The Balmer Spectrum of Stable Module Categories Over Finite Group Schemes

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Abstract

We give an introduction to the Balmer spectrum of a general tensor-triangulated category and a classification of the tensor-thick ideals of such a category. This framework relies on a notion of support for each object. We then summarize the work of Benson, Iyengar, and Krause regarding two equivalent notions of support, namely the triangulated support and the cohomological support. After this, we specialize to the case of the stable module category $\text{stmod}(\mathcal{A}(1))$ and use the previous results to give an explicit classification of the tensor-thick ideals of this category. These tensor-thick ideals are shown to correspond to the vanishing of Margolis homologies.

In the 1980s Hopkins and Smith proved the Thick Subcategory Theorem, which is a crucial component in the Periodicity Theorem in chromatic homotopy theory. The Thick Subcategory Theorem says that the thick subcategories of the stable homotopy category of finite p-local spectra are nested and correspond to the vanishing of Morava K-theories. Hopkins then made the important observation that their method of classification could be applied to a more algebraic context. To this end, he sketched a classification result for the thick subcategories of the derived category $D^b(\text{proj } R)$ of perfect complexes over a commutative noetherian ring R. More precisely, he established a correspondence between specialization closed sets subsets of proj R and the thick subcategories in the derived category. This approach was generalized by Balmer and Thomason to arbitrary *tensor-triangulated* (tt) categories. In his work, Balmer defined the "prime spectrum" of a tt-category. The Balmer spectrum consists of *prime tt-ideals*, which are thick subcategories that behave like prime ideals in a commutative ring. Balmer then formulated an analog of the support of a module. Using this notion of support, he established a bijection between radical tt-ideals in a tt-category and the supports of objects in the category. In this same direction, Benson, Carlson, and Rickard proved an analog of Hopkins's theorem in the stable category stmod(kG) of finitely generated kG-modules in [3], where kG is a finite group algebra. Recently, Benson, Iyengar, Krause, and Pevtsova extended this classification to finite group schemes in [5].

In this project we were interested in applying these results to the specific case of stmod($\mathcal{A}(1)$). The paper is structured as follows. In section 1, we introduce the Balmer spectrum of a tensor-triangulated category following [2]. We also state Balmer's classification of tt-ideals in such a category. After describing this general setting, in section 2 we introduce the triangulated support and the cohomological support of a compactly generated triangulated category and establish an equivalence between them. In section 3 we use this equivalence to give an explicit classification of the tt-ideals in stmod($\mathcal{A}(1)$). More specifically, we show that the Balmer spectrum consists of two prime ideals $(h_{10}, h_{11}, v), (h_{11}, v, w)$ of $\text{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$. We then show that the vanishing of Q_0-Margolis homology corresponds to the prime (h_{11}, v, w), while the vanishing of Q_1-Margolis homology corresponds to the prime (h_{10}, h_{11}, v). We end the paper with a section containing some work toward computing $\text{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, M)$. This is done via a spectral sequence formulated in terms of Q_1-Margolis homology.

1 Balmer Spectrum and Support

In this section we outline a framework that generalizes the Thick Subcategory Theorem of Hopkins-Smith. This framework relies on a notion of *support* for objects in a given category. The original notion of support comes from commutative algebra, in which the support of an R-module M is defined to be the set of primes $\mathfrak{p} \subset R$ such that $M_{\mathfrak{p}} \neq 0$. We will closely follow the exposition of [2].

Definition 1. An *additive category* in which all Hom-sets take values in the category of abelian groups. A *shift functor* Ω^{-1} on a category T is an automorphism $T \to T$. A *triangle* (X, Y, Z, u, v, w) consists of the following data:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Omega^{-1}(X)$$

A *triangulated category* is an additive category T with a shift functor Ω^{-1} and a class of triangles characterized by certain properties.

Given an arbitrary triangulated category T, a *thick subcategory* S is a full subcategory of T which is closed under finite direct sums and summands. We will call a subcategory $S \subset T$ *localizing* if it is thick and closed under small coproducts. Similarly, we will call a subcategory *colocalizing* if it is thick and closed under small products.

Definition 2. A *tensor-triangulated category* is a triple $(\mathcal{K}, \otimes, 1)$ consisting of a triangulated category $\mathcal{K}$, a symmetric monoidal product $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ which is exact in each variable. We will refer to this product as the "tensor product," although in arbitrary categories it may not take the form of the usual tensor product in commutative algebra. The unit is denoted by 1.

We say that a thick subcategory of a tensor-triangulated category is *tensor ideal* if it is closed under tensor product with compact objects. We will also refer to such a thick subcategory as a *tt-ideal* for short, "tt" being shorthand for "tensor-thick." Indeed, tt-ideals behave in a category much like ideals behave in a ring, with closure under tensor product being the analog of multiplicative absorptivity of ideals. A proper tt-ideal $\mathcal{P} \subsetneq \mathcal{K}$ in a $\otimes$-triangulated category $\mathcal{K}$ is *prime* if $a \otimes b \in \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$. We denote by $\mathrm{Spc}(\mathcal{K})$ the set of primes in $\mathcal{K}$. A proper tt-ideal $\mathcal{P} \subsetneq \mathcal{K}$ is *radical* if $a \otimes b \in \mathcal{P}$ implies that some $\otimes$-power of a or b lies in $\mathcal{P}$.

Definition 3. Given an object $a \in \mathcal{K}$, the *support* of a is $\mathrm{supp}(a) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid a \notin \mathcal{P}\}$.

Having defined the support of a given object in our category, we can endow a topology on $\mathrm{Spc}(\mathcal{K})$ just as in algebraic geometry. We define the *Zariski topology* on $\mathrm{Spc}(\mathcal{K})$ to be generated by the basis elements $\{U(a) := \mathrm{Spc}(\mathcal{K}) \setminus \mathrm{supp}(a) \mid a \in \mathcal{K}\}$. With this topology, we call the resulting space $\mathrm{Spc}(\mathcal{K})$ the *prime spectrum* of $\mathcal{K}$. This prime spectrum satisfies a certain universal property:

Theorem 4 (universal property, [2])**.** *(a) $supp(0) = \emptyset$ and $supp(1) = Spc(\mathcal{K})$*

(b) $supp(a \oplus b) = supp(a) \bigcup supp(b)$

(c) $supp(\Omega^{-1}a) = supp(a)$, where Ω^{-1} is the shift or suspension operator in $\mathcal{K}$

(d) $supp(a) \subset supp(b) \bigcup supp(c)$ for an exact triangle $a \to b \to c \to \Omega^{-1}a$

(e) $supp(a \otimes b) = supp(a) \cap supp(b)$

For any pair (X, σ), where X is a topological space and σ an assignment of closed subsets $\sigma(a) \subset X$ to objects $a \in \mathcal{K}$, satisfying the properties above, there exists a unique continuous map $f : X \to Spc(\mathcal{K})$ such that $\sigma(a) = f^{-1}(supp(a))$.

The main result of [2] is the following classification:

Theorem 5 (classification of tt-ideals, [2])**.** *Let $\mathfrak{S}$ be the set of those subsets $Y \subset Spc(\mathcal{K})$ which are unions $Y = \bigcup_{i \in I} Y_i$ of closed subsets Y_i with quasi-compact complement $Spc(\mathcal{K}) \setminus Y_i$ for all $i \in I$. Let $\mathfrak{R}$ be the set of radical tt-ideals of $\mathcal{K}$. Then there is an order-preserving bijection $\mathfrak{S} \to \mathfrak{R}$ given by $Y \mapsto \mathcal{K}_Y := \{a \in \mathcal{K} \mid supp(a) \subset Y\}$ with inverse $\mathcal{J} \mapsto supp(\mathcal{J}) := \bigcup\limits_{a \in \mathcal{J}} supp(a)$.*

In [6] this classification is applied to the case of $\mathrm{stmod}(kG)$ to yield a correspondence between tt-ideals and specialization closed subsets of $\mathcal{V}_G$, which is described in section 3.

2 Equivalence of Supports

In this section we summarize [4] to relate *triangulated support* and *cohomological support*. The main result of this section, theorem 16, is an equivalence between the two notions of support, which allows one to study an object in a tensor-triangulated category by studying its group cohomology. We will closely follow the exposition of [4].

Definition 6. Each R-module M is guaranteed to admit a minimal injective resolution, unique up to quasi-isomorphism. We define the *cohomological support* of M to be

$$\mathrm{supp}_R M = \{\mathfrak{p} \in \mathrm{Spec}\ R \mid \mathfrak{p} \text{ occurs in a minimal injective resolution of } M\}$$

At first glance, this notion of support is vastly different from the usual one in commutative algebra, but Lemma 2.2(1) in [4] shows that they are nearly equivalent:

Lemma 7.

$$\mathrm{supp}_R M \subset \mathrm{cl}(\mathrm{supp}_R M) = \{\mathfrak{p} \in \mathrm{Spec}\ R \mid M_{\mathfrak{p}} \neq 0\},$$

where cl denotes the topological closure. We have equality above when M is finitely generated.

The set of primes defined above can be seen to be the usual support of M in Spec R, so we see that the cohomological support is a subset of this support. Furthermore, in the setting of subsection 3.4, all of the $\mathcal{A}(1)$-modules we consider will be finitely generated, so the two notions of support will be equivalent by the lemma above. Now we define triangulated support, which is formulated in terms of localization functors:

Definition 8. An exact functor $L : T \to T$ is a *localization functor* if there exists a morphism $\eta : \mathrm{id}_T \to L$ such that the morphism $L\eta : L \to L^2$ is idempotent and $L\eta = \eta L$. Let $L : T \to T$ be a localization functor. An object $X \in T$ is *L-local* if ηX is an isomorphism, and it is *L-acyclic* if $LX = 0$.

In particular, we will be interested in the L-acyclic component of objects in our category:

Definition 9. For each $X \in T$, complete the map ηX to an exact triangle

$$\Gamma X \to X \to LX \to .$$

The following lemma shows that the functor $\Gamma : T \to T$ defined in this manner is actually well-defined.

Lemma 10. The functor Γ is exact and has the following properties:

(1) $X \in T$ is L-acyclic if and only if $\mathrm{Hom}_T(X, -) = 0$ on L-local objects

(2) $Y \in T$ is L-local if and only if $\mathrm{Hom}_T(-, Y) = 0$ on L-acyclic objects

(3) Γ is a right adjoint to the inclusion ker $L \to T$

(4) L is a left adjoint to the inclusion im $L \to T$

Proof. (see [4]) $\square$

As a result, ΓX is well-defined for every $X \in T$ and is related to the kernel of the localization functor L. We will describe the connection between localization and support using this relation to the kernel of L, but first we will need to define local cohomology:

Definition 11. The *center* of a compactly generated triangulated category T, denoted $Z(T)$, is a graded-commutative ring with nth component

$$Z(T)^n := \{\eta : \mathrm{id}_T \to (\Omega^{-1})^n \mid \eta\Omega^{-1} = (-1)^n \Omega^{-1}\eta\}$$

Having defined the center $Z(T)$ of our category, we now fix a graded-commutative noetherian ring R and a homomorphism of graded rings $R \to Z(T)$. With these data, we will call T an *R-linear triangulated category.* Now let $C \in T$. For each object $X \in T$, define

$$H^*_C(X) := \mathrm{Hom}^*_T(C, X)$$

It turns out that $H^*_C(-)$ is not entirely dependent upon C itself. Rather, the following lemma shows that $H^*_C(-)$ depends almost entirely on the thick subcategory generated by C, i.e., the (thick) intersection of all thick subcategories of T containing C.

Lemma 12. Let $\{C\}$ be a set of objects in T and C_0 an object contained in the thick subcategory generated by $\{C\}$. Then for each $X \in T$ one has

$$\mathrm{supp}_R H^*_{C_0}(X) \subset \bigcup_{C \in \{C\}} \mathrm{cl}(\mathrm{supp}_R H^*_C(X)),$$

where supp_R denotes the cohomological support defined in definition 6. For our purposes we will take T to be a compactly generated category, generated by $C \in T$.

Recall that a subcategory is localizing (colocalizing) if it is thick and closed under small coproducts (products). We will be interested in localizing subcategories of the form

$$T_U = \{X \in T \mid \mathrm{supp}_R H^*_C(X) \subset U \text{ for each } C \in T^c\},$$

where $U \subset \mathrm{Spec}\ R$ and T^c denotes the full subcategory of compact objects in T. It turns out that if we set $U(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}\ R \mid \mathfrak{q} \subset \mathfrak{p}\}$, then the subcategory $T_{U(\mathfrak{p})}$ is localizing and colocalizing.

Proposition 13. Let $V \subset \mathrm{Spec}\ R$ be specialization closed. There exists a localization functor $L_V : T \to T$ with the property that $L_V X = 0$ if and only if $X \in T_V$.

Proof. (see Proposition 4.5, [4]) □

Definition 14. Let V be a specialization closed subset of Spec R and L_V the associated localization functor given by the previous proposition. Then we get an exact functor Γ_V on T and for each object $X \in T$ a natural exact triangle

$$\Gamma_V X \to X \to L_V X \to .$$

We can now define a notion of triangulated support:

Definition 15. The *triangulated support* of $X \in T$ is defined to be the set

$$\mathrm{tsupp}_R X = \{\mathfrak{p} \in \mathrm{Spec}\ R \mid \Gamma_{\mathfrak{p}} X \neq 0\} \subset \mathrm{Spec}\ R$$

Now we have all the necessary ingredients to state the main theorems of [4]:

Theorem 16 (Theorem 5.15(1), [4])**.** *There exists a unique assignment sending each object $X \in T$ to a subset $tsupp_R X \subset Spec\ R$ such that $cl(tsupp_R X) = cl(supp_R H^*(X))$.*

This theorem tells us that if we want to understand the triangulated support of an object in our category, we can equivalently study the cohomological support of the group cohomology $H^*_C(X)$. We also have the following remarkable result (Corollary 8.3, [4]):

Theorem 17. *Let $\mathfrak{p} \in Spec\ R$. For each object $X \in T$ one has a natural isomorphism*

$$\Gamma_{\mathfrak{p}} X \cong X \otimes \Gamma_{\mathfrak{p}} \mathbb{1},$$

where $\mathbb{1}$ is the unit of the tensor-triangulated category T.

In chromatic homotopy theory, there are localization functors L_n, corresponding to certain spectra $E(n)$, with the property that for any spectrum X,

$$L_n X \cong X \wedge L_n \mathbb{S},$$

where $\mathbb{S}$ is the sphere spectrum. This is referred to as the smash product theorem. Notice that $\mathbb{S}$ serves as the unit in the category of CW spectra, the smash product serves as the tensor, and L_n plays the role of a $\Gamma_{\mathfrak{p}}$. The functors L_n are useful in chromatic homotopy theory specifically because they have this property, which is not true of all such localization functors. The isomorphism in *theorem* 17 explains why the L_n are so special. Each L_n is a localization functor of the form $\Gamma_{\mathfrak{p}}$, where $\mathfrak{p}$ is a point in the prime spectrum of $\mathrm{End}_{\mathrm{cw}_p}(\mathbb{S})$.

3 Applications to stmod($\mathcal{A}(1)$)

In this section we apply Balmer's theory of prime spectra to study the tt-ideals in the stable module category stmod($\mathcal{A}(1)$).

3.1 Stable Module Categories

Definition 18. Let k be a field and G be an affine group scheme. The *group algebra* kG is defined to be the dual of the coordinate ring $k[G]$ of G. This group algebra has a natural ring structure using the structures of k and G.

Furthermore, every group algebra kG naturally has the structure of a *Hopf algebra* $(\Gamma, k, \mu, \eta, \Delta, \varepsilon, S)$. A Hopf algebra is a bialgebra Γ over a field k with algebra structure given by (μ, η) and coalgebra structure given by (Δ, ε), along with an *antipode* $S : \Gamma \to \Gamma$ such that

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\mathrm{id} \otimes S) \circ \Delta$$

Definition 19. Let $\mathrm{Mod}(kG)$ denote the *module category over kG*. The objects in this category are left kG-modules, and the morphisms are kG-module morphisms. We also define $\mathrm{StMod}(kG)$, the *stable module category over kG*. This category also has as objects left kG-modules, but the morphisms are now defined to be *stable morphisms*: we say that $f : M \to N$ and $g : M \to N$ are equivalent morphisms of kG-modules if $f - g : M \to N$ factors through a projective kG-module. An equivalence class defined by this relation will be one such "stable morphism." Within the category $\mathrm{StMod}(kG)$ we define $\mathrm{stmod}(kG)$ to be the full subcategory of finitely generated modules in $\mathrm{StMod}(kG)$.

Notice that "modding out projectives" as we did in defining $\mathrm{StMod}(kG)$ plays a similar role to considering contractible spaces to be equivalent to one-point spaces in algebraic topology. More explicitly, suppose we have a kG-module M and a projective module P admitting a surjective map $p : P \to M$. Using $\Omega(M) = \ker p$, which can be shown to be well-defined and solely dependent on M itself, we get a short exact sequence

$$0 \to \Omega(M) \to P \to M \to 0 \tag{1}$$

In the setting of algebraic topology, given a space X we can similarly construct a Serre fibration $\Omega X \to PX \to X$, where ΩX is the loopspace of X and PX is the pathspace of X. By construction, the pathspace PX will be contractible. Then we can make an analogy between Equation 1 above and our fibration $\Omega X \to PX \to X$ by identifying $\Omega(M)$ with ΩX and P with PX.

By definition, the thick subcategories of $\mathrm{stmod}(kG)$ are in bijective correspondence with the subcategories $\mathfrak{c}$ of $\mathrm{mod}(kG)$ satisfying:

(i) $\mathfrak{c}$ contains all projective modules

(ii) $\mathfrak{c}$ is closed under finite direct sums and summands

(iii) If $M \in \mathfrak{c}$, then so are $\Omega(M)$ and $\Omega^{-1}(M)$, where $\Omega^{-1}(M)$ is the shift functor providing the triangulated structure on $\mathrm{mod}(kG)$

(iv) If $0 \to A \to B \to C \to 0$ is a short exact sequence of modules with two of A, B, and C in $\mathfrak{c}$, then so is the third

Definition 20. The *cohomology ring of G over k* $H^*(G, k)$ is defined to be the ring $\mathrm{Ext}^*_{kG}(k, k)$. This is a finitely generated graded commutative ring, so we define $\mathcal{V}_G$ to be the spectrum of homogeneous prime ideals $\mathfrak{p} \in H^*(G, k)$. A subset $\mathcal{V} \subset \mathcal{V}_G$ is said to be *specialization closed* if $\mathfrak{p} \in \mathcal{V}$, $\mathfrak{q} \in \mathcal{V}_G$, and $\mathfrak{q} \supset \mathfrak{p}$ together imply that $\mathfrak{q} \in \mathcal{V}$.

With these definitions established, we can state the main classification theorem, which is theorem 2.7 in [6]:

Theorem 21 (correspondence)**.** *There is a bijective correspondence between tt-ideals of stmod(kG) and non-empty specialization closed subsets of $\mathcal{V}_G$.*

3.2 The Balmer Spectrum of stmod$(\mathcal{A}(1))$

Recall that $\mathcal{A}(1)$ is defined to be the subalgebra of the Steenrod algebra $\mathcal{A}$ generated by the Steenrod squares Sq^1 and Sq^2. Note also that $\mathcal{A}(1)$ inherits a Hopf algebra structure from $\mathcal{A}$ with unit $\eta : \mathbb{F}_2 \to \mathcal{A}(1)$, multiplication $\mu : \mathcal{A}(1) \otimes \mathcal{A}(1) \to \mathcal{A}(1)$, counit $\varepsilon : \mathcal{A}(1) \to \mathbb{F}_2$, and comultiplication $\Delta : \mathcal{A}(1) \to \mathcal{A}(1) \otimes \mathcal{A}(1)$. The first two maps come from the usual algebra structure on $\mathcal{A}(1)$, and the last two are given by

$$\varepsilon : (a_0 + a_1\mathrm{Sq}^1 + a_2\mathrm{Sq}^2 + \dots) \mapsto a_0$$

and

$$\Delta : \mathrm{Sq}^k \mapsto \sum_{i+j=k} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$$

This comultiplication map Δ is induced by the Cartan formula for Steenrod squares, and we will say that an element $a \in \mathcal{A}(1)$ is *primitive* if $\Delta(a) = 1 \otimes a + a \otimes 1$. The idea behind the term "primitive" is that, since the Cartan formula is a sum over all the decompositions of a square, if the only such decompositions involve the square and the identity, this element is indecomposable in some sense.

Now, it is clear from the correspondence in theorem 21 that if we wish to understand the tt-ideals of stmod$(\mathcal{A}(1))$, we must understand the homogeneous primes in $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$. There are two (equivalent) ways of approaching this computation – we can either compute this Ext over $\mathcal{A}(1)$-modules, or we can compute $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \mathbb{F}_2)$ over $\mathcal{A}(1)_*$-comodules, where $\mathcal{A}(1)_*$ is the linear dual $\mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}(1), \mathbb{F}_2)$ of $\mathcal{A}(1)$. The second method yields the well-known result:

$$\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_{10}, h_{11}, v, w]/(h_{10}h_{11}, h_{11}^3, v^2 + h_{10}^2 w, vh_{11})$$

This Ext is bigraded with

$$|h_{10}| = (1, 1), |h_{11}| = (1, 2), |v| = (3, 7), |w| = (4, 12)$$

Since homogeneous primes in this ring must respect both components of the bidegree on each element, a quick exercise in linear algebra shows that the only such primes are given by

$$(h_{10}, h_{11}, v), (h_{11}, v, w)$$

Technically we should also be considering the irrelevant ideal consisting of positive-degree elements, but this ideal does not support any non-projective modules. Then since we are working in the stable category, we will disregard this ideal. We now claim that these two primes correspond to the vanishing of Q_i-Margolis homology, which will be proven in subsection 3.4.

3.3 Margolis Homology

Before making computations, we first introduce the notion of Margolis homology since it provides a useful criterion for projectivity of $\mathcal{A}(1)$-modules. Define $Q_0 := \mathrm{Sq}^1$ and $Q_1 := \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$. These two elements are primitive in the sense defined in subsection 3.2, namely $\Delta(Q_i) = 1 \otimes Q_i + Q_i \otimes 1$ for $i = 0, 1$. Furthermore, these elements can also be viewed as differentials in a chain complex. Given an $\mathcal{A}(1)$-module M, we can arrange copies of M into a sequence

$$\dots \xrightarrow{Q_i} M \xrightarrow{Q_i} M \xrightarrow{Q_i} M \xrightarrow{Q_i} \dots \tag{2}$$

for fixed i. That this is in fact a chain complex follows from

$$\begin{aligned} Q_0^2 &= \mathrm{Sq}^1\mathrm{Sq}^1 = 0, \\ Q_1^2 = (\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1)^2 &= \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^1 \\ &\quad + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 \\ &= 2\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 \\ &= 0, \end{aligned}$$

where the first cancellation follows from the Steenrod relations $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^2 = 0$ and $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$, and the second comes from the Steenrod relation $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1$ and the fact that everything is 2-torsion. As a result, we know that Equation 2 is indeed a chain complex, so we can use it to compute the following:

Definition 22 (Margolis)**.** Given an $\mathcal{A}(1)$-module M, the Q_i-homology of the corresponding chain complex in Equation 2 is called the *Q_i-Margolis homology* and will be denoted $H(M; Q_i)$.

It should be noted that while the grading on homology groups typically comes from the homological degree, in this case the grading comes from the internal grading of M as an $\mathcal{A}(1)$-module. This notion of homology is extremely useful in determining whether a given $\mathcal{A}(1)$-module is projective, and we will make use of the following result without proof (see [9]).

Proposition 23. An $\mathcal{A}(1)$-module M is projective if and only if $H(M; Q_i) = 0$ for both $i = 0, 1$.

As we are working with *stable* modules, this is equivalent to saying that M lies in the equivalence class of the trivial module 0 in $\mathrm{stmod}(\mathcal{A}(1))$ if and only if $H(M; Q_i) = 0$ for $i = 0, 1$. This notion of homology is especially useful since it admits Künneth isomorphisms:

Proposition 24. We have Künneth isomorphisms for both $i = 0, 1$:

$$H(M \otimes N; Q_i) \cong H(M; Q_i) \otimes H(N; Q_i)$$

3.4 Classification of Thick Subcategories of stmod$(\mathcal{A}(1))$

In this section we classify the tt-ideals in $\mathrm{stmod}(\mathcal{A}(1))$ using theorem 21. Our strategy is as follows. We first establish that $\{$modules with vanishing Q_i-homology$\}$ is a tt-ideal for both $i = 0, 1$, at which point it will be clear that the primes listed in subsection 3.2 correspond to the vanishing of Margolis homology. In order to pin down which prime corresponds to which tt-ideal, we will use a vanishing result of Adams to make one match. Since there are only two primes, this will complete the classification. First, we have the following:

Proposition 25. The subcategories $C_i := \{$modules with vanishing Q_i-homology$\} \subset \mathrm{stmod}(\mathcal{A}(1))$ for $i = 0, 1$ are tt-ideals.

Proof. We first verify that each C_i is a thick subcategory. To that end, note that given $M, N \in \mathrm{stmod}(\mathcal{A}(1))$, we can construct a chain complex

$$\ldots \xrightarrow{d} M \oplus N \xrightarrow{d} M \oplus N \xrightarrow{d} \ldots,$$

where $d = (Q_i, Q_i)$. The $\mathcal{A}(1)$-module $M \oplus N$ is certainly in $\mathrm{stmod}(\mathcal{A}(1))$ as both $M, N \in \mathrm{stmod}(\mathcal{A}(1))$, so this is a well-defined construction in our category. Taking the homology of this chain complex can be seen to give

$$H(M \oplus N; Q_i) \cong H(M; Q_i) \oplus H(N; Q_i)$$

Then $H(M; Q_i) = H(N; Q_i) = 0$ if and only if $H(M \oplus N; Q_i) = 0$. Now note that a finite direct sum of modules in $\mathrm{stmod}(\mathcal{A}(1))$ will again lie in $\mathrm{stmod}(\mathcal{A}(1))$. By induction, this argument extends to any finite direct sum. Then what we have shown is that each C_i is closed under finite direct sums and summands, so each C_i is thick. To see that each C_i is also tensor ideal, we make use of the Künneth isomorphism for Margolis homology described in proposition 24:

$$H(M \otimes N; Q_i) \cong H(M; Q_i) \otimes H(N; Q_i)$$

Then if $M \in C_i$, so is $M \otimes N$, so we see that each C_i is a tt-ideal. □

As a result, the vanishing of Q_0-homology must correspond to one of the primes $(h_{10}, h_{11}, v), (h_{11}, v, w)$ while the vanishing of Q_1-homology must correspond to the other. By the correspondence in theorem 21, we could determine the precise pairing by taking localizations of Ext groups, but we can instead circumvent these calculations by making use of the following results:

Theorem 26 (Adams, [1])**.** *If an $\mathcal{A}(1)$-module M has vanishing Q_0-homology, then*

$$h_{10}^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{F}_2) = 0$$

It should be noted that the M and $\mathbb{F}_2$ are in the “wrong” positions in the equation above, as we are concerned with $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, M)$, but we can get around this small technicality. Namely, if A, B are $\mathcal{A}(1)$-modules, there is a canonical isomorphism

$$\mathrm{Ext}_{\mathcal{A}(1)}(A, B) \cong \mathrm{Ext}_{\mathcal{A}(1)}(A \otimes_{\mathbb{F}_2} B_*, \mathbb{F}_2),$$

where $B_* := \mathrm{Hom}_{\mathbb{F}_2}(B, \mathbb{F}_2)$ is the linear dual of B. In our case, we set $A = \mathbb{F}_2$ and $B = M$ to get the isomorphism

$$\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, M) \cong \mathrm{Ext}_{\mathcal{A}(1)}(M_*, \mathbb{F}_2)$$

Furthermore, the following lemma relates the Q_0-homology of M and its linear dual M_*:

Lemma 27. Given $M \in \mathrm{stmod}(\mathcal{A}(1))$, if we define $M_* := \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$, then $H(M; Q_0) = 0$ if and only if $H(M_*; Q_0) = 0$.

Proof. First note that M_* naturally has the structure of a right $\mathcal{A}(1)$-module on which $\mathcal{A}(1)$ acts “in reverse.” For example, given $m \in M$ such that $m = \mathrm{Sq}^1 m'$, if we denote by $\alpha_m, \alpha_{m'} \in M_*$ the linear duals of m, m', then the action of $\mathcal{A}(1)$ on M_* is by precomposing:

$$\alpha_m \cdot \mathrm{Sq}^1 : m' \mapsto \mathrm{Sq}^1 m' \mapsto \alpha_m(\mathrm{Sq}^1 m') = \alpha_m(m) = 1$$

In other words, if $m = \mathrm{Sq}^1 m'$, then Sq^1 acts on α_m by turning it into $\alpha_{m'}$, essentially “inverting” Sq^1. Now, let $H(M; Q_0) = 0$ so that $\mathrm{Sq}^1 m = 0$ implies $m = \mathrm{Sq}^1 m'$ for every such $m \in M$. Given $\ell \in M_*$ with $\ell \cdot \mathrm{Sq}^1 = 0$, we know that the element in M to which ℓ corresponds, say m_ℓ, does not lie in $\mathrm{Sq}^1 M$ by definition of the Sq^1-action on M_*.

We claim that $\mathrm{Sq}^1 m_\ell \neq 0$. Otherwise, if $\mathrm{Sq}^1 m_\ell = 0$, we'd have $\mathrm{Sq}^1 m_\ell = 0$ *and* $m_\ell \notin \mathrm{Sq}^1 M$, meaning m_ℓ survives to the homology $H(M; Q_0)$, a contradiction. Therefore, $\ell \cdot \mathrm{Sq}^1 = 0$ implies that $\ell \in M_* \cdot \mathrm{Sq}^1$ for every such $\ell \in M_*$, meaning $H(M_*; Q_0) = 0$. The same reasoning gives the converse, so we have the desired equivalence. □

Therefore, if $M \in \mathrm{stmod}(\mathcal{A}(1))$ has vanishing Q_0-homology, then so does M_*, while theorem 26 says that

$$h_{10}^{-1}\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, M) \cong h_{10}^{-1}\mathrm{Ext}_{\mathcal{A}(1)}(M_*, \mathbb{F}_2) = 0$$

This means that for every element in $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, M)$ there is some power of h_{10} which annihilates it. Then we know that, upon localizing $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, M)$ at $\mathfrak{p} = (h_{11}, v, w)$, every element is killed by some element in the complement of $\mathfrak{p}$. In other words, $(\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, M))_{\mathfrak{p}} = 0$, so the kernel of the localization map $\mathrm{Ext} \to \mathrm{Ext}_{\mathfrak{p}}$ is nonzero. Therefore, (h_{11}, v, w) corresponds to the vanishing of Q_0-homology. As we only had two options to begin with, we can then conclude that (h_{10}, h_{11}, v) corresponds to the vanishing of Q_1-local homology.

4 Further Directions

In this section we make use of the Cartan-Eilenberg spectral sequence to compute $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, M)$. Some of the details still need to be worked out.

Theorem 28. *Given Hopf algebras B and C over $\mathbb{F}_2$, there is a spectral sequence, called the Cartan-Eilenberg spectral sequence, with*

$$E_2^{s,t,u} = Ext^s_{C//B}(\mathbb{F}_2, Ext^{t,u}_B(\mathbb{F}_2, M)) \Rightarrow Ext^{s+t,u}_C(\mathbb{F}_2, M)$$

Here, $C//B := C \otimes_B \mathbb{F}_2$ denotes the "quotient Hopf algebra." It can be seen from the form of the E_2 page above that simplifying the computation of the Ext term on the right hand side can be achieved if we pick a convenient splitting of C. In our case, we are considering $C = \mathcal{A}(1)$, so we wanted to find a convenient sub-Hopf algebra $B \subset \mathcal{A}(1)$. We chose $B = E[Q_1]$, where $E[x]$ denotes the exterior algebra $\mathbb{F}_2[x]/(x^2)$. The resulting quotient Hopf algebra is given by $HQ_1 := C//B$. This algebra consists of five generators $\{1, \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^1\mathrm{Sq}^2, \mathrm{Sq}^2\mathrm{Sq}^1\}$ subject to the usual Steenrod relations as well as the relation $\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^2\mathrm{Sq}^1$. We will work from the inside out to compute

$$\mathrm{Ext}^s_{HQ_1}(\mathbb{F}_2, \mathrm{Ext}^{t,u}_{E[Q_1]}(\mathbb{F}_2, M))$$

Then we first build a free resolution of $\mathbb{F}_2$ over $E[Q_1]$:

$$0 \leftarrow \mathbb{F}_2 \leftarrow E[Q_1] \leftarrow \Sigma^3 E[Q_1] \leftarrow \Sigma^6 E[Q_1] \leftarrow \dots,$$

where the maps $\Sigma^{u+3}E[Q_1] \to \Sigma^u E[Q_1]$ are all given by sending $1 \mapsto Q_1$. Applying $\mathrm{Hom}_{E[Q_1]}(-, M)$ to this resolution we get a complex

$$0 \to \mathrm{Hom}_{E[Q_1]}(E[Q_1], M) \xrightarrow{1\mapsto Q_1} \mathrm{Hom}_{E[Q_1]}(\Sigma^3 E[Q_1], M) \xrightarrow{1\mapsto Q_1} \mathrm{Hom}_{E[Q_1]}(\Sigma^6 E[Q_1], M) \xrightarrow{1\mapsto Q_1} \dots \tag{3}$$

Now we claim that the Ext groups obtained from this complex are

$$\mathrm{Ext}^{t,u}_{E[Q_1]}(\mathbb{F}_2, M) \cong \begin{cases} \ker(M \xrightarrow{Q_1} M) & u = t = 0 \\ H(M; Q_1) & u = 3t \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $f \in \ker(\mathrm{Hom}_{E[Q_1]}(E[Q_1], M) \xrightarrow{1\mapsto Q_1} \mathrm{Hom}_{E[Q_1]}(E[Q_1], M))$, where the notation $1 \mapsto Q_1$ means that the map on Hom groups is induced by the map in the resolution sending $1 \mapsto Q_1$. Here we leave out suspensions to simplify notation. Since $f \in \ker$, it must be that $f \in \mathrm{Hom}_{E[Q_1]}(E[Q_1], M)$ sends $E[Q_1] \ni 1 \mapsto m \in M$ with $Q_1 m = 0$. In other words, f sends $1 \in E[Q_1]$ to an element in $\ker(M \xrightarrow{Q_1} M)$. The same reasoning implies that if f is in the image of the map of Hom groups above, then f sends $1 \in E[Q_1]$ to some element in $\mathrm{im}(M \xrightarrow{Q_1} M) = Q_1 M$. Therefore, if f is in the homology of the complex

$$\dots \xrightarrow{1\mapsto Q_1} \mathrm{Hom}_{E[Q_1]}(E[Q_1], M) \xrightarrow{1\mapsto Q_1} \mathrm{Hom}_{E[Q_1]}(E[Q_1], M) \xrightarrow{1\mapsto Q_1} \mathrm{Hom}_{E[Q_1]}(E[Q_1], M) \xrightarrow{1\mapsto Q_1} \dots, \tag{4}$$

then f sends $1 \in E[Q_1]$ to some element in $H(M; Q_1)$. The converse is clear, so we have an isomorphism between the homology of the complex in Equation 4 and the Margolis homology $H(M; Q_1)$. In particular, the complex in Equation 4 is the same as the one in Equation 3 almost everywhere (except $t = 0$), up to suspensions. Once the internal degree is considered we get the Ext groups above.

□

Now that we have the inner Ext term, we compute $\mathrm{Ext}_{HQ_1}(\mathbb{F}_2, M)$. We only write M here to simplify notation, but once this Ext is computed, we will replace the second entry with the $\mathrm{Ext}_{E[Q_1]}(\mathbb{F}_2, M)$ computed above. We begin by building a free resolution of $\mathbb{F}_2$ over HQ_1:

$$0 \leftarrow \mathbb{F}_2 \leftarrow HQ_1 \leftarrow \Sigma HQ_1 \oplus \Sigma^2 HQ_1 \leftarrow \Sigma^2 HQ_1 \oplus \Sigma^4 HQ_1 \leftarrow \Sigma^3 HQ_1 \oplus \Sigma^6 HQ_1 \leftarrow \dots,$$

where the odd-degree maps $\Sigma^{u+1} HQ_1 \to \Sigma^u HQ_1$ are given by $1 \mapsto \mathrm{Sq}^1$ and the even-degree maps $\Sigma^{u+2} HQ_1 \to \Sigma^u HQ_1$ are given by $1 \mapsto \mathrm{Sq}^2$. By the universal property of the direct sum we can consider the odd- and even-degree maps separately. First consider the complex

$$0 \to \mathrm{Hom}_{HQ_1}(HQ_1, M) \xrightarrow{1\mapsto \mathrm{Sq}^1} \mathrm{Hom}_{HQ_1}(\Sigma HQ_1, M) \xrightarrow{1\mapsto \mathrm{Sq}^1} \mathrm{Hom}_{HQ_1}(\Sigma^2 HQ_1, M) \xrightarrow{1\mapsto \mathrm{Sq}^1} \dots$$

The reasoning above implies that the homology of this complex is given by

$$H^{s,u} \cong \begin{cases} \ker(M \xrightarrow{\mathrm{Sq}^1} M) & u = s = 0 \\ H(M; Q_0) & u = s \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Now consider the complex

$$0 \to 0 \to \mathrm{Hom}_{HQ_1}(\Sigma^2 HQ_1, M) \xrightarrow{1\mapsto \mathrm{Sq}^2} \mathrm{Hom}_{HQ_1}(\Sigma^4 HQ_1, M) \xrightarrow{1\mapsto \mathrm{Sq}^2} \mathrm{Hom}_{HQ_1}(\Sigma^6 HQ_1, M) \xrightarrow{1\mapsto \mathrm{Sq}^2} \dots$$

In this case, notice that the reasoning above would have us considering something like $H(M; \mathrm{Sq}^2)$. In general, this is not a well-defined object since $\mathrm{Sq}^2\mathrm{Sq}^2 = 0$ does not hold in general. However, since M is an HQ_1-module, in which $\mathrm{Sq}^2\mathrm{Sq}^2 = 0$ holds, we have that $\mathrm{Sq}^2\mathrm{Sq}^2 M = 0$ as well. Then we will write $H(M; \mathrm{Sq}^2)$ for the Sq^2-homology of M in analogy with Margolis homology. The reasoning above then gives us the following homology groups for our complex:

$$H^{s,u} \cong \begin{cases} \ker(M \xrightarrow{\mathrm{Sq}^2} M) & u = 2, s = 1 \\ H(M; \mathrm{Sq}^2) & u = 2s \geq 4 \\ 0 & \text{otherwise} \end{cases}$$

With both the odd- and even-degree components accounted for, we get the following Ext groups:

$$\mathrm{Ext}^{s,u}_{HQ_1}(\mathbb{F}_2, M) \cong \begin{cases} \ker(M \xrightarrow{\mathrm{Sq}^1} M) & u = s = 0 \\ \ker(M \xrightarrow{\mathrm{Sq}^2} M) & u = 2, s = 1 \\ H(M; Q_0) & u = s \geq 1 \\ H(M; \mathrm{Sq}^2) & u = 2s \geq 4 \\ 0 & \text{otherwise} \end{cases}$$

Now we can write down the E_2 page of our spectral sequence:

$$E_2^{s,t,u} := \mathrm{Ext}^s_{HQ_1}(\mathbb{F}_2, \mathrm{Ext}^{t,u}_{E[Q_1]}(\mathbb{F}_2, M)) \cong \begin{cases} \ker(\ker(Q_1) \xrightarrow{\mathrm{Sq}^1} \ker(Q_1)) & u = t = s = 0 \\ H(H(M;Q_1);Q_0) & u = s = 3t \geq 3 \\ H(H(M;Q_1);\mathrm{Sq}^2) & u = 2s = 3t \geq 6 \\ 0 & \text{otherwise} \end{cases}$$

where $\ker(x)$ denotes $\ker(M \xrightarrow{x} M)$.

Remark 29. This approach was motivated by Katharine Adamyk's unpublished dissertation, some of which is summarized in [1]. She considered $\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{F}_2)$ for Q_0-local M and showed that after localizing at a certain prime in $\mathrm{Ext}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2)$, the problem reduced mostly to computing the corresponding localization of $\mathrm{Ext}_{\mathcal{A}(0)}(M, \mathbb{F}_2)$. This localization then served as the E_1 page of a spectral sequence obtained by filtering the Q_0-localization $L_0M \cong \gamma_\infty \otimes M$ by certain submodules of γ_∞ ([7]). The key to Adamayk's simplification of the E_1 page was in formulating the associated graded of her filtration in terms of the exterior algebra $E[Q_0]$. From there, she built an injective resolution of $\mathbb{F}_2$ that allowed the identification of the resulting Ext groups with the homology $H(M;Q_0)$. In our computation, this is the argument used to simplify Ext over $E[Q_1]$.

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