On a Sharp Sobolev Inequality with Remainder Term
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Abstract

We establish a sharp Sobolev inequality on bounded domians with remainder term in weak L^q norm for a wide range of q. Through this, we can get a lower bound for the exponent in the quantitative sharp Sobolev Inequality.

The key step in the present paper is to show that for radial functions with certain compact support, the distance between the functional and the extremals of the Sobolev Inequality can be bounded from below in terms of weak L^q norm of the functional.

Contents

1 Introduction and Main Theorem

1.1 Introduction

The sharp form of Sobolev Inequality in $\mathbb{R}^n$, proved by Talenti in [1], tells us that, for any $n \geq 2$, $1 < p < n$, the following inequality holds for every $u \in W^{1,p}(\mathbb{R}^n)$:

$$S(p,n)\|u\|_{p^*} \leq \|\nabla u\|_p \tag{1}$$

Here, p^* is the number such that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, that is, $p^* = \frac{np}{n-p}$ and,

$$S(p,n) := \sqrt{\pi} n^{1/p} \left(\frac{n-p}{p-1}\right)^{(p-1)/p} \left(\frac{\Gamma(n/p)\Gamma(1+n-n/p)}{\Gamma(1+n/2)\Gamma(n)}\right)^{1/n} \tag{2}$$

is the sharp Sobolev constant.

As proved by Brothers and Ziemer in [2], the extremals to (1) is the family $\{g_{a,b,x_0}\}_{a\in\mathbb{R},b>0,x_0\in\mathbb{R}^n}$ defined by

$$g_{a,b,x_0}(x) := a\left(1 + (b|x-x_0|)^{p'}\right)^{(p-n)/p} \tag{3}$$

Here, p' is number such that $\frac{1}{p} + \frac{1}{p'} = 1$, that is, $p' = \frac{p}{p-1}$.

It is noteworthy that the extremals are not compactly supported. And therefore it is natural to consider what remainder terms can be introduced to (1) when the integral is taken over certain bounded domain. Some of the most important results and techniques in this field are in [3], [4] [5], and [6].

The following theorem is proved by Egnell, et al. in [6]:

Theorem 1.1. *For open, bounded $\Omega \subset \mathbb{R}^n$, $1 < p < n$, $q < \frac{n(p-1)}{n-p}$, there exists a constant $A = A(p,q,n,\Omega) > 0$, such that for every $u \in W_0^{1,p}(\Omega)$ the following holds :*

$$\|\nabla u\|_p^p - S^p\|u\|_{p^*}^p \geq A\|u\|_q^p \tag{4}$$

Moreover, such inequality does not hold when $q = \frac{n(p-1)}{n-p}$.

Here are some side-notes on our notations:

1. Unless otherwise noted, $S = S(p,n)$ is the sharp Sobolev constant as in (2), where p, n will refer to the proper p, n as in the context.

2. We will refer to the value $\frac{n(p-1)}{n-p}$ as $q^\star(n,p)$. When the values of p, n are clear, we will write it as $q^\star$ for short.

3. When the domain of u is clear under context, we will write $\|u\|_{L^s(\text{domain of } u)}$ as $\|u\|_s$. Alternatively, it can be taken as: we are always defining u to be 0 wherever it is undefined and by $\|u\|_s$ we mean $\|u\|_{L^s(\mathbb{R}^n)}$. Also, we will write $\|\cdot\|_{L^p(\Omega)}$ as $\|\cdot\|_{p(\Omega)}$ for short.

4. Because the present paper is focused on radial and radially decreasing functions, we will call *radial and radially decreasing* as *radially decreasing* for short and write $g_{a,b,0}$ as $g_{a,b}$.

5. ω_n will refer to the volume of the unit n-ball.

In the case $p = 2$, the endpoint inequality of Theorem 1.1 was proved to hold in the weak sense by Brezis and Lieb in [3].

Theorem 1.2. *For open, bounded $\Omega \subset \mathbb{R}^n$, $q = q^\star(n,2)$, there exists a constant $B > 0$, depending only on n, Ω, such that the following holds for every $u \in H_0^1(\Omega)$:*

$$\|\nabla u\|_2^2 - S^2 \|u\|_{2^*}^2 \geq B \|u\|_{q,w}^2 \tag{5}$$

where $\|u\|_{q,w}$ denotes the weak L^q norm defined as:

$$\|u\|_{q,w} := \sup_A |A|^{-1/q'} \int_A |u(x)|dx \tag{6}$$

with A taken over all $A \subset \mathbb{R}^n$ of finite measure.

Unfortunately, the generalization of Theorem 1.2 to other p, to the best of our knowledge, is still open. We will rephrase it as the following:

Conjecture 1.3. *For open, bounded $\Omega \subset \mathbb{R}^n, 1 < p < n$, $q = q^\star(n,p)$, there exists a constant $D > 0$, depending only on n, p, Ω, such that the following holds for every $u \in W_0^{1,p}(\Omega)$:*

$$\|\nabla u\|_p^p - S^2 \|u\|_{p^*}^p \geq D \|u\|_{q,w}^p \tag{7}$$

The motivation for this project was Conjecture 1.3 and our attempt was to use the results proved by Cianchi et al. in [7], which are the following two theorems:

Definition 1.4. Fix $n \geq 2$ and $1 < p < n$, then for every radially decreasing $u \in W^{1,p}(\mathbb{R}^n)$, we define

$$\lambda(u) := \begin{cases} \inf_{a,b} \left\{ \dfrac{\|u - g_{a,b}\|_{p^*}^{p^*}}{\|u\|_{p^*}^{p^*}} : \|g_{a,b}\|_{p^*} = \|u\|_{p^*} \right\} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}$$

Theorem 1.5. *For $n \geq 2$, $1 < p < n$, there exists $\kappa, \beta > 0$, such that the following holds for every radially decreasing $u \in W^{1,p}(\mathbb{R}^n)$:*

$$S \|u\|_{p^*} \left(1 + \kappa \lambda(u)^\beta\right) \leq \|\nabla u\|_p \tag{8}$$

Moreover, (8) *holds for some (β, κ) with $\beta = 3 + 4p - \frac{3p+1}{n}$. When $p = 2$, $\beta = 2/2^* = (n-2)/n$ is such a β as proved by Bianchi and Egnell in [8].*

With a slight abuse of notation, we will define λ for general u:

Definition 1.6. Fix $n \geq 2$ and $1 < p < n$, then for every $u \in W^{1,p}(\mathbb{R}^n)$, we define

$$\lambda(u) := \begin{cases} \inf_{a,b,x_0} \left\{ \dfrac{\|u - g_{a,b,x_0}\|_{p^*}^{p^*}}{\|u\|_{p^*}^{p^*}} : \|g_{a,b,x_0}\|_{p^*} = \|u\|_{p^*} \right\} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}$$

Theorem 1.7. *For $n \geq 2$, $1 < p < n$, there exists $\kappa, \alpha > 0$, such that the following holds for every $u \in W^{1,p}(\mathbb{R}^n)$:*

$$S\|u\|_{p^*}\left(1 + \kappa\lambda(u)^{\alpha}\right) \leq \|\nabla u\|_p \tag{9}$$

When $p = 2$, $\alpha = 2/2^* = (n-2)/n$ is such a α, as proved by Bianchi and Egnell in [8]

Observe that the sharp α in Theorem 1.7 is greater than or equal to the sharp β in Theorem 1.5.

Though we did not solve Conjecture 1.3 completely, we found an inequality similar to (7) for a wide range of q, which could imply Conjecture 1.3 if we can improve the β in Theorem 1.5 to p/p^*.

Reversely, we will prove that (7) does not hold for any $q > q^\star$ and use the Main Theorem to conclude that p/p^* is a lower bound for the β in Theorem 1.5.

1.2 Main Theorem and the organization of the present paper

Main Theorem. For open, bounded $\Omega \subset \mathbb{R}^n$, $1 < p < n, 1 < q < p^*$, if β is a value for which (8) holds, then there exists $\theta = \theta(n, p, q, \beta) > 0$ and $C = C(\theta, p, q, n, \Omega, \beta) > 0$ such that the following inequality holds for every $u \in W^{1,p}(\Omega)$,

$$\|\nabla u\|_p \geq S\|u\|_{p^*} + C\|u\|_{q,w}^{\theta}\|u\|_{p^*}^{1-\theta} \tag{10}$$

More specifically, we can take θ as follows:

$$\theta = \begin{cases} \beta p^* & \text{if } \ q \leq \frac{n(p-1)}{n-p} \\ \dfrac{\beta}{(p-1)\left(\frac{1}{q} - \frac{1}{p^*}\right)} & \text{otherwise} \end{cases}$$

As a corollary, we have that

$$\|\nabla u\|_p^p \geq S^p\|u\|_{p^*}^p + C'(\Omega)\|u\|_{q,w}^{\theta}\|u\|_{p^*}^{p-\theta} \tag{11}$$

We will first focus on radially decreasing functions with compact support and estimate $\lambda(u)$ (as in Definition 1.4) in terms of $\|u\|_{q,w}$ from below. The key in this step is to analyze the relation between $\|g\|_{q,w}$ and $\|g\|_{p^*((\mathbb{R}^n-\Omega))}$, where the latter can be controlled in terms of $\|u-g\|_{p^*}$. Then, we will combine this upper bound on $\|u\|_{q,w}$ with Theorem 1.5 to obtain the Main Theorem. After that, we will prove by direct computation that (7) does not hold for any $q > q^\star$. And finally, by combining the previous step with our Main Theorem, we will conclude that p/p^* is a lower bound for the β in Theorem 1.5.

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2 Proof of the Main Theorem

2.1 Preliminaries

For given p, q, n, Ω, we first fix κ and β for which (8) holds.

By extending u to be 0 outside of Ω and applying the rearrangement inequality (for example, see [1]), we may assume WLOG that u is radially decreasing, nonnegative and is not constantly zero, and Ω is a ball.

Rescaling and normalization arguments allow us to assume $\Omega = B(0,1)$ and $\|u\|_{p^*} = 1$. Note that, even though (10) is not scale-invariant, C is changed only up to a constant independent of u.

Now, for such u, (10) is equivalent to

$$\|\nabla u\|_p \geq S + C\|u\|_{q,w}^{\theta} \tag{12}$$

Because $\|u\|_{q,w} \leq \|u\|_{p^*}|\Omega|^{\frac{1}{q}-\frac{1}{p^*}} = |\Omega|^{\frac{1}{q}-\frac{1}{p^*}}$ and (8) holds, for any $\varepsilon > 0$, we may take the C in (12) to be small enough such that (12) would hold for all $u \in W_0^{1,p}(\Omega)$ with $\lambda(u) \geq \varepsilon$. Therefore, we will assume WLOG that $\lambda(u) < \varepsilon$, where $\varepsilon > 0$ is a small constant to be determined.

2.2 Bound $\|u\|_{q,w}$ using $\lambda(u)$

Claim 2.1. $\lambda(u) > 0$

Proof. We argue by contradiction.

By the definition of λ, if $\lambda(u) = 0$, then there exists a sequence of extremals $\{g_m\}$ such that the L^{p^*}-norm of each $g_m = g_{a_m,b_m}$ is 1, and $g_m \to u$ in L^{p^*}. Note that we have

$$\|u - g_m\|_{p^*} \geq \|\chi_{(\mathbb{R}^n-\Omega)} g_m\|_{p^*} \tag{13}$$

and that $\|g_{a_m,b_m}\|_{p^*} = 1$ implies that we can write g_{a_m,b_m} as a one-parameter family of functions: g_{b_m}. we have $\|\chi_{(\mathbb{R}^n-\Omega)} g_m\|_{p^*} \to 0$, which implies $b_m \to \infty$. However, when $b_m \to \infty$, g_{b_m} point-wise converges to 0 but u cannot be 0. Hence, g_m does not converge to u. Therefore, $\lambda(u)$ is always positive □

Consequently, we can take an extremal g_b such that $\|u - g_b\|_{p^*}^{p^*} \leq 2\lambda(u)$.

Then, we have the following bound on $\|u\|_{q,w}$:

$$\|u\|_{q,w} \leq \|u - g_b\|_q + \|g_b\|_{q,w} \leq C_1 \lambda(u)^{\frac{1}{p^*}} + \|g_b\|_{q,w} \tag{14}$$

where the first inequality holds by the triangle inequality and the fact that weak L^q-norm is bounded by L^q-norm. And the second inequality holds by Hölder's inequality.

Let us first calculate $\|g_b\|_{q,w}$ in terms of b. As g_b is radially decreasing, we know that for all $A \subset \mathbb{R}^n$ with the same measure, $\int_A |g_b| dx$ is maximized when A is a ball centered at the origin. Therefore,

$$\sup_{A \subset \mathbb{R}^n, |A| < \infty} |A|^{-1/q'} \int_A |g_b| dx = \sup_{A = B(0,R)} |A|^{-1/q'} \int_A |g_b| dx$$

Hence,

$$\begin{aligned}
\|g_b\|_{q,w} &= \sup_R (\omega_n R^n)^{-\frac{1}{q'}} \int_0^R a\left(1+(br)^{p'}\right)^{1-\frac{n}{p}} r^{n-1} dr \\
&= \sup_R (\omega_n R^n)^{-\frac{1}{q'}} \frac{a}{b^n} \int_0^{bR} \left(1+r^{p'}\right)^{1-\frac{n}{p}} r^{n-1} dr \\
&\sim \sup_Q \left(\frac{Q}{b}\right)^{-\frac{n}{q'}} \frac{a}{b^n} \int_0^{Q} \left(1+r^{p'}\right)^{1-\frac{n}{p}} r^{n-1} dr \\
&\sim b^{\frac{n}{q'}-n+\frac{n}{p^*}} \sup_Q (Q)^{-\frac{n}{q'}} \int_0^{Q} \left(1+r^{p'}\right)^{1-\frac{n}{p}} r^{n-1} dr \qquad (15) \\
&\sim b^{\frac{n}{q'}-n+\frac{n}{p^*}} \\
&= b^{\frac{n}{q'}-\frac{n}{(p^*)'}} \qquad (16)
\end{aligned}$$

For the supremum in (15), we have the following two estimations:

$$\begin{aligned}
\lim_{Q\to 0} (Q)^{-\frac{n}{q'}} \int_0^{Q} \left(1+r^{p'}\right)^{1-\frac{n}{p}} r^{n-1} dr &\sim \lim_{A=B(0,Q),Q\to 0} |A|^{-1/q'} \int_A |g_b| dx \\
&\leq \lim_{A=B(0,Q),Q\to 0} (\int_A |g_b|^q dx)^{1/q} = 0
\end{aligned}$$

and

$$\lim_{Q\to\infty} (Q)^{-\frac{n}{q'}} \int_0^{Q} \left(1+r^{p'}\right)^{1-\frac{n}{p}} r^{n-1} dr \leq \lim_{Q\to\infty} (Q)^{-\frac{n}{q'}} \int_0^{\infty} \left(1+r^{p'}\right)^{1-\frac{n}{p}} r^{n-1} dr \qquad (17)$$

because the integral $\int_0^\infty \left(1+r^{p'}\right)^{1-\frac{n}{p}} r^{n-1} dr$ converges, the right hand side of (17) is 0.

Therefore, the supremum in (15) is equal to the supremum of the same expression over a certain compact interval. Hence, the supremum is attained at some $Q = Q_0$ and thus it is finite. Moreover, we can see that

$$\sup_{A\subset\mathbb{R}^n, |A|<\infty} |A|^{-1/q'} \int_A |g_b| dx = \left(|A|^{-1/q'} \int_A |g_b| dx\right)\Big|_{A=B(0,Q_0/b)} \qquad (18)$$

Then, it is natural to estimate b in terms of $\lambda(u)$:

$$\begin{aligned}
2\lambda(u) &\geq \|u-g_b\|_{p^*}^{p^*} \geq \|\chi_\Omega^C g_b\|_{p^*}^{p^*} \\
&= n\omega_n \int_1^\infty a^{p^*} \left(1+(br)^{p'}\right)^{-n} r^{n-1} dr \\
&= n\omega_n \frac{a^{p^*}}{b^n} \int_b^\infty \left(1+r^{p'}\right)^{-n} r^{n-1} dr
\end{aligned}$$

Note that $\|g_b\|_{p^*} = 1$ implies

$$a^{p^*} \sim b^n \qquad (19)$$

Hence,

$$\int_b^\infty \left(1+r^{p'}\right)^{-n} r^{n-1} dr \lesssim \lambda(u) \qquad (20)$$

As $\lambda(u) < \varepsilon$ for ε chosen to be sufficiently small, we may assume that $b > 1$. Thus,

$$\int_b^\infty (2r)^{-np'} r^{n-1} dr \le \int_b^\infty \left(1 + r^{p'}\right)^{-n} r^{n-1} dr \lesssim \lambda(u)$$

This implies

$$\int_b^\infty r^{n-1-np'} dr \lesssim \lambda(u),$$

which is to say,

$$b \gtrsim \lambda(u)^{\frac{1}{n-np'}} \tag{21}$$

Combining (21) and (16), we now have:

$$\|g_b\|_{q,w} \lesssim \lambda(u)^{\frac{1}{1-p'}\left(\frac{1}{q'} - \frac{1}{p^{*\prime}}\right)} = \lambda(u)^{(p-1)\left(\frac{1}{q} - \frac{1}{p^*}\right)} \tag{22}$$

Moreover, by (14) and (22),

$$\|u\|_{q,w} \lesssim \lambda(u)^{\frac{1}{p^*}} + \lambda(u)^{(p-1)\left(\frac{1}{q} - \frac{1}{p^*}\right)} \lesssim \lambda(u)^{\min\{\frac{1}{p^*}, (p-1)\left(\frac{1}{q} - \frac{1}{p^*}\right)\}} \tag{23}$$

Denote the above exponent $\min\{\frac{1}{p^*}, (p-1)\left(\frac{1}{q} - \frac{1}{p^*}\right)\}$ by γ, then $\|u\|_{q,w}^{\frac{\beta}{\gamma}} \lesssim \lambda^\beta$. Hence, taking $\theta = \frac{\beta}{\gamma}$ and we can deduce (10) from (8).

This finishes our proof of the Main Theorem.

Note that if (8) holds for $\beta = p/p^*$, we can take $q = q^\star$. Then, $\frac{\beta}{\gamma} = p$, which is to say (11) holds for $q = q^\star, \theta = p$, which coincides with (7). That is, to prove Conjecture 1.3 holds, we only need to show that (8) holds for $\beta = p/p^*$. As this value is achieved when $p = 2$, our Main Theorem offers a new proof for Theorem 1.2.

3 Application: A lower bound for β

In this section, we will show that (7) does not hold for any $q > q^\star$, which will be rephrased as the following lemma. And then, we will apply our Main Theorem to show that p/p^* is a lower bound for the β in Theorem 1.5.

Lemma 3.1. *There exists some open, bounded $\Omega \subset \mathbb{R}^n$ such that for any $C > 0$, there exist some $u \in W_0^{1,p}\Omega$ such that:*

$$\|\nabla u\|_p^p - S^p \|u\|_{p^*}^p < C \|u\|_{q,w}^p$$

Proof. We will prove by construction. Take $\Omega = B(0,1)$ and consider the family $\{u_b\}_{b>0}$ defined by $u_b(x) := (g_b(x) - g_b(1))_+$, where g_b is the unique extremal $g_{a,b}$ such that $\|g_{a,b}\|_{p^*} = 1$ and for $r \in \mathbb{R}$, we define $g_b(r) = g_b(rv)$ where $v \in \mathbb{R}^n$ is arbitrary unit vector. As g_b is radial, there is no ambiguity on the definition of $g_b(r)$. And we claim that this family $\{u_b\}$ will be a counter-example to (7).

Let us first compute $\|u\|_{q,w}^p$. As remarked in (18), the supremum in expression (6) is attained when $A = B(0, Q_0/b)$ where Q_0 is a fixed constant. Therefore, we can fix some sufficiently big $M > 0$ such that $B(0, Q_0/b) \subset B(0, 1/3)$ and $1 + b^{p'} \ge 2^{p'}\left(1 + \left(\frac{b}{3}\right)^{p'}\right)$ holds for every $b > M$.

Then,

$$g_b(r) \geq g_b(1/3) = a\left(1+\left(\frac{b}{3}\right)^{p'}\right)^{(p-n)/p} \geq 2^{(n-p)/(p-1)} g_b(1) \quad \text{for all } r \leq 1/3$$

and hence $g_b(x) - g_b(1) \gtrsim g_b(x)$ for any $|x| \leq 1/3$. Thus we know, as $b \to \infty$,

$$\|u_b\|_{q,w} \gtrsim \|g_b\|_{q,w}$$

As $|u_b(x)| \leq |g_b(x)|$ point-wise, we have $\|u_b\|_{q,w} \leq \|g_b\|_{q,w}$. Therefore,

$$\|u_b\|_{q,w} \sim \|g_b\|_{q,w} \tag{24}$$

By (16) and (24), we can deduce that as $b \to \infty$, $\|u_b\|_{q,w}^p \sim b^{p\left(\frac{n}{q'} - \frac{n}{(p^*)'}\right)}$, which is asymptotically greater than $b^{p\left(\frac{n}{(q^\star)'} - \frac{n}{(p^*)'}\right)} \sim \|u_b\|_{q^\star,w}^p \sim b^{(p-n)/(p-1)}$.

In fact, for this family of u, as $b \to \infty$, we have

$$\|\nabla u_b\|_p - S\|u_b\|_{p^*} \sim \|u_b\|_{q^\star,w}^p \tag{25}$$

but for our purpose, it suffices to prove

$$\|\nabla u_b\|_p - S\|u_b\|_{p^*} \lesssim \|u_b\|_{q^\star,w}^p \tag{26}$$

Using the fact that $\|u_b\|_{q^\star,w} \lesssim \|u_b\|_{p^*} \leq \|g_b\|_{q^*,w} = 1$, we can derive from (26) that

$$\|\nabla u_b\|_p^p - S^p\|u_b\|_{p^*}^p \lesssim \|u_b\|_{q^\star,w}^p$$

And the lemma will follow immediately.

We now compute the left hand side of (26):

$$\begin{aligned}
&\|\nabla u_b\|_p - S\|u_b\|_{p^*} \\
&= \|\nabla g_b\|_{p(\Omega)} - S\|g_b - g_b(1)\|_{p^*(\Omega)} \\
&= \|\nabla g_b\|_{p(\Omega)} - S\|g_b\|_{p^*(\Omega)} + \left(S\|g_b\|_{p^*(\Omega)} - S\|g_b - g_b(1)\|_{p^*(\Omega)}\right) \\
&= \left(\|\nabla g_b\|_{p(\mathbb{R}^n)} - S\|g_b\|_{p^*(\mathbb{R}^n)}\right) + S\left(\|g_b\|_{p^*(\mathbb{R}^n)} - \|g_b\|_{p^*(\Omega)}\right) \\
&\quad - \left(\|\nabla g_b\|_{p(\mathbb{R}^n)} - \|\nabla g_b\|_{p(\Omega)}\right) + \left(S\|g_b\|_{p^*(\Omega)} - S\|g_b - g_b(1)\|_{p^*(\Omega)}\right) \\
&= S\underbrace{\left(1 - \|g_b\|_{p^*(\Omega)}\right)}_{I_b} - \underbrace{\left(\|\nabla g_b\|_{p(\mathbb{R}^n)} - \|\nabla g_b\|_{p(\Omega)}\right)}_{J_b} + S\underbrace{\left(\|g_b\|_{p^*(\Omega)} - \|g_b - g_b(1)\|_{p^*(\Omega)}\right)}_{\delta_b}
\end{aligned} \tag{27}$$

For I_b, we have

$$\|g_b\|_{p^*(\Omega)} = \left(n\omega_n \int_0^1 a^{p^*}\left(1+(br)^{p'}\right)^{-n} r^{n-1}dr\right)^{1/p} = \left(n\omega_n \frac{a^{p^*}}{b^n}\int_0^b \left(1+r^{p'}\right)^{-n} r^{n-1}dr\right)^{1/p}$$

Using the fact that $1 = \|g_b\|_{p^*(\mathbb{R}^n)} = \left(n\omega_n \frac{a^{p^*}}{b^n} \int_0^\infty \left(1+r^{p'}\right)^{-n} r^{n-1}dr\right)^{1/p}$, we have, for b sufficiently large, that

$$\begin{aligned}
1 - \|g_b\|_{p^*(\Omega)} &= 1 - \left(\frac{\int_0^b \left(1+r^{p'}\right)^{-n} r^{n-1}dr}{\int_0^\infty \left(1+r^{p'}\right)^{-n} r^{n-1}dr}\right)^{1/p} \\
&= 1 - \left(1 - \frac{\int_b^\infty \left(1+r^{p'}\right)^{-n} r^{n-1}dr}{\int_0^\infty \left(1+r^{p'}\right)^{-n} r^{n-1}dr}\right)^{1/p} \\
&\sim \frac{1}{p}\left(\frac{\int_b^\infty \left(1+r^{p'}\right)^{-n} r^{n-1}dr}{\int_0^\infty \left(1+r^{p'}\right)^{-n} r^{n-1}dr}\right) &(28)\\
&\sim \int_b^\infty \left(1+r^{p'}\right)^{-n} r^{n-1}dr \\
&\sim \int_b^\infty r^{-p'n+n-1}dr \\
&\sim b^{-n/(p-1)} &(29)
\end{aligned}$$

where the comparability in the third line follows from

$$\lim_{B\to 0} \frac{1-(1-B)^\alpha}{B/\alpha} = 1 \quad \text{for any } \alpha > 0$$

Because we will bound J_b by $-J_b \leq 0$, estimating it is irrelevant to our goal. However, interested readers may take the similar approach as in the estimation for I_b to derive that

$$J_b \sim b^{-n/(p-1)+p'} \tag{30}$$

And finally, as for δ_b, note that by the triangle inequality of L^{p^*} norm, we have

$$0 < \delta_b \leq \|g_b(1)\|_{p^*(\Omega)} \sim g_b(1) = a\left(1+b^{p'}\right)^{(p-n)/p} \sim ab^{(p-n)/(p-1)}$$

as shown in (19), $a \sim b^{n/p^*} = b^{(n-p)/p}$, and hence:

$$0 < \delta_b \lesssim b^{\frac{p-n}{p(p-1)}} \to 0 \quad \text{as } b \to \infty \tag{31}$$

Therefore,

$$\begin{aligned}
\|g_b\|_{p^*(\Omega)}^{p^*} - \|g_b - g_b(1)\|_{p^*(\Omega)}^{p^*} &= \|g_b\|_{p^*(\Omega)}^{p^*} - \left(\|g_b\|_{p^*(\Omega)} - \delta_b\right)^{p^*} \\
&\geq p^*\left(\|g_b\|_{p^*(\Omega)} - \delta_b\right)^{p^*-1}\delta_b &(32)
\end{aligned}$$

where the inequality holds because for any $A > \varepsilon > 0$, $\alpha > 1$,

$$A^\alpha - (A-\varepsilon)^\alpha = \int_{A-\varepsilon}^A \left.\frac{dx^\alpha}{dx}\right|_{x=y} dy \in \left[\alpha(A-\varepsilon)^{\alpha-1}\varepsilon, \alpha A^{\alpha-1}\varepsilon\right]$$

The upper bound in the above line will be used later.

Since the limit of $\|g_b\|_{p^*(\Omega)} - \delta_b$ as $b \to \infty$ is $1 - 0 = 1$, we know from (32) that

$$\delta_b \lesssim \|g_b\|_{p^*(\Omega)}^{p^*} - \|g_b - g_b(1)\|_{p^*(\Omega)}^{p^*} \tag{33}$$

The right hand side of (33) is:

$$\begin{aligned}
& n\omega_n \int_0^1 \left(\left(a\left(1+(br)^{p'}\right)^{(p-n)/p} \right)^{p^*} - \left(a\left(1+(br)^{p'}\right)^{(p-n)/p} - a\left(1+b^{p'}\right)^{(p-n)/p} \right)^{p^*} \right) r^{n-1} dr \\
& \leq n\omega_n \int_0^1 \left(p^* \left(a\left(1+(br)^{p'}\right)^{(p-n)/p} \right)^{p^*-1} a\left(1+b^{p'}\right)^{(p-n)/p} \right) r^{n-1} dr \\
& \sim a^{p^*} \left(1+b^{p'}\right)^{(p-n)/p} \int_0^1 \left(1+(br)^{p'}\right)^{\frac{p-n}{p}(p^*-1)} r^{n-1} dr \\
& = a^{p^*} \left(1+b^{p'}\right)^{(p-n)/p} \frac{1}{b^n} \int_0^b \left(1+r^{p'}\right)^{\frac{p-n}{p}(p^*-1)} r^{n-1} dr \\
& \leq a^{p^*} \left(1+b^{p'}\right)^{(p-n)/p} \frac{1}{b^n} \int_0^\infty \left(1+r^{p'}\right)^{\frac{p-n}{p}(p^*-1)} r^{n-1} dr && (34) \\
& \lesssim \left(1+b^{p'}\right)^{(p-n)/p} && (35) \\
& \sim b^{p' \frac{p-n}{p}} = b^{(p-n)/(p-1)} && (36)
\end{aligned}$$

We can go from (34) to (35) as $a^{p^*} \sim b^n$ (see (19)) and the integral converges by a simple test.

Combining (36), (30), and (28), we have

$$\|\nabla u_b\|_p - S\|u_b\|_{p^*} \lesssim b^{(p-n)/(p-1)}$$

as desired. □

Corollary 3.2. *For $0 < \beta < p/p^*$, there is no κ such that (8) holds.*

Proof. Assume to the contrary that (8) holds for some $\beta_0 < p/p^*$.

By taking $q = \frac{p*p(p-1)}{p^*\beta_0 + p(p-1)} \in (q^\star, p^*)$, which is the value such that $\theta = \frac{\beta_1}{(p-1)\left(\frac{1}{q} - \frac{1}{p^*}\right)} = p$ and applying the Main Theorem, we have (11) holds for this q, which contradicts Lemma 3.1. □

Hence, $p/p*$ is a lower bound for the β in Theorem 1.5, as well as the α in Theorem 1.7.

In particular, we can conclude that when $p = 2$, the sharp exponent β in Theorem 1.5 and the sharp exponent α in Theorem 1.7 are both $2/2^* = (n-2)/n$.

4 Further Questions

As remarked earlier, the Main Theorem can yield Conjecture 1.3 if (8) holds for $\beta = p/p^*$. And we know this is the smallest β possible. Therefore, it is natural to conjecture that p/p^* is the sharp exponent.

Conjecture 4.1. *$\beta = p/p^*$ is the sharp exponent in Theorem 1.5.*

Conjecture 4.2. *$\alpha = p/p^*$ is the sharp exponent in Theorem 1.7.*

And one can also ask if the converse is true. That is:

Question. Assuming that Conjecture 1.3 is true, can we conclude that (8) holds for $\beta = p/p^*$?

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