresponds to an edge in G. By Lemma 4.18, it follows that the number of lines with two mundane edges is at most $1.5b$. As there are $n-1$ lines other than ℓ, the number of mundane edges is at most $n - 1 + 1.5b$.

As each face F in the zone of ℓ has at most five normal edges, a face F with at most one normal edge has at most six sides, hence $\max\{0, \deg F - 6.5\} = 0$. Therefore

$$\begin{aligned}
\sum_{\text{face } F \in \text{zone}(\ell)} \max\{0, \deg F - 6.5\} &= \sum_{F \in B} \max\{0, \deg F - 6.5\} \\
&= \left(\sum_{F \in B} \max\{1.5, \deg F - 5\} \right) - 1.5b \\
&\leqslant \left(\sum_{F \in B} \#\text{ normal edges of } F \right) - 1.5b \\
&\leqslant n - 1,
\end{aligned}$$

and we are done. □

§5 L_I AND $\widetilde{L}_I$

We introduce the quantities L_I and $\widetilde{L}_I$, defined for each $I \subseteq \mathbb{Z}$ to be limits involving the proportion of faces in the zone or half-zone of a pseudoline ℓ whose degree lies in I. In particular, when $I = \{k\}$ is a singleton, the quantities L_k and $\widetilde{L}_k$ gives upper bounds on the number of k-sided faces in a zone or half-zone.

Our main results are given in two theorems. In Theorem 5.5, we determine the values of $\widetilde{L}_I$ for all sets $I \subseteq \mathbb{Z}$. In Theorem 5.6, we determine the values of L_I for various sets $I \subseteq \mathbb{Z}$ and reduce the question of determining L_I for remaining sets I to determining I for eight specific finite sets I. For these eight remaining sets I, we give our best known lower and upper bounds on the value of L_I.

In this section we will work with simple projective arrangements. Let ℓ be a pseudoline in a simple projective arrangement $\mathcal{A}$ of n pseudolines. For each positive integer k, we count the number $u_k(\ell, \mathcal{A})$ of k-sided faces in the zone of ℓ. For each set I of positive integers, we define $u_I(\ell, \mathcal{A}) := \sum_{k \in I} u_k(\ell, \mathcal{A})$ to be the total number of faces in the zone of ℓ whose degree lies in I. We now define L_I as the limit of the maximum possible value of $u_I(\ell, \mathcal{A})$ over simple projective arrangements $\mathcal{A}$ with n pseudolines.

Definition 5.1. For each set $I \subseteq \mathbb{Z}$, we define the quantity L_I by the following limit.

$$L_I := \lim_{n \to \infty} \frac{Mu_I(n)}{n} \quad \text{where} \quad Mu_I(n) := \max_{\substack{\text{simple projective } \mathcal{A} \text{ of } n \text{ pseudolines} \\ \text{pseudoline } \ell \in \mathcal{A}}} u_I(\ell, \mathcal{A}).$$

We define $\widetilde{L}_I$ as an analogue of L_I for half-zones. For each half-zone h in a simple projective arrangement $\mathcal{A}$ of n pseudolines, we count the number $\widetilde{u}_k(h, \mathcal{A})$ of k-sided faces in h. For each set I of positive integers, define $\widetilde{u}_I(h, \mathcal{A}) := \sum_{k \in I} \widetilde{u}_k(h, \mathcal{A})$. The quantity $\widetilde{L}_I$ is defined to be the limit of the maximum possible value of $\widetilde{u}_I(h, \mathcal{A})$ over simple projective arrangements $\mathcal{A}$ with n pseudolines.

Definition 5.2. For each set $I \subseteq \mathbb{Z}$, we define the quantity $\widetilde{L}_I$ by the following limit.

$$\widetilde{L}_I := \lim_{n \to \infty} \frac{M\widetilde{u}_I(n)}{n} \quad \text{where} \quad M\widetilde{u}_I(n) := \max_{\substack{\text{simple projective } \mathcal{A} \text{ of } n \text{ pseudolines} \\ \text{half-zone } h \in \mathcal{A}}} \widetilde{u}_I(h, \mathcal{A}).$$

When $I = \{k\}$ is a singleton, the shorthand L_k and $\widetilde{L}_k$ may be used in place of $L_{\{k\}}$ and $\widetilde{L}_{\{k\}}$.

Our first goal is to show that the limits L_I and $\widetilde{L}_I$ do exist, and therefore can be meaningfully discussed. We start with a lemma.

Lemma 5.3. Let $\mathcal{A}^+$ be a projective pseudoline arrangement, and let ℓ and q be distinct pseudolines in $\mathcal{A}$. Then there is a half-zone h of ℓ with respect to q such that h contains at most two faces supported by both q and ℓ.

Proof. Take a transformation sending q to the line at infinity, and consider the affine pseudoline arrangement $\mathcal{A}$ created by viewing $\mathcal{A}^+ \setminus \{q\}$ as an affine arrangement. Half-zones of ℓ with respect to q in $\mathcal{A}^+$ become half-zones of ℓ in $\mathcal{A}$, and faces supported by q in $\mathcal{A}^+$ become unbounded faces of $\mathcal{A}$. Therefore, it suffices to show that, given a pseudoline ℓ in an affine arrangement $\mathcal{A}$, there is a side of ℓ on which the faces supported by a segment of ℓ are all bounded. (The two extreme faces supported by rays of ℓ are always unbounded.)

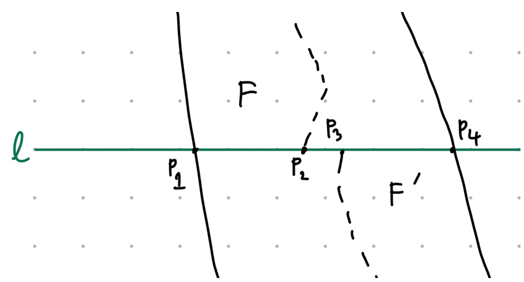

Figure 14: The pseudolines r and r' through P_1 and P_4 can intersect on neither side of ℓ.

Let F be an unbounded face supported by a segment P_1P_2 of ℓ, and suppose that there were another unbounded face F' supported by a segment P_3P_4 of ℓ on the opposite side of ℓ. The segments P_1P_2 and P_3P_4 cannot coincide, so without loss of generality we may assume that P_1, P_2, P_3, P_4 lie in this order. Let r and r', distinct from ℓ, be the pseudolines supporting F and F' passing through P_1 and P_4 respectively. As F and F' is unbounded, it follows that r and r' can intersect on neither side of ℓ, which is impossible. Therefore, for each pseudoline ℓ in an arrangement $\mathcal{A}$, there is a side of ℓ on which the faces supported by a segment of ℓ are all bounded. $\square$

Proposition 5.4. The limits L_I and $\widetilde{L}_I$ exist for all sets I of positive integers.

Proof. We first show that the limit L_I exists for all sets I of positive integers. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be projective arrangements of pseudolines. Let ℓ_1 and q_1 be distinct pseudolines of $\mathcal{A}_1$ and let ℓ_2 and q_2 be distinct pseudolines of $\mathcal{A}_2$. Without loss of generality we may assume that each of q_1 and q_2 is the line at infinity.

By Lemma 5.3, for $i \in \{1,2\}$, there is a half-zone h_i of ℓ_i with respect to q_i such that contains at most two faces supported by both q_i and ℓ_i. Call the half-zone h_i the *good* half-zone of ℓ_i with respect to q_i.

We now create a combined arrangement $\mathcal{A}$ as in Figure 15. We first draw pseudolines ℓ and q with q being the line at infinity. We then place parts corresponding to $\mathcal{A}_1$ and $\mathcal{A}_2$ far apart on ℓ so that the good half-zones h_1 and h_2 coincide, and make the pseudolines in $\mathcal{A}_1 \setminus \{\ell_1, q_1\}$ and $\mathcal{A}_2 \setminus \{\ell_2, q_2\}$ intersect in the good half-zones.

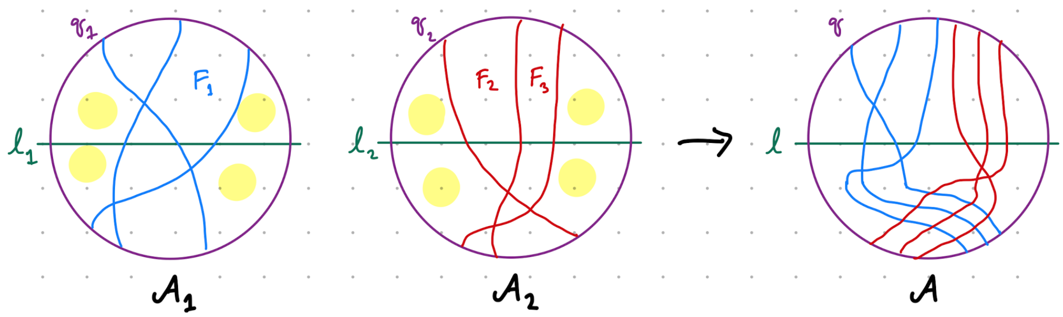

Figure 15: We can combine arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ to create an arrangement $\mathcal{A}$. The zone of $\mathcal{A}$ will contain all faces from the zones of $\mathcal{A}_1$ and $\mathcal{A}_2$ except the faces marked with a yellow dot.

If $\mathcal{A}_1$ has m pseudolines and $\mathcal{A}_2$ has n pseudolines then $\mathcal{A}$ has $m+n-2$ pseudolines. As $\mathcal{A}$ preserves all but eight faces in the zone of ℓ from $\mathcal{A}_1$ and $\mathcal{A}_2$,

$$Mu_I(m+n-2) \geqslant Mu_I(m) + Mu_I(n) - 8.$$

Define $f(n) = Mu_I(n+2) - 8$. It follows that

$$f(m+n) \geqslant f(m) + f(n).$$

As $Mu_I(n) \leqslant 2n$, it follows that $\frac{f(n)}{n} \leqslant 2$ for all positive integers n. Therefore the supremum $s = \sup\left\{\frac{f(n)}{n} : n \in \mathbb{Z}_{>0}\right\}$ exists. From the above inequality it also follows that for all $n \in \mathbb{Z}_{>0}$ and $\varepsilon > 0$, there is an N such that the inequality

$$\frac{f(m)}{m} \geqslant \frac{f(n)}{n} - \varepsilon$$

holds for all $m > N$. Therefore, for all $\varepsilon > 0$, $\frac{f(n)}{n} \geqslant s - \varepsilon$ for all sufficiently large n. Consequently,

$$\lim_{n\to\infty} \frac{f(n)}{n} = s,$$

and it follows that $L_I = \lim_{n\to\infty} \frac{Mu_I(n)}{n} = s$.

The proof for $\widetilde{L}_I$ is similar, with the main difference being that we want lines from $\mathcal{A}_1$ and $\mathcal{A}_2$ intersect on the half-zone of ℓ that is not h instead of the "all-bounded" half-zone. $\square$

We now state our main results on $\widetilde{L}_I$ and L_I. Theorem 5.5 gives a simple procedure to determine the values of $\widetilde{L}_I$ for all sets $I \subseteq \mathbb{Z}_{>0}$.

Theorem 5.5. *For all sets $I \subseteq \mathbb{Z}$, the value of $\widetilde{L}_I$ can be determined as follows.*

a) *For all sets $I \subseteq \mathbb{Z}$, $\widetilde{L}_I = \widetilde{L}_{I\cap\{3,4,5,\dots\}}$. For all sets $I \subseteq \mathbb{Z}_{\geqslant 4}$, we have*

$$\widetilde{L}_I = \widetilde{L}_{\min I} \quad \text{and } \widetilde{L}_{\{3\}\cup I} = \widetilde{L}_{\{3,\min I\}}.$$

b) *For each $k \geqslant 3$, the value of $\widetilde{L}_k$ is given by the following table.*

k	$\widetilde{L}_k$
3	1/2
4	1
5	1
6	2/3
$\geqslant 7$	$1/(k-5)$

c) *For each $k \geqslant 4$, the values of $\widetilde{L}_{\{3,k\}}$ is given by $\widetilde{L}_{\{3,k\}} = \min\left\{1, \frac{1}{2} + \widetilde{L}_k\right\}$.*

Theorem 5.6 gives exact values of L_I for various sets I, and gives lower and upper bounds for L_I for other sets I. The problem of finding the exact value of L_I for all sets I is also reduced to eight remaining cases: $I = \{k\}$ with $5 \leqslant k \leqslant 9$ and $I = \{3, k\}$ with $7 \leqslant k \leqslant 9$.

Theorem 5.6. *Bounds for L_I are given as follows.*

a) *For all sets* $I \subseteq \mathbb{Z}$, $L_I = L_{I \cap \{3,4,5,\ldots\}}$. *For all sets* $I \subseteq \mathbb{Z}_{\geqslant 4}$, *we have*

$$L_I = L_{\min I} \quad \text{and } L_{\{3\} \cup I} = L_{\{3, \min I\}}.$$

Therefore it suffices to determine L_I *for sets* I *of the form* $I = \{k\}$ *with* $k \geqslant 3$ *and* $I = \{3, k\}$ *with* $k \geqslant 4$.

b) *For each* $k \geqslant 3$, *bounds on* L_k *are given by the following table. When there are two numbers listed for* L_k, *the smaller number is the lower bound, and the larger number is the upper bound. When there is a single number listed for* L_k, *it is the exact value of* L_k.

k	*Lower bound of* L_k	*Upper bound of* L_k
3	1	
4	2	
5	14/9	7/4
6	9/8	7/6
7	9/13	5/6
8	1/2	5/8
9	3/8	2/5
$\geqslant 10$	$1/(k-6.5)$	

c) *For each* $k \geqslant 4$, *bounds on* $L_{\{3,k\}}$ *are given by the following table.*

k	*Lower bound of* $L_{\{3,k\}}$	*Upper bound of* $L_{\{3,k\}}$
4	2	
5	2	
6	2	
7	5/3	11/6
8	3/2	13/8
9	11/8	7/5
$\geqslant 10$	$1 + 1/(k-6.5)$	

The rest of this section is dedicated to proving Theorems 5.5 and 5.6. We start by proving a lemma which states, in essence, that for each $k \geqslant 5$ we can reduce all faces F in a zone to k-gons.

Lemma 5.7. Let ℓ be a pseudoline in a projective arrangement $\mathcal{A}$ of n pseudolines. Let F be a face in the zone of ℓ. If $\deg F \geqslant 6$ then there is a projective arrangement $\mathcal{A}'$ of n pseudolines such that the following two conditions hold.

- There are pseudolines $q \in \mathcal{A}$ and $q' \notin \mathcal{A}$ such that $\mathcal{A}' = \mathcal{A} \setminus \{q\} \cup \{q'\}$ for some pseudolines q, q'.
- The zone of ℓ in $\mathcal{A}'$ is identical to the zone of ℓ in $\mathcal{A}$ except for the face F, which becomes a face F' with $\deg F' = \deg F - 1$.

Proof. Let $k = \deg F$ and let $e_0, e_1, \ldots, e_{k-1}$ be the edges of F in order, so that e_0 lies on ℓ, and e_1 and e_{k-1} has an endpoint in ℓ. Let $\ell_1, \ldots, \ell_{k-1}$ be the pseudolines in $\mathcal{A}$ containing $e_1, \ldots, e_{k-1}$ respectively.

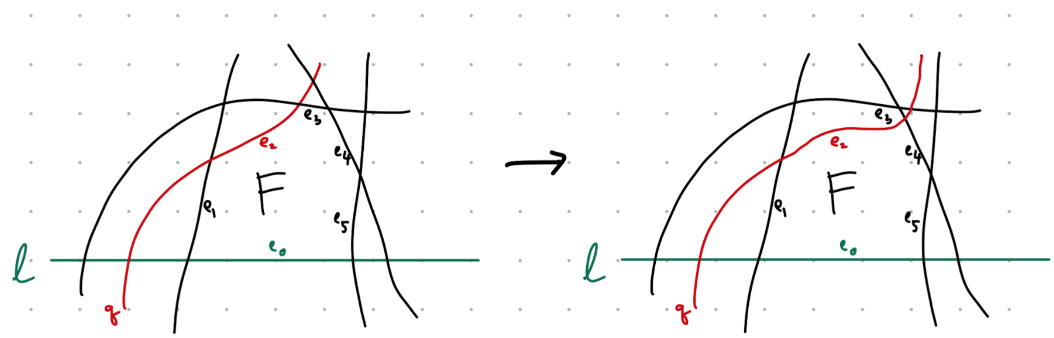

Figure 16: Replacing q by q' reduces the degree of F by 1.

Pick $q = \ell_2$. Draw the pseudoline q' as follows: q' agrees with q for the most part, except q' is moved around the intersection of e_3 and e_4. (See Figure 16.) Replacing q by q' reduces the degree of F by 1, and does not affect other faces in the zone of ℓ because q and q' only differ in the region bounded by ℓ, ℓ_2, and ℓ_{k-2} that contains F, and the only face in the zone of ℓ lying in this region is F. $\square$

We now give a proof of Theorem 5.5.

Proof of Theorem 5.5. a) In a projective pseudoline arrangement with at least 3 pseudolines, all faces have at least three sides. As there are no 1-gons or 2-gons, $\widetilde{L}_I = \widetilde{L}_{I \cap \{3,4,5,\ldots\}}$.

We now show that

$$\widetilde{L}_I = \widetilde{L}_{\min I}$$

for $I \subseteq \mathbb{Z}_{\geqslant 4}$.

- **Case 1:** $\min I = 4$.

  Let J be a set of positive integers with $4 \in J$. Configuration 6.2 shows that $\widetilde{L}_J \geqslant 1$. As there are only $n-1$ faces in a half-zone of a projective arrangement of n pseudolines, $M\widetilde{u}_I(n) \leqslant n-1$ for all $n \geqslant 2$. (Recall that $M\widetilde{u}_I(n)$ is the maximum possible number of faces F with $\deg F \in I$ in a half-zone h in a projective arrangement of n pseudolines.) Therefore

  $$\widetilde{L}_J \leqslant \lim_{n\to\infty} \frac{n-1}{n} = 1,$$

  so $\widetilde{L}_J = 1$ for all sets J with $4 \in J$. Therefore it follows that

  $$\widetilde{L}_I = 1 = \widetilde{L}_4 \quad \text{and} \quad \widetilde{L}_{\{3\}\cup I} = 1 = \widetilde{L}_{\{3,4\}}$$

  for all sets $I \subseteq \mathbb{Z}_{\geqslant 4}$ with $\min I = 4$.

- **Case 2:** $\min I \geqslant 5$.

  Let n be a positive integer, and let $k = \min I$. Let $\mathcal{A}$ be a projective arrangement of pseudolines and let h be a half-zone in $\mathcal{A}$ such that $\widetilde{u}_I(h, \mathcal{A})$ attains the maximum possible value, that is,

  $$\widetilde{u}_I(h, \mathcal{A}) = M\widetilde{u}_I(n).$$

  As $k \geqslant 5$, by repeatedly applying Lemma 5.7, we obtain a projective arrangement $\mathcal{A}'$ of n pseudolines which has a k-sided face for every face in $\mathcal{A}$ with at least k sides. Therefore, $\widetilde{u}_k(h, \mathcal{A}') = \widetilde{u}_I(h, \mathcal{A})$. It follows that $M\widetilde{u}_k(n) \geqslant M\widetilde{u}_I(n)$. However, as $k \in I$, it is clear that $M\widetilde{u}_k(n) \leqslant M\widetilde{u}_I(n)$ as well. Therefore,

  $$M\widetilde{u}_k(n) = M\widetilde{u}_I(n)$$

  for all positive integers n, and hence $\widetilde{L}_k = \widetilde{L}_I$.

b) We divide into cases based on the value of k.

- **Case 1:** $k = 3$.
  We show that in a simple projective pseudoline arrangement $\mathcal{A}$ of at least 4 pseudolines, two triangles cannot share an edge. Suppose for the sake of contradiction that two triangles T_1 and T_2 share an edge e lying on the pseudoline ℓ. Let q_1 and q_2, both distinct from ℓ, be the pseudolines passing through the endpoints of e. (As $\mathcal{A}$ is simple, the choice of q_1 and q_2 is unique.) The pseudolines q_1 and q_2 must both support both T_1 and T_2. It follows that q_1 and q_2 must intersect twice, which is impossible.

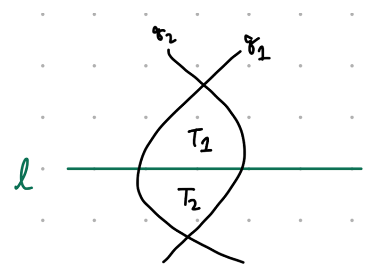

Figure 17: Two triangles cannot share an edge.

  A half-zone h in a projective arrangement of $n \geqslant 4$ pseudolines is a collection of $n-1$ faces $F_1, F_2, \ldots, F_{n-1}$ such that F_i shares an edge with F_{i+1}. It follows that for each i, at most one of $\{F_i, F_{i+1}\}$ can be a triangle. Therefore there are at most $\left\lceil \frac{n-1}{2} \right\rceil$ triangles in h, i.e. $M\widetilde{u}_3 \leqslant \left\lceil \frac{n-1}{2} \right\rceil$. It follows that

$$\widetilde{L}_3 \leqslant \lim_{n\to\infty} \frac{1}{n} \left\lceil \frac{n-1}{2} \right\rceil = \frac{1}{2}.$$

  The value $\widetilde{L}_3 = \frac{1}{2}$ is attained by Configuration 6.1.

- **Case 2:** $k = 4$.
  In Case 1 of part a) we already showed that $\widetilde{L}_4 = 1$.

- **Case 3:** $k = 5$.
  The proof that $\widetilde{L}_J \leqslant 1$ in Case 1 of part a) applies to all sets $J \subseteq \mathbb{Z}_{>0}$, including $J = \{5\}$, so $\widetilde{L}_5 \leqslant 1$. This upper bound is attained by Configuration 6.4.

- **Case 4:** $k = 6$.

Let $n \geqslant 3$. The Half-Zone Theorem for projective arrangements states that for a half-zone h in a projective arrangement $\mathcal{A}$ of n pseudolines,

$$\sum_{F \in h} \deg F \leqslant 5n + O(1).$$

As there are $n-1$ faces in a half-zone, it follows that

$$\sum_{F \in h} (\deg F - 3) \leqslant 2n + O(1).$$

As all faces in $\mathcal{A}$ have at least three sides, every term in the sum $\sum_{F \in h}(\deg F - 3)$ is nonnegative. Each 6-gon contributes 3 to the sum, so the number of 6-gons in h is at most $\frac{2n}{3} + O(1)$. In other words, $M\widetilde{u}_6(I) \leqslant \frac{2n}{3} + O(1)$, so

$$\widetilde{L}_6 \leqslant \lim_{n \to \infty} \frac{1}{n} \left(\frac{2n}{3} + O(1) \right) = \frac{2}{3}.$$

This upper bound is attained by Configuration 6.5.

- **Case 5:** $k \geqslant 7$.

  Let h be a half-zone in a projective arrangement of n pseudolines. Theorem 3.1 gives

  $$\sum_{F \in h} \max\{0, \deg F - 5\} \leqslant n + O(1).$$

  Each term in the sum is nonnegative, and each k-gon contributes $k-5$ to the sum. Therefore $M\widetilde{u}_k \leqslant \frac{n}{k-5} + O(1)$. It follows that $\widetilde{L}_k \leqslant \frac{1}{k-5}$. Configuration 6.6 shows that this upper bound can be reached.

c) The bound $\widetilde{L}_{\{3,k\}} \leqslant 1$ follows from the fact that $\widetilde{L}_J \leqslant 1$ for all $J \subseteq \mathbb{Z}_{>0}$. The bound

$\widetilde{L}_{\{3,k\}} \leqslant \frac{1}{2} + \widetilde{L}_k$ can be derived from the definition of $\widetilde{L}$ as follows:

$$\begin{aligned}
\widetilde{L}_{\{3,k\}} &= \lim_{n\to\infty} \frac{M\widetilde{u}_{\{3,k\}}(n)}{n} \\
&= \lim_{n\to\infty} \max_{\substack{\text{simple proj. } \mathcal{A} \text{ of } n \text{ pseudolines} \\ \text{half-zone } h\in\mathcal{A}}} \widetilde{u}_{\{3,k\}}(h,\mathcal{A}) \\
&\leqslant \lim_{n\to\infty} \left(\max_{\substack{\text{simple proj. } \mathcal{A} \text{ of } n \text{ pseudolines} \\ \text{half-zone } h\in\mathcal{A}}} \widetilde{u}_3(h,\mathcal{A}) + \max_{\substack{\text{simple proj. } \mathcal{A} \text{ of } n \text{ pseudolines} \\ \text{half-zone } h\in\mathcal{A}}} \widetilde{u}_k(h,\mathcal{A}). \right) \\
&= \lim_{n\to\infty} \frac{M\widetilde{u}_3(n) + M\widetilde{u}_k(n)}{n} \\
&= \widetilde{L}_3 + \widetilde{L}_k \\
&= \frac{1}{2} + \widetilde{L}_k.
\end{aligned}$$

The upper bound $\widetilde{L}_{\{3,k\}} \leqslant \min\left\{1, \frac{1}{2} + \widetilde{L}_k\right\}$ can be reached at Configurations 6.2 ($k = 4$), 6.3 ($k = 5$), 6.1 ($k = 6$), and 6.6 ($k \geqslant 7$).

□

We close this section by giving a proof of Theorem 5.6, which uses an approach similar to the proof of Theorem 5.5 in many parts.

Proof of Theorem 5.6. a) The proof of Theorem 5.5(a) also works with L_I in place of $\widetilde{L}_I$ using the bound $L_J \leqslant 2$; for $4 \in J$ this is attained by Configuration 6.2.

b) We first derive the upper bounds given.

- **Case 1:** $k = 3$.
  In the proof of Theorem 5.5(b) we showed that a half-zone h in a simple projective pseudoline arrangement of $n \geqslant 4$ pseudolines contains at most $\left\lceil \frac{n-1}{2} \right\rceil \leqslant \frac{n}{2}$ triangles. A zone can be divided into two half-zones. Therefore a zone in a simple projective pseudoline arrangement of $n \geqslant 4$ pseudolines contains at most n triangles, hence $L_3 \leqslant 1$.
- **Case 2:** $k = 4$.
  The upper bound $L_4 \leqslant 2$ follows from the fact that $L_J \leqslant 2$ for all sets $J \subseteq \mathbb{Z}_{>0}$.

- **Case 3:** $5 \leqslant k \leqslant 6$.
  Let $\mathcal{A}$ be a simple projective arrangement of $n \geqslant 3$ pseudolines, and let ℓ be a pseudoline in $\mathcal{A}$. The Zone Theorem, for projective arrangements, states that $\sum_{F\in\text{zone}(\ell)} \deg F \leqslant \frac{19n}{2} + O(1)$. As there are $2n-2$ faces in the zone of ℓ, it follows that
  $$\sum_{F\in\text{zone}(\ell)} \deg F - 3 \leqslant \frac{7n}{2} + O(1).$$
  As all faces of $\mathcal{A}$ have at least three sides, all terms in the sum above is nonnegative. A k-gon contributes $k-3$ to the sum. Therefore, there are at most $\frac{7n}{2(k-3)} + O(1)$ k-gons in the zone of ℓ, and it follows that $L_k \leqslant \frac{7n}{2(k-3)}$.
- **Case 4:** $7 \leqslant k \leqslant 8$.
  Let $\mathcal{A}$ be a simple projective arrangement of $n \geqslant 3$ pseudolines, and let ℓ be a pseudoline in $\mathcal{A}$. As triangular faces do not contribute to the sum $\sum_{F\in\text{zone}(\ell)} \deg F - 3$, the inequality from the previous case can be written as
  $$\sum_{\substack{F\in\text{zone}(\ell)\\ \deg F\geqslant 4}} (\deg F - 3) = \sum_{F\in\text{zone}(\ell)} (\deg F - 3) \leqslant \frac{7n}{2} + O(1).$$
  As there are at most n triangles in the zone of ℓ, there are at least $(2n-2)-n = n-2$ non-triangular faces in the zone of ℓ. Therefore,
  $$\sum_{\substack{F\in\text{zone}(\ell)\\ \deg F\geqslant 4}} \deg F - 4 \leqslant \left(\frac{7n}{2} + O(1)\right) - (n-2) = \frac{5n}{2} + O(1).$$
  We finish using the same method as the previous case. All terms in the sum above are nonnegative, and each k-gon contributes $k-4$ to the sum. Therefore, there are at most $\frac{5n}{2(k-4)} + O(1)$ k-gons the zone of ℓ, and it follows that $L_k \leqslant \frac{5n}{2(k-4)}$.
- **Case 5:** $k \geqslant 9$.
  Let ℓ be a pseudoline in a projective arrangement of n pseudolines. Theorem 4.1 gives
  $$\sum_{F\in\text{zone}(\ell)} \max\{0, \deg F - 6.5\} \leqslant n + O(1).$$
  Each term in the sum is nonnegative, and each k-gon contributes $k-6.5$ to the sum. Therefore, there are at most $\frac{n}{k-6.5} + O(1)$ k-gons in the zone of ℓ, and it follows that $L_k \leqslant \frac{1}{k-6.5}$.

The lower bounds listed are given by Configurations 6.1 ($k = 3$), 6.2 ($k = 4$), 6.7 ($k = 5$),6.8 ($k = 6$),6.9 ($k = 7$),6.11 ($k = 8$),6.12 ($k = 9$), and 6.13 ($k \geqslant 10$). For $k = 3$, $k = 4$, and $k \geqslant 10$, the lower bounds given by the configurations agree with the upper bounds proved, so we obtain the exact value of L_k.

c) The upper bounds listed in the table for $L_{\{3,k\}}$ is exactly $\min\{2, L_3 + L_k\}$. An argument similar to 5.5(c) shows that these values are indeed upper bounds on $L_{\{3,k\}}$.

The configurations giving the lower bounds are Configurations 6.2 ($k = 4$), 6.3 ($k = 5$), 6.1 ($k = 6$), 6.10 ($k = 7$), 6.11 ($k = 8$), 6.12 ($k = 9$), and 6.13 ($k \geqslant 10$).

□

§6 Configurations

In this section we give explicit configurations attaining the lower bounds mentioned in Theorems 5.5 and 5.6. In the diagrams for these configurations, ℓ is always a horizontal black line. In configurations for $\widetilde{L}_I$, we always consider the half-zone above ℓ. We start with well-known configurations based on the hexagonal and the square lattices.

Configuration 6.1. Choosing ℓ as the middle diagonal of pattern in which three diagonals cut through a square lattice gives $\widetilde{L}_3 = \frac{1}{2}, L_3 = 1, \widetilde{L}_{\{3,6\}} = 1, L_{\{3,6\}} = 2$.

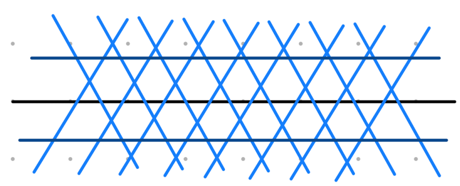

Figure 18: Configuration 6.1

Configuration 6.2. A line ℓ in a square lattice gives $\widetilde{L}_4 = 1, L_4 = 2, \widetilde{L}_{\{3,4\}} = 1, L_{\{3,4\}} = 2$.

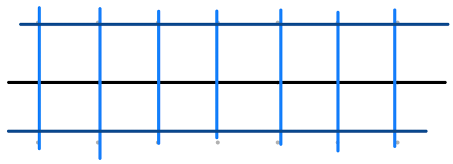

Figure 19: Configuration 6.2

Configuration 6.3. A pattern in which the ℓ is a diagonal line cutting through a square lattice gives $\widetilde{L}_{\{3,5\}} = 1, L_{\{3,5\}} = 2$.

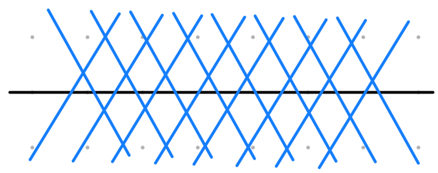

Figure 20: Configuration 6.3

We now describe a class of what we call *multi-stage configurations*. Each of these configurations contains a *foundation* and a finite number of *stages*.

- The *foundation* is a configuration of pseudolines which contains a high number of 'outgoing' parallel pseudolines, which are light blue lines in the box labeled 'foundation' in Figure 22 These parallel pseudolines will be used to form a *rolling pattern* (see Figure 21) for the first stage.

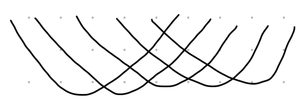

Figure 21: A rolling pattern of five pseudolines.

- Each *stage* is a configuration of pseudolines which utilizes 'incoming' parallel pseudolines from the previous stage to form a rolling pattern, and also includes 'outgoing' parallel pseudolines which will form the rolling pattern for the next stage.

A general depiction of a multi-stage configuration is given in Figure 22. In a multi-stage configuration, odd-numbered stages will be located on one side of the foundation, and even-numbered stages will be on the other side of the foundation.

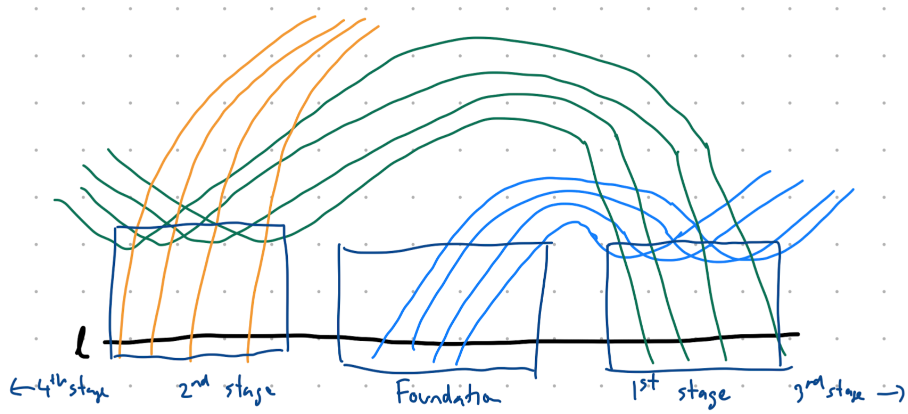

Figure 22: In a multi-stage configuration, 'outgoing' parallel pseudolines from a stage is used to form a rolling pattern in the next stage. In this diagram, the pseudolines in green are outgoing pseudolines of the first stage, and form a rolling pattern in the second stage.

We now give an analysis of the lower bounds on L_I and $\widetilde{L}_I$ obtained from multi-stage configurations. In our multi-stage configurations, each stage will have the same periodic pattern, though the number of times this pattern repeats in each stage may vary.

Suppose that in each stage, for each incoming pseudoline,

- b additional pseudolines are used,
- t faces with the required number of sides are created (that is, each stage contributes t to the value of u_I or $\widetilde{u}_I$ per each incoming pseudoline), and
- c outgoing pseudolines are created.

We now divide into two cases based on the value of c, which is the most important parameter because the value of c represents the ratio between the number of pseudolines in consecutive stages. For all of our configurations, $c \leqslant 1$, so we will only consider these values of c.

- **Case 1:** $c = 1$.

  When $c = 1$, our strategy will be to simply take some m parallel pseudolines as our foundation, and thus in configurations where $c = 1$ we will only provide a diagram for the stages.

  As $c = 1$, all stages will contain m incoming and outgoing pseudolines. Each stage will contain $bm + O(1)$ non-incoming pseudolines, and contributes $tm + O(1)$ to the value of u_I or $\widetilde{u}_I$. The $O(1)$ terms account for rounding errors and faces at the fringes of the stage.

  By taking m to also be the number of stages, we have a configuration of $bm^2 + m$ pseudolines giving $u_I = tm^2 + O(m)$. We therefore obtain a lower bound of

  $$\lim_{m\to\infty} \frac{bm^2 + O(m)}{tm^2 + O(m)} = \frac{b}{t}$$

  for L_I or $\widetilde{L}_I$.

- **Case 2:** $c < 1$.

  In our configurations where $c < 1$, we will also provide a specific foundation configuration. Suppose that in the foundation, for each *outgoing* pseudoline,

  - b_0 pseudolines are used, including the outgoing pseudolines, and
  - t_0 faces with the required number of sides are created. (In other words, the foundation contributes t_0 to the value of u_I or $\widetilde{u}_I$ for each outgoing pseudoline.)

  Suppose our foundation contains m outgoing pseudolines. These pseudolines will be incoming pseudolines for the first stage, which will contain cm outgoing pseudolines. In general, in the j^{th} stage, there will be

  - $c^{j-1}m + O(1)$ incoming pseudolines,
  - $bc^{j-1}m + O(1)$ additional pseudolines,
  - $tc^{j-1}m + O(1)$ faces contributing to u_I or $\widetilde{u}_I$, and
  - $c^j m + O(1)$ outgoing pseudolines,

where the $O(1)$ terms account for rounding errors and faces at the fringes of the stage. Take $s = -\frac{\log m}{\log c} - O(1)$ stages. In our entire configuration, there will be

- $b_0 m + b(1 + c + \cdots + c^{s-1})m + O(s) = \left(b_0 + \frac{b}{1-c} - o(1)\right) m$ pseudolines and
- $t_0 m + t(1 + c + \cdots + c^{s-1})m + O(s) = \left(t_0 + \frac{t}{1-c} - o(1)\right) m$ faces with the required number of sides.

Therefore we will obtain a lower bound on L_I or $\widetilde{L}_I$ of

$$\lim_{m \to \infty} \frac{\left(t_0 + \frac{t}{1-c} - o(1)\right) m}{\left(b_0 + \frac{b}{1-c} - o(1)\right) m} = \frac{t_0(1-c) + t}{b_0(1-c) + b}.$$

We now give multi-stage configurations attaining our lower bounds for L_I and $\widetilde{L}_I$ for various sets I. For configurations with $c = 1$, we will only provide a single diagram representing all stages. For configurations with $c < 1$, we will provide two diagrams: one for the foundation, and one for the stages. In all diagrams, the incoming pseudolines forming a rolling pattern will be in red, and outgoing pseudolines will be in green.

Configuration 6.4. A multi-stage configuration for $\widetilde{L}_5$ based on a square lattice has $c = 1$ and $(b, t) = (1, 1)$, which gives $\widetilde{L}_5 \geqslant 1$. The overall configuration including the foundation and stages will look like Figure 22.

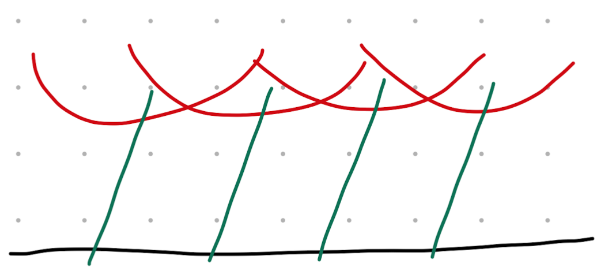

Figure 23: A stage in Configuration 6.4

Configuration 6.5. A multi-stage configuration for $\widetilde{L}_6$ has $c = 1$ and $(b, t) = (3/2, 1)$, which gives $\widetilde{L}_6 \geqslant 2/3$.

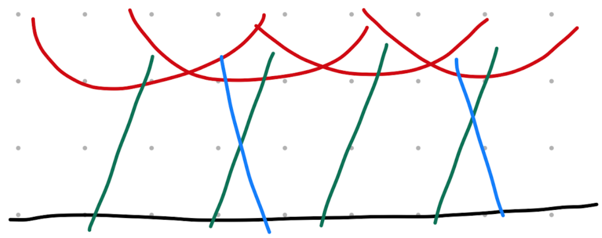

Figure 24: A stage in Configuration 6.5

Configuration 6.6. A multi-stage configuration for $\widetilde{L}_7$ has $c = 1$ and $(b,t) = (2,1)$, which gives $\widetilde{L}_7 = 1/2$. This configuration is also a configuration for $\widetilde{L}_{\{3,7\}}$ with $c = 1$ and $(b,t) = (2,2)$; therefore, $\widetilde{L}_{\{3,7\}} \geqslant 1$.

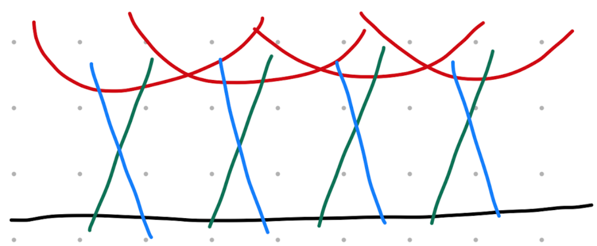

Figure 25: A stage in Configuration 6.6

This configuration can be modified into a configuration for $\widetilde{L}_k$ by adding $k-7$ lines per each 7-gon. This results in a configuration with $c = 1$ and $(b,t) = (k-5, 1)$, giving a bound of $\widetilde{L}_k \geqslant 1/(k-5)$ for all $k \geqslant 7$.

The modified configuration also works for $\widetilde{L}_{\{3,k\}}$ with $c = 1$ and $(b,t,c) = (k-5, k-4)$. Therefore, $\widetilde{L}_k \geqslant \frac{k-4}{k-5} = 1 + \frac{1}{k-5}$ for all $k \geqslant 7$.

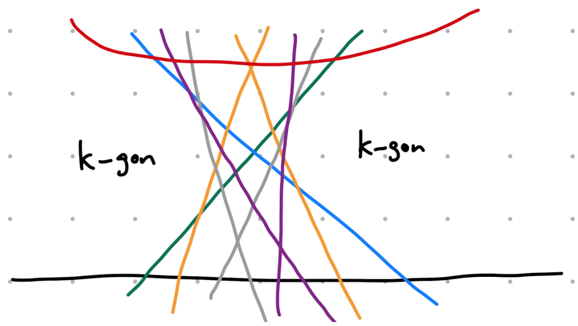

Figure 26: To modify a configuration for $\widetilde{L}_7$ into a configuration for $\widetilde{L}_k$ with $k > 7$, we add $k-7$ pairs of pseudolines for each incoming pseudoline: first we add the pseudolines marked in yellow, then we add the pseudolines in gray and purple. More pseudolines can be added in the same way as the purple pair.

Configuration 6.7. A multi-stage configuration for L_5 has $c = \frac{2}{3}$, $(b,t) = \left(1, \frac{5}{3}\right)$, and $(b_0, t_0) = \left(\frac{3}{2}, 2\right)$. This gives the bound

$$L_5 \geqslant \frac{2 \cdot \frac{1}{3} + \frac{5}{3}}{\frac{3}{2} \cdot \frac{1}{3} + 1} = \frac{14}{9}.$$

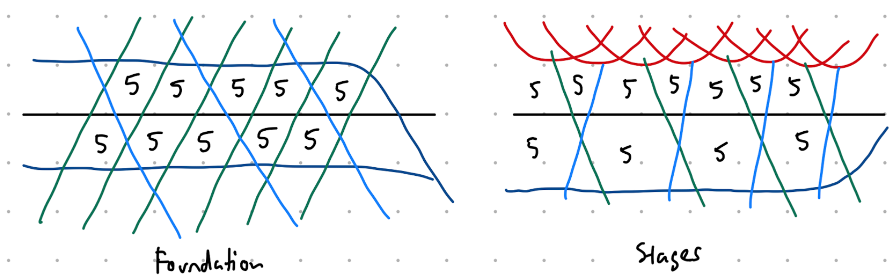

Figure 27: The foundation and a stage in Configuration 6.7. The pattern for stages repeats for every two incoming pseudolines.

Configuration 6.8. A multi-stage configuration for L_6 has $c = \frac{1}{2}$, $(b,t) = \left(2, \frac{5}{2}\right)$, and

$(b_0, t_0) = (2, 2)$. This gives the bound

$$L_6 \geqslant \frac{2 \cdot \frac{1}{2} + \frac{5}{4}}{2 \cdot \frac{1}{2} + 1} = \frac{9}{8}.$$

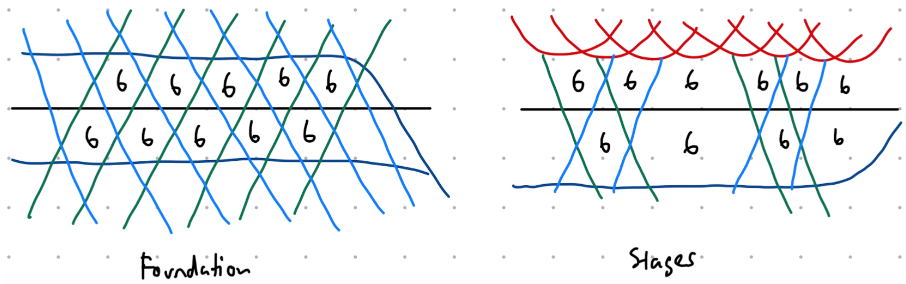

Figure 28: The foundation and a stage in Configuration 6.8. The pattern for stages repeats for every four incoming pseudolines.

Configuration 6.9. A multi-stage configuration for L_7 has $c = 1$, $(b, t) = (2.6, 1.8)$, giving the bound $L_7 \geqslant \frac{9}{13}$.

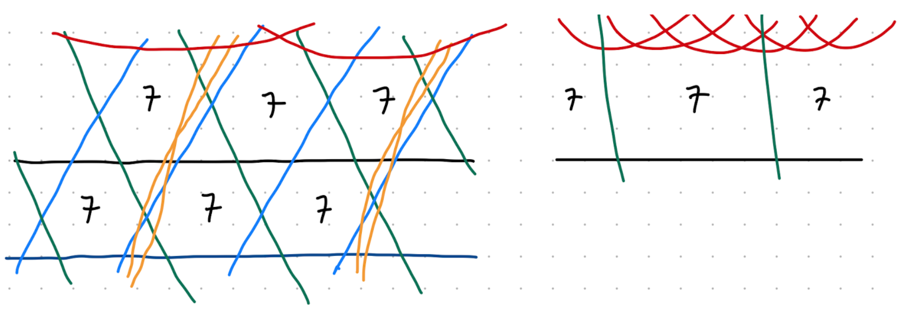

Figure 29: A stage in Configuration 6.9 consists of two patterns: when there are m incoming edges, the pattern on the left uses $0.4m$ incoming edges, and the pattern on the right uses $0.6m$ incoming edges.

Configuration 6.10. A multi-stage configuration for $L_{\{3,7\}}$ has $c = 1$, $(b, t) = (3, 5)$, giving the bound $L_7 \geqslant \frac{5}{3}$.

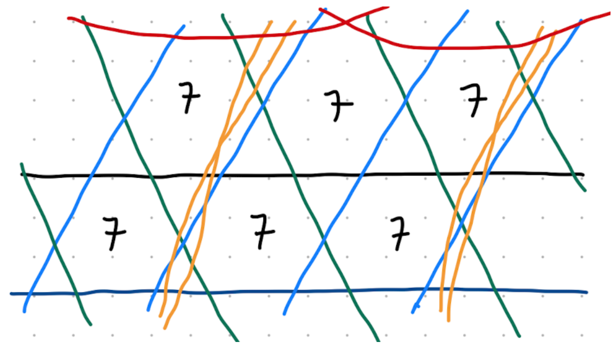

Figure 30: A stage in Configuration 6.10. To ensure $c = 1$, only half of the outgoing edges from each stage are used as incoming edges for the next stage.

Configuration 6.11. A multi-stage configuration for L_8 has $c = 1$ and $(b, t) = (4, 2)$, giving $L_8 \geqslant \frac{1}{2}$. This configuration also works for $L_{\{3,8\}}$ with $c = 1$ and $(b, t) = (4, 6)$, giving $L_{\{3,8\}} \geqslant \frac{3}{2}$.

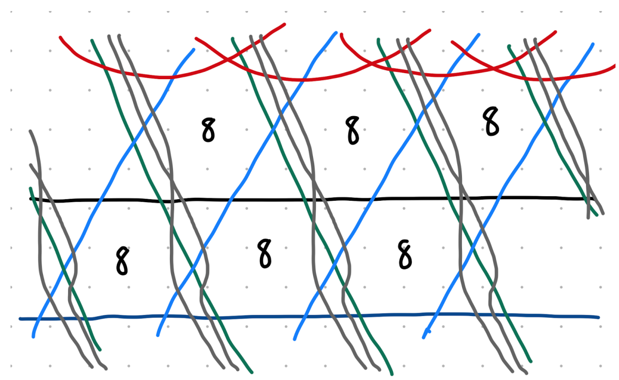

Figure 31: A stage in Configuration 6.11. The pattern repeats for every pseudoline in the rolling pattern.

Configuration 6.12. A multi-stage configuration for L_9 has $c = 1$ and $(b, t) = \left(\frac{16}{3}, 2\right)$, giving $L_9 \geqslant \frac{3}{8}$. This configuration also works for $L_{\{3,9\}}$ with $c = 1$ and $(b, t) = \left(\frac{16}{3}, \frac{22}{3}\right)$, giving $L_{\{3,8\}} \geqslant \frac{11}{8}$.

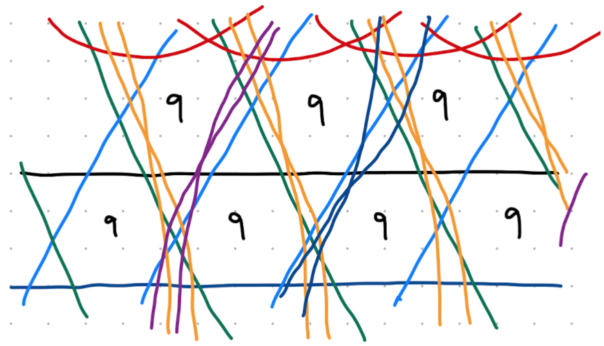

Figure 32: A stage in Configuration 6.12. The pattern repeats for every three incoming pseudolines (in the rolling pattern). There is a pair of yellow pseudolines for each incoming pseudoline, and a pair each of blue and purple pseudolines for every three incoming pseudolines.

Configuration 6.13. A multi-stage configuration for L_{10} has $c = 1$ and $(b, t, c) = (7, 2)$, which gives $L_{10} \geqslant \frac{2}{7}$. This configuration is also a configuration for $L_{\{3,10\}}$ with $c = 1$ and $(b, t) = (7, 9)$; therefore, $\widetilde{L}_{\{3,10\}} \geqslant \frac{9}{7}$.

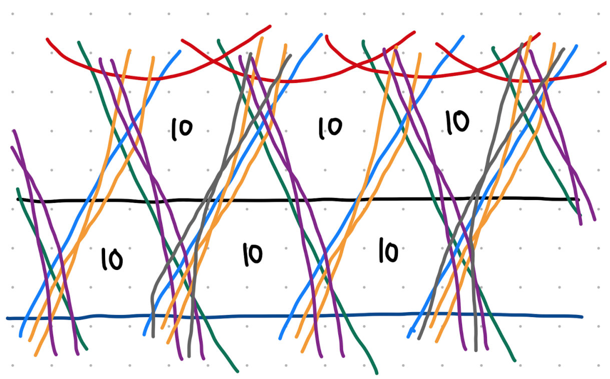

Figure 33: A stage in Configuration 6.13. The pattern repeats for every two incoming pseudolines. There is a pair of yellow pseudolines and a pair of purple pseudolines for every incoming pseudoline, and a pair of gray pseudolines for every two incoming pseudolines.

This configuration can be modified into a configuration for L_k by adding $k - 10$ lines per each 10-gon in a way similar to Configuration 6.6. This results in a configuration with $c = 1$ and $(b, t) = (2k - 13, 2)$, which gives $L_k \geqslant \frac{1}{k-6.5}$ for all $k \geqslant 7$.

This modified configuration also works for $\widetilde{L}_{\{3,k\}}$ with $c = 1$ and $(b, t) = (2k - 13, 2k - 11)$; therefore, $\widetilde{L}_{\{3,k\}} \geqslant \frac{2k-11}{2k-13} = 1 + \frac{1}{k-6.5}$ for all $k \geqslant 10$.

§7 Extendable polygonal chains

In previous sections we established bounds on the values $u_k(\ell, \mathcal{A})$ and $\widetilde{u}_k(h, \mathcal{A})$, which are global properties of pseudoline arrangements. In this section we seek to establish some local criteria: whether a pseudoline arrangement can contain certain local patterns. It will be easier to work with *zonotopal tilings* of $2n$-gons, which are duals of affine arrangements of n pseudolines. A pseudoline in an arrangement is dual to a set of parallel edges in a zonotopal tiling.

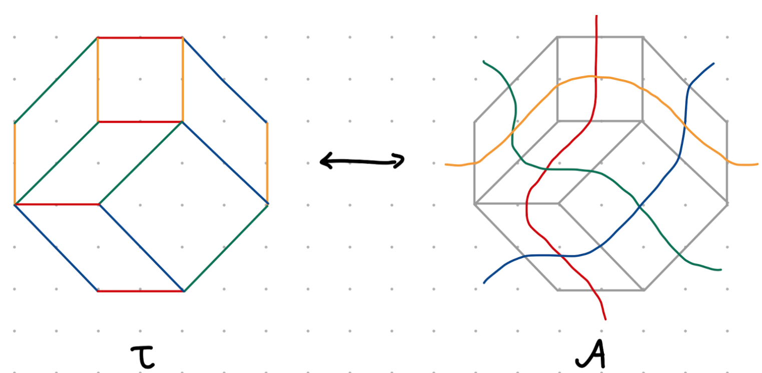

Figure 34: Zonotopal tilings are duals of affine pseudoline arrangements.

We study a class of closed polygonal chains corresponding to permutations of $\{1, 2, \dots, 2n\}$. Our main result, Theorem 7.8, is a simple criterion to determine whether a closed polygonal chain can be embedded in the 1-skeleton of a zonotopal tiling.

Throughout this section, let σ be a permutation of $\{1, 2, \dots, 2n\}$, and let $V = (\vec{v_1}, \dots, \vec{v_n}) \in \mathbb{R}^{2\times n}$ be an n-tuple of vectors satisfying $v_i = (x_i, 1)$ and $x_1 > x_2 > \cdots > x_n$. For brevity define $\vec{v}_{n+k} = -\vec{v}_k$ for each $k = 1, \dots, n$.

The set V and the permutation σ together define a *closed polygonal chain* $\Upsilon_{V,\sigma} : [0, 1] \rightarrow$

$\mathbb{R}^2$ given by linearly interpolating between the points $\Upsilon_{V,\sigma}\left(\frac{k}{2n}\right)$ defined by

$$\Upsilon_{V,\sigma}\left(\frac{k}{2n}\right) = \sum_{i=1}^{k} \vec{v}_{\sigma(k)},$$

for $k = 0, 1, \ldots, 2n$. For brevity we may use P_k to refer to $\Upsilon_{V,\sigma}\left(\frac{k}{2n}\right)$.

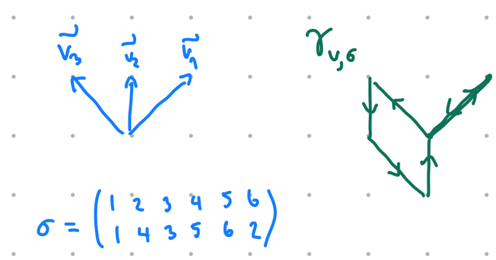

Figure 35: An example of a closed polygonal chain.

We now formally define the notion of embeddability.

Definition 7.1. A closed polygonal chain $\Upsilon_{V,\sigma}$ is said to be *embeddable* if there is a fine zonotopal tiling τ of $\mathrm{Zon}(V)$ such that (V, σ) is a closed curve in the 1-skeleton of τ. In this case we say $\Upsilon_{V,\sigma}$ is *embeddable in* τ.

Remark 7.2. Let τ be a zonotopal tiling and $\mathcal{A}$ be an affine pseudoline arrangement such that τ and $\mathcal{A}$ are dual to each other. If $\Upsilon_{V,\sigma}$ is embeddable in τ then $\Upsilon_{V,\sigma}$ corresponds to a closed curve in $\mathcal{A}$ which intersects every pseudoline exactly twice.

We introduce the notion of properness of closed polygonal chains to capture a class of closed polygonal chains that are close to a Jordan curve oriented counter-clockwise.

We first give an informal definition. Imagine that the segments of $\Upsilon_{V,\sigma}$ are walls and a person is walking along them while keeping their left hand on the wall, so that at time t the person's left hand is at position $\Upsilon_{V,\sigma}(t \bmod 1)$. A closed polygonal chain $\Upsilon_{V,\sigma}$ is said to be *proper* if the aforementioned person traces a simple closed curve, and the region bounded by this curve is on their left-hand side. We now give a formal definition of properness.

Definition 7.3. Let $\Upsilon_{V,\sigma}$ be a closed polygonal chain. For each $\varepsilon > 0$, define the oriented closed curve γ_ε as follows.

- Draw circles of radii ε around each vertex of $\Upsilon_{V,\sigma}$ and draw straight lines parallel to the edges of $\Upsilon_{V,\sigma}$ lying ε^2 to the left of each edge, oriented in the same way as the edge of $\Upsilon_{V,\sigma}$.
- Join these straight lines by arcs going counter-clockwise on the vertex circles; the resulting curve is γ_ε.

We call $\Upsilon_{V,\sigma}$ *proper* if for all sufficiently small $\varepsilon > 0$, the curve γ_ε is a Jordan curve with a counter-clockwise orientation.

Figure 36: The closed polygonal chain in this figure is proper because the curve γ_ε in green is a Jordan curve oriented counter-clockwise.

We now define a function θ_σ which roughly corresponds to the total angle turned by a person walking on $\Upsilon_{V,\sigma}(t)$ from $t = 0$ to $t = \frac{k}{2n}$.

Definition 7.4. The *discrete angle function* $\theta_\sigma : \mathbb{Z} \to \mathbb{Z}$ is defined by $\theta_\sigma(1) = \sigma(1)$ and

$$\theta_\sigma(k) = \begin{cases} \theta_\sigma(k-1) + \sigma(k) - \sigma(k-1) + 2n & \text{if } \sigma(k) \leqslant \sigma(k-1) - n, \\ \theta_\sigma(k-1) + \sigma(k) - \sigma(k-1) - 2n & \text{if } \sigma(k) > \sigma(k-1) + n, \text{ and} \\ \theta_\sigma(k-1) + \sigma(k) - \sigma(k-1) & \text{otherwise.} \end{cases}$$

for all $k \in \mathbb{Z}$, where $\sigma(k)$ is shorthand for $\sigma(k \bmod 2n)$.

It follows that $\theta_\sigma(k) \equiv \sigma(k) \pmod{2n}$ for all k, and $\left\lfloor \frac{\theta_\sigma(k) - \sigma(k)}{2n} \right\rfloor$ denotes the number of full counter-clockwise rotations made by the person. If $\Upsilon_{V,\sigma}$ is proper then the curve $\Upsilon_{V,\sigma}$

rotates exactly once counter-clockwise, which gives $\theta_\sigma(k+2n) = \theta_\sigma(k) + 2n$, implying that θ_σ is a bijection.

Finally, we define the quantity $\psi_\sigma(i,j)$ which encapsulates the order of $i, j, n+i, n+j$ in σ when viewed as a circular permutation. See Figure 37 for an example.

Definition 7.5. Given a permutation σ of $\{1, 2, \ldots, 2n\}$, the *baseball diagram* of σ is created by writing $\sigma(1), \ldots, \sigma(2n)$ in this order, equally spaced, clockwise around a circle, then drawing a chord labeled i joining the points i and $n+i$ for each $i = 1, 2, \ldots, n$.

Definition 7.6. For each $\{i, j\} \subseteq \{1, 2, \ldots, n\}$ with $i < j$, consider the counter-clockwise order of $i, j, n+i, n+j$ in the baseball diagram of σ, and define

$$\psi_\sigma(i,j) = \begin{cases} +1 & \text{if } i, j, n+i, n+j \text{ lie in this order,} \\ -1 & \text{if } j, i, n+j, n+i \text{ lie in this order, and} \\ 0 & \text{otherwise.} \end{cases}$$

Also define $\psi(\sigma) = \sum_{1 \leqslant i < j \leqslant n} \psi_\sigma(i,j)$.

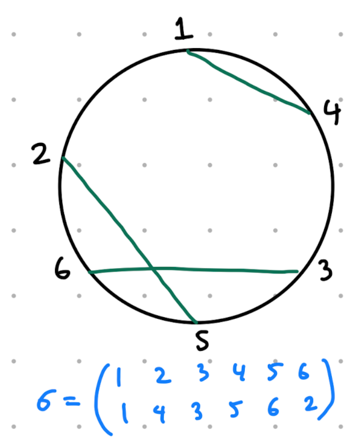

Figure 37: An example of a baseball diagram.

Observe that $\psi_\sigma(i,j) = 0$ if the chords labeled i and j in the baseball diagram of σ do not intersect. We will see later, in Proposition 7.13, that $\psi(\sigma)$ is equal to the number of tiles within $\Upsilon_{V,\sigma}$.

We can now characterize embeddable polygonal chains. We do so by introducing another notion–*neatness*–of proper closed polygonal chains $\Upsilon_{V,\sigma}$ based only on properties of the permutation σ, then showing that neatness is equivalent to embeddability.

Definition 7.7. We say a proper closed polygonal chain $\Upsilon_{V,\sigma}$ is *neat* if

- $\psi_\sigma(i,j) \geqslant 0$ for all $1 \leqslant i < j \leqslant n$, and
- $\theta_\sigma(k) > \theta_\sigma(\ell) - n$ for all $k > \ell$.

Theorem 7.8. *A proper closed polygonal chain is embeddable if and only if it is neat.*

The rest of this section is dedicated to proving Theorem 7.8. We start by showing that an embeddable proper closed polygonal chain must satisfy the first condition for neatness.

Proposition 7.9. If $\Upsilon_{V,\sigma}$ is proper and embeddable then $\psi_\sigma(i,j) \geqslant 0$ for all $1 \leqslant i < j \leqslant n$.

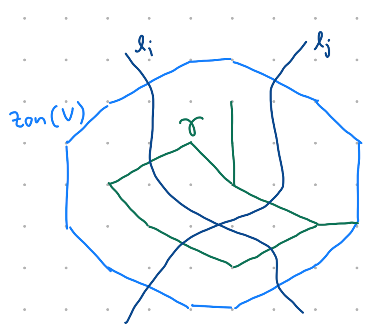

Figure 38: In this figure, ℓ_i and ℓ_j intersect in the interior of $\Upsilon_{V,\sigma}$. This implies $\psi_\sigma(i,j) = 1$.

Proof. Suppose $\Upsilon_{V,\sigma}$ is embeddable in a fine zonotopal tiling τ of $\mathrm{Zon}(V)$. Consider the pseudoline arrangement $\mathcal{A}$ which is the dual of τ, constructed by joining the midpoints of opposite sides of each tile of τ.

For each pair (i,j) with $1 \leqslant i < j \leqslant n$, consider the pseudolines ℓ_i, ℓ_j corresponding to edges parallel to $\vec{v}_i, \vec{v}_j$ in τ. The lines ℓ_i and ℓ_j intersect the boundary of $\mathrm{Zon}(V)$ at segments corresponding to vectors $\vec{v}_i, \vec{v}_j, \vec{v}_{n+i}$, and $\vec{v}_{n+j}$ in this counter-clockwise order. Consider the unique intersection of ℓ_i and ℓ_j. If this intersection lies in the interior of $\Upsilon_{V,\sigma}$ then ℓ_i and ℓ_j intersect $\Upsilon_{V,\sigma}$ in the same order as $\mathrm{Zon}(V)$, hence $\psi_\sigma(i,j) = 1$. If ℓ_i and ℓ_j intersect outside $\Upsilon_{V,\sigma}$ then the lines joining $\psi_\sigma(i,j) = 0$. $\square$

We now show that $\Upsilon_{V,\sigma}$ rotates by exactly π between each pair of parallel segments (which are segments corresponding to vectors $\vec{v}_k$ and $\vec{v}_{n+k}$ for some k).

Lemma 7.10. If $\Upsilon_{V,\sigma}$ is proper and embeddable then $\theta_\sigma^{-1}(k+n) > \theta_\sigma^{-1}(k)$ for all $k \in \mathbb{Z}$.

Proof. Consider the segments of $\Upsilon_{V,\sigma}$ corresponding to a pair of opposite vectors $\vec{v}_k$ and $\vec{v}_{n+k}$. Suppose that $\Upsilon_{V,\sigma}$ is embeddable in τ, and let ℓ_k be the pseudoline in the dual of τ corresponding to $\vec{v}_k$ and $\vec{v}_{n+k}$. The pseudoline ℓ_k only intersects segments of τ parallel to v_k. Therefore, ℓ_k intersects $\Upsilon_{V,\sigma}$ only at the segments corresponding to v_k and v_{n+k}.

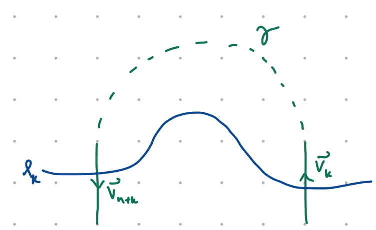

Figure 39: The curve $\Upsilon_{V,\sigma}$ must turn counter-clockwise by exactly π between the segments corresponding to vectors $\vec{v}_k$ and $\vec{v}_{n+k}$.

As $\Upsilon_{V,\sigma}$ is proper, it turns counter-clockwise around its interior. It follows that from the segment corresponding to $\vec{v}_k$ to the segment corresponding to $\vec{v}_{n+k}$, the curve $\Upsilon_{V,\sigma}$ must turn counter-clockwise by exactly π. Therefore $\theta_\sigma^{-1}(k+n) > \theta_\sigma^{-1}(k)$. $\square$

We also show that $\Upsilon_{V,\sigma}$ cannot rotate by more than π without containing two parallel segments.

Lemma 7.11. Let V be a collection of non-parallel vectors, and let P be a path in the 1-skeleton of a zonotopal tiling τ of $\mathrm{Zon}(V)$ consisting of non-parallel segments. Then the path P cannot turn by more than π.

Proof. Suppose for the sake of contradiction that there is a path in a tiling τ of $\mathrm{Zon}(V)$ which turns a total of $\alpha > \pi$, but which consists of nonparallel segments. For each such path P, consider the length k_P and the angle α_P of the path. As there are finitely many possible values of α for each fixed k, there must be an 'minimal' path P such that for all paths Q turning more than π, either

$$k_Q > k_P \quad \text{or} \quad \big(k_Q = k_P \text{ and } \alpha_Q < \alpha_P.\big)$$

Let $P : p_0 \to p_1 \to \cdots \to p_k$. Without loss of generality, assume P turns clockwise by more than π. The segment p_0p_1 must turn clockwise to p_1p_2, otherwise we may pick $Q = p_1 \to \cdots \to p_k$.

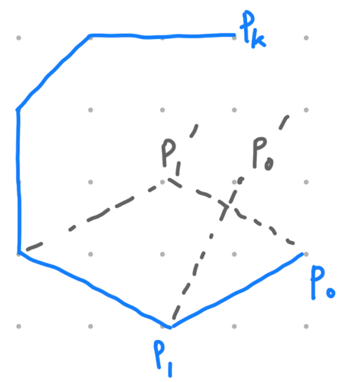

Figure 40: The path $P : p_0 \to \cdots \to p_k$ is supposed to be the minimal path turning by more than π, but either replacing p_0 by p_0' or replacing p_0 and p_1 by p_1' will violate the minimality of P.

Consider edges p_1p' of τ incident to p_1 such that $\overrightarrow{p_1p_2}, \overrightarrow{p_1p'}$, and $\overrightarrow{p_1p_0}$ lie in this order clockwise.

- If there are no such edges, then p_0p_1 and p_1p_2 must be sides of the same tile in τ. Consider the point p_1' opposite p_1 on this tile. It follows that $Q = p_1' \to p_2 \to \cdots \to p_k$ is shorter than P and still turns by more than π.
- On the other hand, if there is such an edge p_1p_0', we may replace p_0 by p_0' creating a path Q with $\alpha_Q > \alpha_P$.

In either case it follows that P cannot be the minimal path described. This is a contradiction. Therefore, such a path P cannot exist in the first place. □

Lemma 7.10 and Lemma 7.11 together imply that an embeddable proper closed polygonal chain must satisfy the second condition of neatness.

Proposition 7.12. If $\Upsilon_{V,\sigma}$ is proper and embeddable then $\theta_\sigma(k) > \theta_\sigma(\ell) - n$ for all $k > \ell$.

Proof. Let $\Upsilon_{V,\sigma}$ be an embeddable proper closed polygonal chain. Lemma 7.10 tells us that $\theta_\sigma(k)$ cannot be exactly $\theta_\sigma(\ell) - n$ for all $k > \ell$.

Assume for the sake of contradiction that there are $k > \ell$ such that $\theta_\sigma(k) < \theta_\sigma(\ell) - n$. Pick such a pair (k, ℓ) which minimizes $k - \ell$. We must have $k - \ell < 2n$ else we may replace k by $k - 2n$.

- If there are $t_1, t_2 \in \{\ell, \ell+1, \dots, k\}$ such that $\vec{v}_{\sigma(t_1)}, \vec{v}_{\sigma(t_2)}$ are parallel with $t_1 < t_2$, then $\theta_\sigma(t_2) = \theta_\sigma(t_1) + n$. It follows that either $\theta_\sigma(k) < \theta_\sigma(t_2) - n$ or $\theta_\sigma(t_1) < \theta_\sigma(\ell) - n$, and both cases violate the minimality of $k - \ell$.

- On the other hand, if the vectors $\vec{v}_{\sigma(\ell)}, \vec{v}_{\sigma(\ell+1)}, \dots, \vec{v}_{\sigma(k)}$ are pairwise nonparallel then Lemma 7.11 implies that $\Upsilon_{V,\sigma}$ cannot turn by more than π between $\Upsilon_{V,\sigma}\left(\frac{\ell}{2n}\right)$ and $\Upsilon_{V,\sigma}\left(\frac{k}{2n}\right)$. However, a path with $|\theta_\sigma(k) - \theta_\sigma(\ell)| > n$ necessarily turns by more than π, which is a contradiction.

It follows that for all pairs k, ℓ with $k > \ell$, the inequality $\theta_\sigma(k) > \theta_\sigma(\ell) - n$ holds. $\square$

It remains to prove that a neat proper closed polygonal chain must be embeddable. Our strategy is as follows. In Proposition 7.13, we show that we can divide the interior of $\Upsilon_{V,\sigma}$ into $\psi(\sigma)$ tiles. We then show in Lemma 7.14 that we can extend the interior of $\Upsilon_{V,\sigma}$ tile by tile until we obtain the entire zonotope $\mathrm{Zon}(V)$. This gives a tiling of $\mathrm{Zon}(V)$ which contains $\Upsilon_{V,\sigma}$ embedded in, and therefore $\Upsilon_{V,\sigma}$ must be embeddable.

Proposition 7.13. Let $\Upsilon_{V,\sigma}$ be a neat proper closed polygonal chain. Then the interior of $\Upsilon_{V,\sigma}$ can be divided into $\psi(\sigma)$ rhombic tiles with exactly one tile congruent to $\mathrm{Zon}(v_i, v_j)$ for each i, j with $\psi_\sigma(i, j) = 1$.

Proof. Consider the baseball diagram of σ. (See Figure 41.) Without loss of generality, suppose the chord c_1 joining 1 and $n+1$ is shortest. Let m be the minor arc subtended by c_1; assume it is counterclockwise from 1 to $n+1$. By the minimality of c_1, all other chords with an endpoint on m must have the other endpoint on the opposite side of c_1.

Without loss of generality let $\sigma(1) = 1$, so that $\theta(1) = 1$. Let k be such that $\sigma(k) = n+1$. By Lemma 7.10, $\theta(k) = n+1$. We claim that $1 < \theta(j) < n+1$ for all j such that $1 < j < k$. In fact, if $\theta(j) > n+1$ or $\theta(j) < 1$ then $\psi_\sigma(1, \sigma(j) \mod n) = -1$, which contradicts $\Upsilon_{V,\sigma}$ being neat. Therefore $\theta(j)$ lies between 1 and $n+1$ for all j between 1 and k.

Consider the path Q which is created by translating down the path from P_1 to P_{k-1} in $\Upsilon_{V,\sigma}$ by $-\vec{v}_1$. This path splits the interior of $\Upsilon_{V,\sigma}$ into two parts. (See Figure 42.)

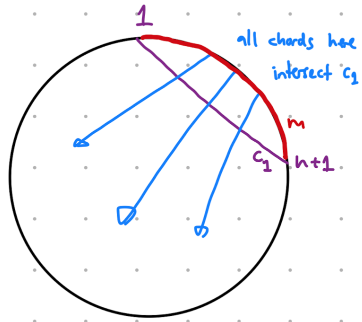

Figure 41: As c_1 is the shortest chord, all chords with an endpoint in m must intersect c_1.

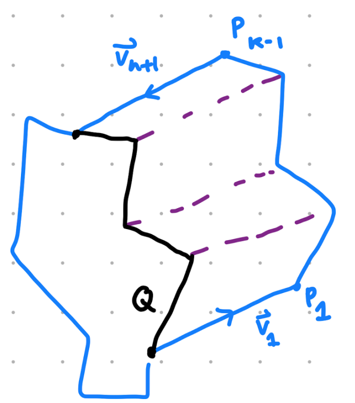

Figure 42: The path from Q divides the interior of $\Upsilon_{V,\sigma}$ into two parts.

- The part containing the segment parallel to $\vec{v}_1$ can easily be divided into tiles congruent to $\mathrm{Zon}(v_1, v_\ell)$ for $\ell = \sigma(2), \ldots, \sigma(k-1) \bmod n$, which are exactly the values of ℓ for which $\psi_\sigma(1, \ell) = +1$.
- The remaining part is a neat proper closed polygonal chain of $V' = V \setminus \{v_1\}$ and a permutation σ' of $\{2, \ldots, n\}$ with $\psi'_\sigma(i, j) = \psi_\sigma(i, j)$ for all $2 \leqslant i < j \leqslant n$.

We have effectively removed $\vec{v}_1$. We may repeat this process with V' until we have eliminated all vectors, thus completely dividing the interior of $\Gamma_{V,\sigma}$ into tiles, and there will be exactly one tile congruent to $\mathrm{Zon}(v_i, v_j)$ for each $i < j$ with $\psi_\sigma(i, j) = 1$. $\square$

Lemma 7.14. Let $\Upsilon_{V,\sigma}$ be a neat proper closed polygonal chain, and let $k \in \{1, \dots, 2n\}$ satisfy $\theta_\sigma(k+1) < \theta_\sigma(k)$. Define $\sigma' := \sigma \circ (k \;\; k+1)$, where $(k \;\; k+1)$ is the transposition swapping k and $k+1$. Then the following holds:

(i) $\Upsilon_{V,\sigma'}$ is also a neat proper closed polygonal chain,

(ii) the interior of $\Upsilon_{V,\sigma'}$ is formed by joining the tile $\mathrm{Zon}(\vec{v}_{\sigma(k)}, \vec{v}_{\sigma(k+1)})$ to the interior of $\Upsilon_{V,\sigma}$, and

(iii) $\psi(\sigma') = \psi(\sigma) + 1$.

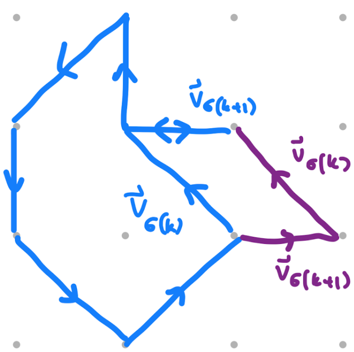

Figure 43: Swapping $\sigma(k)$ and $\sigma(k+1)$ adds a tile to the interior of $\Upsilon_{V,\sigma}$.

Proof. Swapping $\sigma(k)$ and $\sigma(k+1)$ only affects the parallelogram region bounded by $\vec{v}_{\sigma(k)}, \vec{v}_{\sigma(k+1)}$ in $\Upsilon_{V,\sigma}$. For $\Upsilon_{V,\sigma}$ to remain proper, it suffices to show that no other edge of $\Upsilon_{V,\sigma}$ can intersect this region. Indeed, if an edge $\vec{v}_{\sigma(t)}$ intersects this region then one of the paths joining $\vec{v}_{\sigma(t)}$ and $\vec{v}_{\sigma(k)}$ or $\vec{v}_{\sigma(k+1)}$ must turn clockwise by more than π, violating the condition that $\theta_\sigma(i) > \theta_\sigma(j) - n$ for all $i > j$. From this it also follows that the interior of $\Upsilon_{V,\sigma'}$ is formed by joining the tile $\mathrm{Zon}(\vec{v}_{\sigma(k)}, \vec{v}_{\sigma(k+1)})$ to the interior of $\Upsilon_{V,\sigma}$.

Now let $\sigma(k) \equiv a \bmod n$ and $\sigma(k+1) \equiv b \bmod n$. Swapping $\sigma(k)$ and $\sigma(k+1)$ in σ does not affect the order of $i, j, i+n, j+n$ for all $\{i,j\} \neq \{a,b\}$. Therefore, $\psi_{\sigma'}(i,j) = \psi_\sigma(i,j)$ for $\{i,j\} \neq \{a,b\}$. For $\psi_{\sigma'}(a,b)$ it is easy to verify that $\psi_{\sigma'}(a,b) = 1$ and $\psi_\sigma(a,b) = 0$. From these results it clearly follows that $\psi_{\sigma'}(i,j) \geqslant 0$ for all $1 \leqslant i < j \leqslant n$ and $\psi(\sigma') = \psi(\sigma) + 1$.

It remains to show that $\theta_{\sigma'}(i) > \theta_{\sigma'}(j) - n$ for all $i > j$. Observe that $\theta_\sigma(i) = \theta_{\sigma'}(i)$ except at $i \equiv k, k+1 \mod 2n$, in which case the values get swapped, so it suffices to check for pairs $\{i,j\} \equiv \{k, k+1\} \mod 2n$, which is straightforward. $\square$

We now give a proof of Theorem 7.8.

Proof of Theorem 7.8. Proposition 7.9 and Proposition 7.12 imply that an embeddable proper closed polygonal chain must be neat. Now suppose that $\Upsilon_{V,\sigma}$ is a neat proper closed polygonal chain. Proposition 7.13 implies that the interior of $\Upsilon_{V,\sigma}$ can be divided into tiles $\mathrm{Zon}(v_i, v_j)$ for pairs i, j with $\psi_\sigma(i, j) = 1$. If $\Upsilon_{V,\sigma}$ is not the boundary of the zonotope $\mathrm{Zon}(V)$ then θ_σ is not strictly increasing, so there is a k such that $\theta_\sigma(k+1) < \theta_\sigma(k)$. We apply Lemma 7.14, which adds one more tile to $\Upsilon_{V,\sigma}$. As this tile corresponds to a pair (i, j) with $\psi_\sigma(i, j) = 0$, it is a new tile. This process can be done until $\theta_\sigma(k)$ becomes strictly increasing, at which point we have a zonotopal tiling of $\mathrm{Zon}(V)$ which contains $\Upsilon_{V,\sigma}$ in its 1-skeleton. □

§8 Comments and remarks

While we stated our comprehensive results Theorem 5.5 and Theorem 5.6 for projective arrangements, most results also hold for affine arrangements.

A key difference between affine and projective arrangements is perhaps best demonstrated as follows. Let $\mathcal{A}$ be an affine arrangement of n pseudolines, and consider the projective pseudoline arrangement $\mathcal{A}^+$ of $n+1$ lines: the lines in $\mathcal{A}$, and the line at infinity ℓ_∞. Then the cells of $\mathcal{A}$ and $\mathcal{A}^+$ are almost exactly the same—an edge is missing for unbounded cells in $\mathcal{A}$ compared to their counterpart in $\mathcal{A}^+$ due to the existence of ℓ_∞.

The result is that unbounded cells in affine arrangements can behave weirdly. An example that is relevant to our results is that in affine arrangements, two 3-sided cells can be adjacent if at least one is unbounded. This affects the values of $\widetilde{L}_I$ and L_I for sets I containing 3. While for projective arrangements, $\widetilde{L}_3 = \frac{1}{2}$ and $L_3 = 1$, for affine arrangements, we instead have $\widetilde{L}_3 = 1$ and $L_3 = \frac{3}{2}$. The increased values of $\widetilde{L}_3$ and L_3 means that our bounds for $L_{\{3,k\}}$ and $\widetilde{L}_{\{3,k\}}$ become much weaker, and therefore the values of $L_{\{3,k\}}$ and $\widetilde{L}_{\{3,k\}}$ for affine arrangements are still open.

Nevertheless, our bounds on values of L_I for sets I such that $3 \notin I$ are the same for affine and projective arrangements, because these bounds are derived from Theorems Theorem 2.3, Theorem 2.2, Theorem 3.1, and Theorem 4.1. For the purpose of these theorems, the bounds for affine and projective arrangements only differ by constant terms, and a simple proof is given in Corollary 3.3.

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