Irreducible Components and Dimension of the Spaltenstein Variety in Type A

UROP+ FINAL PAPER, SUMMER 2018

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Abstract. We provide a complete classification of the irreducible components of the Spaltenstein variety when the corresponding partial flag variety is of the Grassmann type. We also classify the components when the corresponding partial flag variety is arbitrary, for the special cases of the nilpotent N being a hook or a 2-block. We also discuss the dimension of these spaces.

Contents

1. Introduction

Let $\mathbb{K}$ be a field, n a positive integer and $\mu = (\mu_1, ..., \mu_r)$ a partition of n. Let $\mathcal{F}_\mu$ be the partial flag variety of type μ and N a nilpotent transformation of $\mathbb{K}^n$ with Jordan type λ. The Spaltenstein variety $\mathcal{F}_\mu^N$ corresponding to N is the fixed point subvariety of N on $\mathcal{F}_\mu$. Namely, $\mathcal{F}_\mu^N = \{F = (V_i) \in \mathcal{F}_\mu | NV_i \subseteq V_i \text{ for } i = 1, ..., r\}$. In the case $\mu = (1, 1, ..., 1)$, it has been proven ([Spal]) that $\mathcal{F}_\mu^N$ is a pure-dimensional space whose irreducible components are in bijection with the standard Young tableau of type λ. In the case of an arbitrary partition μ, [Shim] gives a correspondence between some irreducible subvarieties of $\mathcal{F}_\mu^N$ that partition the space and minimal, semistandard μ-tableaux of type λ. However, this correspondence does not take into account inclusion relations that often occur between the closures of these irreducible subvarieties.

In this paper, we explicitly compute those inclusion relations and provide a bijective correspondence between a subset of the minimal, semistandard μ-tableaux of type λ and the irreducible components of $\mathcal{F}_\mu^N$, in the case where $\mathcal{F}_\mu$ is a Grassmann variety, as well as the cases where N is of the hook and 2-block type. As a corollary we also derive the dimenson of $\mathcal{F}_\mu^N$ in the above cases.

This paper is organized in 3 main sections. Section 2 presents some basic definitions and preliminary results that will be used throughout the paper, intruduces the basic notation and reviews existing results on the topic of Spaltenstein varieties. Section 3 presents the complete classification of the irreducible components of the Spaltenstein variety in the case where $\mathcal{F}_\mu$ is a Grassmann variety. Section 4 describes the irreducible components of the Spaltenstein variety of any partial flag variety $\mathcal{F}_\mu$ in two special cases; when the nilpotent N is a 2-block and when N is a Hook.

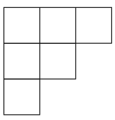
Acknowledgements This paper is a result of the UROP+ program of the MIT Mathematics Department under the guidance of mentor Pablo Boixeda Alvarez. At this point, I would like to thank Pablo for all the guiding discussions and advice and for helping me build the necessary background to be able to tackle this problem. I would also like to thank prof. Roman Bezrukavnikov for suggesting this fascinating problem that introduced me to many different areas of mathematics. Finally, I am grateful to prof. Slava Gerovitch and the MIT Mathematics Department for organizing the UROP+ program.

2. Preliminaries

Let N be a nilpotent transformation of $\mathbb{K}^n$ for some positive integer n. By identifying N with the sizes λ_i of its Jordan blocks in descending order, we create a bijection between nilpotent conjugacy classes of $GL(n,\mathbb{K})$ and ordered partitions of n.

We can identify ordered partitions of n with Young diagrams. The Young diagram corresponding to a partition $\lambda = (\lambda_1, ..., \lambda_r)$ of n, is an arrangement of n boxes in left aligned rows, such that the i^{th} row contains λ_i boxes.

Example 2.1. The diagram of the partition $(3,2,1)$ of 6 is:



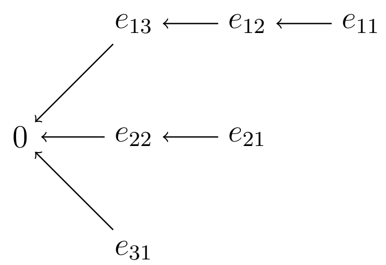
This allows us to identify nilpotent transformations with Young diagrams. In fact, we can encode all the information of the nilpotent matrix by assignig a Jordan basis element to each box of the Young diagram, so that if box i is to the right of box j then for the corresponding basis elements e_i and e_j we have that $Ne_i = e_j$ and if box i is on the leftmost column then $Ne_i = 0$.

Remark 2.2. Throughout this paper we will use a doubly indexed Jordan basis (e_{ij}) for a nilpotent N corresponding to a partition $\lambda = (\lambda_1, ..., \lambda_r)$, so that:

$$\begin{cases} 1 \le i \le r, \\ 1 \le j \le \lambda_i, \\ Ne_{ij} = e_{i(j+1)} \quad \text{if} \quad j < \lambda_i \\ Ne_{i\lambda_i} = 0 \end{cases}$$

We will use the same indexing to label the squares of the Young diagram.

Example 2.3. For the nilpotent corresponding to the partition $(3,2,1)$, we have:



where the arrows represent the action of N.

We will now introduce the main combinatorial object used throughout this paper, borrowing the language of [Shim].

Definition 2.4. Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a young diagram of size n and $\mu = (\mu_1, ..., \mu_r)$ be a partition of n. A $\boldsymbol{\mu}$**-tableau of type** $\boldsymbol{\lambda}$ is a numbering of the boxes of λ with the numbers $1, ..., r$ so that i has multiplicity μ_i.

If α is a μ-tableau of type λ, then:

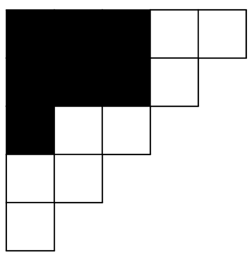
(1) α is **semistandard** if the numbers along each row are weakly increasing.
(2) α is **minimal** if the numbers along each column are weakly increasing.
(3) α is **standard**, if it is minimal, semistandard and the numbers along each column are strictly increasing.

Remark 2.5.

(1) When $\mu = (1, 1, ..., 1)$, a standard μ-tableau of type λ is percisely a standard Young Tableau.
(2) When $\mu = (k, n-k)$, then a tableau is just a numbering of the boxes with the numbers 1 and 2. In accordance with [Shim], we will instead represent the boxes numbered with 1 as ■ and the boxes numbered with 2 as □. Then, being minimal, semistandard is equivalent to the condition that the ■s form a Young diagram starting at the top left corner of λ.
(3) When it is clear from the context, we might refer to a minimal, semistandard μ-tableau of type λ simply as a tableau.

Example 2.6. The following diagrams are minimal semistandard:

1	1	1	2	2	3
1	1	2	3	3	
1	2				
3					

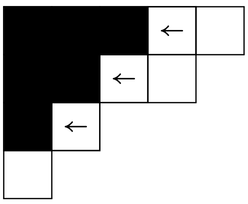


Let α be a μ-tableau of type λ. We will denote by α^i the tableau formed when we replace the boxes that are labeld with a number $j \leq i$ with $\blacksquare$ and the boxes that are labeld with a number $j > i$ with $\square$. Then, the condition that α is minimal, semistandard is equivalent to the condition that α^i is minimal, semistandard for every i. [Shim]

Let N be a nilpotent and $\lambda = (\lambda_1, ..., \lambda_r)$ be the corresponding Young diagram. We will use the basis for N presented in 2.2. For now, let $\mu = (k, n-k)$. In this case, $\mathcal{F}_\mu$ is a Grassmann variety which we will denote by $Gr(k, n)$.

Let $\alpha = (\alpha_1, ..., \alpha_s)$ be a μ-tableau of type λ, where $\alpha_i > 0$ is the number of $\blacksquare$s in the i^{th} row of α. We will call every $\blacksquare$ of α that is to the left of a $\square$ a **starting box**. Using the labeling of 2.2, the coordinates of the starting boxes are $(i, \lambda_i - \alpha_i + 1)$ for $i = 1, ..., s$.

Example 2.7. Here, the arrows point at the starting boxes of $\alpha = (3, 2, 1)$. Their coordinates are $(1, 3)$,$(2, 3)$ and $(3, 2)$.



To every starting box, we will assign a variable vector, the **starting vector** v_i, such that:
$v_i = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_i} a^i_{jk} e_{jk}$, where:

$$a^i_{jk} = \begin{cases} 1, \text{ if } (j,k) \text{ is the starting box.} \\ 0, \text{ if } (j,k) \text{ is any other } \blacksquare. \\ 0, \text{ if } (j,k) \text{ is a } \square \text{ to the right of the starting box.} \\ \text{free, otherwise.} \end{cases}$$

Now, we fix the free variables $\{a^i_{jk}\}$. For every starting vector $v_i|_{\{a^i_{jk}\}}$, we have that $v_i|_{\{a^i_{jk}\}} \in Ker(N^{\alpha_i}) \smallsetminus Ker(N^{\alpha_i - 1})$. So, the space $W_i|_{\{a^i_{jk}\}} = \langle \{N^j v_i\} \rangle$ is an α_i dimensional vector space. Because $\sum_{i=1}^{s} \alpha_i = k$, the space

$$W_\alpha|_{\{a^i_{jk}\}} = \langle \{W_i|_{\{a^i_{jk}\}} | i = 1, ...s\} \rangle$$

is a k dimensional vector space and by construction $NW_\alpha|_{\{a^i_{jk}\}} \subseteq W_\alpha|_{\{a^i_{jk}\}}$. Thus, the flag $F_\alpha|_{\{a^i_{jk}\}} = (W_\alpha|_{\{a^i_{jk}\}} \subset \mathbb{K}^n)$ is a point in $Gr^N(k, n)$.

Note: We used the notation $|_{\{a^i_{jk}\}}$ to emphasize that the variables $\{a^i_{jk}\}$ were fixed. In the rest of the paper we will simply state whether the variables are fixed, or free and ommit such notation.

Now, consider the space $Y_\alpha = \{(W_\alpha|_{\{a^i_{jk}\}} \subset \mathbb{K}^n)|$ for $\{a^i_{jk}\}$ free$\}$. [Shim] proves the following proposition:

Proposition 2.8. *$\overline{Y_\alpha}$ is an irreducible subvariety of $Gr^N(k,n)$ and $Gr^N(k,n) = \bigcup\limits_{\alpha\in A} \overline{Y_\alpha}$, where A is the set of all minimal, semistandard μ-tableaux of type λ.*

Remark 2.9. The dimension $d(\alpha)$ of Y_α is equal to the number of free variables a^i_{jk}.

Example 2.10. Going back to the example 2.7, we have that:

$$\begin{aligned} v_1 &= [0\ 0\ 1\ 0\ 0|\ 0\ a^1_{22}\ 0\ 0|\ a^1_{31}\ 0|a^1_{41}] \\ v_2 &= [0\ 0\ 0\ 0\ 0|\ 0\ \ 0\ \ 1\ 0|\ a^2_{31}\ 0|a^2_{41}] \\ v_3 &= [0\ 0\ 0\ 0\ 0|\ 0\ \ 0\ \ 0\ 0|\ \ 0\ \ 1|a^3_{41}] \end{aligned}$$

Returning to the case of an arbitrary partition $\mu = (\mu_1, ..., \mu_r)$, we can define Y_α in a similar way.

Let $Y_\alpha = \{(V_1 \subset ... \subset V_{r-1} \subset \mathbb{K}^n) \in \mathcal{F}^N_\mu | (V_i \subset \mathbb{K}^n) \in Y_{\alpha^i}\}$. Then, the same proposition (proved in [Shim]) holds in the general case:

Proposition 2.11. *$\overline{Y_\alpha}$ is an irreducible subvariety of $\mathcal{F}^N_\mu$ and $\mathcal{F}^N_\mu = \bigcup\limits_{\alpha\in A} \overline{Y_\alpha}$, where A is the set of all minimal, semistandard μ-tableaux of type λ.*

Remark 2.12. In this case, [Shim] shows that we can compute the dimension of Y_α if we sum over all boxes of the tableau, the number of boxes that are above them in their column and are numbered with a strictly smaller integer.

Remark 2.13. Because of inclusion relations that occur between the closures of the irreducible subvarieties Y_α for $\alpha \in A$, not every $\overline{Y_\alpha}$ is a component of $\mathcal{F}^N_\mu$. To understand these inclusion relations, it is useful to introduce an ordering on Young diagrams. Since by 2.5, the ■ part of a minimal, semistandard μ-tableau of type λ can also be viewed as a Young diagram in the Grassmann case, this ordering extends to an ordering of minimal, semistandard μ-tableaux of type λ when $\mu = (k, n-k)$.

Let $\lambda = (\lambda_1, ..., \lambda_k)$ and $\mu = (\mu_1, ..., \mu_k)$ be two Young diagrams of size k.

We will say that λ **dominates** μ and we wil write $\lambda \geq \mu$ if $\sum\limits_{j=1}^{i} \lambda_j \geq \sum\limits_{j=1}^{i} \mu_j$ $\forall i \in \{1, ..., k\}$.

The **dominance ordering** is a partial ordering on the set of Young diagrams. We will use it extensively throughout this paper as it dictates which inclusion relations are possible between irreducible components of the Steinberg variety.

We will say that λ is **lexicographically** greater than μ and we wil write $\lambda \geq \mu$ if $\lambda_j = \mu_j$ for $j < i$ and $\lambda_i \geq \mu_i$ for some $i \in \{1, ..., k\}$.

This is a total ordering on the set of Young Diagrams. It immediately follows from the two definitions that $\lambda \geq \mu$ implies $\lambda \geq \mu$.

Before we move to the main section of the paper, we will discuss two techniques that we will use extensively in the proofs of this paper.

Remark 2.14 ("Going to ∞")**.** In order to prove inclusions of the form $Y \subseteq X$ for $X, Y \in \{\overline{Y_\alpha} | \alpha \in A\}$, we will use the following trick. We will define maps

$$f : \mathbb{A}^1 \to X$$

by assigning values to all but one of the free variables of the flags in X. We will then extend f to a map

$$\overline{f} : \mathbb{P}^1 \to X.$$

Because X is a closed subvariety, $\overline{f}(\mathbb{P}^1) \subseteq X$ and $\overline{f}$ is well defined. By selecting f appropriately, we can arrange for "the point at infinity" of $\mathbb{P}^1$ to also land in Y. This way we can construct points in $X \cap Y$. If we can define enough maps to hit every point in Y, we will have that $Y \subseteq X$.

Remark 2.15 (Closed Conditions)**.** In order to understand the defining equations of a subvariety $Y \in \{\overline{Y_\alpha} | \alpha \in A\}$ in the Grassmann case, we will look at conditions of the form

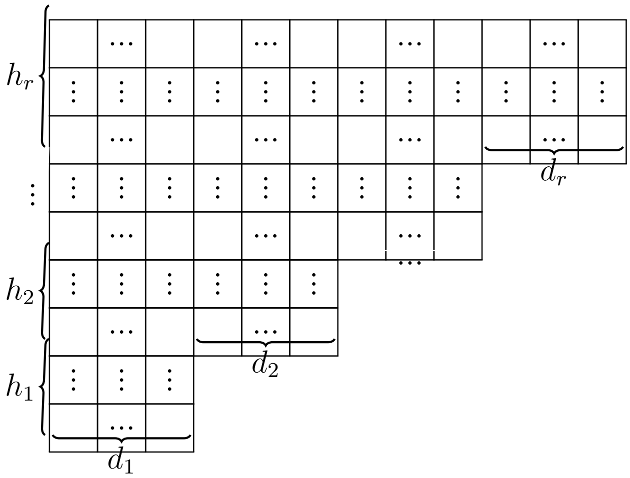
$$dim(V \cap W) \geq k$$

for a fixed space W and an integer k, that are satisfied by every point $(V \in \mathbb{K}^n) \in Y$. This condition is equivalent to the vanishing of the $(k - n + 1) \times (k - n + 1)$ minors of the matrix M of equations defining V and W. These are polynomial equations in the entries of M, so $dim(V \cap W) \geq k$ is a closed condition.

3. Grassmann Variety

In this section, we will look at one of the simplest types of partial flags, the Grassmann Variety. Fix two positive integers k, n and a nilpotent transformation N of $\mathbb{K}^n$ of type λ. Let $\mu = (k, n - k)$ and denote by A, the set of minimal, semistandard μ-tableaux of type λ.

Define the quantities h_i and d_i for the Young diagram λ as in the following diagram:



Let $D_i = \sum_{j=1}^{i} d_i$ for $i = 1,..,r$. Starting from the left, we will call the columns $\{D_{i-1}+1,...,D_i\}$ the i^{th} block B_i of the nilpotent. We can define a map

$$f : A \to \mathbb{Z}^r$$
$$\alpha \mapsto (\#\{\blacksquare \in B_i\})$$

For $\alpha \in A$ we can explicitly compute $f(\alpha) = (f_1,...,f_s)$. Namely, we have that $f_1 = \sum_{j=1}^{k} min(D_1, \alpha_j)$ and $f_i = \sum_{j=1}^{k} min(D_i, \alpha_j) - f_{i-1}$ for $i > 1$.

Given a point $p = (p_1,...,p_r) \in f(A)$ we can write $p_i = d_i\pi_i + r_i$ for $\pi_i, r_i \in \mathbb{Z}$ and $0 \le r_i < d_i$ (note the strict inequality). Let α_p be the tableau in $f^{-1}(p)$ whose $\blacksquare$ fill the first π_i rows of block B_i and the first r_i boxes of the next row, for $i = 1,...,r$ and let $\mathcal{M} = \{\alpha_p | p \in f(A)\}$.

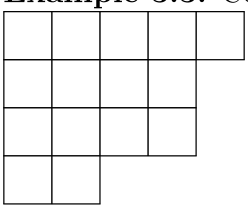
Finally, let $\mathcal{S} = \{\alpha_p \in \mathcal{M} | \pi_i - \pi_{i+1} + \epsilon_i \le h_i \text{ for } i = 1,...,r-1\}$, where $\epsilon_i = \begin{cases} 0, & \text{if } r_i = r_{i+1} = 0 \\ 1, & \text{otherwise} \end{cases}$

Remark 3.1. Note that $\mathcal{S} \neq \varnothing$ because the maximal tableau with respect to the dominance ordering is always contained in $\mathcal{S}$. The reason behind this is that the maximal tableau always fills the $\square$s of each row with $\blacksquare$s completely before moving onto the next row. Thus, it "follows" the shape of λ and it can never violate the conditions for being in $\mathcal{S}$.

Lemma 3.2. *The tableau α_p dominates β for every $\beta \in f^{-1}(p)$.*

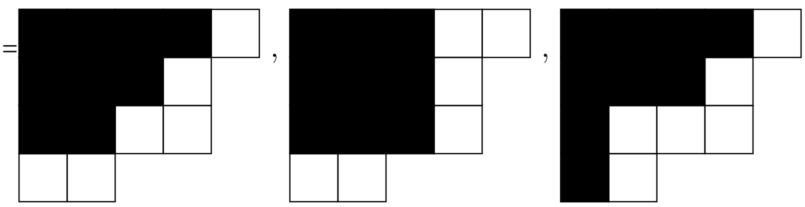
Proof. The proof of this lemma will become obvious after 3.13. □

Example 3.3. Consider the following example for $n = 15$ and $k = 11$;



Here, $D_1 = 2, D_2 = 4, D_3 = 5$. Consider the point $p = (6,3,0) \in f(A)$.

The tableaux in $f^{-1}(p)$ are the following:

$\alpha_p =$ 

Then, the main proposition of tihs section is the following:

Proposition 3.4. *For every minimal, semistandard tableau $\alpha \in \mathcal{S}$, $\overline{Y_\alpha}$ is an irreducible component of $Gr^N(k,n)$. All the irreducible components of $Gr^N(k,n)$ are of the form $\overline{Y_\alpha}$, for some $\alpha \in \mathcal{S}$.*

Lemma 3.5. *Let $(V \in \mathbb{K}^n)$ be a point in Y_α for some $\alpha \in A$. Then:*

(1) *$dim(V \cap ImN^i) \geq \sum_{j=1}^{k} max(0, \alpha_j - i)$*
(2) *$dim(V \cap KerN^i) \geq \sum_{j=1}^{k} min(i, \alpha_j)$.*
(3) *There always exists a point $(V_\alpha \in \mathbb{K}^n)$ in Y_α that achieves $dim(V_\alpha \cap KerN^i) = \sum_{j=1}^{k} min(i, \alpha_j) \quad \forall i \in \{1, ..., k\}$.*

Proof. (1) By the definition of Y_α, we have that $V = \langle\{N^x v_y | x = 1, 2, ...\}\rangle$ for a set of starting vectors $\{v_y\}$. By construction we have that

$$\langle\{N^x v_y | x = i, i+1, ...\}\rangle \subseteq V \cap ImN^i$$

. But $dim(\langle\{N^x v_y | x = i, i+1, ...\}\rangle) = \sum_{j=1}^{k} max(0, \alpha_j - i)$. This proves (1).

(2) With V as in (1), we have that for any particular starting vector v_y, we have that $N^{\alpha_y} v_y = 0$. Thus, $N^{\alpha_y - j} v_y \in KerN^i$ for $j = 0, 1, ..., min(i, a_y)$. Thus, $dim(V \cap KerN^i) \geq \sum_{y=1}^{k} min(i, \alpha_y)$.

(3) Consider the point obtained when setting all the free variables equal to zero. Then, $V_\alpha = \langle\{e_{ij} | i = 1, ..., k,\ j = \lambda_i - \alpha_i, ..., \lambda_i\}\rangle$. This obviously has the desired property.

□

Remark 3.6. (1) As explained in 2.15, the inequalities 3.5.1 and 3.5.2 are closed conditions that are satisfied by every point in Y_α, so they must also be satisfied by every point in $\overline{Y_\alpha}$.

(2) Note that 3.5.3 is actually very weak. In fact, every point in Y_α satisfies this property. However, this is not necessarily true about $\overline{Y_\alpha}$.

Lemma 3.7. *Let α and β be minimal, semistandard μ-tableau of type λ. Then,*

(1) *If $\alpha > \beta$ then $Y_\alpha \notin \overline{Y_\beta}$.*
(2) *If α and β are not comparable with the dominance ordering, then $Y_\alpha \notin \overline{Y_\beta}$ and $Y_\beta \notin \overline{Y_\alpha}$.*

Proof. (1) If $\alpha > \beta$ lexicographically, for some $p \in \{1, ..., k\}$ we have that $\alpha_i = \beta_i$ for $i < p$ and $\alpha_p > \beta_p$. Now, every point $(V \in \mathbb{K}^n)$ in $\overline{Y_\beta}$ satisfies: $dim(V \cap KerN^{\beta_p}) \geq \sum_{i=1}^{k} min(\beta_p, \beta_i) = p \cdot \beta_p + \sum_{i=p+1}^{k} \beta_i = p \cdot \beta_p + k - \sum_{i=1}^{p} \beta_i > p \cdot \beta_p + k - \sum_{i=1}^{p} \alpha_i \geq dim(V_\alpha \cap KerN^{\beta_p})$, where V_α as in 3.5.3. So, $(V_\alpha \in \mathbb{K}^n) \notin Y_\beta$ and thus $Y_\alpha \notin \overline{Y_\beta}$.

(2) Even though α and β are not comparable with the dominance ordering, they are still comparable lexicographically, so wlog suppose that $\alpha > \beta$. By part (1) of this Lemma, we have that $Y_\alpha \notin \overline{Y_\beta}$.
Since α and β are not comparable with the dominance ordering, there exists some $q > p$ such that $\sum_{i=1}^{q} \alpha_i < \sum_{i=1}^{q} \beta_i$. Thus, $\sum_{i=q+1}^{k} \alpha_i > \sum_{i=q+1}^{k} \beta_i$. Now, every point $(V \in \mathbb{K}^n)$ in $\overline{Y_\alpha}$ satisfies:

$dim(V \cap KerN^{\alpha_q}) \geq \sum_{i=1}^{k} min(\alpha_q, \alpha_i) = q \cdot \alpha_q + \sum_{i=q+1}^{k} \alpha_i > q \cdot \alpha_q + \sum_{i=q+1}^{k} \beta_i \geq dim(V_\beta \cap KerN^{\alpha_q})$, where V_β as in 3.5.3. So, $(V_\beta \in \mathbb{K}^n) \notin Y_\alpha$ and thus $Y_\beta \nsubseteq \overline{Y_\alpha}$.

□

Remark 3.8. This lemma immediately implies that if $Y_\alpha \subseteq \overline{Y_\beta}$, then $\beta \geq \alpha$.

Lemma 3.9. *Let α be a minimal, semistandard tableau in $\mathcal{S}$. Then, there exists a point $(V \in \mathbb{K}^n) \in Y_\alpha$ such that $dim(V \cap ImN^i) = \sum_{j=1}^{k} max(0, \alpha_j - i)$ for every $i \in \{D_1, ..., D_{r-1}\}$.*

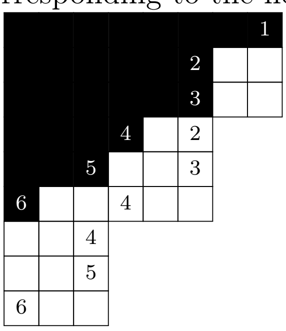
Proof. We will focus on $dim(V \cap ImN^i)$ for some $i \in \{D_1, ..., D_{r-1}\}$. In every row of the nilpotent that has length at least i, the basis vectors corresponding to the first i boxes of the row (from the right), are not in ImN^i. So, if a starting vector v_y of V with $\alpha_y \geq i$ has a nonzero free variable a^y_{1x} corresponding to the first box of such a row, then $N^j v_y \notin ImN^i$ for $j < i$. Moreover, all the basis vectors that correspond to a row of the nilpotent with length less than i are not in ImN^i. So, if a starting vector v_z of V with $\alpha_z < i$ has a nonzero free variable in the same column as its starting box and in a row of length less than i, then $N^j v_y \notin ImN^i$ for $j < \alpha_z$.

In order to exclude the possibility of linear combinations of the starting vectors being in ImN^i, we will place the nonzero free variable of every starting vector at a different row of the nilpotent. Since $\alpha \in \mathcal{S}$, the condition $\pi_j - \pi_{j+1} + \epsilon_j \leq h_j$ for every j guarantees that for every starting vector v_y with $\alpha_y = D_j$ for some j, we can find a row in N with lenght exactly D_j. And for every starting vector with v_z with $\alpha_z = D_j + r_{j+1}$ for some j (there can be at most one such vector for every j), we can find two rows in N with lenghts exactly D_j and D_{j+1}.

□

We will attempt to clarify this proof with an example.

Example 3.10. It can be checked that $\alpha \in \mathcal{S}$. Here, we want to find a point in Y_α that minimizes the dimension of its intersection with ImN^3 and ImN^6. We have numbered the starting boxes of α. We also labeled with the same number, the boxes corresponding to the nozero free variables of each starting vector.



If we set the nonzero free variables equal to 1, we get:
$v_1 = e_{11}$
$v_2 = e_{23} + e_{41}$
$v_3 = e_{33} + e_{51}$
$v_4 = e_{43} + e_{63} + e_{71}$

$v_5 = e_{54} + e_{81}$
$v_6 = e_{66} + e_{93}$

Lemma 3.11. *Let $\alpha \in \mathcal{S}$ and $\beta \in A$ with $\beta > \alpha$. Then,*

$$\sum_{j=1}^{k} max(0, \alpha_j - i) < \sum_{j=1}^{k} max(0, \beta_j - i)$$

for some $i \in \{D_1, ..., D_{r-1}\}$.

Proof. Note that $\sum_{j=1}^{k} max(0, \alpha_j - i)$ is the number of ■s that remain when we completely delete the leftmost i columns of α.

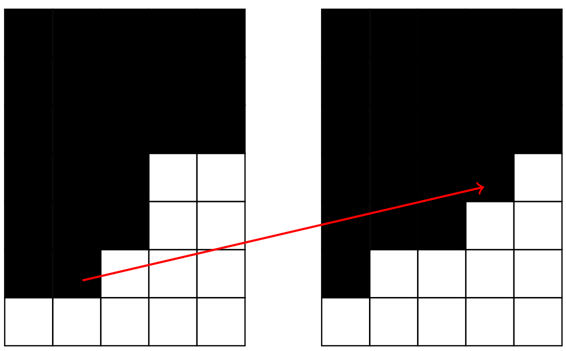
Let x be the smallest index for which $\alpha_x < \beta_x$ and suppose that the starting box of α in row x lies inside the block B_y and set $i = D_{y-1}$. Then, $\sum_{j=1}^{k} max(0, \alpha_j - i)$ is the number of ■s remaining when deleting blocks $B_1, ..., B_{i-1}$ and ditto for $\sum_{j=1}^{k} max(0, \beta_j - i)$. In the remaining blocks, every row of α and β has the same number of ■s, except for the x^{th} row in which α has fewer ■s than β. This completes the proof. □

Remark 3.12. By 3.9, for every $\alpha \in \mathcal{S}$ a point in Y_α, minimizes $dim(V \cap ImN^i)$ for every $i \in \{D_1, ..., D_{r-1}\}$ and by 3.11 none of the tableaux that dominate α can achieve this minimum for every $i \in \{D_1, ..., D_{r-1}\}$. So, combining the two results tells us that $Y_\alpha \notin \overline{Y_\beta}$ for any $\beta > \alpha$. Thus, $\overline{Y_\alpha}$ is a component of $Gr^N(k, n)$.

Let α and β be two tableau in $f^{-1}(p)$ for some $p \in f(A)$, such that β is obtained from α in the following way; For some i, we move the lowest starting box that is contained in the block B_i to the firt row of B_i that is not full.

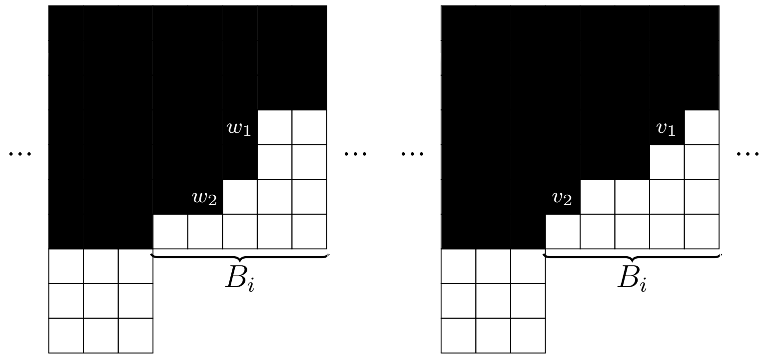
Remark 3.13. Since we are moving ■s within B_i, α remains in $f^{-1}(p)$. Moreover, by construction $\beta \geq \alpha$. The only case where we cannot find any block B_i for which this move results in a different tableau is when $\beta = \alpha_p$. In other words, we cannot maximize α_p further while remaining in $f^{-1}(p)$. On the other hand, repeatedly applying this move to any tableau in $f^{-1}(p)$ will lead us to α_p.

Here, we are only drawing the block B_i.



Lemma 3.14. *If β is obtained from α with the process described above, then $Y_\alpha \subseteq \overline{Y_\beta}$.*

Proof. When performing this move, the starting vectors that are affected are the ones labeled in the diagram below. For simplicity, we are using the indexes 1 and 2 rather than the actual ones;



Initially, we might try to send the vector $Nv_1 \to w_1$. However, the problem with this is that being in ImN, causes Nv_1 to have fewer free variables than w_1. Moreover, we need to get w_2 as the image of some starting vector.

Our strategy is to use "the infinity trick" 2.14, to prove the desired inclusion. In particular, we will argue that for a particular choice for the free variables of v_1 and v_2 we have:

$$\frac{1}{c} \cdot v_1 \to w_2,$$

$$Nv_1 - \frac{k_1}{c} \cdot v_1 - k_2 \cdot v_2 \to w_1,$$

as $c \to \infty$ for some constants k_1, k_2 which we will later specify.

To achieve the first relation, we will give the following values to the free variables of v_1. To the free variable corresponding to the starting box of w_2, we will give the value c. To all the free variables of v_1 that also occur in w_2, which are the ones that correspond to boxes below and to the left of w_2, we will give values proportional to c. The constant of proportionality will be equal to the value of the corresponding variable of w_2. We will let all other variables be finite. Then, the first relation is achieved.

Since we are using $\frac{1}{c} \cdot v_1$ to play the role of w_2, we have not yet used v_2, which has the same freedom that v_1 has in all the blocks prior to v_1. Thus, we can use its free variables to cancel the parts of Nv_1 that are going to infinity, and to account for the free variables that Nv_1 lacks in comparison to w_1 due to being in ImN. We can also adjust k_1 and k_2 to eliminate unecessary free variables that Nv_1 had on the row of v_2 that now lie on $\blacksquare$s. This achieves the second relation.

Since no other rows of the two tableaux differ, it is easy to send all other starting vectors of β to the corresponding starting vectors of α. Repeatedly applying N to the obtained starting vectors will give us the whole vector space.

□

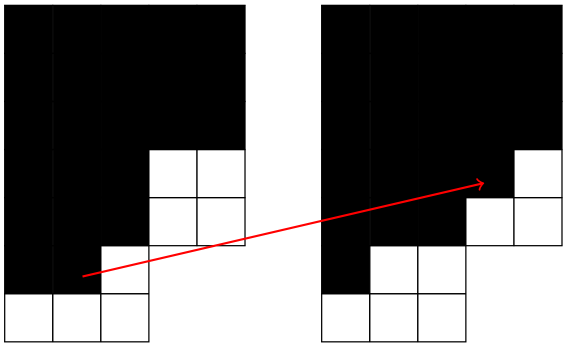
Corollary 3.15. *Let $\alpha \in f^{-1}(p)$ for some $p \in f(A)$. Then, $Y_\alpha \subseteq \overline{Y_{\alpha_p}}$.*

Proof. The proof of this can be obtained immediately by repeatedly applying 3.14 until we reach α_p. □

Now that we have proven these inclusion relations, the only possible candidates for irreducible components of $Gr^N(k,n)$ are the subvarieties $\overline{Y_\alpha}$ for $\alpha \in \mathcal{M}$. In order to complete the proof of proposition 3.4 we need to show that for any $\beta \in \mathcal{M} \setminus \mathcal{S}, Y_\beta \subseteq \overline{Y_\alpha}$ for some $\alpha \in \mathcal{S}$.

Let $\alpha \in \mathcal{M} \setminus \mathcal{S}$ and let i be the smallest index for which $\pi_i - \pi_{i+1} + \epsilon_i > h_i$. Consider the tableau $\beta \in \mathcal{M}$ that is obtained from α by moving the last starting box that is contained in the block B_i to the last unfilled row of the block B_{i+1}.

Here, we are only drawing the blocks B_i and B_{i+1}.

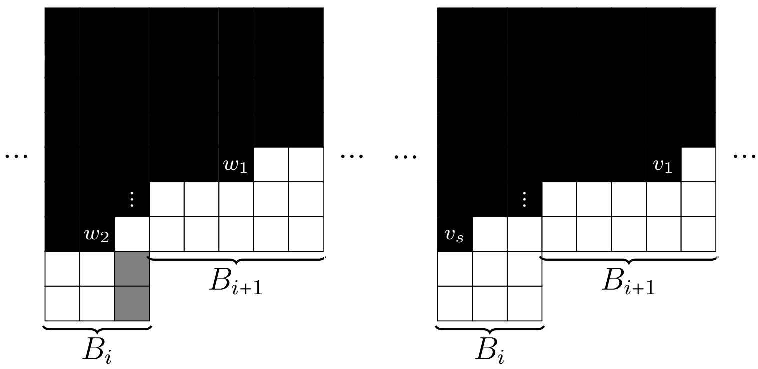


Lemma 3.16. *If β is obtained from α with the process described above, then $Y_\alpha \subseteq \overline{Y_\beta}$.*

Proof. This proof is similar to that of 3.14. However, here we will not have any unused vectors in the block B_{i+1} to compensate for all the free variables (shown in gray) that Nv_1 loses in the block B_i due to being in ImN. Since $\alpha \notin \mathcal{S}$, the number of rows from w_1, to w_2 is greater than the height difference h_i between the blocks B_{i+1} and B_i. Thus, the corresponding number of rows in β is at least equal to h_i. So, we can use linear combinations of the starting vectors of these rows to compensate for the missing free variables. The relations we will achieve are the following:

$$\frac{1}{c^m} \cdot v_1 \to w_2,$$
$$Nv_1 - \frac{1}{c} \cdot v_1 - \sum_{j=2}^{s} k_j \cdot v_j \to w_1,$$

as $c \to \infty$ for some constants $m, k_1, ..., k_s$ which we will later specify.



Since we want $\frac{1}{c^m} \cdot v_1 \to w_2$, we must set the free variable of v_1 that corresponds to the starting box of w_2 equal to c^m. Moreover, we will let all of the free variables of v_1 that correspond to boxes below and to the left of the starting box of w_2 be proportional to c^m with constant of proportionality equal to the corresponding free variable of w_2 plus terms that tend to infinity at slower rates than c^m. We will also let all the free variables of v_1 that correspond to boxes in B_{i+1} be finite and we will let all other free variables of v_1 go to infinity at rates slower than c^m. This achieves the first relation.

In order to achieve the second relation, we must account for the following:

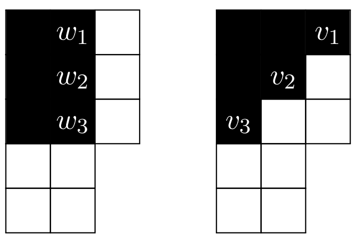
- Since the free variables of v_1 in B_i are finite, the only term of the sum

$$Nv_1 - \frac{1}{c} \cdot v_1 - \sum_{j=2}^{s} k_j \cdot v_j$$

  that contributes to the free variables in B_i is Nv_1. Thus, we can set all these free variables equal to the corresponding ones of w_1.
- We will let the values of the variables k_j be opposite of the free variables that Nv_1 has on the starting box of the corresponding starting vector v_j. These variables are unnecessary since they correspond to $\blacksquare$s on α
- Since we are using $N^j(\frac{1}{c^m} \cdot v_1)$ to account for the vectors in the row of w_2, we can use the free variables of v_s to account for the variables that Nv_1 loses due to being in ImN. We will use the other free variables of v_s to counteract the variables that tend to infinity in the blocks B_j for $j < i$ and in the part of the block B_i that is to the left of w_2 .
- We will now look at the free variables in the part of the block B_i to the left of w_2. Let $a_1, ..., a_{d_i}$ be the free variables of v_1 corresponding a row of the block B_i. Then, if we let $a_i = cb_i + ca_{i-1}$ and $a_1 = cb_1$ for some $b_1, ..., b_{d_i}$, we will have that the variables of $Nv_1 - \frac{1}{c} \cdot v_1$ at that row will be $-b_1, ..., -b_{d_i}$. Note, however that we do not have complete freedom in our choice of the b_j, since we also want $\frac{1}{c^m} \cdot v_1$ to satisfy the first relation. This is why we also include the starting vectors v_j for $j < s$ to the linear combination. By our assumption, the number of such vectors v_j is at least h_i. Thus demanding that the projections of the v_j to the subspace spanned by the last h_i rows of the block B_i are a spanning set of that subspace guarantees that we can achieve any combination of free variables. By making this linear independence assumption we land in an open part of our target, since by 2.15, linear dependence is a closed condition. Thus, taking the closure will give us the whole subvariety.

□

Example 3.17. The details described above will be illustrated better with an example. We will prove an inclusion between the following components.



$$v_1 = [1\ 0\ 0|a^1_{21}\ 0\ 0\ |a^1_{31}a^1_{32}0|a^1_{41}a^1_{42}|a^1_{51}a^1_{52}] \qquad w_1 = [0\ 1\ 0|0\ 0\ 0\ |0\ 0\ 0|b^1_{41}b^1_{42}|b^1_{51}b^1_{52}]$$
$$v_2 = [0\ 0\ 0|\ 0\ 1\ 0\ |\ 0\ a^2_{32}\ 0\ |a^2_{41}a^2_{42}|a^2_{51}a^2_{52}] \qquad w_2 = [0\ 0\ 0|0\ 1\ 0\ |0\ 0\ 0|b^2_{41}b^2_{42}|b^2_{51}b^2_{52}]$$
$$v_3 = [0\ 0\ 0|\ 0\ 0\ 0\ |\ 0\ \ 0\ \ 1\ |\ 0\ a^3_{42}|\ 0\ a^3_{52}] \qquad w_3 = [0\ 0\ 0|0\ 0\ 0|0\ 1\ 0|b^3_{41}b^3_{42}|b^3_{51}b^3_{52}]$$

For $[a^1_{41}a^1_{42}|a^1_{51}a^1_{52}] = a^1_{32} \cdot [b^3_{41}b^3_{42}|b^3_{51}b^3_{52}]$, and for $a^1_{32} \to \infty$ we get that $\frac{1}{a^1_{32}}v_1 \to w_3$. We can also set $v_2 = w_2$ (which implies $a^2_{32} = 0$)

Now, the full linear combination we want to concider is

$$v = Nv_1 + \frac{k_1}{a^1_{32}}v_1 + k_2v_2 + k_3v_3.$$

We have that

$$Nv_1 = [0\ 1\ 0|0\ a^1_{21}0|0\ a^1_{31}a^1_{32}|0\ a^1_{41}|0\ a^1_{51}]$$

We will set $k_3 = -a^1_{32}$ and $k_2 = -a^1_{21}$ in order to eliminate the unecessary free variables that Nv_1 has. Then, we will have:

$$Nv_1 + k_2v_2 + k_3v_3 = [0\ 1\ 0|0\ 0\ 0|0\ a^1_{31}\ 0|\ *\ *\ |\ *\ *\ ]$$

As a last step, we will set $k_1 = -a^1_{31}$. This will eliminate the last unecessary free variables. Thus, we will have that

$$v = [0\ 1\ 0|0\ 0\ 0\ |0\ 0\ 0|\ *\ *\ |\ *\ *\ ].$$

We want to be able to assign any value to the $*$s of v. Indeed, under the assumption that the last 4 coordinates of the vectors $v_1, v_2 and v_3$ are a spanning set (as mentioned in the proof), this is achievable.

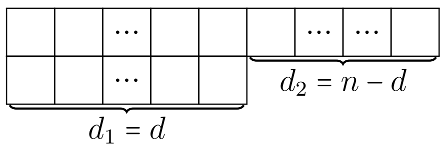
Now we are ready to prove the main proposition.

Proof of 3.4. For every $\alpha \in \mathcal{M} \setminus \mathcal{S}$, we can repeatedly apply the process described above, until we achieve $\pi_i - \pi_{i+1} + 1 \leq h_i$ for every $i \in \{1, ..., r-1\}$. Thus, we have that $Y_\alpha \subseteq \overline{Y_\beta}$ for some $\beta \in \mathcal{S}$. Combinibg this with 3.12, we have that the irreducible components of $Gr^N(k,n)$ are percisely the subvarieties Y_α for $\alpha \in \mathcal{S}$.

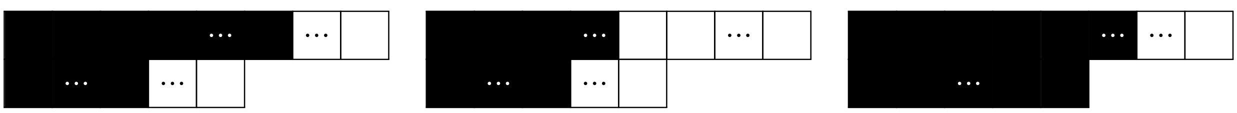
□

We will now explicitly describe the components in two special cases, which we will use in the next section of the paper.

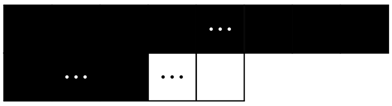
Example 3.18 (The 2-block).
When the nilpotent N has just 2 Jordan blocks, the Young diagram of N is:



and every tableau apart from the maximal one can look as follows:



In every case, we get that $\pi_1 \geq 1$ and $\pi_2 = 0$, so $\pi_1 - \pi_2 + 1 > 1$. As noted in 3.1, the maximal component is always contained in $\mathcal{S}$. Indeed, in this case the maximal tableau can look as follows, depending on whether $k > n - d$ or $k \leq n - d$;

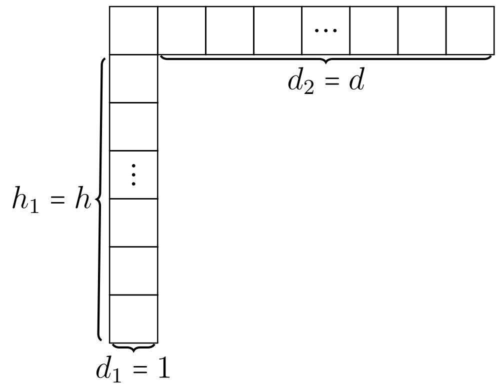


or

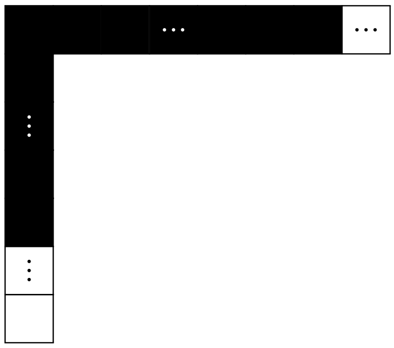
As expected, in any case $\pi_1 - \pi_2 \leq 1$.

To conclude, in the 2-block case $Gr^N(k,n) = \overline{Y_{\alpha_{max}}}$ is an irreducible subvariety of $Gr(k,n)$. By 2.9, $dim(Gr^N(k,n)) = dim(Y_{\alpha_{max}}) = min(k,d,n-k)$.

Example 3.19 (**The Hook**).
The case of the hook is in some way opposite to that of the 2-block. Here, the Young diagram of N is:



When $h \geq k - 1$, every tableaux looks as follows;



Thus, every tableaux $\alpha \in A$ is in $\mathcal{S}$ and correspods to an irreducible component of $Gr^N(k,n)$. In this case, for each tableau π_1 is simply the number of ■s in the column of the hook. By 2.9, for $\alpha \in A$, $dim(Y_\alpha) = \pi_1(h - \pi_1 + 1)$. The dimension becomes maximal for the components with $\pi_1 \in \{\lfloor \frac{h+1}{2} \rfloor, \lceil \frac{h+1}{2} \rceil\}$. Since $Gr^N(k,n) = \bigcup_{\alpha \in A} \overline{Y_\alpha}$, $dim(Gr^N(k,n)) = \max_{\alpha \in A} dim(Y_\alpha) = \lceil \frac{h+1}{2} \rceil \cdot \lfloor \frac{h+1}{2} \rfloor$.

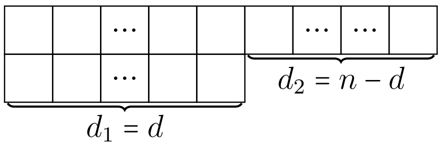
The only case when we do not get complete correspondence between the elements of A and the irreducible components is when $h+1 \leq k$. Then, the minimal component with respect to the dominance ordering looks as follows;

In this case $Y_{\alpha_{min}}$ is a single point and it is not contained in $\mathcal{S}$. By applying Lemma 3.16, we get that $Y_{\alpha_{min}} \subset \overline{Y_\beta}$, where β is the tableau with a single $\square$ in the column of the hook.

4. General Spaltenstein Variety

In this section, we will compute the irreducible components of the general Spaltenstein variety when the nilpotent N is a 2-block and a hook. Once again, we fix a positive integer n and a nilpotent transformation N of $\mathbb{K}^n$ of type λ, where λ is of the 2-block type in 4.1 and of the hook type in 4.2. Let $\mu = (\mu_1, ..., \mu_k)$ be a partition of n and denote by A, the set of minimal, semistandard μ-tableaux of type λ.

4.1. **The 2-block.** As in 3.18, when the nilpotent N has just 2 Jordan blocks, the Young diagram of N is:



Given $\alpha \in A$, by 2.12 we can compute the dimension of Y_α by counting the number of boxes in the second row of the tableau that have boxes numbered with strictly smaller integers directly above them.

Example 4.1.

1	1	2	3	3	4	4
1	2	2	4			

In this example, the dimension of the component corresponding to the above tableau is 2.

We can find the maximal possible dimension among the subvarieties Y_α for $\alpha \in A$ by following the procedure described next to obtain a tableau. We can start filling the Young diagram row by row with the numbers $i = 1, ..., k$ with multiplicity μ_i, and exhausting i before movinig to $i + 1$. There are two main cases for the resulting tableau:

(1) If $\mu_i \leq n - d$ for every i, then the resulting tableau looks as follows;

1	⋯	1	2	⋯	2	⋯	i	⋯	i
i	⋯	i	⋯	⋯	k	⋯	k		

Namely, every box of the second row has a number strictly greater above it. Thus, the dimension is obviously maximal and equal to d. Note that for any other tableau α whose columns look like $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$ with $i < j$, $dim(Y_\alpha) = d$.

(2) If $\mu_i > n - d \geq \lfloor \frac{n}{2} \rfloor$, for some i (note that this is only possible for a single value of i), then the resulting tableau looks as follows;

1	⋯	⋯	$i-1$	i	⋯	i	⋯	⋯	⋯	i
i	⋯	⋯	⋯	i	⋯	i	$i+1$	⋯	k	

Namely, there are exactly $\mu_i - (n - d)$ columns that have i on both the top and the bottom box. Thus, the dimension of this subvariety is

$$d - (\mu_i - (n - d)) = n - \mu_i < d.$$

We will call this tableau α_m. Note that because $\mu_i > n - d$ it is impossible to have fewer than $\mu_i - (n - d)$ columns with the same number in both boxes. Thus, this subvariety has maximal dimension and in fact it is the only one that achieves this bound.

Remark 4.2.

(1) Subvarieties Y_α that achieve the maximal dimension are guaranteed to be components since they cannot be contained in the closure of any other Y_β for $\beta \in A$. Our goal in this subsection is to prove that in the 2-block case, those subvarieties are percisely the components of $\mathcal{F}^N_\mu$.

(2) Knowing the dimension of the maximal components also gives us the dimension of $\mathcal{F}^N_\mu$. We have that

$$dim(\mathcal{F}^N_\mu) = \max_{\alpha \in A} dim(Y_\alpha) = min(d, n - m),$$

where $m = \max_{i \in \{1,\dots,k\}} \mu_i$.

Let $\alpha \in A$ be a tableau that has a column of the form $\begin{array}{|c|}\hline x \\ \hline x \\ \hline\end{array}$ for some $x \in \{1, \dots, k\}$. Then, α will look as follows:

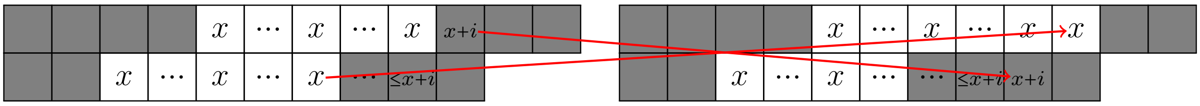
				x	⋯	x	⋯	x				
		x	⋯	x	⋯	x						

,

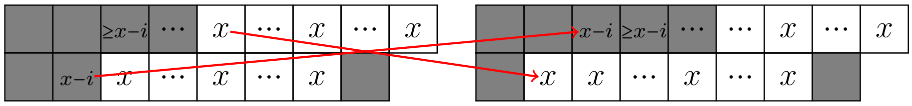
with the gray regions containing numbers strictly smaller or strictly greater than x respectively.

Consider, now the following moves;

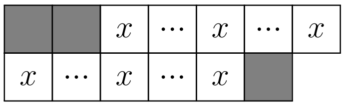
- If there are numbers to the right of x in the first row, take the first such number $x + i$ and exchange it with the rightmost x of the bottom row;



- If there are no numbers to the right of x in the first row, but there are numbers to the left of x in the bottom row, take the first such number $x-i$ and exchange it with the leftmost x of the top row;



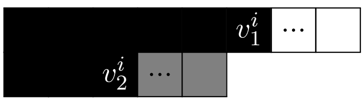
- If none of these moves are possible, then $\mu_x > n-d$ and $\alpha = \alpha_m$ is the tableau corresponding to the only maximal component described in case (2) above.



Remark 4.3. Every time we perform one of the moves described above, the dimension of the subvariety that corresponds to the tableau increases by 1. Thus we can repeatedly apply this procedure until we are unable to perform these moves. This will either mean that there are no columns of the form $\begin{smallmatrix} x \\ x \end{smallmatrix}$ left, or that we have reached the tableau α_m. In either case, the final tableau will correspond to a component with maximal dimension.

Lemma 4.4. *Let $\alpha \in A$ be a tableau that has a column of the form $\begin{smallmatrix} x \\ x \end{smallmatrix}$ for some $x \in \{1, ..., k\}$, such that $\alpha \neq \alpha_m$. Let β be the tableau obtained from α by performing the moves described above. Then, $Y_\alpha \subset \overline{Y_\beta}$.*

Proof. We will begin by noting the following; Let $\tau \in A$ be a tableau and let $F = (0 \subset V_1 \subset ... \subset V_k = \mathbb{K}^n) \in Y_\tau$. For some V_i such that $\tau^i = (\tau_1^i, \tau_2^i)$ has ■s in both rows of N, consider the starting vectors v_1^i and v_2^i of V_i.
Then, v_2^i is completely fixed and equal to the basis vector corresponding to its starting box, whereas v_1^i has a free variable for every box shaded in gray below. From this, it is easy to see that $KerN^{\tau_2^i} \subseteq V_i$, for any choice of free variables.

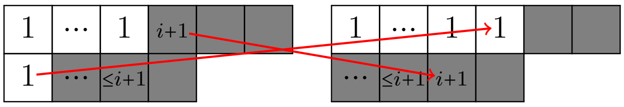
$\tau^i =$ 

Now if the first starting vector v_1^{i-1} of V_{i-1} is in $KerN^{\tau_2^i}$, then we can "decompose" F in two parts that do not interact with each other. Namely, $(0 \subset V_1 \subset ... \subset KerN^{\tau_2^i})$ and $(KerN^{\tau_2^i} \subseteq V_i \subset ... \subset V_k = \mathbb{K}^n)$. Let us now return to the main proof;

By our assumption, α always contains a column of the form $\begin{smallmatrix} x \\ x \end{smallmatrix}$ for some $x \in \{1, ..., k\}$. Thus, we can always "decompose" its flags in two parts. By construction,

β either contains a block of the form $\begin{array}{|c|c|}\hline \leq x & x \\ \hline \leq x & >x \\ \hline\end{array}$ or of the form $\begin{array}{|c|c|}\hline x< & \geq x \\ \hline x & \geq x \\ \hline\end{array}$. In either case, any flag of β is also decomposable. Now, let (V_i) and (W_i) be two flags in Y_α and Y_β respectively.

- Consider a move of the first type. In this case we are making alterations to the second part of the flag, so we can forget all the vector spaces before V_x and W_x and quotient by $KerN^{\beta_2^x} \subset KerN^{\alpha_2^x}$. We can also shift the numbers in every box by $x-1$. Then, the proof reduces to the following case:



In this case, we have that $\alpha_j = \beta_j$ for all $j > i$. Since in the first row of α, we have the adjacent boxes $\begin{array}{|c|c|}\hline 1 & i+1 \\ \hline\end{array}$, all the numbers between 2 and i can only appear in the second row and to the left of $i+1$. By construction, this is also true for β. This means that the starting box of the top row of α^j is the same for every $j < i$. Due to this, for $j < i$ the conditions imposed by the fact that $V_j \supset V_{j-1}$, give the following structure to the flag;

The free variables of the starting vector v_1^j of V_j, are equal to the free variables of the starting vector v_1^{j-1} of V_{j-1} that are to the right of the second starting box of V_{j-1}. The same property also holds for the W_j. Let $p = (1, n-d-\alpha_1^1)$ be the position of the first starting box of V_1 and q be the position of the box below it. Suppose that the starting vector

$$v_1^1 = e_p + \sum_{j=q}^{(2,d-1)} a_j e_j.$$

Now, the first starting vector of W_1 is of the form

$$w_1^1 = e_{p-(0,1)} + \sum_{j=q-(0,1)}^{(2,d)} b_j e_j,$$

for some free variables b_j. If we set $b_j = -ca_j - cb_{j-1}$ and $b_q = -ca_q$, then as $c \to \infty$, we get that

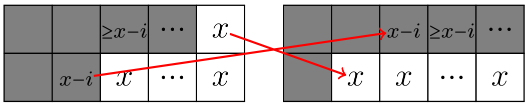
$$Nw_1^1 + \frac{1}{c} w_1^1 \to v_1^1 \text{ and } w_1^1 \to v_2^1.$$

Moreover, by the property discussed above, it is easy to see that for every $j \leq i$ we also get that

$$Nw_1^j + \frac{1}{c} w_1^j \to v_1^j \text{ and } w_1^j \to v_2^j.$$

Since we have that $\alpha_j = \beta_j$ for all $j > i$, the conditions imposed on V_i and W_i by the next subspaces are the same. Thus, we can reach every point of Y_α by taking such a limit. So, $Y_\alpha \subset \overline{Y_\beta}$.

- Consider a move of the second type. In this case we are making alterations to the first part of the flag, so we can forget all the vector spaces after V_x and W_x and replace V_x and W_x with $KerN^{\beta_1^{x-1}}$



In this case, since in the second row of α, we have the adjacent boxes $\boxed{x-i}\boxed{x}$, all the numbers between $x-i+1$ and x can only appear in the first row and to the right of $x-i$. By construction, this is also true for β. This time it is the starting box of the bottom row of α^j that remains the same for every $j \geq x-i$. Due to this, for $j > x-i$ the conditions imposed by the fact that $V_j \in V_{j+1}$, give the following structure to the flag;

Suppose that the starting vector v_1^j has y free variables. Then, those free variables are equal to the y rightmost (with respect to the position of their corresponding boxes in N) free variables of the starting vector v_1^{j+1}. This is also true for the free variables of the vectors w_1^j.

We will now try to prove the inclusion by working backwards. Let r be the position of the leftmost box in the bottom row that contains an x. Suppose that the starting vector

$$v_1^{x-1} = e_{1,2} + \sum_{j=2,2}^{r} a_j e_j.$$

Now, the first starting vector of W_{x-1} is of the form

$$w_1^{x-1} = e_{1,1} + \sum_{j=2,1}^{r} b_j e_j,$$

for some free variables b_j. If we set $b_j = -ca_j - cb_{j-1}$ and $b_{2,2} = -ca_{2,2}$, then as $c \to \infty$, we get that

$$Nw_1^{x-1} + \frac{1}{c} w_1^{x-1} \to v_1^{x-1} \text{ and } w_1^{x-1} \to v_2^{x-1}.$$

By the property discussed above, the leftmost free variable of w_1^j for $j \geq x-i$ will always contain the highest power of c in it. Since the position of the leftmost free variable is constant for $j \geq x-i$, we will always have that as $c \to \infty$, $w_1^j \to v_2^j$. It is also easy to see that $Nw_1^j + \frac{1}{c} w_1^j \to v_1^j$. Since we have that $\alpha_j = \beta_j$ for all $j < x-i$, the conditions imposed on V_{x-i} and W_{x-i} by the previous subspaces are the same. Thus, we can reach every point of Y_α by taking such a limit. So, $Y_\alpha \subset \overline{Y_\beta}$.

□

We can now prove the main result of this subsection;

Proposition 4.5. *The irreducible components of $\mathcal{F}_\mu^N$ when N is a 2-block are equidimensional.*

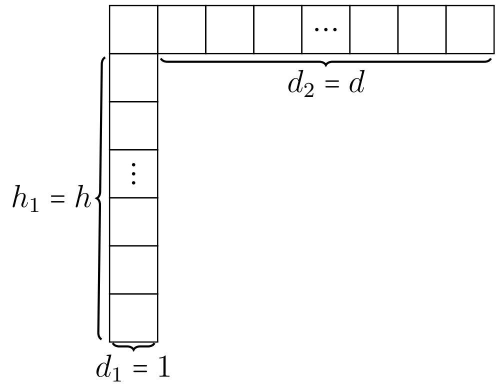
(1) *When* $m = \max_{i \in \{1,\ldots,k\}} \mu_i \leq n - d$, *they are of the form* $\overline{Y_\alpha}$ *for tableaux* $\alpha \in A$ *whose columns are of the form* $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$ *with* $i < j$.

(2) *When* $m > n - d$, $\mathcal{F}_\mu^N = \overline{Y_{\alpha_m}}$ *is an irreducible subvariety of* $\mathcal{F}_\mu$.

Proof. By remark 4.2 we konw that the subvarieties $\overline{Y_\alpha}$ with maximal dimension correspond to components of $\mathcal{F}_\mu^N$. Now, we can repeatedly apply lemma 4.4 to any tableau $\beta \in A$ until we reach one of the tableau corresponding to such a component $\overline{Y_\alpha}$. We will then have that $Y_\beta \subset \overline{Y_\alpha}$. This completes the proof.

□

4.2. **The hook.** As in 3.19, we have that the Young diagram of N is:



In this case, in order to understand the inclusion relations that may occur between the subvarieties Y_α for $\alpha \in A$, we will look at the maps:

$$\phi_i : \mathcal{F}_\mu^N \to Gr^N(M_i, n)$$

$$(\{0\} \subset V_1 \subset \ldots \subset V_k = \mathbb{K}^n) \mapsto (V_i \subset V_k = \mathbb{K}^n),$$

where $M_i = \sum_{j=1}^{i} \mu_j$. In particular, we want to understand when $\phi_i(Y_\alpha) = Y_{\alpha^i}$.

Let f be the map defined in section 3. We can let f act on the sets $A^i = \{\alpha^i | \alpha \in A\}$. In the case of the hook, we have that $f = (f_1, f_2)$, where f_1 is the number of $\blacksquare$s of α^i in the first column of the hook and $f_2 = M_i - f_1$. From this it is apparent that f in this case is an injection.

In this setting, the following lemma holds:

Lemma 4.6. *Let* $\alpha \in A$ *and* $i \in \{1, \ldots, k-1\}$. *If* $f_1(\alpha^{i+1}) > f_1(\alpha^i)$, *then* $\phi_i(Y_\alpha) = Y_{\alpha^i}$.

Proof. For simplicity, we will drop the double indexing for the Jordan basis of N in the rows that contain a single box. Out basis will become $(e_{11}, \ldots, e_{1(d+1)}, e_2, \ldots, e_{h+1})$.

Let $p_1 = f_1(\alpha^i)$, $p_2 = f_2(\alpha^i)$, $q_1 = f_1(\alpha^{i+1})$ and $q_2 = f_2(\alpha^{i+1})$. Also, let $\{v_j\}, j = 1, \ldots, p_1$ be the starting vectors of V_i and $\{w_j\}, j = 1, \ldots, q_1$ be the starting vectors of V_{i+1}. Then, we have that:

$v_1 = e_{1(d-p_2)} + \sum_{j=p_1+1}^{h+1} a_j^1 e_j$ and $v_y = e_y + \sum_{j=p_1+1}^{h+1} a_j^y e_j$ for $y > 1$ and similarly,
$w_1 = e_{1(d-q_2)} + \sum_{j=q_1+1}^{h+1} b_j^1 e_j$ and $w_y = e_y + \sum_{j=q_1+1}^{h+1} b_j^y e_j$ for $y > 1$.

The condition that $V_i \subset V_{i+1}$ allows the a_j^y to be free for $j = p_1 + 1, ..., q_1$ and forces $a_j^y = \sum_{k=p_1+1}^{q_1} a_k^y b_j^k$ for $j = q_1 + 1, ..., h + 1$.

If $i > 1$ we can let $V_{i-1} = \langle e_{1(d-s_2)}, v_2, ..., v_{s_1} \rangle$, where $s_1 = f_1(\alpha^{i-1})$, $s_2 = f_2(\alpha^{i-1})$. This way, the condition $V_{i-1} \subset V_i$ does not impose any restrictions on the a_j^y.

By using the freedom of the a_k^y and inductively assuming that the b_j^k can take any value for $k = p_1 + 1, ..., q_1$ and $j = q_1 + 1, ..., h + 1$, we can hit any point of Y_{α^i} with ϕ.

Note that as explained in the remark below, the only free variables of a subspace V_j that can potentially be set to zero identically are those of the first starting vector of α_j. Since the b_j^k for $k = p_1 + 1, ..., q_1$ and $j = q_1 + 1, ..., h + 1$ do not belong to the first starting vector, the inductive assumption is sound. □

Remark 4.7. Let us now explain the necessity of the condition $f_1(\alpha^{i+1}) > f_1(\alpha^i)$. If we did not have $q_1 > p_1$, then the condition $V_i \subset V_{i+1}$ would have forced $v_1 = e_{1(d-p_1)}$ and $v_j = w_j$ for $j > 1$, so we would not be able to hit every point of Y_{α^i} with ϕ_i.

Lemma 4.8. *Let α and β in A be two tableaux.*

1. *If for some $i \in \{1, ..., k-1\}$, we have that $|f_1(\alpha^i) - f_1(\beta^i)| \geq 2$, then $Y_\alpha \notin \overline{Y_\beta}$ and $Y_\beta \notin \overline{Y_\alpha}$.*
2. *If for some $i \in \{1, ..., k-1\}$, we have that $f_1(\alpha^i) - f_1(\beta^i) = -1$, then $Y_\alpha \notin \overline{Y_\beta}$*
3. *If for some $i \in \{1, ..., k-1\}$, we have that $f_1(\alpha^i) - f_1(\beta^i) = 1$ and $f_1(\alpha^{i+1}) > f_1(\alpha^i)$, then $Y_\alpha \notin \overline{Y_\beta}$.*

Proof. Let $p_1 = f_1(\alpha^i)$, $p_2 = f_2(\alpha^i)$, $q_1 = f_1(\beta^i)$ and $q_2 = f_2(\beta^i)$. Let $\{v_j\}, j = 1, ..., p_1$ be the starting vectors of $V_{\alpha,i}$ and $\{w_j\}, j = 1, ..., q_1$ be the starting vectors of $V_{\beta,i}$.

1. Wlog suppose that $p_1 - q_1 \geq 2$. Then, also $q_2 - 2 \geq p_2$.
   We have that $w_1 = e_{1(d-q_2)} + \sum_{j=q_1+1}^{h+1} b_j^1 e_j$. So,
   $$Nw_1 = e_{1(d-q_2+1)} \notin KerN^{q_2-2} \supseteq KerN^{p_2} \supseteq V_{\alpha,i}.$$
   Namely, $\phi_i(Y_\alpha)$ and $\phi_i(Y_\beta)$ land in disjoint components of $Gr^N(M_i, n)$. Thus, no inclusion can occur between Y_α and Y_β.
2. If $p_1 - q_1 = -1$, then $p_2 = q_2 + 1$. We can let $v_1 = e_{1(d-p_2)}$, because the flag where all the free variables of all subspaces are set to zero is always a point of Y_α. Then, $v_1 \notin KerN^{p_2-1} = KerN^{q_2} \supseteq V_{\beta,i}$. Namely, there exist points in $\phi_i(Y_\alpha)$ that do not land in $\phi_i(Y_\beta)$. Thus, $Y_\alpha \notin \overline{Y_\beta}$.
3. Since $f_1(\alpha^{i+1}) > f_1(\alpha^i)$, we have by 4.6 that $\phi_i(Y_\alpha) = Y_\alpha^i$. So, there exists a flag in Y_α with $v_1 = e_{1(d-p_2)} + c \cdot e_{h+1}$ for an arbitrary constant c. If we let $c \to \infty$, we will have $v_1 = e_{h+1}$ and thus $V_{\alpha,i} \subseteq KerN^{p_2-1} \not\supseteq V_{\beta,i}$ for any choice of $V_{\beta,i}$. Namely, there exist points in $\phi_i(Y_\alpha)$ that do not land in $\phi_i(Y_\beta)$. Thus, $Y_\alpha \notin \overline{Y_\beta}$.

□

Remark 4.9. With the help of 4.8, we have shown that to get an inclusion relation of the form $Y_\alpha \subset \overline{Y_\beta}$ for α and β in A it is necessary to have that for every $i \in \{1, \ldots, k-1\}$ for which $\alpha_i \neq \beta_i$:

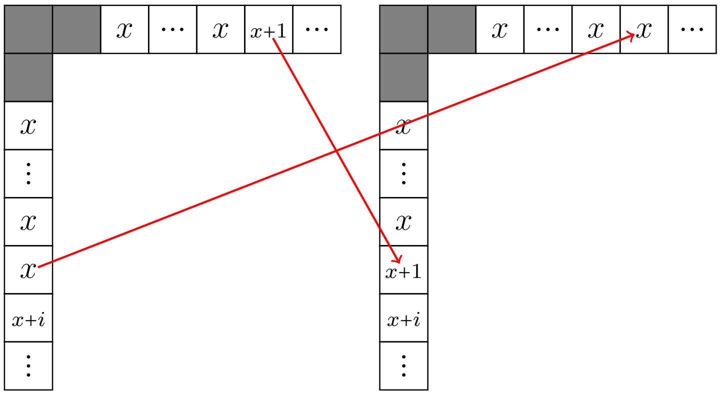
(1) $f_1(\alpha^i) - f_1(\beta^i) = 1$.
(2) $f_1(\alpha^{i+1}) = f_1(\alpha^i)$.

The following construction will investigate the sufficient conditions, and will thus complete the classification of the irreducible components in the hook case.

Let $\alpha \in A$ be a tebleau such that for some $x \in \{1, \ldots, k-1\}$, we have that:

$$\begin{cases} f_1(\alpha^x) - f_1(\alpha^{x-1}) \geq 2 \\ f_1(\alpha^x) = f_1(\alpha^{x+1}) = \ldots = f_1(\alpha^{x+i-1}), \text{ for some } i > 1 \\ f_1(\alpha^{x+i}) > f_1(\alpha^x) \end{cases}$$

This means that α looks as follows;



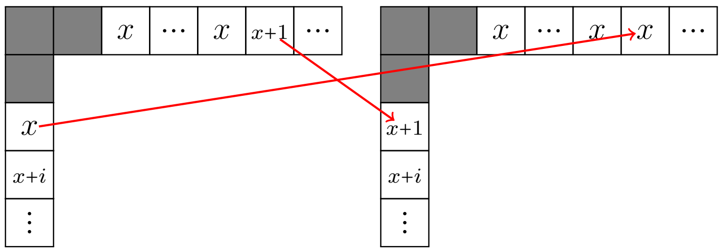
Now, consider the move that swaps the last x of the first column with the leftmost $x+1$ of the first row. Since no $(x+1)$s appear in the column, there must exist at least one in the first row. Call the resulting tableau β.

Lemma 4.10. *Let α and β be the tableaux described above. Then, $Y_\alpha \subset \overline{Y_\beta}$.*

Proof. We have that $\alpha^j = \beta^j$ for every $j \neq x$. Let us now look at α^x and β^x. Let $p_1 = f_1(\alpha^x)$, $p_2 = f_2(\alpha^x)$, $q_1 = f_1(\beta^x)$ and $q_2 = f_2(\beta^x)$. Also, let $\{v_j\}, j = 1, \ldots, p_1$ be the starting vectors of $V_{\alpha,x}$ and $\{w_j\}, j = 1, \ldots, q_1$ be the starting vectors of $V_{\beta,x}$. As discussed in 4.7, we have that $v_1 = e_{1(d-p_1)} = Nw_1$. Moreover, we can certainly set $w_j = v_j$ for $j = 2, \ldots, q_1$, since w_j has a free variable in every box of N for which v_j has a free variable. We can now use $w_1 = e_{1(d-q_2)} + \sum_{j=q_1+1}^{h+1} b_j^1 e_j$ to obtain $v_{p_1} = e_{p_1} + \sum_{j=p_1+1}^{h+1} a_j^{p_1} e_j$ by taking a limit at infinity. To do this, we set $b_j^1 = a_j^{p_1} b_{p_1}^1$ for $j = p_1 + 1, \ldots, h+1$ and let $b_{p_1}^1 \to \infty$. Letting the $b_j^1 \to \infty$ does not have any effect on the vector spaces $V_{\beta,i}$ for $i < x$, as they are not involved in any linear combination of the free variables of $V_{\beta,i}$. The conditions imposed on any subsequent vectorspace $V_{\beta,i}$ for $i > x$ are identical to those imposed by $V_{\alpha,x}$ on $V_{\alpha,i}$. Thus, every point of Y_α can be obtained as "a point at infinity" of $\overline{Y_\beta}$. Hence the lemma.

□

If we did not have the condidion $f_1(\alpha^x) - f_1(\alpha^{x-1}) \geq 2$, then this swap would look as follows:



This would result in $f_1(\beta^x) - f_1(\beta^{x-1}) = 0$, which would make $\phi_{i-1}(Y_\beta) \subsetneq Y_{\beta^{i-1}}$. However, we still have that $\phi_{i-1}(Y_\alpha) = Y_{\alpha^{i-1}}$, so we would not get the inclusion $Y_\alpha \subset \overline{Y_\beta}$.

The exact same problem would arise if we atempted to make the swap between x and $x+j$ for any $j \in \{1, ..., i-1\}$. Note that the fact that $f_1(\alpha^{x+i}) > f_1(\alpha^{x+i-1})$ forces us to only consider swaps with the numbers $\{1, ..., i-1\}$. Attempting to swap with $x+i$ would deffinitely not result in an inclusion by 4.8.3. Thus, none of the possible candidates based on 4.9 result in an inclusion in this case. This leads us to the following conclusion;

Corollary 4.11. *Let $\mathcal{S}_1 \subseteq A$ be the set of tableaux that satisfy the property that for every "gap" $\begin{array}{|c|}\hline x \\ \hline x+i \\ \hline \end{array}$ for $i > 1$ that occurs in the first column of N, $f_1(\alpha^x) - f_1(\alpha^{x-1}) = 1$. Then, for every $\alpha \in \mathcal{S}_1$, $\overline{Y_\alpha}$ is a component of $\mathcal{F}_\mu^N$.*

Proof. By the discussion above, we know that wherever such a "gap" occurs, none of the possible candidates β based on 4.9 result in an inclusion $Y_\alpha \subset \overline{Y_\beta}$. If all the gaps of α are of this form, then Y_α cannot be in the closure of any other component. Thus, $\overline{Y_\alpha}$ is a component of $\mathcal{F}_\mu^N$. □

With this corrolary, we are now ready to state the full result for the hook case.

First, let $\mathcal{S}_2 \subseteq A$ be the set of tableaux that satisfy the property that every $x \in \{1, ..., k\}$ appears in the first column of N. Note that $\mathcal{S}_2$ could be the empty set.

Proposition 4.12. *The irreducible components of $\mathcal{F}_\mu^N$ when N is of hook type are of the form $\overline{Y_\alpha}$ for $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2$.*

Proof. We already know that the tableaux in $\mathcal{S}_1$ correspond to components. By 4.9, we know that in order for an inclusion relation to occur, it is necessary to have a "gap" at some point in the first column of N. This is obviously not satisfied by the tableaux in $\mathcal{S}_2$. Thus, they also correspond to components.

For any other tableau in $A \smallsetminus \mathcal{S}_1 \cup \mathcal{S}_2$, we can apply the swap described above at every point of its first column where a "gap" occurs until we reach a tableau in $\mathcal{S}_1$. Thus, we get the desired inclusions.

□

Finally, we want to discuss the dimension of $\mathcal{F}_\mu^N$. Fix a tableau $\alpha \in A$. Let y_i be the multiplicity of the number i in the first column of N. The y_i must satisfy the following conditions;

$$(*) \begin{cases} \sum_{i=1}^{k} y_i = h+1 \\ 0 \leq y_i \leq \mu_i, \text{ for } i = 1, ..., k. \end{cases}$$

This is a set of convex constraints. By 2.12, we have that

$$d_\alpha = dim(Y_\alpha) = \sum_{i=1}^{k} y_i \sum_{j=1}^{i-1} y_j = \sum_{i>j} y_i y_j = \frac{1}{2}[(h+1)^2 - \sum_{i=1}^{k} y_i^2].$$

In order to find the dimension of $\mathcal{F}_\mu^N$, we want to find the component Y_α for which d_α is maximal. This is equivalent to minimizing the convex function $\sum_{i=1}^{k} y_i^2$ under the constraints $(*)$.

To find the minimum, it is enough to note that due to the convexity, if for some $y_i < y_j$ we replace y_i and y_j by $y_i + c$ and $y_j - c$ for some $0 < c < \frac{y_i - y_j}{2}$, then $\sum_{i=1}^{k} y_i^2$ decreases. Thus, our goal is to move the y_i "as close as possible". This is achieved by setting;

$$\begin{cases} y_i = \mu_i \text{ if } \mu_i \leq M \\ y_i \in \{\lfloor \frac{m - \sum_{i=1}^{k} a_i}{n-k} \rfloor, \lceil \frac{m - \sum_{i=1}^{k} a_i}{n-k} \rceil\} \text{ if } \mu_i > M \end{cases}$$

for the minimum M for which these equalities do not vionate $(*)$.

Following this process gives us the desired α for which d_α is maximal. Then, we have that $\mathcal{F}_\mu^N = d_\alpha$.

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