The Steenrod Square on Khovanov Homologies

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Abstract

Khovanov homology is a bi-graded abelian group (or ring module) associated to any knot or oriented link in S^3. Lipshitz and Sarkar strengthened this inviariant in [LSa], constructing a homotopy type whose cohomology was the Khovanov homology of a link. Thus, Khovanov homology inherits Steenrod operations. In [CS], Seed proved the Steenrod square alone is a stronger invariant than integral Khovanov homology. This paper tackles the question of a Steenrod square on Bar-Natan homology, as well as gives a combinatorial proof of the well-definedness and link invariance of the Steenrod square on Khovanov homology.

Contents

1 Introduction

Given an oriented link projection L in S^2, one can compute its *Jones polynomial*, which is a link invariant ([Jon]), one of several simple invariants used to classify links. Khovanov noticed in [Kh] that it was the (graded) Euler characteristic of a certain chain complex $KC(L)$, whose homology was also a link invariant, the *Khovanov homology*, denoted $Kh^*(L)$, of a link (with respect to any given ring of coefficients). $KC(L)$ is bi-graded by the homological grading and by an extra grading called the *quantum grading*, which the differential preserves, and thus inherits onto the homology groups. Thus, there are homology groups $Kh^{i,j}(L)$ for each homological grading i and quantum grading j. Moreover, this invariant is strictly stronger than the Jones polynomial; as Bar-Natan computed in [BNa] several examples of pairs of links with the same Jones polynomial but different (rational) Khovanov homology, such as: $(4_1, 11^n_{19})$, $(5_1, 10_{132})$, $(5_2, \overline{11^n_{57}})$, $(7_2, \overline{11^n_{88}})$, $(8_1, \overline{11^n_{70}})$, $(9_2, \overline{11^n_{13}})$, $(9_{42}, \overline{9_{42}})$, $(9_{43}, 11^n_{12})$, $(10_{125}, \overline{10_{125}})$, $(10_{130}, \overline{11^n_{61}})$, $(10_{133}, \overline{11^n_{27}})$, $(10_{136}, 11^n_{92})$, $(11^n_{24}, \overline{11^n_{24}})$, $(11^n_{28}, 11^n_{64})$, $(11^n_{50}, \overline{11^n_{133}})$, $(11^n_{79}, \overline{11^n_{138}})$, $(11^n_{82}, \overline{11^n_{82}})$, and $(11^n_{132}, \overline{11^n_{133}})$. (This list uses the numbering of knots as in [BNa], which is somewhat standard).

Lipshitz and Sarkar constructed in [LSa] a CW complex Y whose cohomology, up to a certain grading shift, corresponds to $Kh^*(L)$. More accurately, for all quantum gradings j, they obtained a CW complex Y^j such that $\widetilde{H}^{i+C}(Y^j) \cong Kh^{i,j}(L)$ for a fixed constant C. The resulting space is then $Y := \bigvee_j Y^j$. By desuspending C times, i.e. setting $\mathcal{X}_{Kh}(L) := \Sigma^{-C} Y$, we obtain a suspension spectrum whose cohomology matches $Kh^*(L)$. It can also be decomposed as a wedge sum:

$$\mathcal{X}_{Kh}(L) = \bigvee_v \mathcal{X}^j_{Kh}(L),$$

and $\widetilde{H}^i(\mathcal{X}^j_{Kh}(L)) \cong Kh^{i,j}(L)$. This spectrum is denoted the *Khovanov homotopy type* of the link, and it's also a link invariant. A natural question arises: is this homotopy type a strictly stronger invariant than integral Khovanov homology?

To tackle this question, one notices this construction endows $Kh^*(L)$ with Steenrod operations (when working over $\mathbb{F}_2$-coefficients) $Sq^k : Kh^{i,j} \to Kh^{i+k,j}$. Partial information about these Steenrod operations could therefore be sufficient to differentiate links without having to compute the full homotopy type. Lipshitz and Sarkar used this idea in [LSb] to give a combinatorial description of Sq^1 and Sq^2 that is feasible to compute and results in a link invariant that is indeed strictly stronger than merely the integral Khovanov homology $Kh_{\mathbb{Z}}(L)$, as was verified by Seed in [CS]. In particular, the previous question has an affirmative answer. For instance, Seed finds the triple $(K14n5017, K14n11311, K14n11629)$ of knots which all have the same Khovanov homology but different Steenrod squares. (Seed's notation for links is different than Bar Natan's, and it can be consulted in [CS]).

This paper gives a different, fully combinatorial proof of Lipshitz and Sarkar's aforementioned main result in [LSb]. (In order to state it, we will reference some objects which we will define in Section 2.4). Namely,

Theorem 1 ([LSb])**.** Given any cycle $c \in KC^{i,j}(L)$ in the Khovanov chain complex of a given link diagram, it has some *boundary matching*. With respect to it, one can define numbers

$\#\mathfrak{G}_c(x), f(\mathfrak{G}_c(x))$, and $g(\mathfrak{G}_c(x)) \in \mathbb{Z}/2$ for some elements $x \in KC^{i+2,j}(L)$. The operation $\mathfrak{sq}^2 : Kh^{i,j} \to Kh^{i+2,j}$ defined as

$$\mathfrak{sq}^2([c]) := \left[\sum_{x \in KG^{i+2,j}} \left(\#|\mathfrak{G}_c(x)| + f(\mathfrak{G}_c(x)) + g(\mathfrak{G}_c(x))\right) x \right] \tag{1}$$

is a well-defined link-invariant.

There are other homology theories associated to a link, such as odd Khovanov homology ([ORSz]) and Bar-Natan homology ([BNb]). This paper studies a hypothetical Steenrod square on Bar-Natan homology, and concludes the following technical result, which again references objects defined later in the paper.

Theorem 2. Let $F_{BN} : Cob^3_{e/l} \to f\mathscr{B}$ be the 1-functor defined in Section 5.3. Given a link diagram L, the composition $2^n \xrightarrow{F_L} Cob^3_{e/l} \longrightarrow f\mathscr{B}$ describes a 1-functor. There does not exist an assignment $X \to \{Standard, Nonstandard\}$ (where X is the set of 2-faces of the cube), such that the above composition extends to a Bar-Natan functor $2^n \to f\mathscr{B}$ agreeing with the assignment. Similarly, there does not exist an assignment $X \to \{Standard, Nonstandard\}$ (where X is the category of squares in $Cob^3_{e/l}$) such that the above composition extends to a Bar-Natan functor $2^n \to f\mathscr{B}$ agreeing with the assignment.

The paper is structured as follows. In Section 2 we define Khovanov generators and Khovanov homology, as well as its reduced version. We also define the combinatorial Steenrod operations, $\mathfrak{sq}^1$ and $\mathfrak{sq}^2$, and their reduced versions. In Section 3 we present the combinatorial proof for Theorem 1. In Section 4 we introduce some topological and categorial concepts needed to understand the rest of the paper. In Section 5 we try to find a Steenrod square on Bar-Natan homology, and reduce the task to finding what we call the *Bar-Natan Ladybug matching*. In Section 6 we discuss further questions.

1.1 Acknowledgements

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2 Khovanov homology

For a given oriented link projection L in S^2, one can classify crossings as $+1$ or -1-crossings according to the following rule: if we rotate the over-strand counterclockwise to reach the under-strand pointing in the same direction, we classify it as a $+1$-crossing. Otherwise, it is a -1-crossing. In this process, we sweep two regions. The resolution of the crossing which unites these regions is called a 0-resolution, and the other resolution is called a 1-resolution. This process generates another diagram. For convenience, we keep track of 0-resolutions by

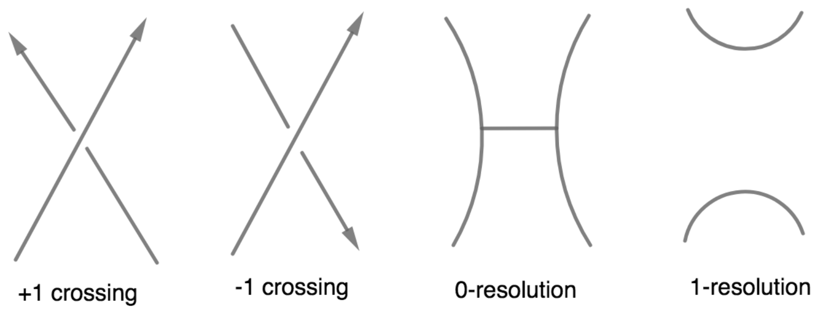


Figure 1: Resolutions

drawing a small arc, as shown above. We call any such collection of (oriented) cycles and arcs a *resolution configuration*. Hence, If L has n crossings, one can order them arbitrarily, and thus to each vector $v \in \{0,1\}^n$ one associates a resolution configuration $D_L(u)$. If n_- and $n_+ = n - n_-$ are the number of -1 and $+1$-crossings respectively, then

$$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \left(\sum_{v \in \{0,1\}^n} (q + q^{-1})^{c(D_L(v))} (-q)^{|v|} \right),$$

the *unnormalized Jones polynomial*, is a link invariant, where $|v| = \sum v_i$ is the magnitude of the vector and $c(D_L(v))$ denotes the number of cycles in a resolution configuration. The proof is a straightforward application of Reidemeister's Theorem, by verifying $\hat{J}$ is preserved by the three Reidemeister moves.

2.1 The Khovanov complex of a link projection

Definition 3. A *Khovanov generator* of a link projection L is a resolution configuration $D_L(v)$ together with an assignment x of either the symbol $+$ or $-$ to each cycle. The set of all Khovanov generators is denoted by $KG(L)$. It is bigraded:

$$\begin{aligned} \mathrm{gr}_h(D_L(v), x) &= -n_- + |v| \\ \mathrm{gr}_q(D_L(v), x) &= n_+ - 2n_- + |v| + \sum x \end{aligned}$$

where $\sum x$ denotes the sum of all the signs assigned to the cycles (where a $+$ contributes a $+1$ and a $-$ contributes -1).

(The constant grading shifts $-n_-$ and $n_+ - 2n_-$ are there to keep Khovanov homology invariant under Reidemeister moves). The Khovanov generators are the edges of a graph, also denoted KG, whose edges can be described as follows. Whenever $v \leq_1 w$ in the obvious poset structure (and the subscript 1 denotes that the vectors differ in exactly 1 entry), we draw an arrow $(D_L(v), x) \to (D_L(w), y)$ if, when turning the corresponding 0-resolution into a 1-resolution, we either

- split a circle, labeled $+$, into two circles, labeled $+$ and $-$ in some order.
- split a circle, labeled $-$, into two circles labeled $-$.
- fuse two circles together, both labeled $+$, into a circle labeled $+$, or
- fuse two circles, labeled $+$ and $-$ in some order, into a circle labeled $-$.

Notice all of these operations increase gr_h by 1 and preserve gr_q. In case this arrow exists, we set $\delta((D_v(L), x), (D_w(L), y)) = \pm 1$ (we will determine the sign later). Otherwise, set $\delta = 0$. The origin of these seemingly arbitrarily rules can be explained algebraically as follows. Fix a ring or field R. (In this paper we always choose $R = \mathbb{Z}/2$ for the Steenrod operations to exist). Let $V = \langle v_+, v_- \rangle$ be the $\mathbb{Z}$-graded free R-module (or vector space if working over a field) generated by 2 elements v_+, v_-, of gradings 1 and -1 respectively. Then to each resolution configuration $D_L(v)$ assign the vector space $V(D_L(v)) := V^{\otimes c(D_L(v))}\{|v|\}$, where the curly brackets indicate the grading is shifted upwards by $|v|$. Let

$$KC^i := \bigoplus_{\mathrm{gr}_h(D_L(v), x) = i} V(D_L(v)).$$

We define an (unsigned) "differential"' map $d : KC^i \to KC^{i+1}$ as the direct sum of differentials $d_{v,w} : V(D_L(v)) \to V(D_L(w))$. These differentials are defined as follows. If $v \not\leq w$, then $d_{v,w} = 0$. Otherwise, $v \leq_1 w$, and if when turning the corresponding 0-resolution into a 1-resolution we split a circle, the differential keeps all tensor factors corresponding to the cycles that don't participate unchanged, and employs $\Delta : V \to V \otimes V$ on the circle that is split:

$$\Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

On the other hand, if when turning the corresponding 0-resolution into a 1-resolution we merge two circles, we employ $m : V \otimes V \to V$ defined as follows:

$$m : \begin{cases} v_+ \otimes v_- \mapsto v_- \quad v_- \otimes v_+ \mapsto v_- \\ v_+ \otimes v_+ \mapsto v_+ \quad v_- \otimes v_- \mapsto 0 \end{cases}$$

Notice the Khovanov generators correspond to basis elements of the vector spaces $V(D_L(v))$, and there is an arrow $(D_L(v), x) \mapsto (D_L(w), y)$ if and only if the differential $V(D_L(v)) \to V(D_L(w))$ applied to $(D_L(v), x)$ contains an $(D_L(w), y)$-term in its summation.

Defining $KC = \bigoplus KC^i$ gives a (bi-graded) abelian group endowed with a differential that increases gr_h by 1 and preserves gr_q. To make $d^2 = 0$, i.e. to obtain a chain complex, we use the signed differential instead, which is simply $\delta : (-1)^{s(v,w)} d_{v,w}$, where $s(v,w) := \sum_{i=1}^{a} v_i$, where a is the index where v and w differ: $v_a = 0, w_a = 1$. (This is known as the *standard 1-sign assignment* on the edges of the cube, and can be seen as a 1-cochain of the cellular cohomology groups of the cube with $\mathbb{F}_2$-coefficients.) For notational convenience later on, when $(D_v(L), x) \to (D_w(L), y)$ in KG, we define $\delta((D_v(L), x), (D_w(L), y)) = (-1)^{s(v,w)}$. Khovanov homology is simply the homology of this chain complex (KG, δ), i.e.

$$Kh^{i,j}(L) := \frac{\ker\ KC^{i,j} \xrightarrow{\delta} KC^{i+1,j}}{\operatorname{im}\ KC^{i-1,j} \xrightarrow{\delta} KC^{i,j}}.$$

We also define the following useful 2-cochain, for use later-on.

Definition 4. The *standard 2-sign assignment* f on the edges of the cube is defined on 2-faces of the cube. For each face $v \geq_2 w$, its value is

$$f(v,w) = \left(\sum_{i=1}^{a-1} v_i\right) \left(\sum_{j=a+1}^{b-1} v_j\right),$$

where $v = v_1 \cdots v_n$ and a, b are the positions where v and w differ. If x, y are Khovanov generators whose corresponding vectors are v, w, then we also denote $f(x,y) := f(v,w)$.

Remark 5. For the interested reader: the Khovanov homotopy type is obtained as the realization of the *Khovanov flow category*, which is a framed flow category assigned to a link projection. This formula arises from the need to give an explicit framing of the 1-dimensional moduli spaces in the Khovanov flow category which will extend to a framing of the whole category. A thorough discussion can be found in [LSa].

2.2 Reduced Khovanov homology

Fix a point $p \in L$ in the link diagram. In this section, L will denote the information of this pointed link. (Notice two previously isotopic links may be different if we choose different basepoints). This allows us to partition the vertices of the graph of Khovanov generators $KG(L) = KG_+(L) \sqcup KG_-(L)$ into the subgraph with the generators having a $+$ sign at p and a $-$ sign at p, respectively. It is easy to see that if $x \to y$ is an arrow in KG, and $x \in KG_- \Rightarrow y \in KG_-$ and $y \in KG_+ \Rightarrow x \in KG_+$. (Notice, however, there can be arrows from KG_+ to KG_-). Therefore, we get a short exact sequence of chain complexes

$$0 \to \widetilde{KC}_-(L) \to KC(L) \to \widetilde{KC}_+(L) \to 0,$$

where $\widetilde{KC}_-$ denotes the the subcomplex generated by KG_- and $\widetilde{KC}_+$ is the quotient complex generated by KG_+. The obvious bijection between $KG_- \to KG_+$ (which increases quantum grading by 2) allows us to define a unique *reduced Khovanov complex* $\widetilde{KC}_-^{i,j-1} =: \widetilde{KC}^{i,j}(L) := \widetilde{KC}_+^{i,j+1}(L)$, and the above exact sequence gives us, by taking cohomology, the reduced long exact sequence:

$$\cdots \to \widetilde{Kh}^{i,j+1}(L) \to Kh^{i,j}(L) \to \widetilde{Kh}^{i,j-1}(L) \to \widetilde{Kh}^{i+1,j+1}(L) \to \cdots$$

2.3 Combinatorial Steenrod operations

(In this section we work entirely in $\mathbb{F}_2$ coefficients, unless a subindex suggests otherwise). We will define operations $\mathfrak{sq}^1 : Kh^{i,j}(L) \to Kh^{i+1,j}(L)$ and $\mathfrak{sq}^1 : Kh^{i,j}(L) \to Kh^{i+2,j}(L)$ on Khovanov homology. These will match the Steenrod operations Sq^1, Sq^2 inherited from the Khovanov homotopy type. The short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ gives a short exact sequence of chain complexes

$$0 \to KC_{\mathbb{F}_2} \to KC_{\mathbb{Z}/4} \to KC_{\mathbb{F}_2} \to 0.$$

We will later trivially verify that the corresponding Bockstein homomorphism, $\beta : Kh^{i,j}(L) \to Kh^{i+2,j}(L)$, matches Sq^1. In order to describe it, we will need to define further combinatorial data on the graph KG. For any chain $c \in KC^{i,j}$, we will say $y \in KG^{i,j}$ is *in* c, denoted $y \in c$, if y appears in c with coefficient $\neq 0$ (in this case, 1). For $z \in KG^{i+1,j}$, we denote $\mathscr{G}_c(z) := \{y \in c | \delta(z, y) = \pm 1\}$. (Since we're over $\mathbb{F}_2$ coefficients, we don't care about the sign of the δ). Then, β can be described as follows. Let $c \in KC^{i,j}$ be a cycle. This implies $|\mathscr{G}_c(z)|$ is always even, so that $\sum_{y\in\mathscr{G}_c(x)} \delta(z, y)$ is even too. Then it is straightforward by standard homological algebra that

$$\beta([c]) := \left[\sum_z \left(\frac{\sum_{y\in\mathscr{G}_c(z)} \delta(z,y)}{2}\right) z\right]$$

is a well-defined map (i.e. the RHS is a cycle and its class is independent of the choice of representative of $[c]$), and indeed is the Bockstein map.

Definition 6. $\mathfrak{sq}^1 : Kh^{i,j}(L) \to Kh^{i+1,j}(L)$ is defined as

$$\mathfrak{sq}^1([c]) := \left[\sum_z \left(\frac{\sum_{y\in\mathscr{G}_c(z)} \delta(z,y)}{2}\right) z\right] \tag{2}$$

Remark 7. If we used $\mathbb{Z}/p$-coefficients instead, we get a similar Bockstein map. The formula divides by p instead of by 2, and multiplies $\delta(z, y)$ by c_y, the coefficient of y in c.

Now, we proceed to define $\mathfrak{sq}^2$ for any cycle $c \in KC^{i,j}$. We recall the definition of the *ladybug matching* on KG. (For a full discussion, the reader may consult [LSa]). For notational convenience, if x, y are Khovanov generators, we denote $x \geq_n y$ if there is a chain of n arrows taking x to y in the graph KG. Given any two Khovanov generators $x \geq_2 y$, the set $\{z \in KG | x \geq_1 z \geq_1 y\}$ has either 2 or 4 elements. (The latter case happens yo only occurs in 1 case, called the *ladybug*). The ladybug matching is a prescribed involution (or pairing) $\mathfrak{l}_{x,y}$ of this set. Furthermore, as observed before, for $z \in KG^{i+1,j}$, $|\mathscr{G}_c(z)|$ is even, so we can pair up its elements in some arbitrary fashion via some fixed point free involution $\mathfrak{b}_z : \mathscr{G}_c(z) \to \mathscr{G}_c(z)$. Define some sign assignment $\mathfrak{s}_z : \mathscr{G}_c(z) \to \{0, 1\} \subset \mathbb{Z}$ so that $\mathfrak{s}_z(y) + \mathfrak{s}_z(\mathfrak{b}_z(y)) = 0 \bmod 2$ if $s(z, y) \neq s(z, \mathfrak{b}_z(y))$, and 1 otherwise. Call a collection of such choices $\mathfrak{m} = \{(\mathfrak{b}_z, \mathfrak{s}_z)\}_{z\in KG^{i+1,j}}$ a *boundary matching* for c.

Remark 8. Notice $\mathfrak{sq}^1([c]) = [\sum_z \sum_{y\in\mathscr{G}_c(x)} \mathfrak{s}_x(y)x \bmod 2]$ for any boundary matching. This is an alternative definition for $\mathfrak{sq}^1$.

For any $x \in KG^{i+2,j}$, define

$$\mathcal{G}_c(x) := \{(z, y) \in KG^{i+1,j} \times KG^{i,j} | y \in \mathcal{G}_c(z), \delta(x, z) = \pm 1\}.$$

Choose a boundary matching $\mathfrak{m}$ for c. We construct a graph $\mathfrak{G}_c(x)$ (depending on $\mathfrak{m}$) as follows. Its vertices are $\mathcal{G}_c(x)$. There's an unoriented edge $(z, y) - (z', y)$ if the ladybug matching $\mathfrak{l}_{x,y}$ matches z and z'. This edge is labeled $f(x, y) \in \mathbb{F}_2$. There is also an oriented edge (labeled 0) from (z, y) to (z, y') if $\mathfrak{b}_z$ matches y with y' and $\mathfrak{s}_z(y) = 0, \mathfrak{s}_z(y') = 1$. Otherwise if $\mathfrak{s}_z(y) = \mathfrak{s}_z(y')$ then the edge is unoriented.

Let $f(\mathfrak{G}_c(x)) \in \mathbb{F}_2$ denote the sum of edge labels in the graph. This is independent of the boundary matching (f increases by 1 for every pair (x, y) with $|\mathcal{G}_y(x)| = 2$ and $f(x, y) = 1$). Notice there's two types of edges: those (like $(z, y) - (z', y)$) that pivot on a y, which we call y-pivots, and z-pivots. By construction, every vertex is attached to two edges: one y-pivot and one z-pivot. Thus, each component of the graph is an even cycle. Temporarily label each vertex (z, y) with $s(x, z) + s(z, y) \in \mathbb{F}_2$. An unoriented edge must switch vertex labels while an oriented edge won't. Therefore, the number of oriented edges in each cycle is even. Let $\#|\mathfrak{G}_c(x)|$ denote the number of components. Define $g(\mathfrak{G}_c(x)) \in \mathbb{F}_2$ as follows: traverse each component cycle once in some direction. Each time you traverse an edge contrary to the way it's oriented, add 1 to g. It is straightforward to verify that the final tally $g(\mathfrak{G}_c(x))$ is well-defined. Thus, we are finally able to understand equation (1). We have the following important theorem.

Theorem 9 ([LSa])**.** The combinatorial first and second Steenrod operations coincide with the first and second Steenrod operations coming from $\mathcal{X}_{Kh}(L)$. In other words, $\mathfrak{sq}^2 = \mathrm{Sq}^2 : Kh^{i,j} \to Kh^{i+2,j}$ and $\mathfrak{sq}^1 = \mathrm{Sq}^1 : Kh^{i,j} \to Kh^{i+1,j}$.

2.4 Combinatorial reduced Steenrod operations

Let $c \in \widetilde{KC}_-^{i,j}$ be a cycle. Notice the equation $dc = 0$ has the same meaning in both $\widetilde{KC}_-$ and KC, so it is unambiguous. For any $x \in KG_-^{i+1,j}$, notice $\mathscr{G}_c(x) \subset KG_-$ by definition. Further, $\mathscr{G}_c(x) = \varnothing$ whenever $x \in KG_+$. Therefore, the same definition as before can define the first reduced Steenrod operation:

Definition 10. Choose any boundary matching for c, seen as a cycle in KC. Then

$$\widetilde{\mathfrak{sq}}^1([c]) := \left[\sum_x \left(\frac{\sum_{y \in \mathscr{G}_c(x)} \delta(x, y)}{2}\right) x\right] = \left[\sum_x \sum_{y \in \mathcal{G}_c(x)} \mathfrak{s}_x(y) x \bmod 2\right].$$

The next important observation is that for $x \in KG_+^{i+2,j}$, $\mathcal{G}_c(x) = \varnothing$. Moreover, when $x \in KG_-^{i+2,j}$, we have $\mathcal{G}_c(x) \subset KG_- \times KG_-$. Then (1) also extends as a carbon copy for the definition of $\widetilde{\mathfrak{sq}}^2$. (The ladybug $\mathfrak{l}_{x,y}$ matching can potentially pair elements from KG_+ with KG_-, but this only happens if $x \in KG_+, y \in KG_-$). We must also prove Theorem 1 in this case. This turns out to be pretty easy. The only change to the proof lies in the fact that boundaries used in the proof must lie in $d\widetilde{KC}_-$ and not just dKC, which is straightforward from the proof. Notice the first Steenrod operations commute with the map $\widetilde{Kh}^{i,j+1}(L) \to Kh^{i,j}(L)$ from the long exact sequence.

Doing these constructions starting from KG_+ works as well, albeit the proofs are slightly more elaborate. We no longer have $\mathcal{G}_c(x) = \varnothing$ when $x \in KG_-$, so we must restrict the above formula in Definition 10 to $x \in KG_+$. $c \in \widetilde{KC}_+^{i,j}$ might no longer be a cycle inside KC. However, $dc \in KC_-$, and so the formula is still well-defined once we restrict to $x \in KG_+$. When $x \in KG_+^{i+2,j}$, we get $\mathcal{G}_c(x) \subset KG_+ \times KG_+$. However, when $x \in KG_-$, we no longer get $\mathcal{G}_c(x) = \varnothing$, so we must also restrict the summation in (1) to $x \in KG_+$. The following observation saves us the trouble of having to prove Theorem 1 again.

Let $S : KG_- \leftrightarrow KG_+$ and $S : \widetilde{KC}_- \leftrightarrow \widetilde{KC}_+$ (shifting quantum grading by 2 but preserving homological grading) be the obvious bijection and isomorphism, respectively. Notice S carries cycles to cycles. It is trivial to verify $S : \mathscr{G}_c(x) \to \mathscr{G}_{S(c)}(S(x))$ is a bijection for any cycle in $c \in KC_-^{i,j}$ and $x \in KG_-^{i+1,j}$. If we use the same boundary matchings for c, then the two definitions for $\widetilde{\mathfrak{sq}}^1$ agree with respect to S. Moreover, for $z \in KG_-^{i+2,j}$, we also have that $S \times S : \mathcal{G}_c(z) \to \mathcal{G}_{S(c)}(S(z))$ is also a bijection. Hence the two definitions for $\widetilde{\mathfrak{sq}}^2$ also agree with respect to S. Thus, it is justified to think of these operations as

$$\widetilde{\mathfrak{sq}}^1 : \widetilde{Kh}^{i,j} \to \widetilde{Kh}^{i+1,j}$$
$$\widetilde{\mathfrak{sq}}^2 : \widetilde{Kh}^{i,j} \to \widetilde{Kh}^{i+2,j}.$$

(Because they agree with the topological reduced Steenrod operations, they are link invariant). It is straightforward to verify that the two Steenrod operations commute with $Kh^{i,j}(L) \to \widetilde{Kh}^{i,j-1}(L)$.

Remark 11. In the case of Khovanov homology, there is no need to prove directly, as I do in the proof of Theorem 1, that these operations are well-defined and link invariant; it suffices to prove that they agree with the operations coming from the homotopy type. However, if we attempted to define similar combinatorial definitions for other homology theories (such as Bar-Natan homology) which lack a homotopy type, a direct proof of Theorem 1 would be needed, as well as a proof of link invariance.

2.5 Bar-Natan homology

In this section, we define Bar-Natan homology. We construct the graph KG_{BN} similarly as before, with the only difference that we draw an arrow between two generators $(D_L(v), x) \to (D_L(w), y)$ if either the previously described scenarios occur, or we either

- split a circle, labeled $+$, into two circles both labeled $+$.
- fuse two circles, both labeled $-$, into a circle labeled $-$.

All arrows $x \to y$ will be labeled $\sigma(x, y) = +1$ except for arrows of the first type described above (spliting a + circle into two + circles), which we will label -1. In other words, we use the operators

$$\Delta_{BN} : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ - v_+ \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

and

$$m_{BN} : \begin{cases} v_+ \otimes v_+ \mapsto v_+, v_+ \otimes v_- \mapsto v_- \\ v_- \otimes v_+ \mapsto v_-, v_- \otimes v_- \mapsto v_-. \end{cases}$$

They generate a chain complex KC_{BN}, with the same underlying abelian group as KC but with a different differential. A proof that this is indeed a chain complex, whose homology is link invariant, can be found in [BNb].

3 Combinatorial Proof for the Steenrod Square

3.1 Well-definedness of the Steenrod Square

Before presenting the proof of the well-definedness of (1), we note that the same proof can be used to verify the following slightly more general technical lemma.

Lemma 12. Let X be a subset of KG (for some link L), such that you cannot use the arrows in KG to start in X, leave X, then return to X. Then the elements of X generate a free $\mathbb{Z}/2$-module equipped with a differential that makes into a chain complex, $C(X)$, with homology groups $H^*(X)$. In this definition, all the definitions (the ladybug matching and boundary matchings) needed for equation (1) still make sense. Then, $\mathfrak{sq}^2 : H^i(X) \to H^{i+2}(X)$ defined by

$$\mathfrak{sq}^2([c]) = \left[\sum_{x \in X} \left(\#|\mathfrak{G}_c(x)| + f(\mathfrak{G}_c(x)) + g(\mathfrak{G}_c(x)) \right) x \right]$$

is well-defined.

We outline the sketch and then prove each step separately.

Step 1. We start by proving it is independent of the sign assignments $\mathfrak{s}_z$ involved in choosing $\mathfrak{m}$. By our above observations, $f(\mathfrak{G}_c(x))$ is fixed. Also, changing the sign assignment only changes the edge orientations in the graph, so $\#|\mathfrak{G}_c(x)|$ is also unchanged. Thus we focus on $g(\mathfrak{G}_c(x))$.

Step 2. Then we prove (1) is independent of the involutions $\mathfrak{b}_z$. Again fix $z \in KG^{i+1,j}$ and $y_1, y_2, y_3, y_4 \in \mathcal{G}_c(z)$ paired up by $\mathfrak{b}_z$ in the following way: $y_1 \text{ ——— } y_2$, $y_3 \text{ ——— } y_4$. The new boundary matching will consist of the same information but with the pairing $y_1 \text{ ——— } y_3$, $y_2 \text{ ——— } y_4$, and change $\mathfrak{s}_z(y_i)$ for $i = 1, 2, 3, 4$ in any way that satisfies the axioms for boundary matchings. Any other boundary matching can be reached by a finite number of these moves.

Step 3. Next we prove $\mathfrak{sq}^2$ is a homomorphism.

Step 4. Well-definedness will follow from the above, together with verifying $\mathfrak{sq}^2(dw) = 0$ for any generator w.

Step 5. Finally, we will prove the RHS of (1) is a cycle.

This will complete the proof of this first part of the theorem. The following proposition proves Step 1.

Proposition 13. Fix $z \in KG^{i+1,j}$ and $y, y' \in \mathcal{G}_c(z)$ paired up by $\mathfrak{b}_z$. If $(\mathfrak{s}_z(y), \mathfrak{s}_z(y')) = (0,0)$ then there is no choice to make. Otherwise it is either $(0,1)$ or $(1,0)$, and we can get a new assignment $\mathfrak{s}'_z$ be switching between these two choices. (Any other boundary matching $\mathfrak{m}'$ with the same involutions can be reached from $\mathfrak{m}$ via a finite number of these switches). The formula (1) is unchanged by this switch.

Proof. When making such a change, it is easy to check that $g(\mathfrak{G}_c(x))$ changes if and only if $\delta(z,x) = \pm 1$. Thus, $\mathfrak{sq}^2([c])$ changes by a total of dz. This is a boundary, so the cohomology class remains unchanged. $\square$

The following proposition proves the second step.

Proposition 14. The formula (1) is unchanged by the switch described in Step 2 above.

Proof. If $\delta(z,x) = 0$ then the coefficient of x in (1) is unchanged, so we focus solely on the case $\delta(z,x) = \pm 1$. For $i, j = 1,2,3,4$, let $(ij) = 1$ if we have an edge $(z, y_i) \longrightarrow (z, y_j)$ in $\mathfrak{G}_c(x)$ (oriented in that fashion) for the original boundary matching, and 0 otherwise. Notice $(ij) + (ji) = 1$. Similarly define $(ij)' = 1$ or 0 for the new boundary matching. This is justified because these numbers are independent of x, as long as $\delta(z,x) = \pm 1$. There are two cases:

$$\textbf{Case 1: } (12) + (43) = (13)' + (42)' \bmod 2, \text{ or}$$
$$\textbf{Case 2: } (12) + (43) \neq (13)' + (42)' \bmod 2$$

For each x, there are three further cases.

A. 2 cycles to 1 cycles. In Case 1, g doesn't change, so $\mathfrak{sq}^2([c])$ becomes $\mathfrak{sq}^2([c]) + x$. In Case 2, $g(\mathfrak{G}_c(x))$ changes, so $\mathfrak{sq}^2([c])$ stays the same.

B. 1 cycle to 2 cycles. In Case 1, g doesn't change, so $\mathfrak{sq}^2([c])$ becomes $\mathfrak{sq}^2([c]) + x$. In Case 2, $g(\mathfrak{G}_c(x))$ changes, so $\mathfrak{sq}^2([c])$ stays the same.

C. 1 cycle to 1 cycles. In Case 1, g changes, so $\mathfrak{sq}^2([c])$ becomes $\mathfrak{sq}^2([c]) + x$. In Case 2, $g(\mathfrak{G}_c(x))$ doesn't change, so $\mathfrak{sq}^2([c])$ stays the same.

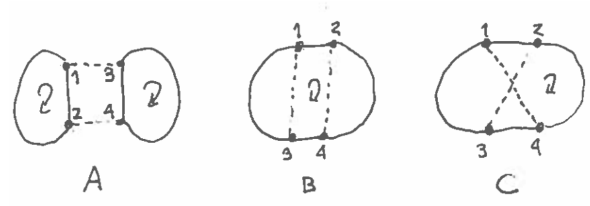
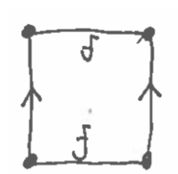


Figure 2: Cases A,B,C

Therefore, in case 1, $\mathfrak{sq}^2([c])$ changes by dz, a boundary, and hence its cohomology class remains the same. In case 2, clearly $\mathfrak{sq}^2([c])$ also doesn't change. This proves the proposition. $\square$

Before continuing, notice we can "cheat" a little. For any $d \in KC^{i,j}$ disjoint of c, we can replace c by $c+2d$, which is the same chain but contains different y's when viewed as a multiset. Notice $2d$ is obviously a cycle. We can view this in the graph KG as duplicating those vertices in d. We can then choose a boundary matching for $c+2d$ that respects this partition, so that $\mathfrak{sq}^2(c+2d) = \mathfrak{sq}^2(c) + \mathfrak{sq}^2(2d)$. But it is straightforward to see that $\mathfrak{sq}^2(2d) = 0$, by inducting on the number of vertices in d and choosing the obvious boundary matching for $2d$: therefore $\mathfrak{sq}^2(c+2d) = \mathfrak{sq}^2(c)$. (When d has n vertices, $\mathfrak{G}_{2d}(x)$ consists of between n or $2n$ squares, depending on how many ladybugs occur, of the following type:)



Proposition 15. $\mathfrak{sq}^2$ is a homomorphism.

Proof. Let $c, c' \in KC^{i,j}$ be cycles, and let $c+c'$ denote the cycle thought of as a multiset that admits double elements, as explained above. Choose a boundary matching for this $c+c'$ that respects the partition between c and c'. Then this restricts to a boundary matching for c and another for c'. This proves that $\mathfrak{sq}^2(c+c') = \mathfrak{sq}^2(c) + \mathfrak{sq}^2(c')$, so that $\mathfrak{sq}^2$ is a homomorphism. $\square$

Now we prove Step 4.

Proposition 16. We have $\mathfrak{sq}^2(dw) = 0$ for any $w \in KG^{i-1,j}$.

Proof. To compute the coefficient of any x in $\mathfrak{sq}^2(dw)$, we can restrict to the case when $w \leq_3 x$, and focus on the 3-cube they form. Set a boundary matching for $c = dw$ as follows. For any $z \in KG^{i+1,j}$, the involution in $\mathscr{G}_c(z) = \{y \in dw | z \in dy\}$ is the same as the ladybug matching for (y, w). If $\delta(y,z) = \delta(y',z)$ for $y, y' \in \mathcal{G}_w(z)$ paired up, then we will have

$\delta(w,y)=0, \delta(w,y')=1$ (or vice-versa). Then choose $\mathfrak{s}_z(y)=0, \mathfrak{s}_z(y')=1$. (Else, choose both $\mathfrak{s}_z(y)=\mathfrak{s}_z(y')=0$). Then, by Lemmas 5.14, 5.15, and 5.17 in [LSa], or by [Zi, pp. 31-33], the graph $\mathfrak{G}_c(x)$ is a disjoint union of hexagons. Suppose the cube vertices corresponding to w and x are $A0B0C0D$ and $A1B1C1D$, where A, B, C, D denote binary strings (by abuse of notation, we will also denote by A, B, C, D the sum of their digits). Restrict momentarily to one of these hexagons, which must be of the following form:

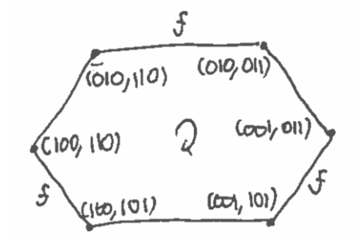


Figure 3: Hexagon in the graph

(We suppress the digits where x and w coincide). We notice first $\#|\mathfrak{G}_c(x)|=1$. Now,

$$\begin{aligned} f(\mathfrak{G}_c(x)) &= f(100,111)+f(010,111)+f(001,111) \\ &= (A+B+1)C + A(B+C+1) + AB \\ &= A + BC + C \end{aligned}$$

Lastly, traverse the hexagon in the manner observed in the above diagram. Let $x = s(000,100), y = s(000,010), z = s(000,001)$. We traverse an oriented edge, say $(100,110) \to (010,110)$, in its direction if and only if, in this example, $x=0$ and $y=1$. In other words,

$$\begin{aligned} g(\mathfrak{G}_c(x)) &= (x+1)y + (y+1)z + (z+1)x = xy + yz + zx + x + y + z \\ &= (A+1)(A+B) + (A+B+1)(A+B+C) + (A+B+C+1)A \\ &= B + BC + C \end{aligned}$$

All in all, $\#|\mathfrak{G}_c(x)| + f(\mathfrak{G}_c(x)) + g(\mathfrak{G}_c(x)) = A+B+1$. Thus, we get:

$$\#|\mathfrak{G}_c(x)| + f(\mathfrak{G}_c(x)) + g(\mathfrak{G}_c(x)) = \begin{cases} A+B+1, & \text{if } \mathfrak{G}_c(x) \text{ is 1 hexagon.} \\ 0, & \text{if it's 2 hexagons} \end{cases}$$

Let G denote the set of $z \geq_2 w$ such that if $w = A0B0C$ and $z = A1B1C$, then the sum of digits in A is 0. For $x = A1B1C1D$ and $w = A0B0C0D$ as before, if $x \geq_3 w$ then there are three or six generators $x \geq_1 z \geq_2 w$. Their vectors will be $A1B1C0D, A1B0C1D$, and $A0B1C1D$. If $A+B+1=1$ then either 1 or 3 of them will be in G. Otherwise, either 0 or 2 of them will be in G. Hence

$$\mathfrak{sq}^2(dw) = d\left(\sum_{z \in G} z\right),$$

which is 0 in cohomology, as desired. $\square$

Finally, we verify Step 5.

Proposition 17. If we fix $w \in KG^{i+3}$ and a cycle c of (homological) grading i, the coefficient of w in $d\mathfrak{sq}^2(c)$ is 0.

Proof. We create a graph $\mathfrak{G}_c$ as follows. Its vertices will be paths y —— z —— x —— w such that $y \in c$, and $z \in dy, x \in dz, w \in dx$. Two vertices will be joined by a blue edge if they differ in the z coordinate with respect to the ladybug $\mathfrak{l}_{x,y}$ matching (such edges will also be called xy-pivots); they will be joined by a black edge if they differ in the y coordinate with respect to the boundary matching (such edges will also be called xz-pivots); finally, they will be joined by a red edge if they differ in the x coordinate with respect to the ladybug $\mathfrak{l}_{w,z}$ matching (such edges will also be called zy-pivots). Notice if we erase all red edges, we're left with the disjoint union of all $\mathfrak{G}_c(x)$ (for $w \in dx$). Black edges are oriented as they are in $\mathfrak{G}_c(x)$. Blue edges are $\mathbb{F}_2$-labeled similarly. We have the following result.

Definition 18. In $\mathfrak{G}_c$, all vertices have degree 3. Blue and black edges give even cycles (called NA cycles), each of which has an even number of oriented black edges. Red and blue edges give hexagons (called RA cycles), and red and black edges give squares (called RN cycles). An RN cycle either have 2 or 0 oriented black edges; in the former case, they are oppositely oriented. Call the former “oriented RN cycles”, and the latter “unoriented RN cycles”.

Thus, the number of hexagons equals (mod 2) the number of red edges, which is even since there's 2 red edges per RN cycle. We see easily that the coefficient of w in $d\mathfrak{sq}^2(c)$ equals the sum of all labels on the blue edges, plus the number of NA cycles, plus the number of oriented black edges we traverse in their orientation when we traverse each NA cycle in an arbitrary fashion:

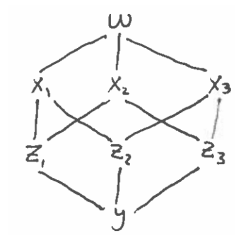
$$0 \stackrel{?}{=} f(\mathfrak{G}_c) + g(\mathfrak{G}_c) + \#_{NA}|\mathfrak{G}_c(x)| \bmod 2. \tag{3}$$

Next, we establish a useful proposition, proven later.

Proposition 19. The graph $\mathfrak{G}_c$ is planar. Its faces (including the outermost faces of each connected component) are precisely the NA, RN, and RA cycles.

When computing g in (3), traverse each NA cycle clockwise, except for the outermost face (if it happens to be a NA cycle), which will be traversed counterclockwise. Each oriented black edge lies in exactly one oriented RN cycle. It is easy to see that, for each oriented RN cycle, one of its black edges will be traversed with its proper orientation and the other one will not. Thus, $g(\mathfrak{G}_c)$ equals the number of oriented RN cycles.

Each red edge $(w, x, y, z) - (w, x', y, z)$ can also be labeled by $f(w, y)$. By [LSb, Lemma 2.1] (a simple combinatorial calculation), we have that in each RA cycle, the sum of the blue labels plus the sum of the red labels equals $s(z_1, y) + s(z_2, y) + s(z_3, y)$, if the hexagon corresponds to the following cube:



Label each vertex $V = (w, x, y, z)$ of the graph by $s(z, y) = S(V)$. The above paragraph implies $f_{blue}(\mathfrak{G}_c) = f_{red}(\mathfrak{G}_c) + \frac{1}{2}\left(\sum_{V\in\mathfrak{G}_c} S(V)\right)$ mod 2. However, in each RN cycle, both red edges have the same labeling, and so $f_{red}(\mathfrak{G}_c) = 0$. Furthermore, the latter sum is 0 for each oriented RN cycles, and 1 for unoriented RN cycles. Thus, if NA, RN, RA denotes the number of NA, RN, and RA cycles respectively,

$$\#_{NA}|\mathfrak{G}_c| + f(\mathfrak{G}_c) + g(\mathfrak{G}_c) = NA + RN = NA + RN + RA \text{ mod } 2.$$

(The last equality comes from the beginning of the proof, since RA is even). We are left with proving the number of faces of $\mathfrak{G}_c$ is even. We prove this for each connected component. Each connected component is a polyhedron, so we may apply Euler's formula: $V - E + F = 0$ mod 2, for V, E, F the number of vertices, edges, and faces respectively. But each RN cycle has 2 edges and 4 vertices, and so $V = E = 0$ mod 2. Thus, the number of faces is also even, as desired. $\square$

Proof of Proposition 19. Planarity follows from Kuratowski's theorem. Indeed, assume it were not planar. It clearly can't contain a subgraph that is a subdivision of K_5, since all vertices have degree 3. Thus, it must contain a subgraph that is a subdivision of $K_{3,3}$. Let the 6 relevant vertices be v_1, v_2, v_3 and w_1, w_2, w_3. Then there are 3 paths coming from v_1, to w_1, w_2, w_3, respectively. One of these paths must start with a red edge and another one with a black edge. By the previous lemma, these two paths must intersect again to give an RN cycle, which prohibits the formation of the $K_{3,3}$, as desired.

If we drew the graph and the faces didn't correspond to the NA, RN, and RA cycles, then take a cycle that wasn't a face (represented below as the black cycle). For convenience, say it were an NA cycle (but any other colors would do). Then all other vertices have exactly one other (red) edge. If they all go "in" or they all go "out", then the cycle would be a face. Therefore, there must be two consecutive vertices in the cycle, where one red edge goes inside and the next one goes outside, as depicted below. These edges will form an RN or RA cycle (depicted as the red cycle), which must intersect the previous cycle in at least 4 vertices.

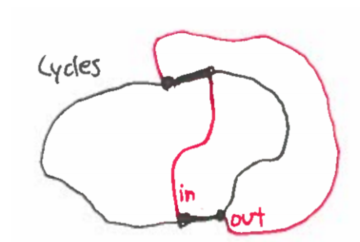


Figure 4: Proof that the cycles must be faces

But this situation can easily be discarded. Indeed, if an RN and NA cycle intersect in 4 vertices, then all the vertices of the RN cycle will have the same x-coordinate, which is absurd. If an RN and RA cycle intersect in 4 vertices, then all the vertices of the RN cycle will have the same y-coordinate, again absurd. Finally, if an NA and RA cycle intersect in 4 vertices, then 4 vertices of the RA hexagon will have the same x-coordinate, which is absurd since only consecutive vertices of an RA cycle can have the same x-coordinate. $\square$

We provide an example of a graph $\mathfrak{G}_c$ to illustrate the lemma, along with the graph KG inducing it.

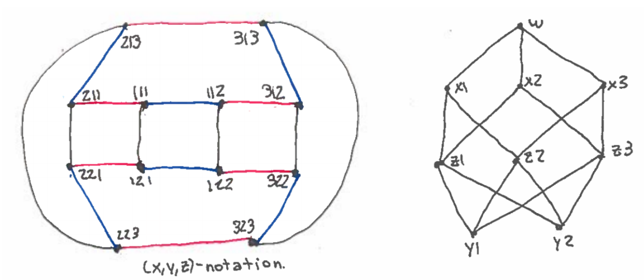


Figure 5: Graph $\mathfrak{G}_c$

3.2 Link Invariance of the Steenrod Square

We proceed to prove the invariance of the formula under the Reidemeister moves.

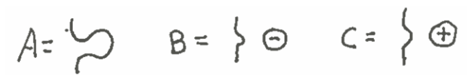
Proposition 20. Let L' and L'' be related to the link diagram L via the Reidemeister moves:

Figure 6: Reidemeister move 1

Then under the isomorphisms $Kh(L) \cong Kh(L')$ and $Kh(L) \cong Kh(L'')$ in [BNa], the formula (1) is invariant.

Proof. Ignoring notation for degree shift, we have $KC(L')$ is:

Let $A, B, C \subset KG(L')$ be the following subsets:



and KC_A, KC_B, KC_C the corresponding subcomplexes of $KC(L')$. Then $KC(L') \cong KC_A \oplus KC_B \oplus KC_C$ as groups, but with extra differentials $KC_A \to KC_B, KC_A \to KC_C$. Notice KC_C is a subcomplex. Further, the quotient complex $KC(L')/KC_C$ is acyclic, since the differential $KC_A \to KC_B$ is an isomorphism. Thus, the quasi-isomorphism is $KC(L) \overset{\cong}{\to} KC_C \hookrightarrow KC(L')$. Given a cycle $c \in KC(L)$, if we consider it as an element of KC_C, all the elements in δc and $\delta\delta c$ will be in KC_C. Hence, clearly formula (1) is preserved: i.e. the following commutes:

$$\begin{tikzcd} Kh^{i,j}(L) \arrow[r, "\cong"] \arrow[d, "sq^2"] & Kh^{i,j}(L') \arrow[d, "sq^2"] \\ Kh^{i+2,j}(L) \arrow[r, "\cong"] & Kh^{i+2,j}(L') \end{tikzcd}$$

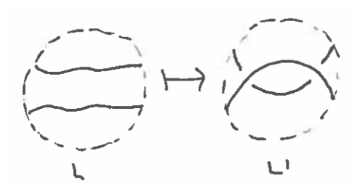
On the other hand, we have $KC(L'')$ is:

Define $A, B, C, KC_A, KC_B, KC_C$ as before. Then $KC(L'') \cong KC_A \oplus KC_B \oplus KC_C$ as groups, but with extra differentials $KC_B \to KC_A, KC_C \to KC_A$. The subcomplex $KC_C \to KC_A$ is acyclic since that differential is an isomorphism. The quotient complex, KC_B, is clearly isomorphic to $KC(L)$, by forgetting the small circle. Thus, the quasi-isomorphism is $KC(L'') \twoheadrightarrow KC_B \overset{\cong}{\to} KC(L)$, given by forgetting the A and C components. Given a cycle $c = c_a + c_b + c_c \in KC(L'')$ broken up into its corresponding parts, it gets mapped to $c_b \in KC(L)$ (by a small abuse of notation). For any generator $x \in KC_B$, its coefficient in $sq^2(c) \in Kh^{i+2,j}(L'')$ and in $sq^2(c_b) \in Kh^{i+2,j}(L)$ will be the same, since the coefficient only depends on paths from the cycle to x, and since the differential only goes from $B \to A$ and $C \to A$, these paths can never leave B. Thus, the C-part of $sq^2(c)$ is equal to $sq^2(c_c)$. It might have A and B-parts, but that doesn't matter: it's enough to guarantee the following commutes:

$$\begin{tikzcd}
Kh^{i,j}(L'') \arrow[r, "\cong"] \arrow[d, "sq^2"] & Kh^{i,j}(L) \arrow[d, "sq^2"] \\
Kh^{i+2,j}(L'') \arrow[r, "\cong"] & Kh^{i+2,j}(L)
\end{tikzcd}$$

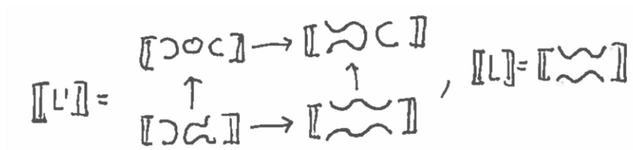
as desired. □

Proposition 21. Let L and L' be link diagrams related by the following Reidemeister move 2:



Then under the isomorphism $Kh(L) \cong Kh(L')$ in [LSa], the formula (1) is invariant.

Proof. Ignoring notation for degree shifts, we have $KC(L')$ and $KC(L)$ are:



Let $A, B, C, D, E \subset KG(L')$ be the following subsets:

and let $X := A \cup B \cup D$. Notice that the only differentials between these sets are the ones portrayed above; hence, $X \subset KG(L')$ satisfies the conditions of Lemma 12, so it defines a chain complex KC_X and homology groups $H^*(X)$ equipped with a Steenrod square. Furthermore, KC_X is a quotient complex of $KC(L')$, and $KC(L) \hookrightarrow KC_X$ is a subcomplex (by injecting into the B-coordinate). It is verified in [LSa] that this quotient map and inclusion are quasi-isomorphisms, which give

$$Kh(L') \xrightarrow{\cong} H^*(X) \xleftarrow[\cong]{} Kh(L)$$

$$[a, b, c, d, e] \longmapsto [a, b, d]$$

$$[0, b, 0] \longleftarrow [b]$$

We prove the Steenrod square agrees with *both* of these maps. For the first map, let $\gamma = (a, b, c, d, e)$ be a cycle representing a class in $Kh(L')$, where a, b, c, d, e represent the corresponding coordinates in A, B, C, D, E. Somewhat informally, we wish to prove that

$$\mathfrak{sq}^2_{L'}[a, b, c, d, e]|_X = \mathfrak{sq}^2_X[a, b, d],$$

where the subindices indicate with respect to which complex we are taking the Steenrod square, and the $|_X$ denotes that we are forgetting the C- and E-coordinates. We can make this formal by choosing specific boundary matchings for both cycles, obtaining specific cycles $c' \in KC(L')$ and $c_X \in KC_X$ representing both Steenrod squares, and asking for $c'|_X = c_X$.

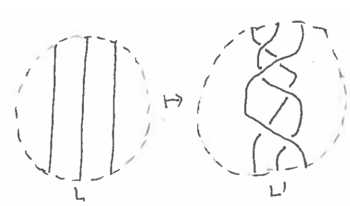
Notice any boundary matching for γ "respects X": in other words, if $x \in X$ has degree 1 higher than γ, then the boundary matching of γ for x pairs up elements of X with other elements of X (namely simply because $\mathscr{G}_\gamma(x) \subset X \times X$). Choose any boundary matching for γ and notice that this observation means it restricts to a boundary matching for (a, b, d). Using these boundary matchings, the coefficient for any $y \in X$ (of degree 2 higher than γ) in $\mathfrak{sq}^2_{L'}(\gamma)$ (with respect to this boundary matching) equals its coefficient in $\mathfrak{sq}^2_X(\gamma)$. This is precisely what the above formula asks for, as desired.

Similarly, for the second map, let $\gamma = (b)$ be a cycle representing a class in $Kh(L)$. Informally, we wish to prove

$$\mathfrak{sq}^2_X[0, b, 0]|_B = \mathfrak{sq}^2_L[b].$$

Again, to make this formal we choose a specific boundary matching for $(0, b, 0)$, and notice that it restricts to a boundary matching for $b \in KC(L)$ (namely, since the cycle simply has no elements in A or D). Using these two boundary matchings, we notice that the B-coordinate of $\mathfrak{sq}^2_X[0, b, 0]$ equals $\mathfrak{sq}^2_L[b]$, which is what the above formula asks for, as desired. □

Proposition 22. Let L and L' be link diagrams related by the following braid-like Reidemeister move 3:



Then under the isomorphism $Kh(L) \cong Kh(L')$ in [LSa], the formula (1) is invariant.

Proof. In [LSa, pp. 51-52], a partition is described: $KG(L') = A \cup B \cup C$, where B is the "top half" of the cube, A represents the 000111-vector, and C represents the bottom half of the cube. We pictorially recall these definitions for the reader's convenience:

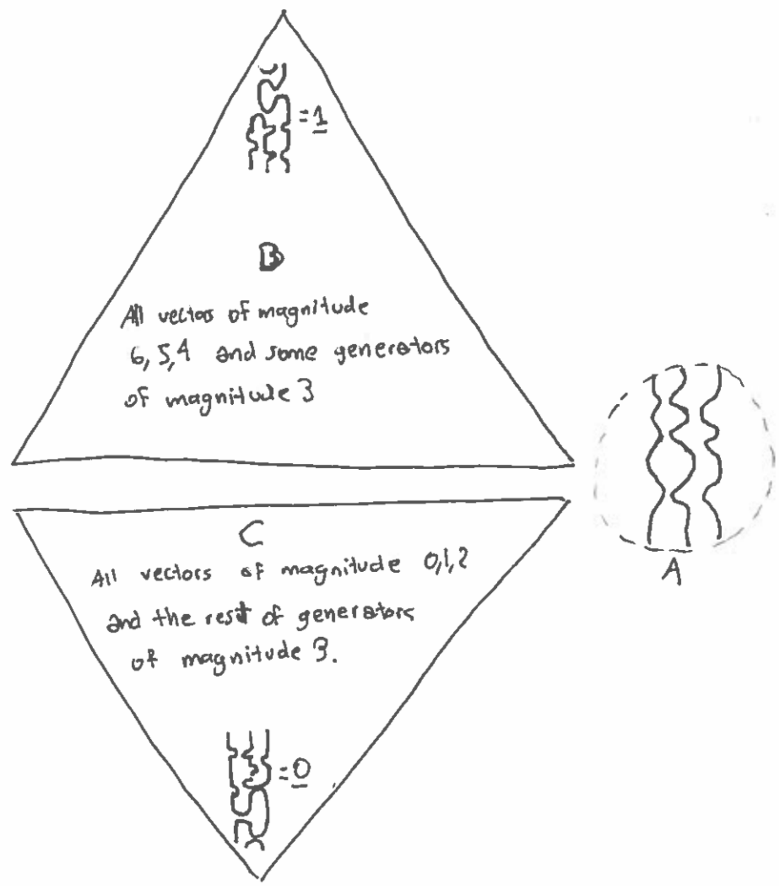


Figure 7: The cube of resolutions for the braid-like Reidemeister move 3, c.f. [LSa, Fig. 15]

The differentials are $C \to B, C \to A$, and $A \to B$. It is then verified that the subcomplex generated by B is acyclic, and the quotient complex generated by C is also acyclic. Let $X := A \cup C$, which satisfies the conditions of Lemma 12, so it defines a chain complex KC_X with homology groups $H^*(X)$ equipped with a Steenrod square. The isomorphism is described as the following composition:

$$\begin{tikzcd} Kh(L') \arrow[r, "\cong"] & H^*(X) & Kh(L) \arrow[l, "\cong"'] \\ {[a,b,c]} \arrow[r, mapsto] & {[a,c]} & \\ & {[a,0]} & {[a]} \arrow[l, mapsto] \end{tikzcd}$$

The proof can now be finished as in the previous proposition. □

4 Box Maps and the Homotopy Colimit Construction

This construction of the Khovanov homotopy type $\mathcal{X}_{Kh}(L)$ of a link projection consists of 3 main steps. First, one constructs a *flow category* (which is a small category with extra information, such that the Hom-sets are manifolds with faces) $\mathscr{C}_{Kh}(L)$ for any link diagram L. The flow category for a link diagram $\mathscr{C}_{Kh}(L)$ is built as a (trivial) cover of the *cube flow category* $\mathscr{C}_C(n)$. (The reader familiar with Morse theory can recognize this as the flow category for the function $\sum_{i=1}^n 3x_i^2 - 2x_i^3$ on $\mathbb{R}^n$). For two vectors $v \geq w$ (or two generators $x \geq y$), the Hom-space forms a manifold (with faces), called the *moduli space* $\mathcal{M}(v, w)$.

One then "frames" this category, by embedding it in a sufficiently big Euclidean space (i.e. embedding the Hom-set manifolds in a nice enough way) and choosing bases for the normal bundle. Finally, for any framed flow category, they give a realization (which differs from the usual simplicial realization of small categories) $|\mathscr{C}_{Kh}(L)| = Y$. One must verify that different embeddings lead to CW complexes that are homotopy equivalent "up to suspension" by the appropriate constant. Moreover, one must also verify the construction (up to homotopy) does not depend on the many choices involved in the construction, and, most importantly, that it is invariant under the three Reidemeister moves, so that $\mathcal{X}_{Kh}(L)$ is a link invariant, the *Khovanov homotopy type* of a link. The reader interested in this construction may consult [LSa].

However, there is another alternative construction of $\mathcal{X}_{Kh}(L)$ (which is homotopy equivalent to the aforementioned one). To discuss it, we will need to define homotopy-coherent diagrams and homotopy colimits. The main goal is to define this alternative construction of $\mathcal{X}_{Kh}(L)$. We will construct a homotopy-coherent cube (which is a homotopy-coherent diagram with index category $\mathscr{D} = 2^n$) in $\mathbf{Top}_*$ whose homotopy colimit is $\mathcal{X}_{Kh}(L)$. To this end, one must first introduce box maps.

4.1 Box Maps

Fix a sufficiently big integer k (vitally, $k \geq 3$), along with a permanent identification of the pointed sphere $S^k = B^k/\partial B^k$, where B^k is the k-dimensional box, and its boundary is identified with the basepoint of the sphere. Any diagram of n disjoint smaller boxes inside a bigger box induces a map $S^k \to S^k$:

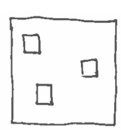


Figure 8: Little-box diagram

by mapping the interior of each small box to B^k and then collapsing the remainder to the basepoint. Letting B be a box, we denote $E(B, n)$ the space of n (ordered) disjoint boxes inside B. This gives a map $E(B,n) \to \mathrm{Map}_*(S^k, S^k)$ to the pointed mapping space. Let's now consider a slight generalization by allowing the smaller boxes to be labelled $+$ or $-$, such that negative boxes are considered to be flipped horizontally (i.e. along the first coordinate). We get a map $E_\pm(B, n_+, n_-) \to \mathrm{Map}_*(S^k_B, S^k)$ of configurations of n disjoint signed boxes to maps from the sphere to itself. Its degree as a map of spheres equals the sum of the signs, $n_+ - n_-$. Now, consider the space $E_m(B, n_+, n_-) \supset E_\pm(B, n_+, n_-)$, which allows the boxes not to be disjoint if (and only if) the configuration is the following:

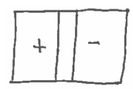


Figure 9: Overlapping boxes

The plus box must be on the left and the minus box on the right (or viceversa), they must have the same dimensions and the same percentage of their volumes must be overlapping. In particular, they are allowed to completely overlap. We also get a map $E_m(B, n_+, n_-) \to \mathrm{Map}_*(S^k_B, S^k)$ described as follows:

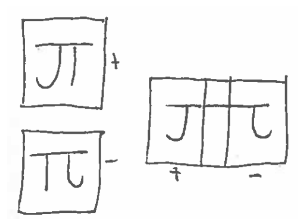


Figure 10: Overlapping boxes mapping to the sphere

Definition 23. It is thus possible to homotope the map represented by two disjoint opposite-sign boxes to the trivial map. There are two ways to do this: either by placing the $+$ box on the left, or on the right. For convenience, we now define the former to be the *standard* way, and the latter to be the *nonstandard* way. We call this process *killing* two boxes (i.e. a *death*). We call the reverse process *creating* two boxes (i.e. a *birth*).

(Notice the resulting sphere maps need not be surjective, for instance if there are only two overlapping boxes). Notice if, to a particular configuration, we add two completely-overlapping oppositely-signed boxes, the resulting map in $\mathrm{Map}_*(S^k_B, S^k)$ is the same. Let:

$$E_m(B, d) = \bigcup_{n_+ - n_- = d} E_m(B, n_+, n_-) \Big/ \sim$$

where the equivalence relation identifies a configuration in $E_m(B, n_+ + 1, n_- + 1)$ with two completely-overlapping oppositely-signed boxes with the corresponding one (with the boxes erased) in $E_m(B, n_+, n_-)$. Thus, there's a map $E_m(B, d) \to \mathrm{Map}_*(S^k_B, S^k)$ into degree d maps, which is a homeomorphism onto its image.

4.2 The Homotopy Coherent Cube

A *homotopy-coherent diagram* (in $\mathbf{Top}_*$) is a "functor" $F : \mathscr{D} \to \mathbf{Top}_*$ from an index category $\mathscr{D}$, but which doesn't necessarily commute: i.e. we don't necessarily have $F(g) \circ F(f) = F(g \circ f)$ for composable morphisms f, g. However, F comes with a specified homotopy between these morphisms. Similarly, for any chain of composable morphisms, one defines a higher-dimensional homotopy between all the resulting morphisms. For example, for a chain $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, we have a "square" of morphisms, whose corners correspond to $F(hgf), F(h)F(gf), F(hg)F(f)$, and $F(h)F(g)F(f)$. For such a functor F, one can define its *homotopy limit* and *colimit*, as one would do for an actual functor. For a full definition, the reader may consult [Vog].

As in [LLSa, Rem. 3.6], a homotopy coherent cube (in $\mathbf{Top}_*$) is equivalent to a functor $\mathcal{F} : \mathscr{C}_C(n) \to \mathbf{Top}_*$. For each Khovanov generator $x \in KG$, assign a box B_x to it, and $S^k_x := B_x/\partial B_x$. We produce the functor $\mathcal{F}$ as follows. For each $v \in \{0,1\}^n$, we set $\mathcal{F}(v) = \bigvee_{x \to v} S^k_x$. (The subscript runs over all generators x which correspond to the vector v). For each $x \geq_1 y$ in KG, we fix a small box $B_{y,x} \subset B_x$; such that for each x, all these small boxes are disjoint. This gives a box map $S^k_x \to S^k_y$. For fixed $v \geq_1 w$, the collection of all these maps defines a map $\mathcal{F}(v) \to \mathcal{F}(w)$, which we define to be the image (under $\mathcal{F}$) of the unique point in $\mathcal{M}(v, w)$. Defining $\mathcal{F}$ on the 1-dimensional moduli spaces amount to specifying homotopies between these maps, and defining the functor on the 2-dimensional moduli spaces amounts to specifying "homotopies of homotopies", and so on. This is the main objective in [LLSa].

For Bar-Natan homology (associated to the graph KG_{BN}), we can attempt to define the functor on 1-dimensional moduli spaces. We proceed to describe a sketch of the construction. For $x \geq_1 y$, the map $S_x \to S_y$ is defined as previously, except if the arrow $x \to y$ is labeled -1 (i.e. it comes from a splitting $+ \to ++$), in which case the map is defined as the box map where $B_{y,x}$ is labeled with a $-$ in B_x (and thus flipped horizontally). For a fixed face $v \geq_1 u_1, u_2 \geq_1 w$ of the cube, we must define a homotopy between the two maps $F(v) \to F(w)$, described as the compositions $F(v) \to F(u_1) \to F(w)$ and $F(v) \to F(u_2) \to F(w)$. By analyzing all the possible cases (which amounts to analyzing all basic decorated index-2 resolution configurations), we find that these homotopies can be described as translations of boxes (as before), except in one case, which must be a birth or a death, and we must assign a standard or non-standard label to these. We proceed with this discussion in Section 5.3.

4.3 Functors into the Burnside category

Let 2^n be the 2-category where there is exactly one 1-morphism $\varphi_{v,u} : v \to u$ whenever $v \geq u$, and the only 2-morphisms are the identities. The Burnside $\mathscr{B}$ category is the weak 2-category whose objects are sets, 1-morphisms are correspondences (i.e. a set A together with source and target maps $s : A \to X$ and $t : A \to Y$), and 2-morphisms are bijections between correspondences (which commute with the source and target maps). Compositions of 1-morphisms $A : X \to Y$ and $B : Y \to Z$ are denoted as fiber products $A \times_Y B : X \to Z$.

Remark 24. For historical reasons, even though originally Khovanov homology used arrows

$v \to u$ whenever $v \leq_1 u$ (i.e. 0-resolutions going to 1-resolutions, and not vice-versa), it is convenient to use this notation instead when working with the homotopical construction.

An important result in [LLSa] is that the information of a cubical flow category (i.e. a flow category equipped with a "nice" functor into 2^n) is the same as that of a (strictly unitary lax 2-)functor from the cube into the Burnside category, $F : 2^n \to \mathscr{B}$. In the case of the Khovanov homology of a given ink diagram, the functor F_{Kh} associates to each binary vector the set of Khovanov generators over it, and to each morphism $v \geq_1 w$ it associates the correspondence $\{(x, y) \in F(v) \times F(w) | \delta(x, y) \neq 0\}$ with the obvious source and target maps. To each 2-face of the cube $v \geq_1 w_1, w_2 \geq_1 u$ we associate a bijection $F(\varphi_{v,w_1}) \times_{F(w_1)} F(\varphi_{w_1,u}) \to F(\varphi_{v,w_2}) \times_{F(w_2)} F(\varphi_{w_2,u})$. This data amounts to providing the Ladybug Matching.

In order for this information to extend to a complete functor F_{Kh}, the bijections must satisfy the following condition. For each 3-face:

$$\begin{tikzcd} & v \arrow[dl] \arrow[d] \arrow[dr] & \\ w_1 \arrow[d] \arrow[dr] & w_2 \arrow[dl] \arrow[dr] & w_3 \arrow[dl] \arrow[d] \\ t_1 \arrow[dr] & t_2 \arrow[d] & t_3 \arrow[dl] \\ & u & \end{tikzcd}$$

the following hexagon must commute.

$$\begin{tikzcd} & F(v, w_1, t_2, u) \arrow[r] & F(v, w_3, t_2, u) \arrow[dr] & \\ F(v, w_1, t_1, u) \arrow[ur] & & & F(v, w_3, t_3, u) \arrow[dl] \\ & F(v, w_2, t_1, u) \arrow[ul] & F(v, w_2, t_3, u) \arrow[l] & \end{tikzcd}$$

(For convenience, we write $F(v, w_1, t_1, u)$ instead of $F(\varphi_{v,w_1}) \times_{F(w_1)} F(\varphi_{w_1,t_1}) \times_{F(t_1)} F(\varphi_{t_1,u})$. For the hexagon to commute means that traversing it exactly once results in the identity map). The reason lies in the fact that the moduli spaces of the cubical flow categories must be trivial covers of the cube flow category. Apart from the hexagon, no $6n$-gon can trivially cover a hexagon. In later sections, [LLSa] verify that any such functor $F : 2^n \to \mathscr{B}$ can be, essentially uniquely up to homotopy, realized as a homotopy coherent cube in the pointed spaces category (called its *little-box realization*, $\widetilde{F}$). The homotopy colimit of this diagram is thus defined as the Khovanov homotopy type of the link. One must verify that it is indeed a link invariant, and that it is equivalent to the previous construction of the Khovanov homotopy type. When viewed through this lens, the "hexagon condition" for the functor $F : 2^n \to \mathscr{B}$ has another explanation. Suppose we are trying to construct a little box realization $\widetilde{F} : 2^n \to \mathbf{Top}_*$. For any 2-face $v \geq w_1, w_2 \geq u$, the ladybug matching allows us to specify a homotopy $\widetilde{F}(\varphi_{w_1,u}) \circ \widetilde{F}(\varphi_{v,w_1}) \simeq \widetilde{F}(\varphi_{w_2,u}) \circ \widetilde{F}(\varphi_{v,w_2}) : \widetilde{F}(v) \to \widetilde{F}(u)$. Restricting to any two given $x \in F(v)$ and $y \in F(u)$, the above hexagon is effectively a loop in $\mathrm{Map}_*(S_x, S_y)$. We must specify how to "fill-in" this hexagon, and for this we need the loop to be homotopic to a constant loop. This is clearly possible if the hexagon is commutes, as illustrated below:

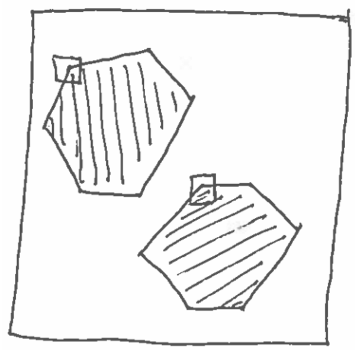


Figure 11: Filling-in a loop in the mapping space

Remark 25. If the resulting disks aren't "disjoint", the "filling-in" process is not so clear. However, in higher dimensions it's clearly possible.

5 Steenrod Square on Bar-Natan Homology

Bar-Natan homology does not come from a strictly unitary lax 2-functor $F_{BN}(L) : 2^n \to \mathscr{B}$. (In other words, there is no way to assign such a functor to any link diagram, such that we can construct from the functor a chain complex equal to the Bar-Natan chain complex). Equivalently, there is no cubical flow category associated to it (whose chain complex *is* the Bar-Natan chain complex). Therefore, if a nontrivial Steenrod square existed on Bar-Natan homology, it would probably come from the little-box homotopy colimit construction. Namely, one would attempt to construct a partial homotopy-coherent cube $\widetilde{F}_{BN} : 2^n \to \mathbf{Top}_*$, where by "partial" we mean that only the homotopies on the 2-faces have been specified. One would then look at 3 given "layers" of the cube (i.e. the set of vectors with magnitude in $\{k, k+1, k+2\}$ for a given k), and take the homotopy colimit of $\widetilde{F}_{BN}$ restricted to these 3 layers. The resulting space would endow the Bar-Natan homology with a Steenrod square in this dimension: $Kh_{BN}^{k-n_-}(L) \to Kh_{BN}^{k+2-n_-}(L)$. One will also need the partial homotopy-coherent cube $\widetilde{F}_{BN}$ to have "fillable" 3-faces. Namely, for given vectors $v \geq_3 w$ and Khovanov generators x, y corresponding to v, w, the homotopy information gives us a map

$$\begin{tikzcd} \partial\mathcal{M}_n(v,w) \arrow[r] \arrow[d, hook] & \mathrm{Map}_*(\widetilde{F}(x), \widetilde{F}(y)) \\ \mathcal{M}_n(v,w) \arrow[ur, dashed] & \end{tikzcd}$$

which we wish to extend, as displayed in the diagram above.

Remark 26. For the confused reader: the boundary of the moduli spaces corresponds to maps coming from a composition $v \geq u \geq w$ for some u. In other words, we have the composition map

$$\coprod_{v \geq u \geq w} \mathcal{M}_n(v,u) \times \mathcal{M}_n(u,w) \xrightarrow{\circ} \partial \mathcal{M}_n(v,w) \hookrightarrow \mathcal{M}_n(v,w).$$

This gives us the aforementioned map $\partial \mathcal{M}_n(v,w) \to \mathrm{Map}_*(\widetilde{F}(x), \widetilde{F}(y))$. We will see why need this complicated condition in Remark 32.

The partial homotopy-coherent cube would be a little-box realization of a "fake functor" $F_{BN}(L) : 2^n \to f\mathscr{B}$ into a category which, for lack of a better name, we denote the "fake Burnside" category, or $f\mathscr{B}$. It is a 1-category with 2-morphisms. (However, the 2-morphisms don't compose, either vertically nor horizontally). Notice, however, that the Bar-Natan differential contains signs which don't come from the cube. Therefore, 1-morphisms in $f\mathscr{B}$ are *signed correspondences*: a 1-morphism $A : X \to Y$ is a correspondence $s : A \to X, t : A \to Y$ together with a sign assignment $A \to \{1, -1\}$. Composing 1-morphisms is the same as corresponding correspondences, with the additional condition that the signs of the elements in the correspondences are multiplied. The 2-morphisms must be as follows. Given signed correspondences $A, B : X \to Y$, a 2-morphism $A \to B$ is an injective map $\varphi : A' \to B$ which respects signs, for some subset $A' \subset A$. We also have a sign-reversing involution $A - A' \to A - A'$ called "killings" or "deaths", each of which is specified to be either *standard* or *non-standard* (the reason for this comes from Definition 23). Similarly, there is a sign-reversing involution $B - \varphi(A') \to B - \varphi(A')$ called "births" or "creations", each of which is specified to be either standard or non-standard. A 2-morphism is illustrated as follows:

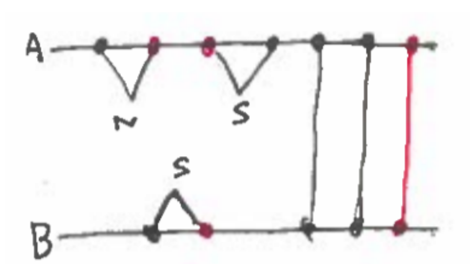


Figure 12: 2-morphism in $f\mathscr{B}$

In the Khovanov case, we required a certain hexagon associated to a 3-face to commute. In this case, since 2-morphisms don't compose, we cannot ask for a similar condition. Therefore, the natural question to ask is what conditions we need to enforce on these hexagons. This is the same problem as the extension problem described above. If the 3-face is $v \geq_3 w$, since $(\mathcal{M}_n(v,w), \partial \mathcal{M}_n(v,w)) \cong (D^2, S^1)$, it reduces to requiring that the (un-pointed) loop $S^1 \to \mathrm{Map}_*(\widetilde{F}(x), \widetilde{F}(y))$ be homotopic to a constant loop. As we are realizing these functors via the little-box construction, this motivates the study of the homotopy type of $\mathrm{Map}_*(S^k, S^k)$.

5.1 Fundamental Group of Mapping Spaces

For convenience, set $X := \mathrm{Map}_*(S^k, S^k)$. Set X_d to be the degree-d path component of X, so that $X = \bigsqcup_{d \in \mathbb{Z}} X_d$. Any map $S^k \to S^k$ can be represented by a box B (equipped with an identification $B/\delta B = S^k$) and a map $(B, \delta B) \to (S^k, *)$. Notice we have maps $X_d \to X_{d+1}, X_{d+1} \to X_d$ as follows:

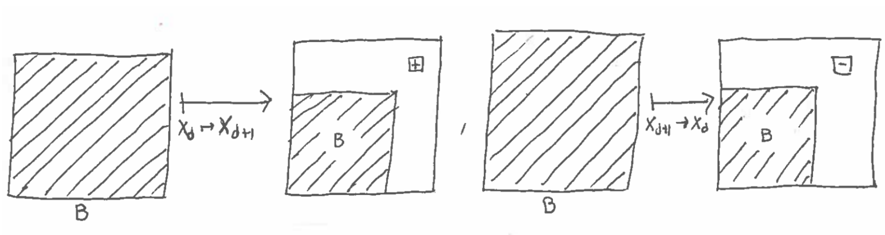


Figure 13: Homotopy equivalences $X_d \simeq X_{d+1}$

by shrinking the box B and adding a far-away small-box, labeled either $+$ or $-$. These are homotopy-equivalences. Indeed, the composition $X_d \to X_{d+1} \to X_d$ is homotopic to the identity, by always standardly killing the two boxes and then zooming in on the original box B, as follows:

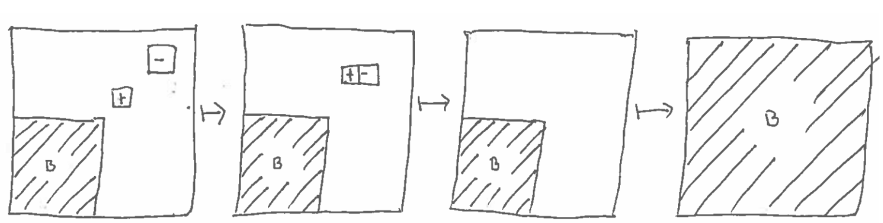


Figure 14: Homotopy to the identity

(Similarly, the composition $X_{d+1} \to X_d \to X_{d+1}$ is homotopic to the identity). Now, notice X is a pointed space: its basepoint is the trivial map lying in X_0, which is path-connected, since $\pi_0(X) = \pi_0(\Omega^k S^k) = \pi_k(S^k) = \mathbb{Z}$. Therefore, if $k \geq 3$,

$$\pi_1(X_d) \cong \pi_1(X_0) = \pi_1(X) = \pi_1(\Omega^k S^k) = \pi_{k+1}(S^k) = \mathbb{Z}/2.$$

(The spaces X_d are not canonically pointed for $d \neq 0, 1$ (one can assign the identity as the basepoint of X_1), but they are path-connected and so we can still define their fundamental group). The 0-element corresponds to loops homotopic to a constant loop. By abuse of notation, call a loop in these spaces "nullhomotopic" if it's homotopic to a constant loop. We give the following description of the (essentially unique) non-nullhomotopic loops in X_0 and X_2, represented in the following diagram:

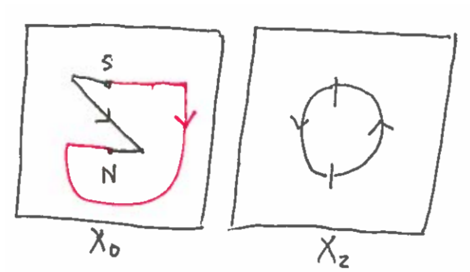


Figure 15: Non-nullhomotopic loops in X_0, X_2

Proposition 27. The loops in Figure 15 are non-nullhomotopic.

Proof. Fix an identification $B/\partial B = S^2$, where B is a 2-dimensional box endowed with the following orthonormal frame:

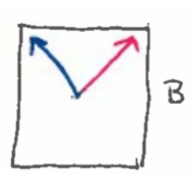


Figure 16: Box B with orthogonal frame

Fix a half-plane and its boundary line in $\mathbb{R}^3$, and consider the box embedded inside the half-plane. Given a loop in $SO(2)$, rotate the half-plane along its boundary line, and rotate the box along with it, so that its orthonormal frame matches the given loop. We get a solid "torus" with cross-sections identified canonically with B. We get a map $S^3 \to S^2 = B/\partial B$ as follows: consider $\mathbb{R}^3$ as the punctured 3-sphere. Then, map everything outside the "torus" to the basepoint, and map the inside of the torus to the corresponding points in B. This procedure describes a map j:

$$\pi_1(SO(2)) \xrightarrow{j} \pi_3(S^2) \xleftarrow{h} \pi_1(\mathrm{Map}_*(S^2, S^2))$$

(which is an isomorphism since both are infinite cyclic groups). We also have an isomorphism $h : \pi_1(\mathrm{Map}_*(S^2, S^2)) \to \pi_3(S^2))$, since the mapping space is $\Omega^2 S^2$. Explicitly, h can be given by a similar procedure: Fix a "standard (square) torus" in $\mathbb{R}^3$ with cross-sections identified with B. A loop in the mapping space gives us a new way to identify each cross-section with B; which gives us a map $S^3 \to S^2$. Thus, the loops in Figure 16 can be described by the following images:

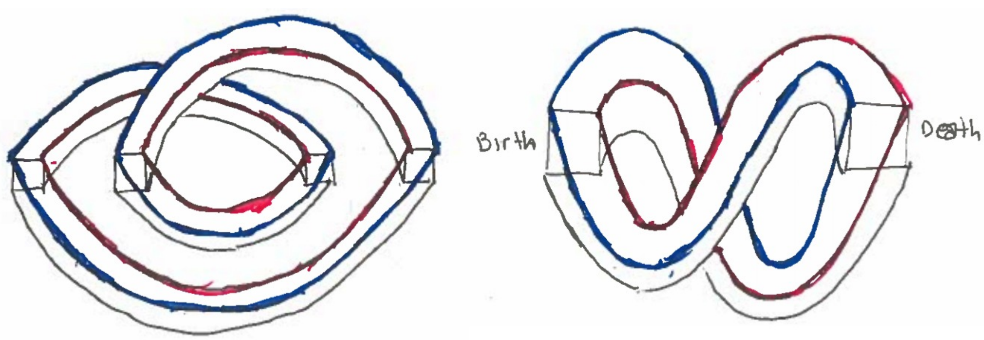


Figure 17: The solids corresponding to the loops in Figure 15

where the "blue strand" is always identified with the blue vector in Figure 16 and similarly for the red strand. We can "untangle" these solids to form the standard torus, but the blue and red curves might now be linked. The corresponding number in $\pi_1(SO(2))$ will equal the linking number of these two curves. But this linking number can be computed from the original images as follows:

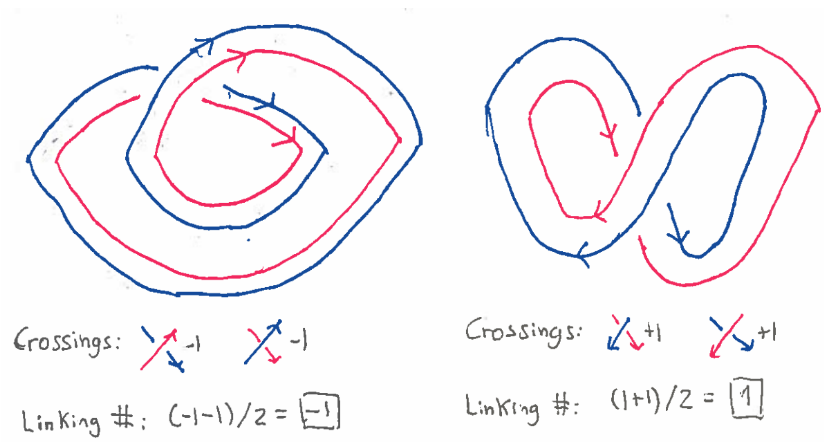


Figure 18: Computation of the Linking number

and so both linking numbers are odd. That means when we suspend (to get higher dimensional boxes), the map $\pi_1(\mathrm{Map}_*(S^2, S^2)) \to \pi_1(\mathrm{Map}_*(S^k, S^k))$ is just the mod 2 map $\mathbb{Z} \to \mathbb{Z}/2$, and the element will be odd. Thus, these loops are non-nullhomotopic, as desired. $\square$

Lemma 28. Assume given a loop $\gamma : S^1 \to E_m(B, d) \hookrightarrow X_d \subset \mathrm{Map}_*(S^k, S^k)$ of degree-d box maps. It can be described, possibly up to perturbation and dimension change, as a disjoint union of "loops" of boxes, as in Figure 19:

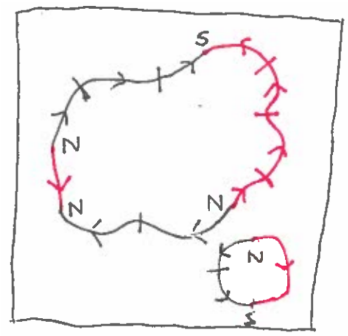
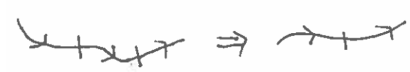


Figure 19: Loop in the mapping space

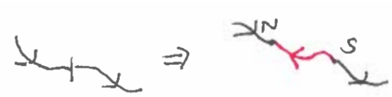
Each directed segment represents a little box's path, from time 0 to 1. Black segments represent plus boxes and red segments represent minus boxes. Each birth and death is either labeled N or S, for non-standard and standard, respectively. The homotopy class (non-nullhomotopic or nullhomotopic) of the loop is changed by the following operations:

1. **Move 1:** Changing a birth/death from type N to S, or viceversa.

2. **Move 2:** Replacing two same-sign boxes with three, as follows:



3. **Move 3:** Inserting an opposite-colored box (with a birth and death of opposite type) between two different same-colored boxes, as follows:



Proof. Because the fundamental group is a $\mathbb{Z}/2$, we know adding one of the disjoint cycles in Figure 11 must change the homotopy class of the loop. For the first move, we have the following homotopy:

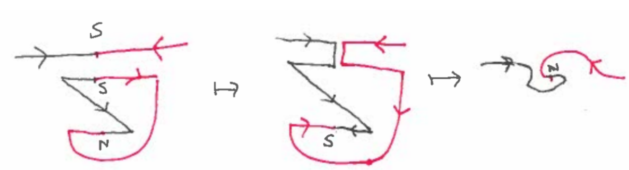


Figure 20: Move 1

For the third move, notice the loop must be of the same type as its image under $X_d \to X_{d+1}$, by adding a disjoint constant red box. We have the following homotopy:

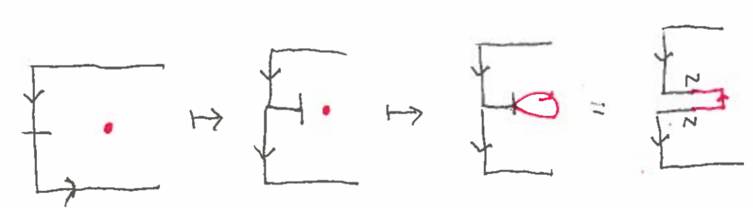
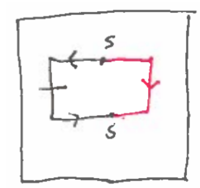


Figure 21: Move 3

Now we can use Move 1 to change the N/S-type of the birth or death in the last loop to change the homotopy class of the loop. Before we prove Move 2, notice the following loop is non-nullhomotopic:



This is because we can apply Move 3 to get to the second loop from Figure 15. For Move 2, first we use Move 3 and disjointly add the above loop, which altogether doesn't change the type of the loop. Then we perform the following homotopy:

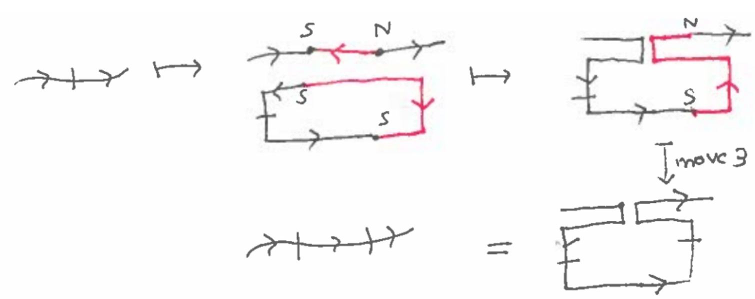


Figure 22: Move 2

This completes the proof of the lemma. $\square$

Remember that the loops in Figure 15 are non-nullhomotopic, and notice that the following loops are clearly nullhomotopic:

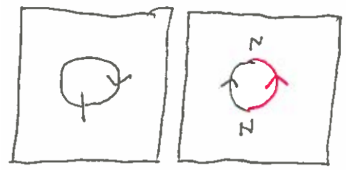


Figure 23: Nullhomotopic loops

The lemma, together with this small observation, guarantee the following corollary:

Corollary 29. For such a loop γ, assume it is made up of k cycles, $C_1, \ldots, C_k$. For each cycle C, let $In(C)$ be the number of "points" separating same-colored boxes (we will call these *interruptions*), and $NS(C)$ the number of N-type births or deaths in the cycle (which is equal mod 2 to the number of S-type births or deaths). Finally, let $PM(C)$ be the number of black arcs in the cycle (which is equal to the number of red arcs, if the whole circle isn't monochromatic). If the whole circle is monochromatic, set $\sigma(C) := In(C) + 1$. Otherwise, set $\sigma(C) = In(C) + PM(C) + NS(C) + 1$. Then, if

$$\sigma(\gamma) := \sum_{i=1}^{k} \sigma(C_i) \tag{4}$$

is odd, γ is non-nullhomotopic. If it's even, it's nullhomotopic. In other words, (with a small abuse of notation), $\sigma(\gamma) = [\gamma] \in \pi_1(X_d) = \mathbb{Z}/2$.

Proof. Any loop can be reduced to either the second one of Figure 15 or the first one of Figure 23, as follows. First, get rid of all interruptions using Move 3. Then get rid of all the red arcs again by Move 3, using Move 1 too if necessary. This will leave a monochromatic black cycle. Use Move 2 repeatedly to reduce the number of interruptions to 1 or 2. If we keep track of all the moves we made, we will get precisely (4). $\square$

Thus, we're finally able to define what a *Bar-Natan functor* $F : 2^n \to f\mathscr{B}$ means.

Definition 30. A *Bar-Natan functor* $F : 2^n \to f\mathscr{B}$ comprises the following information.

- A set $F(v)$ for every $v \in 2^n$.
- For every edge $v \geq_1 w$, a signed correspondence $F(\varphi_{v,w})$ from $F(v)$ to $F(w)$.
- For every face $v \geq_1 u_1, u_2 \geq_1 w$, a (fake) 2-morphism

  $$F_{v,u_1,u_2,w} : F(\varphi_{u_1,w}) \circ F(\varphi_{v,u_1}) \to F(\varphi_{u_2,w}) \circ F(\varphi_{v,u_2})$$

  and its inverse $F_{v,u_2,u_1,w}$.

Such that the following condition is satisfied: For any 3-face:

$$\begin{tikzcd}
& v \arrow[dl] \arrow[d] \arrow[dr] & \\
w_1 \arrow[d] \arrow[dr] & w_2 \arrow[dl] \arrow[dr] & w_3 \arrow[dl] \arrow[d] \\
t_1 \arrow[dr] & t_2 \arrow[d] & t_3 \arrow[dl] \\
& u &
\end{tikzcd}$$

the following hexagon will describe a cycle diagram, such as the one represented in Figure 12:

$$\begin{tikzcd}
& F(v, w_1, t_2, u) \arrow[r] & F(v, w_3, t_2, u) \arrow[dr] & \\
F(v, w_1, t_1, u) \arrow[ur] & & & F(v, w_3, t_3, u) \arrow[dl] \\
& F(v, w_2, t_1, u) \arrow[ul] & F(v, w_2, t_3, u) \arrow[l] &
\end{tikzcd}$$

The cycle diagram γ must satisfy that $\sigma(\gamma) = 0 \bmod 2$, where $\sigma(\gamma)$ is defined as in equation (4).

5.2 Little-box refinement of a Bar-Natan functor

A *partial refinement* of a Bar-Natan functor is a partial homotopy-coherent cube $\widetilde{F} : 2^n \to \mathbf{Top}_*$ (where the homotopies are only defined on the 2-faces, and are at all times given by box maps which refine the correspondence).

Lemma 31. If $k \geq 2$ then there exists a k-dimensional refinement. If $k \geq 3$ then any two k-dimensional refinements are homotopic (as in [Vog]) as partial homotopy-coherent cubes.

Proof. For proof of existence, we choose any collection of boxes where $B_y \subset B_x$ whenever $x \in F(v), y \in F(w)$ and $v \geq_1 w$. To specify a homotopy $\widetilde{F}_{v,u_1,u_2,w}$ for any 2-face, we use the corresponding 2-morphism. The homotopy will happen simultaneously in each big box B_x for $x \in F(v)$. However, to make sure little boxes corresponding to elements of $F(w)$ don't overlap, we choose any total order of $F(w)$, and homotope the small boxes in this order, using the specified 2-morphism.

For proof of uniqueness, for any two refinements $\widetilde{F}, \widetilde{F}'$, we give a partial homotopy-coherent cube $G : 2^{n+1} \to \mathbf{Top}_*$, where the homotopies are now defined on 3-faces as well. We wish $G|_1 = \widetilde{F}'$ on the top layer and $G|_0 = \widetilde{F}$ on the bottom layer, and furthermore we want $G(\varphi_{1,0} \times \mathrm{Id}_u) : \widetilde{F}'(u) \to \widetilde{F}(u)$ to be a homotopy equivalence for any $u \in 2^n$. To prove G exists, we simply define $G(\varphi_{1,0} \times \mathrm{Id}_u) : \widetilde{F}'(u) \to \widetilde{F}(u)$ to be the identity. If a 2-face doesn't have a "vertical" component, then we already know how to "fill it in". If it does, as in the following case:

$$\begin{tikzcd} v1 \arrow[r, "f'"] \arrow[d] & w1 \arrow[d] \\ v0 \arrow[r, "f"] & w0 \end{tikzcd}$$

then we need a homotopy $\widetilde{F}'(f) \simeq \widetilde{F}(f)$. But these two are box maps which refine the same correspondence, and so clearly such a homotopy must exist (indeed we only need $k \geq 2$ here), using only translations. For non-vertical 3-faces, we already know that they *can* be filled in (even if we haven't specified an explicit way to do so). Otherwise, assume we have a vertical 3-face:

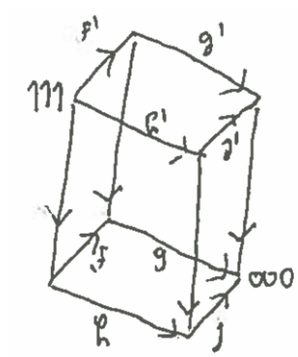


Figure 24: Vertical 3-face

We give an illustrative example of a face and two refinements:

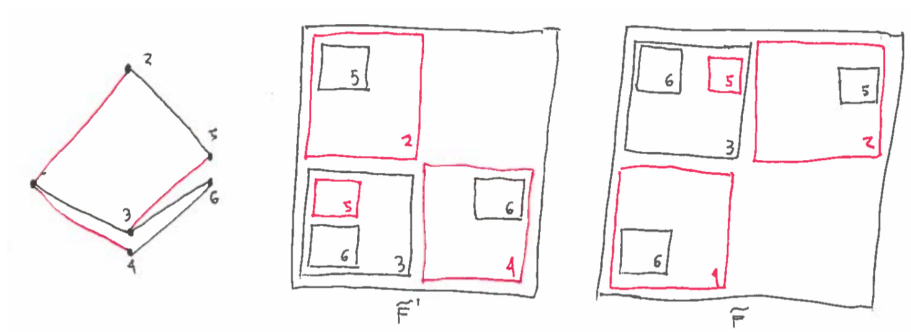


Figure 25: Two possible refinements for the 2-face. One corresponds to the top 2-face and the other to the bottom 2-face.

The hexagon to be filled-in is as follows:

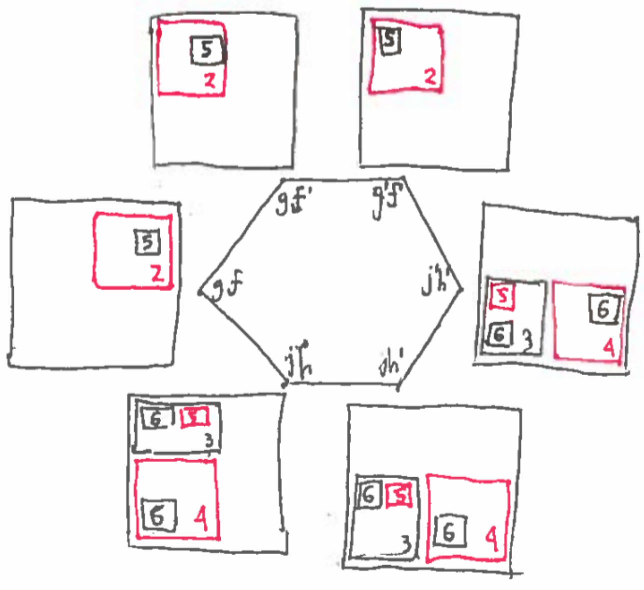
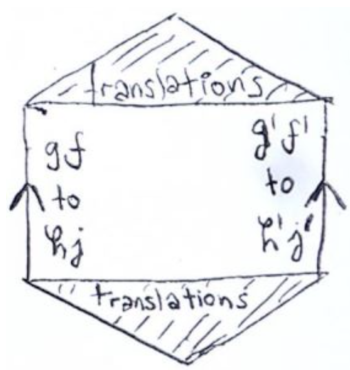


Figure 26: A hexagon in $\mathrm{Map}_*(S^k, S^k)$ corresponding to the 3-face.

In any case, this hexagon can be described as follows:



where the two arrows represent the "same" homotopy: the same pairs of boxes are born in the same fashion, and the same pairs of boxes die in the same fashion. Thus the loop is a disjoint union of those specified in Figure 17. This loop is obviously nullhomotopic, so the result follows. (We use $k \geq 3$ to be allowed to use Corollary 15). $\square$

Remark 32. The need to guarante that non-vertical 3-faces can be filled in is the reason why we have the extension problem discussed in Remark 26.

The existence of G implies that for any 3 consecutive "levels" of magnitude $\ell, \ell+1, \ell+2$ in the cube 2^n, the (full) homotopy-coherent diagrams $\widetilde{F}, \widetilde{F}' : 2^n|_{\ell \leq |v| \leq \ell+2} \to \mathbf{Top}_*$ are homotopic.

Thus, their homotopy colimits are homotopic, by [Vog, Prop. 4.6]. Therefore, there is a unique homotopy type associated to these 3 levels, independent of the choice of little-box refinement.

5.3 The Bar-Natan Ladybug Matching

Defining a Bar-Natan functor $F_{BN}(L) : 2^n \to f\mathscr{B}$ associated to a link requires 3 steps. On objects, the functor takes a vector $F(v)$ to the set of Khovanov generators associated to it, as before. On 1-faces $v \geq_1 w$, we define the signed correspondence $F(\phi_{v,w})$ as the subset $\{(x, y) \in F(v) \times F(w) | \delta_{BN}(x, y) \neq 0\}$. An element is labeled -1 if and only if its sign is negative in the Bar-Natan differential (ignoring the signs coming from the cube); in other words, an element is labeled -1 if and only if it comes from a + circle splitting into two + circles.

The hardest step, of course, is to define the 2-morphisms for every 2-face $v \geq_2 w$ of the cube. By analyzing the basic index-2 decorated resolution configurations (in Appendix A), we notice most of them are bijections identical to those given by the Khovanov functor F_{Kh}. Some of them are forced bijections between 1-element correspondences (that were empty in the Khovanov case). The only case requiring a birth or death is the following:

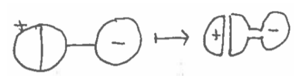


Figure 27: Exceptional 2-face

whose diagram after taking F_{BN} is the following:

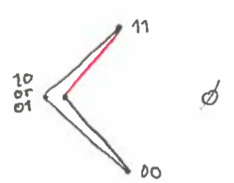


Figure 28: A birth/death. The above is a subset of the graph KG. The set of dots next to a vector v represent the set of generators $F(v)$. (For instance, $|F(11)| = 1$ but $\{|F(10)|, |F(01)\} = \{0, 2\}$.) Two generators x, y are joined if by a red edge if the corresponding element of the signed correspondence is negative, and by a black edge if it's positive.

As we said, this is either a birth or a death, depending on the direction of the 2-morphism. Thus, finding the Bar-Natan ladybug matching reduces to deciding which births and deaths are standard, and which are non-standard. (Notice that if a 2-face traversed in one direction gives a standard birth, then it being traversed in the opposite direction must give a standard death, and viceversa). The following result holds:

Lemma 33. The Bar-Natan Ladybug Matching, if it exists, cannot depend solely on the cube. Formally, there does not exist an assignment $X \to \{Standard, Nonstandard\}$ (where X is the set of 2-faces of the cube), such that $F_{BN}(L) : 2^n \to f\mathscr{B}$ extends to a Bar-Natan functor $2^n \to f\mathscr{B}$ agreeing with the assignment.

(By "agreeing with the assignment", we mean that all births and deaths happening on that 2-face must be of the type specified by the assignment). Before proving it, we make a small observation. Suppose we either have a birth or death happening inside a bigger box, or a birth and death concerning two boxes which contain smaller boxes, as follows:

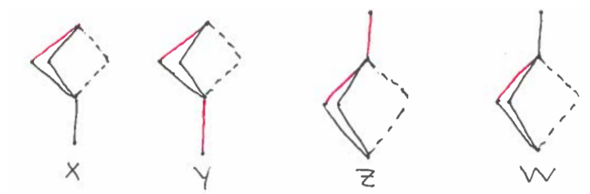
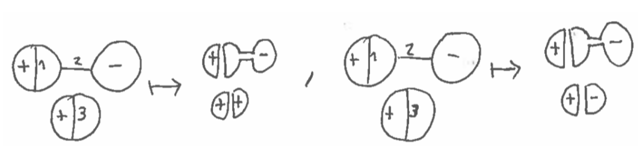


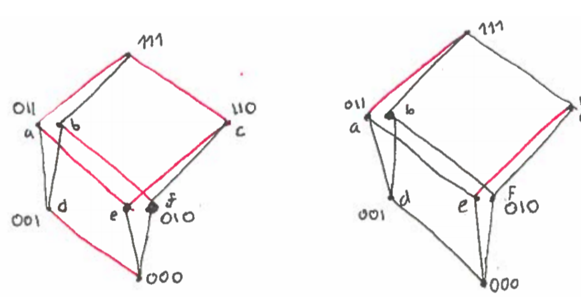
Figure 29: Births and deaths in a 3-face. Only the Y situation reverses its type.

Clearly, in situation X or W, the type of birth/death is maintained inside the bigger box, i.e. if the plus box was on the left, then it will still be on the left inside the bigger box. In situation Z, the two boxes flip sides but they also flip signs, and so the type of birth/death is also maintained. However, in situation Y, the two boxes flip sign but they don't flip sides, and so situation Y actually reverses the type of birth/death happening.

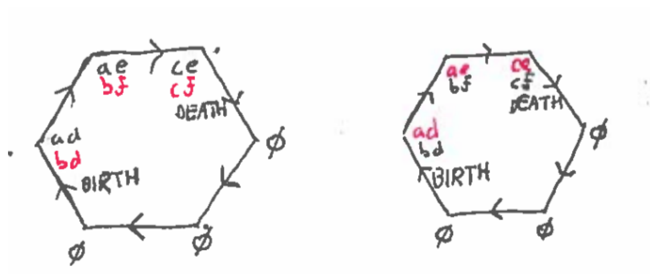
Proof of Lemma 33. Consider the following index-3 decorated resolution configurations:



whose Bar-Natan graphs look as follows:



The commutative hexagons thus look as follows:



Notice on the left hexagon, the birth is in situation Y while the death is in situation Z. On the right hexagon, the birth is in situation X and the death in situation W. If the ladybug matching relied solely on the cube, then the two births (inside the 2-faces) would be of the same type, as would the two deaths. However, by our observation, the birth on the left hexagon above is mirrored (since it's in situation Y). Thus, the two births in the two hexagons are actually of opposite type, while the two deaths on both hexagons are of the same type. Thus, exactly one of these hexagons is a loop of the type described in Figure 23, the other one must be non-nullhomotopic (of the type described in Figure 15) a contradiction. □

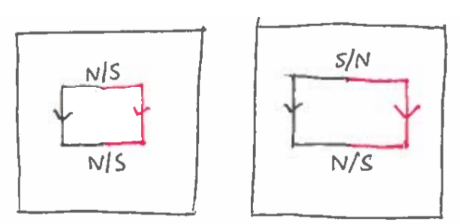


Figure 30: Two loops in X_0: one nullhomotopic and one not.

It turns out that there is a "functor" $F_{BN} : Cob^3_{e/l} \to f\mathscr{B}$ from the embedded cobordism category (whose formal definition is beyond the scope of this paper) to the fake Burnside category which "described" Bar-Natan homology. Essentially, the objects of $Cob^3_{e/l}$ are sets of disjoint circles embedded in S^2, and morphisms are cobordisms (embedded in $S^2 \times [0, 1]$) between them. We define F_{BN} on objects by sending a set of circles to the set of sign assignments to them. (For instance, a single circle gets sent to $\{+, -\}$). A pair of paints gets sent to the signed correspondence coming from the map Δ_{BN}, and an upside-down pair of pants gets sent to the signed correspondence coming from the map m_{BN}.

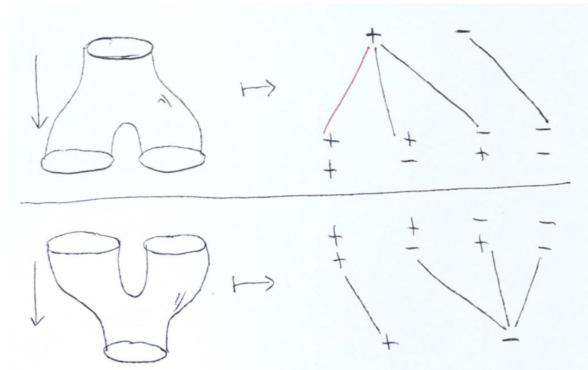


Figure 31: The functor $F_{BN} : Cob^3_{e/l} \to f\mathscr{B}$.

(Note: it won't matter where the cup and cap cobordisms get mapped). From any link diagram L, we can construct a functor $F_L : 2^n \to Cob^3_{e/l}$. (For more information on this construction, see [BNb]). We have $F_{BN} \circ F_L = F_{BN}(L) : 2^n \to f\mathscr{B}$. This means that the above lemma implies the first half of Theorem 2. Furthermore, index-2 resolution configurations correspond to squares in $Cob^3_{e/l}$. Thus, the second part of Theorem 2 is consequent from the following lemma.

Lemma 34. The Bar-Natan Ladybug Matching, if it exists, cannot depend solely on the resolution configuration of the 2-face. Formally, there does not exist an assignment $X \to \{Standard, Nonstandard\}$ (where X is the category of squares in $Cob^3_{e/l}$), such that $F_{BN}(L) : 2^n \to f\mathscr{B}$ extends to a Bar-Natan functor $2^n \to f\mathscr{B}$ agreeing with the assignment.

Proof. Consider the following index-3 decorated resolution configuration:

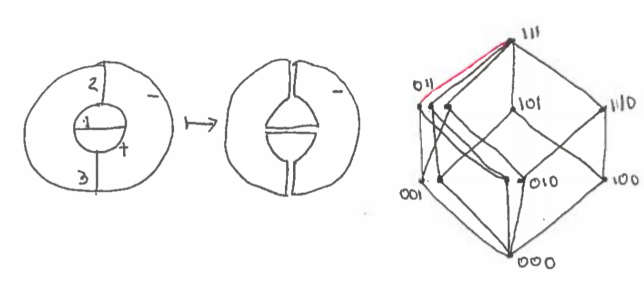


Figure 32: Counterexample: the configuration and its 3-cube

whose Bar-Natan graph looks as shown above. The loop in X_1 is thus as follows:

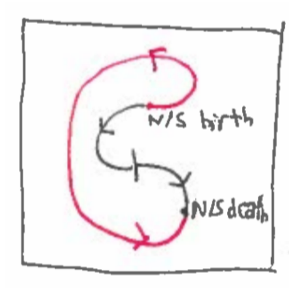


Figure 33: Non-nullhomotopic loop in X

Notice the birth and death are both in situation X. Thus, for it to be nullhomotopic, the birth and death must be of opposite types in their respective 2-faces. But their resolution configurations are isomorphic:

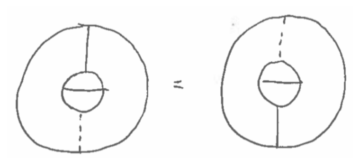


Figure 34: Isomorphic resolution configurations

Hence, for the Bar-Natan matching to assign them different types, it must depend on the cube's information too. ☐

Fortunately, this "flipping" of the N/S-type of births and deaths that occurs in 3-faces because of the Y situation only occurs in the following basic index-3 decorated resolution configurations:

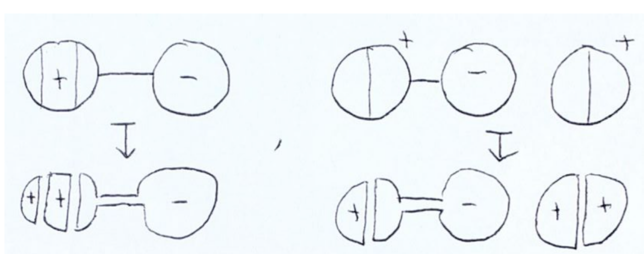


Figure 35: Configurations with Y situation

whose corresponding loops are simply of the type described in Figure 30 (but, of course, the type of births and deaths is undetermined).

6 Further Questions

There are many interesting question still open to tackle in the topic of Steenrod operations on Khovanov-type homologies, most of which can tentatively lead to distinguishing links. This paper concludes that a possible Steenrod square on Bar-Natan homology would be hard to describe and to compute, or else no such natural square exists.

However, the question of a Steenrod square on odd Khovanov homology is still open and, to the knowledge of the author, has not yet been tackled. So are the questions of higher (at least 4-dimensional, since $Sq^3 = Sq^1 \circ Sq^2$) Steenrod operations on (even) Khovanov homology, and of Steenrod operations on $\mathbb{Z}/p$-coefficients. As we remarked previously, the existence of a fully combinatorial proof for the existence of a natural Steenrod square on Khovanov homology suggests that a similar proof could potentially be used for other Steenrod operations - as long as one has a combinatorial definition - without the need to define a homotopy type.

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A Appendix

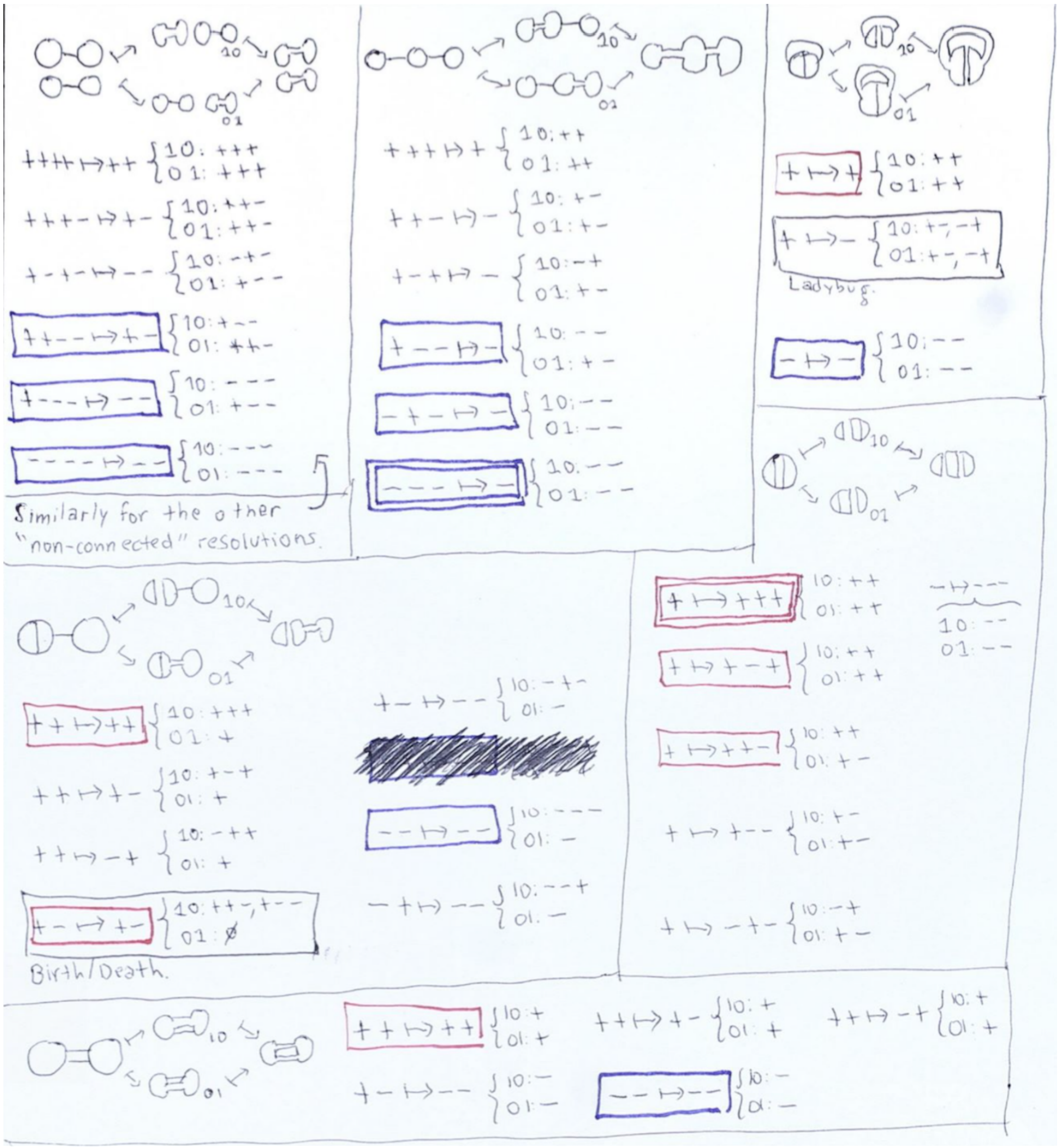

Figure 36: A list of all index-2 basic decorated resolution configurations, along with their Bar-Natan ladybug matching. All configurations that were not present in the Khovanov ladybug matching are circled; i.e. configurations that use the differentials present in the Bar-Natan differential but not in the Khovanov differential. Configurations that use an arrow of the type $+ \mapsto ++$ are circled in red. Configurations that use an arrow of the type $-- \mapsto -$ are circled in blue. (If a configuration uses two of these arrows, it's circled twice). The most noteworthy of the configurations are the ladybug and the birth/death, which are circled in black. The non-canonical choices are the pairing up of the ladybug configuration, and the standard/non-standard assignment of the birth/death configuration.