Almost Conservation Laws for KdV and Cubic NLS

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Abstract

In this paper, we explore the method of almost conservation laws proposed in [1] for integrable systems. In particular, we consider the Korteweg-de Vries initial value problem and give a rigorous proof on how the "algorithm" that generates these almost conservation laws can be used to recover infinitely many conserved integrals that make the KdV an integrable system. In addition, we applied the same idea of almost conservation laws to the cubic nonlinear Schrödinger equation.

1 Introduction

As presented in [1], the method of almost conservation laws can be used to extend local well-posedness results of this equation to global-wellposedness ones, which the authors apply to the KdV equation,

$$\begin{cases} \partial_t u + \partial_x u + \frac{1}{2}\partial_x(u^2) = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

This method is based on studying some norms like $\|Iu\|_{H^s}$, where I is a functional given by a certain Fourier multiplier, and u is the solution of our equation. In practice, it requires a careful analysis of how conservation laws are proved in frequency space, in order to gain understanding on how different frequencies interact. A simple example of this idea is the following: consider the L^2 conservation law for the solution of the KdV equation above. One can use Plancherel to write:

$$\int |u_t(x)|^2\, dx = \int_{\xi_1+\xi_2=0} \widehat{u_t}(\xi_1)\widehat{u_t}(\xi_2)\, d\xi_1 d\xi_2$$

By using the Fourier transform of the equation, one can get the following identity

$$\begin{aligned} \frac{d}{dt}\int |u_t(x)|^2\, dx = &-i\int_{\xi_1+\xi_2=0} (\xi_1^3+\xi_2^3)\, \widehat{u_t}(\xi_1)\widehat{u_t}(\xi_2)\, d\xi_1 d\xi_2 \\ &- i\int_{\xi_1+\xi_2+\xi_3=0} (\xi_1+\xi_2)\, \widehat{u_t}(\xi_1)\widehat{u_t}(\xi_2)\widehat{u_t}(\xi_3)\, d\xi_1 d\xi_2 d\xi_3 \end{aligned} \tag{2}$$

The first term is clearly zero, and the second term becomes zero after writing $\xi_1+\xi_2 = -\xi_3$ and symmetrizing.

As explained in [1] and [2], an interesting idea in order to prove these global results is to define a hierarchy of modified energies $E_I^i(t)$ for the solution of our dispersive equation that are comparable to the norm $\|u\|_{H^s}$ that we are trying to control, but whose increments decrease as generations evolve. Whether such a hierarchy can be found depends on the cancellations that occur on a more general type of integrals resembling (2) above. However, these cancellations are not yet well-understood, and therefore we will start with a simpler choice of the multiplier I, with which one may recover the classical conservation laws. A guess for what this choice would be and the proof for the first three conservation laws appears in [2], and in this paper we will provide the full proof.

Another goal of this paper is to study the analogous case of the cubic NLS equation

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u, \\ u(x,0) = u_0(x) \end{cases}$$

for $x \in \mathbb{T}$, which also enjoyes infinitely many conserved quantities, and to try to accomplish the same objectives as with the KdV equation.

In Section 2 we provide some definitions and introduce some notation that will be useful throughout the paper. In Section 3 we provide the main results and proofs. In Section 4 we set out the analogous problem for the cubic NLS equation. And finally, in Section 5 we discuss possible future directions of research on these topics.

The results presented in this paper were developed in collaboration with Ricardo Grande Izquierdo.

2 Preliminary Definitions

Definition 2.1. Throughout this paper, we define the spatial Fourier transform of $f(x)$ to be

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$

Note that with this definition of the Fourier transform, we have

$$\widehat{\partial_x^n f(x)} = (i\xi)^n \hat{f}(\xi).$$

In this paper, we consider the conservation laws of the KdV equation, defined as the following

Definition 2.2. We define the KdV equation to be the following:

$$\begin{cases} \partial_t u + \partial_x^3 u + u(\partial_x u) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

We then define the operator I.

Definition 2.3. Suppose u is a solution to the KdV equation, then we define an operator I, usually called Fourier multiplier, such that

$$\widehat{Iu}(\xi) = m(\xi)\hat{u}(\xi).$$

As shown in the introduction, the symmetrization describes the nonlinear interaction of the frequencies of the solution u to the KdV equation. Through this method, we keep track of the various pieces of $\hat{u}$. Now we introduce some notation that will be frequently used in the rest of the paper.

Definition 2.4. A k-multiplier m is a function $m : \mathbb{R} \to \mathbb{C}$. A k-multiplier is symmetric if $m(\xi) = m(\sigma(\xi))$ for all $\sigma \in S_k$. The symmetrization of a k-multiplier is

$$[m]_{sym}(\xi) = \frac{1}{k!} \sum_{\sigma \in S_k} m(\sigma(\xi))$$

A k-multiplier generates the k-linear functional via the integration

$$\Lambda_k(m) = \int_{A_k} m(\xi_1, ..., \xi_k)\hat{u}(\xi_i)...\hat{u}(\xi_k)\, d\xi_1 \, \dots \, d\xi_k$$

where $A_k = \{(\xi_1, ..., \xi_k) : \xi_1 + \cdots + \xi_k = 0\}$.

Remark. This notation does not refer to integrating over a k-dimensional set of measure zero, instead it refers to $k-1$ integrals, i.e.

$$\Lambda_k(m) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} m(\xi_1, \dots, \xi_k)\hat{u}(\xi_i)...\hat{u}(\xi_{k-1})\hat{u}(-\sum_{i=1}^{k-1} \xi_i) d\xi_1 \cdots d\xi_{k-1}$$

By observation, we find that

$$||Iu(t)||_{L_2}^2 = \int_{A_2} m(\xi_1)m(\xi_2)\hat{u}(\xi_1)\hat{u}(\xi_2)d\xi_1 d\xi_2 = \Lambda_2(m(\xi_1)m(\xi_2))$$

Hence we have the following useful proposition, which can be found in [1].

Proposition 2.1. Suppose u satisfies the KdV equation, and m is a symmetric k-multiplier and $\Lambda_k(m)$ is the k-linear functional generated by m, then

$$\frac{d}{dt}\Lambda_k(m) = \Lambda_k(\alpha_k m) - i\frac{k}{2}\Lambda_{k+1}(\tilde{m}(\xi_1, ..., \xi_{k+1})).$$

where

$$\tilde{m}(\xi_1, \dots, \xi_{k+1}) = (\xi_k + \xi_{k+1}) \cdot m(\xi_1, \dots, , \xi_{k-1}, \xi_k + \xi_{k+1})$$

Proof.

$$\begin{aligned} \frac{d}{dt}\Lambda_k(m) &= \frac{d}{dt}\int_{A_k} m(\xi_1, ..., \xi_k)\hat{u}(\xi_1) \cdots \hat{u}(\xi_k) d\xi_1 \cdots d\xi_k \\ &= \int_{A_k} m(\xi_1, ..., \xi_k)\frac{d}{dt}(\hat{u}(\xi_1) \cdots \hat{u}(\xi_k)) d\xi_1 \cdots d\xi_k \end{aligned}$$

Substitute in the equation

$$\partial_t \hat{u} = -(i\xi)^3 \hat{u} - \frac{1}{2} i\xi \widehat{u^2}$$

then we have

$$\frac{d}{dt}\Lambda_k(m) = -\sum_{i=1}^{k} \int_{A_k} (i\xi_i)^3 m(\xi_1, ..., \xi_k) \hat{u}(\xi_1) \cdots \hat{u}(\xi_k)$$
$$- \sum_{i=1}^{k} \int_{A_k} \frac{1}{2}(i\xi_i) \hat{u}(\xi_1) \cdots \widehat{u^2}(\xi_i) \cdots \hat{u}(\xi_k) m(\xi_1, ..., \xi_k)$$

Since

$$\widehat{u^2}(\xi_i) = \hat{u}(\xi_i) * \hat{u}(\xi_i) = \int_{\mathbb{R}} \hat{u}(\xi_i - \xi_{k+1}) \hat{u}(\xi_{k+1}) d\xi_{k+1}$$

By symmetry on the set A_k we obtain the desired identity. □

With the above notation, we now define the chain of modified energies following the notation in [1],

Definition 2.5. We recursively define the chain of energies $E^k(t)$ as the following:

$$E_I^2(t) = ||Iu(t)||_{L^2}^2 = \Lambda_2(m(\xi_1)m(\xi_2))$$

where $u(t)$ is the solution to the KdV equation and I is an operator such that $\widehat{If(\xi)} = m(\xi)\hat{f}(\xi)$, then

$$E_I^k(t) = E_I^{k-1}(t) + \Lambda_k(\sigma_k),$$

where σ_k is an operator chosen such that

$$\frac{d}{dt} E_I^k(t) = \Lambda_{k+1}(M_{k+1}).$$

Note that by Proposition 2.1, we have the explicit form

$$M_{k+1} = -i\frac{k}{2}[\sigma_k(\xi_1, ..., \xi_{k-1}, \xi_k + \xi_{k+1}) \cdot (\xi_k + \xi_{k+1})]_{sym}$$

It was proposed in [1] that by setting the operator $I = \partial_x^n$, i.e. $m(\xi) = (i\xi)^n$ and putting $E^2(t) = ||Iu||_{L_2}^2 = \Lambda_2(m(\xi_1)m(\xi_2))$, we can recover all the conservation laws of the KdV equation. It was also shown explicitly in [1] that the first two conservation laws can be recovered by the proposed method, yet a rigorous proof was not given. Before illustrating this construction with an example, we first note some properties of the symmetrization operation that will help with the calculations involved in the example. The proof of the following lemmas are straight-forward.

Lemma 2.1. (Linearity) Suppose $m : \mathbb{R}^k \to \mathbb{C}, n : \mathbb{R}^k \to \mathbb{C}$ are two k-multipliers, then we have

$$[m+n]_{sym} = [m]_{sym} + [n]_{sym}$$

Lemma 2.2. Suppose $m : \mathbb{R}^k \to \mathbb{C}$ is a k-multiplier and $\sigma \in S_k$, then

$$[m(\xi)]_{sym} = [m(\sigma(\xi))]_{sym}$$

Lemma 2.3. Let m, n be two k-multipliers, if m is symmetric, then

$$[mn]_{sym} = m[n]_{sym}$$

We now provide a detailed example to explicitly illustrate how the chain of modified energies recover a conservation law of the KdV equation.

Example 2.1. In the case where $n = 3$, we have

$$E^2(t) = ||\partial_x^3 u||_{L_2}^2 = \Lambda((i\xi_1^3)(i\xi_2^3)) \quad m(\xi) = i\xi^3$$

Then

$$\sigma_3 = -\frac{[\xi_1^7]}{3[\xi_1^3]} = -\frac{7}{3}[\xi_1^2\xi_2^2]$$

$$\begin{aligned} M_4 &= \frac{i}{2}[\frac{m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 + m^2(\xi_3+\xi_4)(\xi_3+\xi_4)}{3\xi_1\xi_2}] \\ &= -\frac{i}{2}[\frac{\xi_1^7 + \xi_2^7 + (\xi_3+\xi_4)^7}{3\xi_1\xi_2}] \\ &= -\frac{7i}{3}([\xi_1^5] + 3[\xi_1^4\xi_2] + 5[\xi_1^3\xi_2^2]) \\ &= -\frac{35i}{3}[\xi_1^3\xi_2^2] \end{aligned}$$

then

$$\sigma_4 = -\frac{M_4}{\alpha_4} = \frac{35}{12}\frac{[\xi_1^3\xi_2^2]}{[\xi_1^3]} = \frac{35}{12}(\frac{4}{3}[\xi_1^3][\xi_1^2] - \frac{1}{3}[\xi_1^5]) = \frac{35}{18}[\xi_1\xi_2]$$

Note that we have the following identities

$$\begin{aligned} \xi_1^5 + \xi_2^5 + \xi_3^5 + (\xi_4+\xi_5)^5 &= \xi_1^5 + \xi_2^5 + \xi_3^5 - (\xi_1+\xi_2+\xi_3)^5 \\ &= -5(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)(\xi_1^2+\xi_2^2+\xi_3^2+\xi_1\xi_2+\xi_1\xi_3+\xi_2\xi_3) \end{aligned}$$

$$\begin{aligned} \xi_1^3 + \xi_2^3 + \xi_3^3 + (\xi_4+\xi_5)^3 &= \xi_1^3 + \xi_2^3 + \xi_3^3 - (\xi_1+\xi_2+\xi_3)^3 \\ &= -3(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3) \end{aligned}$$

Therefore we have

$$\begin{aligned} M_5 &= -2i\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5) \cdot (\xi_4 + \xi_5) \\ &= 2i \cdot \frac{35}{18} \cdot \frac{1}{6}(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3 - (\xi_1 + \xi_2 + \xi_3)^2)(\xi_1 + \xi_2 + \xi_3) \\ &= -\frac{35}{54} \cdot 3[\xi_1\xi_2\xi_3] \end{aligned}$$

Note that

$$[\xi_1\xi_2\xi_3] = -\frac{1}{3}[\xi_1\xi_2(\xi_1 + \xi_2)] = -\frac{2}{3}[\xi_1^2\xi_2] = \frac{1}{6}[\xi_1^3]$$

i.e. $[\xi_1\xi_2\xi_3] = \frac{1}{30i}\alpha_5$ Therefore

$$\sigma_5 = -\frac{M_5}{\alpha_5} = \frac{35}{54} \cdot \frac{1}{10} = \frac{7}{108}$$

Notice that $E^5(t) = E^2(t) + \Lambda(\sigma_3) + \Lambda(\sigma_4) + \Lambda(\sigma_5)$ is exactly the conservation law according to [4]

$$\int_{\mathbb{R}} ((\partial_x^3 u)^2 - \frac{7}{4}u(\partial_x^2 u)^2 + \frac{35}{18}u^2(\partial_x u)^2 - \frac{7}{108}u^5)dx = C$$

3 Results

We now provide a proof for the correspondence between the chain of modified energies and the conservation laws. We first start with three useful lemmas.

Lemma 3.1. Suppose $\Lambda_k(p) = 0$, then $p = 0$ on the set A_k.

Proof. By definition

$$\Lambda_k(p) = \int_{A_k} p(\xi_1, ..., \xi_k)\hat{u}(\xi_1) \cdots \hat{u}(\xi_k) d\xi_1 \cdots d\xi_k$$

Suppose by contradiction that $p \neq 0$, using Plancherel's theorem and taking the inverse Fourier transform, we have

$$\Lambda_k(p) = \int_{\mathbb{R}} p'(u_0, u_1, ..., u_{deg(p)})dx = 0$$

where p' is a homogeneous differential polynomial of u of degree k, and u_j denotes the j-th derivative of u. Then since this is true for every time t, this implies that $\int_{\mathbb{R}} p' dx$ is a conservation law, hence it suffices to argue that $\int_{\mathbb{R}} p' dx$ cannot be a conservation law of the KdV equation. Note that every conservation law C_n can be written as

$$C_n = \int_{\mathbb{R}} P_n dx = \sum_{k=2}^{n} \Lambda_k(p_k)$$

where P_n is a differential polynomial of u homogenous of rank n according to [4]. The rank of each term in the form $u_0^{a_0} u_1^{a_1} \cdots u_l^{a_l}$ is defined to be

$$r := \sum_{j=0}^{l} (1 + \frac{1}{2} j) a_j$$

Note that for the polynomial p', each monomial has $\sum_{j=0}^{l} a_j = k$. Suppose p' is a conservation law of rank n, $n \geqslant k$, then for each monomial in p', we must have $\sum_{j=0}^{l} j \cdot a_j = 2(n-k)$, $j \leqslant deg(p)$, therefore the choice of a_j must corresponds to a proper subset of the monomials in C_n. Since the conservation law of rank n for the KdV equation is unique, p' cannot be a conservation law, contradiction. Hence we can conclude that $p = 0$ on the set A_k. □

Lemma 3.2. Suppose that for all solutions of the KdV equation $\Lambda_n(p) + \Lambda_m(q) = 0$, $n < m$. Then $\Lambda_n(p) = 0, \Lambda_m(q) = 0$. In particular, $p = 0$ and $q = 0$ on the set A_n and A_m , respectively.

Proof. Suppose that u is a solution of KdV, then (with the corresponding scaling of the initial data) the following is also a solution of KdV for all $\lambda \in \mathbb{R}$

$$u_\lambda(t,x) = \lambda^{-2} u(\lambda^3 t, \lambda x)$$

Then taking the Fourier transform with respect to x, we obtain

$$\widehat{u_\lambda}(t,\xi) = \lambda^{-3} \hat{u}(\lambda^3 t, \lambda^{-1} \xi)$$

If $\Lambda_n(p) + \Lambda_m(q) = 0$ holds for all solutions of KdV, it holds for u_λ and note that

$$\Lambda_n(p) = \int_{A_n} p(\xi_1, \ldots, \xi_n) \widehat{u_\lambda}(\xi_1) \cdots \widehat{u_\lambda}(\xi_n) d\xi_1 \cdots d\xi_n$$

Then by change of variables sending ξ_i to $\lambda \xi_i$ we obtain

$$\Lambda_n(p) = \int_{A_n} p'(\xi_1, \ldots, \xi_n) \hat{u}(\xi_1) \cdots \hat{u}(\xi_n) d\xi_1 \cdots d\xi_n$$

where

$$p'(\xi_1, \ldots, \xi_n) = \lambda^{deg(p)-2n+1} \sum_i c_i \frac{\xi_1^{a_1} \cdots \xi_n^{a_n}}{\lambda^{\sum_{i=1}^n a_i - deg(p)}}$$

Similarly

$$\Lambda_m(q) = \int_{A_m} q'(\xi) \hat{u}(\xi_1) \cdots \hat{u}(\xi_m) d\xi_1 \cdots d\xi_m$$

where

$$q'(\xi_1, \ldots, \xi_n) = \lambda^{deg(q)-2m+1} \sum_i d_i \frac{\xi_1^{b_1} \cdots \xi_m^{b_m}}{\lambda^{\sum_{i=1}^m b_i - deg(q)}}$$

Since $n < m$, then $deg(p) < deg(q)$, and therefore

$$\frac{1}{\lambda^{deg(p)-2n+1}} (\Lambda_n(p') + \Lambda_m(q')) = 0$$

Then we have

$$\int_{A_n} (\sum_i c_i \frac{\xi_1^{a_1} \cdots \xi_n^{a_n}}{\lambda^{\sum_{i=1}^n a_i - deg(p)+2n-1}}) \hat{u}(\xi_1) \cdots \hat{u}(\xi_n) d\xi_1 \cdots d\xi_n$$

$$+ \int_{A_m} (\lambda^{deg(q)-deg(p)+2(m-n)} \sum_i d_i \frac{\xi_1^{b_1} \cdots \xi_m^{b_m}}{\lambda^{\sum_{i=1}^m b_i - deg(q)}}) \hat{u}(\xi_1) \cdots \hat{u}(\xi_m) d\xi_1 \cdots d\xi_m = 0$$

We now consider the case $t = 0$ and reduce the solution to the initial data u_0.

By taking $\lambda \to 0$, the second term vanishes and we have

$$\int_{A_n} (\sum_i c_i \frac{\xi_1^{a_1} \cdots \xi_n^{a_n}}{\lambda^{\sum_{i=1}^n a_i - deg(p)+2n-1}}) \hat{u}_0(\xi_1) \cdots \hat{u}_0(\xi_n) d\xi_1 \cdots d\xi_n = 0$$

Note that $\sum_{i=1}^n a_i \leqslant deg(p) - 2n + 1$, therefore we have

$$\int_{A_n} \sum_{i \text{ s.t. } \sum_{i=1}^n a_i = deg(p)} c_i \xi_1^{a_1} \cdots \xi_n^{a_n} \hat{u}_0(\xi_1) \cdots \hat{u}_0(\xi_n) d\xi_1 \cdots d\xi_n = 0$$

Since this is true for every initial data, the polynomial

$$\sum_{i \text{ s.t. } \sum_{i=1}^n a_i = deg(p)} c_i \xi_1^{a_1} \cdots \xi_n^{a_n}$$

must be zero on the set A_n. However, this produces a contradiction about the degree of p unless $deg(p) = 0$, in which case by the same reasoning as above, $p = 0$. Then $\Lambda_n(p) = 0$, and therefore the initial hypothesis becomes $\Lambda_m(q) = 0$. By Lemma 3.1, $q = 0$.

□

Lemma 3.3. Suppose that S is a finite subset of positive integers and $\sum_{k\in S}\Lambda_k(p_k)=0$. Then $p_k=0$ on the set A_k for every $k\in S$.

Proof. The same scaling trick as in Lemma 3.2 should prove that $p_{k_0}=0$ for $k_0=\min_S k$. From then on, one simply iterates the idea, the key fact being that the exponent of λ in each integral is positive, i.e. $deg(p_k)-deg(p_{k_0})+2(k-k_0)>0$ for $k\neq k_0$, which allows every other item to vanish as $\lambda\to 0$. □

We are now ready to give the main theorem of this paper

Theorem 3.4. Let C_n denote the n-th conservation law of the KdV equation, then

$$C_{n+1}=E^{n+2}_{I=\partial_x^n}(t)\quad n\geqslant 0$$

Proof. Note that for $n\geqslant 0$

$$C_{n+1}=\int_{\mathbb{R}}((\partial_x^n u)^2+r_{n+1})dx,$$

where r_{n+1} is a differential polynomial in u and its first $n-1$ derivatives with each term no greater than degree $n+2$. Taking the Fourier transform of C_{n+1}, we can write the following

$$\widehat{C_{n+1}}=\sum_{k=2}^{n+2}\int_{A_k}[p_k(\xi_1,...,\xi_k)]\hat{u}(\xi_1)\cdots\hat{u}(\xi_k)\xi_1\cdots d\xi_k$$

Note that by the Plancherel theorem, $\widehat{C_{n+1}}=C_{n+1}$.

By differentiating with respect to time and with Proposition 2.1, we have

$$\begin{aligned}\partial_t C_{n+1}=\sum_{k=2}^{n+2}(&\int_{A_k}\alpha_k[p_k(\xi_1,...,\xi_k)]\hat{u}(\xi_1)\cdots\hat{u}(\xi_k)d\xi_1\cdots d\xi_k\\ &-\frac{k}{2}i\int_{A_{k+1}}[\tilde{p_k}(\xi_1,...,\xi_{k+1})])\hat{u}(\xi_1)\cdots\hat{u}(\xi_{k+1})d\xi_1\cdots d\xi_{k+1}),\end{aligned}$$

where

$$\tilde{p_k}(\xi_1,...,\xi_{k+1})=p_k(\xi_1,...,\xi_{k-1},\xi_k+\xi_{k+1})\cdot(\xi_k+\xi_{k+1}).$$

Since $\partial_t C_{n+1}=0$ and for all $3\leqslant k\leqslant n-2$, each integral on A_k has the form $\int_{A_k}(-i\frac{k-1}{2}[\tilde{p}_{k-1}]+\alpha_k[\tilde{p}_k])\hat{u}(\xi_1)\cdots\hat{u}(\xi_k)d\xi_1\cdots d\xi_k$

$$\begin{aligned}\partial_t C_{n+1} &= \int_{A_2}\alpha_2[p_2(\xi_1,\xi_2)]\hat{u}(\xi_1)\hat{u}(\xi_2)d\xi_1 d\xi_2\\ &+\sum_{k=3}^{n+2}\int_{A_k}(-i\frac{k-1}{2}[\tilde{p}_{k-1}]+\alpha_k[p_k])\hat{u}(\xi_1)\cdots\hat{u}(\xi_k)d\xi_1\cdots d\xi_k\\ &+\int_{A_{n+3}}(-i\frac{n+2}{2}[\tilde{p}_{n+2}])\hat{u}(\xi_1)\cdots\hat{u}(\xi_{n+3})d\xi_1\cdots d\xi_{n+3}=\\ &\Lambda_2(\alpha_2[p_2])+\sum_{k=3}^{n+2}\Lambda_k\left(-i\frac{k-1}{2}[\tilde{p}_{k-1}]+\alpha_k[p_k]\right)+\Lambda_{n+3}(-i\frac{n+2}{2}[\tilde{p}_{n+2}]) \quad =0\end{aligned}$$

Notice that $\alpha_2=0$ on the set A_2, hence the integral $\int_{A_2}\alpha_2 p_2(\xi_1,\xi_2)\hat{u}(\xi_1)\hat{u}(\xi_2)d\xi_1 d\xi_2 = 0$. Since the term with the highest degree in C_{n+1} consists no derivatives of u, i.e. p_{n+2} is a constant, therefore $\tilde{p}_{n+2}=0$ on the set A_{n+3}, therefore $\int_{A_{n+3}}-i\frac{n+2}{2}[\tilde{p}_{n+2}]\hat{u}(\xi_1)\cdots\hat{u}(\xi_{n+3})d\xi_1\cdots d\xi_{n+3}=0$. Hence

$$\sum_{k=3}^{n+2}\Lambda_k\left(-i\frac{k-1}{2}[\tilde{p}_{k-1}]+\alpha_k[p_k]\right)=0$$

By Lemma 3.3, for every $k=3,\dots,n+2$ we have

$$-i\frac{k-1}{2}[\tilde{p}_{k-1}]+\alpha_k[p_k]=0 \text{ on } A_k\,. \tag{3}$$

By construction of the chain of modified energies

$$E^k_{I=\partial_x^n}(t)=E^{k-1}_{I=\partial_x^n}(t)+\Lambda_k(\sigma_k) \quad E^2_{I=\partial_x^n}(t)=\Lambda_2((i\xi_1)^n(i\xi_2)^n)=\Lambda(\sigma_2),$$

where $\sigma_k=-\frac{M_k}{\alpha_k}$ and $M_k=-i\frac{k-1}{2}[\tilde{\sigma}_{k-1}]$. By definition we have $\sigma_2=[p_2]$ and therefore $M_3=\tilde{\sigma}_2=[\tilde{p}_2]$. Therefore continuing this process and taking into account (3), notice that $M_k=-i\frac{k-1}{2}[\tilde{p}_{k-1}]$ and we choose σ_k such that $M_k+\alpha_k\sigma_k=0$, then $\sigma_k=[p_k]$.

Remember than the algorithm created a chain of modified energies satisfying:

$$\begin{aligned}E^{n+2}_{I=\partial_x^n}(t)&=E^{n+1}_{I=\partial_x^n}(t)+\Lambda_{n+2}(\sigma_{n+2})\\ &=E^2_{I=\partial_x^n}(t)+\sum_{k=3}^{n+2}\Lambda_k(\sigma_k)\end{aligned}$$

By the explanation above, it is now clear that

$$C_{n+1}=E^{n+2}_{I=\partial_x^n}(t) \quad n\geqslant 0.$$

□

4 Cubic NLS

From the results above, it is natural to consider applying the same idea to other integrable equations. Another frequently studied integrable equation is the cubic nonlinear Schrödinger (NLS) equation, which also satisfies infinitely many conservation laws, defined as the following

Definition 4.1. We define the cubic nonlinear Schrödinger (NLS) equation as the following

$$\begin{cases} i\partial_t\phi + \partial_x^2\phi + 2|\phi|^2\phi = 0 \\ \phi(x,0) = \phi_0(x) \end{cases}$$

According to [3], the conservation laws of the cubic NLS equation are given by the following recursive formula

Definition 4.2. The Cubic NLS equation is an integrable system, its conservation laws can be recursively found as follows

$$w_1 = \phi(x)$$

$$w_{n+1} = -i\frac{dw_n}{dx}(x) + \bar{\phi}(x)\sum_{k=1}^{n-1} w_k(x)w_{n-k}(x)$$

The conserved quantities are

$$C_n = \int_{-\infty}^{\infty} \bar{\phi}(x)w_n(x)$$

Hence the first four conservation laws are

$$C_1 = \int |\phi(x)|^2 dx, \quad C_2 = \int -i\bar{\phi}(x)\partial_x\phi(x)dx$$

$$C_3 = \int (-\bar{\phi}(x)\partial_x^2\phi(x) + |\phi(x)|^4)dx$$

$$C_4 = \int (i(\bar{\phi}\partial_x^3\phi - |\phi|^2\phi\partial_x\bar{\phi} - 4|\phi|^2\bar{\phi}\partial_x\phi)dx$$

Let ϕ be a solution to the cubic NLS equation, then ϕ satisfies

$$\partial_t\phi = i(\partial_x^2\phi + 2|\phi|^2\phi)$$

Take the fourier transform with respect to space, then

$$\partial_t\hat{\phi} = i((i\xi)^2\hat{\phi} + 2\widehat{\phi^2\bar{\phi}}) = -i\xi^2\hat{\phi} + 2i\hat{\phi} * \hat{\phi} * \hat{\bar{\phi}}$$

Note that a significant difference between the KdV equation and the cubic NLS equation is that the solutions to the KdV equation are real functions but the solutions to the cubic NLS equation are complex functions. Hence we make the following modification to our previous definitions.

Definition 4.3. Suppose m is a k-multiplier for k an even number, we define the k-linear functional generated by m via integration to be the following

$$\Lambda_k(m) = \int_{A_k} m(\xi_1, ..., \xi_k)\hat{\bar{\phi}}(\xi_1) \cdots \hat{\bar{\phi}}(\xi_{k/2})\hat{\phi}(\xi_{k/2+1}) \cdots \hat{\phi}(\xi_k) d\xi_1 \cdots d\xi_k$$

Similarly, we have the following proposition

Proposition 4.1. Suppose ϕ is a solution to the cubic NLS equation, then

$$\frac{d}{dt}\Lambda_k(m) = -\Lambda_k(\alpha_k m) + 2ki\Lambda_{k+2}(\tilde{m}(\xi_1, ..., \xi_{k+2}))$$

where

$$\alpha_k = i(\xi_1^2 + \cdots + \xi_k^2)$$
$$\tilde{m}(\xi_1, ..., \xi_{k+2}) = m(\xi_1, ..., \xi_{k-1}, \xi_k + \xi_{k+1} + \xi_{k+2})$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\Lambda_k(m) &= \int_{A_k} m(\xi_1, ..., \xi_k)\frac{d}{dt}\hat{\bar{\phi}}(\xi_1) \cdots \hat{\bar{\phi}}(\xi_{k/2})\hat{\phi}(\xi_{k/2+1}) \cdots \hat{\phi}(\xi_k) d\xi_1 \cdots d\xi_k \\
&= \sum_{j=1}^{k/2} m(\xi_1, ..., \xi_k)\hat{\bar{\phi}}(\xi_1) \cdots (-i\xi_j^2 \hat{\bar{\phi}}(\xi_j) + 2i\hat{\bar{\phi}} * \hat{\bar{\phi}} * \hat{\phi}(\xi_j)) \cdots \hat{\phi}(\xi_k) d\xi_1 \cdots \xi_k \\
&+ \sum_{j=k/2+1}^{k} m(\xi_1, ..., \xi_k)\hat{\bar{\phi}}(\xi_1) \cdots (-i\xi_j^2 \hat{\phi}(\xi_j) + 2i\hat{\phi} * \hat{\phi} * \hat{\bar{\phi}}(\xi_j)) \cdots \hat{\phi}(\xi_k) d\xi_1 \cdots \xi_k \\
&= -\sum_{j=1}^{k} \int_{A_k} m(\xi_1, ..., \xi_k)(i\xi_j^2)\hat{\bar{\phi}}(\xi_1) \cdots \hat{\bar{\phi}}(\xi_{k/2})\hat{\phi}(\xi_{k/2+1}) \cdots \hat{\phi}(\xi_k) d\xi_1 \cdots d\xi_k \\
&+ \sum_{j=1}^{k/2} \int_{A_k} m(\xi_1, ..., \xi_k) 2i\hat{\bar{\phi}}(\xi_1) \cdots \widehat{\phi\bar{\phi}^2}(\xi_j) \cdots \hat{\phi}(\xi_k) d\xi_1 \cdots \xi_k \\
&+ \sum_{j=k/2+1}^{k} \int_{A_k} m(\xi_1, ..., \xi_k) 2i\hat{\bar{\phi}}(\xi_1) \cdots \widehat{\phi^2\bar{\phi}}(\xi_j) \cdots \hat{\phi}(\xi_k) d\xi_1 \cdots \xi_k \\
&= -\Lambda_k(\alpha_k m) + 2ki\Lambda_{k+2}(\tilde{m}(\xi_1, ..., \xi_{k+2}))
\end{aligned}$$

where

$$\alpha_k = i(\xi_1^2 + \cdots + \xi_k^2)$$

$$\tilde{m}(\xi_1, ..., \xi_{k+2}) = m(\xi_1, ..., \xi_{k-1}, \xi_k + \xi_{k+1} + \xi_{k+2})$$

□

Similarly, we provide an example for illustration

Example 4.1. In order to recover the conservation law C_3, we set

$$E^2(t) = \Lambda_2((i\xi_1)^2)$$

Then

$$M_4 = -2i(4[\xi_1^2] + 6[\xi_1\xi_2]) = -4i[\xi_1^2]$$

$$\sigma_4 = \frac{M_4}{\alpha_4} = \frac{-4i[\xi_1^2]}{4i[\xi_1^2]} = -1$$

Hence

$$E^4(t) = E^2(t) + \Lambda_4(\sigma_4) = \int (\bar{\phi}(x)\partial_x^2\phi(x) - |\phi(x)|^4)dx$$

Therefore by similar intuition, we have the following conjecture for the cubic NLS equation, which has not been rigorously proven

Conjecture 4.1. Let C_k denote the k-th conservation law of the cubic NLS equation, then

$$C_{k+1} = E_I^{k+2}(t) \quad k \geqslant 0, k \text{ even}$$

where

$$E_I^2(t) = \int_{\mathbb{R}} \bar{\phi}\partial_x^k\phi dx = \Lambda_k((i\xi_1)^k)$$

5 Future Research

In the future, it would be interesting to work out the details for the case of cubic NLS and rigorously prove how the chain of modified energies recover conservation laws. In addition, it would be interesting to consider different multipliers m, for example the multiplier

$$m(\xi) = \begin{cases} 1, & |\xi| < N \\ N^{-s}|\xi|^s, & |\xi| > 10N \end{cases}$$

in order to obtain some results on extending local well-posedness to global well-posedness. One important obstacle when tackling this problem would be to better understand the cancellations that take place as one computes more elements in the chain of modified energies. Even though we were expecting to advance towards this goal when studying this method applied to the recovery of conservation laws, the proofs we ended up producing don't seem to shed much light on these cancellations and new ideas might be necessary.

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