Derived Functors for Hom and Tensor Product: The Wrong Way to do It

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Abstract

In this paper we study the properties of the *wrong* derived functors LHom and $\overset{R}{\otimes}$. We will prove identities that relate these functors to the classical Ext and Tor. With these results we will also prove that the functors LHom and $\overset{R}{\otimes}$ form an adjoint pair. Finally we will give some explicit examples of these functors using spectral sequences that relate them to Ext and Tor, and also show some vanishing theorems over some rings.

1 Introduction

In this paper we will discuss derived functors. Derived functors have been used in homological algebra as a tool to understand the lack of exactness of some important functors; two important examples are the derived functors of the functors Hom and Tensor Product ($\otimes$). Their well known derived functors, whose cohomology groups are Ext and Tor, are their right and left derived functors respectively.

In this paper we will work in the category R-mod of a commutative ring R (although most results are also true for non-commutative rings). In this category there are different ways to think of these derived functors. We will mainly focus in two interpretations. First, there is a way to concretely construct the groups that make a derived functor as a (co)homology. To do this we need to work in a category that has enough injectives or projectives, R-mod has both. The second way is a categorical construction that defines the derived functors as left or right Kan Extension for homotopy categories. To see that these definitions agree see [1].

Now we present some of the reasons why people are mostly interested on the left (right) derived functor LF (RF) of a right (left) exact functor F; there is a result that shows the equality of functors

$$L_0F = F \quad (R^0F = F)$$

this result is equivalent to saying that the left (right) derived functor will give information related to the exactness of the functor F. However, in this paper we focus on giving an insight on the properties that the left (right) derived functor LF (RF) of a left (right) exact functor F has.

In this paper we will study derived functors that are not as natural, we study the explicit examples of the functors LHom, *LHom and $\overset{R}{\otimes}$, the left and right derived functors of the functors $\mathrm{Hom}(A, -)$, $\mathrm{Hom}(-, A)$ and $- \otimes A$ respectively. To study these functors we prove some identities that will relate them to the traditional RHom and $\overset{L}{\otimes}$. One of these results, surprisingly, shows that $\mathrm{LHom}(M, -) \dashv -\overset{R}{\otimes} M$ is an adjoint pair whenever M is a finitely generated module.

To summarize some of the similarities and differences between the classical derived functors of Hom and $\otimes$ and the derived functors studied in this paper we present the following table:

	RHom	LHom	*LHom	$\overset{L}{\otimes}$	$\overset{R}{\otimes}$
Type of Resolution	Projective or Injective	Projective	Injective	Flat	Injective
Type of Kan Extension	Left	Right	Right	Right	Left

2 Statement of Results

The main results in this paper are the following three theorems. These relate the *wrong* derived functors we study in this paper, LHom and $\overset{R}{\otimes}$ to the functors Tor and Ext, when evaluated at some dual modules. In particular, we will look at the dual module $M^\vee = \mathrm{Hom}(M, R)$, and also consider for an injective cogenerator E the modules $M^+ = \mathrm{Hom}(M, E)$ and $M^e = \mathrm{Hom}(E, M)$. With this notation, we have the following results:

Theorem 2.1. *For a finitely presented module M and any module N, we have the following isomorphism in the derived category of R-mod:*

$$LHom(M, N) \simeq M^\vee \overset{L}{\otimes} N.$$

Theorem 2.2. *For a finitely presented module M and any module N, we have the following isomorphism in the derived category of R-mod:*

$$M \overset{R}{\otimes} N \simeq RHom(M^\vee, N).$$

From Theorem 2.1 (6.1) and Theorem 2.2 (6.2) we get the following corollary:

Corollary 2.1. *For a finitely generated module M, the functors*

$$LHom(M, -) \dashv - \overset{R}{\otimes} M,$$

form an adjoint pair.

We do not know of any reference where these functors have been studied explicitly. We only know of general results for derived functors and these theorems give more explicit properties for the functors LHom and $\overset{R}{\otimes}$ that resemble properties of the functors RHom and $\overset{L}{\otimes}$; the most surprising of these results being that the *wrong* derived functors form an adjoint pair.

These results may suggest that there may be other interesting properties for the *wrong* derived functors of other well-known functors, particularly that they might form an adjoint pair. In the case of $\otimes$ and Hom it was necessary that the module was finitely presented–which can be related to some conditions of commutativity with certain colimits. If there is a way to extend these results to general adjoint pairs it cannot be done following the proofs of the theorems above as they are dependent on properties of Hom and $\otimes$.

In the next section we show a set of three spectral sequences, one for each of LHom, *LHom and $\overset{R}{\otimes}$, that converges to the modules $\mathrm{Ext}(M, N)$, $\mathrm{Ext}(M, N)$ and $M \otimes N$ respectively. We get this result by one of the possible filtrations; while in the other filtration we get a different E_2 page which contains the functors LHom, *LHom and $\overset{R}{\otimes}$. We give as an illustrative example the E_2 page for LHom.

Proposition 2.1. *Let M, N be two R modules. Then the spectral sequence that comes from a sign change of the double complex $Hom(P_M, Q_N)$ where $P_M := \cdots P_2 \to P_1 \to P_0 \to 0$ and $Q_N := \cdots Q_2 \to Q_1 \to Q_0 \to 0$ are projective resolutions of M and N respectively, converges to $E^{\infty}_{p,0} = Ext^p(M,N)$ under one of the filtrations and under the other filtration we get the following E_2 page:*

$$\begin{array}{cccc}
LHom^0(M,N) & L_0Ext^1(M,N) & L_0Ext^2(M,N) & \cdots \\
LHom^1(M,N) & L_1Ext^1(M,N) & L_1Ext^2(M,N) & \cdots \\
LHom^2(M,N) & L_2Ext^1(M,N) & L_2Ext^2(M,N) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Furthermore, if M is finitely generated and R is noetherian we can use that $Hom(M,N) \simeq M^{\vee} \otimes N$ and get the E_2 page:

$$\begin{array}{cccc}
Tor_0(Ext^0(M,R),N) & Tor_0(Ext^1(M,R),N) & Tor_0(Ext^2(M,R),N) & \cdots \\
Tor_1(Ext^0(M,R),N) & Tor_1(Ext^1(M,R),N) & Tor_1(Ext^2(M,R),N) & \cdots \\
Tor_2(Ext^0(M,R),N) & Tor_2(Ext^1(M,R),N) & Tor_2(Ext^2(M,R),N) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

This result shows that the *wrong* derived functors are related to the classical derived functors and that we do not need any assumption as finite generation as it is needed for theorems 2.1 and 2.2. The second part of the result is related to theorems 2.1 and 2.2 as we consider the case of M finitely generated and use the same lemmas to get another result.

The next section is motivated by the idea that we would wish that the derived functor *LHom had similar properties as those we have proved for LHom and $\overset{R}{\otimes}$. However, this functor is more related to the injective cogenerator instead of the projective generator (as R); unfortunately, the injective cogenerator E does not behave that well in this context and for our results we are forced to work with modules E that satisfy special conditions.

Theorem 2.3. *For any module M and N finitely copresented by E, an injective cogenerator with endomorphism ring $End(E) = S$, we have the following isomorphism in the* *derived category of R-mod:*

$$^{*}LHom(M,N) \simeq M^{+} \overset{L}{\underset{S}{\otimes}} N^{e}.$$

Finally, in sections 9 and 10 we work to understand how these functors behave in some particular rings. In section 9 we show that the higher (co)homology groups of the *wrong* derived functors vanish for rings of projective (or injective) dimension 2 or lower. In contrast, in section 10 we prove that we can find examples where the (co)homology groups do not vanish. We prove this fact with an explicit construction using regular local rings of dimension 3 or higher.

Theorem 2.4. *Let R be a regular local ring of dimension $d \geq 3$ and maximal ideal $\mathfrak{m}$, let $k := R/\mathfrak{m}$ be its residue field. Then for any integer $3 \leq \ell \leq d-1$ there exists a module N such that:*

$$N \overset{R}{\otimes} R \simeq k[-\ell]$$

in the derived category.

We see that the *wrong* derived functors also have some interesting properties and can be, in some conditions, related to the derived functions that are already studied. The results proved here cannot be straightforwardly extended to general derived functors, but we hope it can give a little more understanding of this subject.

3 Acknowledgements

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4 Notation and Conventions

In this paper we will use the following notation and conventions. The ring of interest will be a commutative ring R and all functors we consider will be functors from R-mod to R-mod unless specified otherwise. We will omit the subindex R in functors $\mathrm{Hom}_R(A,B)$ or $\otimes_R$ and so we will only write $\mathrm{Hom}(A,B)$ or $\otimes$ respectively. If it is based in another ring it will be specified.

Another module that we will use is an injective cogenerator E of R-mod. There are many ways to find an injective cogenerator an example is taking the character module $R^{+} = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$, this result comes from the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of $\mathbb{Z}$-mod. As a note, we add that in section 8 *Injective Cogenerator* we will assume that the ring $S = \mathrm{End}(E)$ is noetherian or that E is a Morita duality module.

There will be three *dual* modules that we use for a module M. First, we take

$$M^\vee := \mathrm{Hom}(M, R).$$

Motivated from the notion of character module we write

$$M^+ := \mathrm{Hom}(M, E),$$

and finally we let

$$M^e := \mathrm{Hom}(E, M).$$

In the paper we will define the derived functors LHom, *LHom and $\overset{R}{\otimes}$, which are left and right derived functors of $\mathrm{Hom}(M, -)$, $\mathrm{Hom}(-, M)$ and $- \otimes M$ respectively. So if we let P_N and I_M be projective and injective resolutions of N and M respectively, we could describe these functors as:

$$\mathrm{LHom}(M, N) \simeq \underline{\mathrm{Hom}}(M, P_N) \qquad \underline{\mathrm{Hom}} \text{ stands for Hom of chain complexes}$$
$$^*\mathrm{LHom}(M, N) \simeq \underline{\mathrm{Hom}}(I_M, N)$$
$$M \overset{R}{\otimes} N \simeq I_M \otimes N \qquad \text{for tensor products of chain complexes}$$

This description is explained in [2]. We will add that for the i^{th} (co)homology group of the derived functors we will use the notation:

$$\mathrm{L}_i\mathrm{Hom}(M, N) \simeq H_i(\mathrm{Hom}(M, P_N))$$
$$^*\mathrm{L}_i\mathrm{Hom}(M, N) \simeq H_i(\mathrm{Hom}(I_M, N))$$
$$M \overset{R^i}{\otimes} N \simeq H^i(I_M \otimes N)$$

In the case that our category has enough projectives and injectives, all of these functors are indeed well defined.

5 Preliminaries

In this section we give a small review about derived functors, mainly focusing on how to construct them and some basic properties that they have. For a more extensive coverage of the topic we refer to [2], [3].

To construct derived functors we need the notion of an injective or projective resolution. For an object A in a category $\mathcal{C}$ an injective resolution $I_\bullet$ consists of an exact sequence

$$0 \to A \to I_0 \to I_1 \to ...$$

where $I_i \in \mathcal{C}$ are injective objects. A projective resolution $P_\bullet$ is the dual construction made by taking projective objects. The resolutions I_A and P_A are the resolutions $I_\bullet$ and $P_\bullet$ where we delete A, so for example

$$I_A = 0 \to I_0 \to I_1 ...$$

Definition 5.1 (Derived Functor (Homology))**.** For any covariant functor $F : \mathcal{C} \to \mathcal{D}$, any object $A \in C$ and a (projective) injective resolution (P_A) I_A, we define the (right) left derived functor, (RF) LF, as the (co)homology of FI so that:

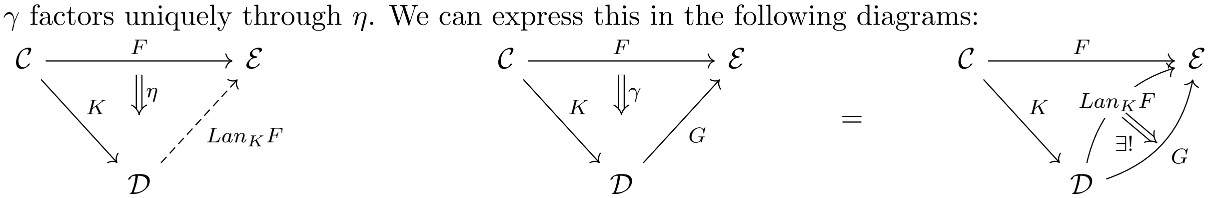
$$RF(A) = H(F(P_A))$$

$$LF(A) = H(F(I_A)).$$

Similarly, we can define the derived functors for contravariant functors. In the definition it is not obvious that the functor RF is well defined, but we refer to [2, Theorem 6.16] for a proof of this fact.

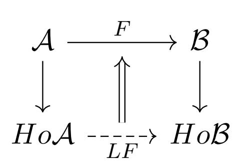
Now, we give a categorical construction of derived functors using Kan Extensions. We use the definition given in Riehl [4, Chapter 6].

Definition 5.2 (Kan Extension)**.** Given functors $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$, we define the the left Kan Extension as a functor $Lan_K F : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta : F \Rightarrow Lan_K F \circ K$ such that for any other pair $(G : \mathcal{D} \to \mathcal{E}, \gamma : F \to G \circ K)$, γ factors uniquely through η. We can express this in the following diagrams:



Let us write $K(\mathcal{C})$ as the category of chain complexes of the category $\mathcal{C}$. Then we will have that homotopically equivalent chain complexes form a weak equivalence in the category $K(C)$. To define a derived functor we need to talk about homotopy categories. For a category $\mathcal{A}$ together with a set of weak equivalences (or quasi-isomorphisms) we define its homotopy category $Ho\mathcal{A}$ as the initial object of categories $\mathcal{B}$ that have a homotopy map $\mathcal{A} \to \mathcal{B}$ such that weak equivalences in $\mathcal{A}$ are isomorphisms in $\mathcal{B}$. So if we consider $K(\mathcal{C})$ then the derived category $D(\mathcal{C})$ is equivalent to the homotopy category $Ho\mathcal{C}$. Now we can give a categorical definition of a derived functor.

Definition 5.3 (Derived Functor [4])**.** For a functor $F : \mathcal{A} \to \mathcal{B}$ of homotopical categories, the left derived functor, LF, is defined as the right Kan Extension of the following diagram (if it exists):



Similarly we can define the right derived functor as the right Kan Extension. And for the case of chain complexes we would have that the derived functor is a functor from $D(\mathcal{C})$ to $D(\mathcal{D})$.

Now, we look at an important result that tells us why we are mostly interested in the left derived functor of a right exact functor (analogously, why we care for the right derived functor of a left exact functor). This is can be found in [2, Theorem 6.29].

Theorem 5.1. *For a right exact additive covariant functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories, and if $\mathcal{C}$ has enough projectives, then F is naturally isomorphic to L_0F.*

This theorem is relevant to the setting we are working in the category of modules of a commutative ring, which is an abelian category and has enough projectives. However, Quillen adjunction tells us that for a pair of adjoint functors $F \dashv G$ their derived functors are an adjoint pair as well.

Theorem 5.2. *Let $F \dashv G$ be a pair of adjoint functors between homotopical categories, assuming that LF and RG exist and are absolute Kan extensions, then $LF \dashv RG$ is an adjoint pair of functors between homotopical categories.*

6 Describing the Wrong Derived Functors

In this section we prove formulas that relate LHom and $\overset{R}{\otimes}$ with Tor and Ext, respectively. For this we will need some finite presentation assumptions on the modules we use. Finally, we prove that LHom and $\overset{R}{\otimes}$ form an adjoint pair of functors from R-mod to R-mod.

We begin by proving a lemma that will show that $\mathrm{Hom}(M, N)$ is isomorphic to $M^\vee \otimes N$ in some special cases. This will be useful when considering the homology calculated for our derived functor LHom.

Lemma 6.1. *For modules M and N over a ring R we have that the map*

$$\underset{(f(-)\otimes n)\mapsto(nf(-))}{M^\vee \otimes N \xrightarrow{\psi} Hom(M, N).}$$

is a natural map of bifunctors.

Moreover, ψ is an isomorphism if M is finitely presented and N is a flat module.

Proof. If we consider maps $\varphi : M \to M'$ and $\phi : N \to N'$ we can see that the naturality of ψ follows of a simple diagram chase in the following diagrams:

$$\begin{array}{ccc} \underset{(f\circ\varphi)\otimes n)}{M^\vee \otimes N} & \longrightarrow & \underset{(n(f\circ\varphi))}{\mathrm{Hom}(M, N)} \\ \uparrow & & \uparrow \\ \underset{(f\otimes n)}{(M')^\vee \otimes N} & \longrightarrow & \underset{(nf)}{\mathrm{Hom}(M', N)} \end{array} \qquad \begin{array}{ccc} \underset{(f\otimes n)}{M^\vee \otimes N} & \longrightarrow & \underset{(nf)}{\mathrm{Hom}(M, N)} \\ \downarrow & & \downarrow \\ \underset{(f\otimes\phi(n))}{M^\vee \otimes N'} & \longrightarrow & \underset{(\phi(nf))}{\mathrm{Hom}(M, N')} \end{array}$$

Now we prove that ψ is an isomorphism when M is finitely presented and N is a flat module.

Since M is finitely presented, we can get the following exact sequences:

$$\begin{array}{rrrrr} R^{\oplus n} \to & R^{\oplus m} \to & M \to & 0 & \text{for integers } n \text{ and } m. \\ 0 \to & \mathrm{Hom}(M,R) \to & \mathrm{Hom}(R^{\oplus m},R) \to & \mathrm{Hom}(R^{\oplus n},R) & \text{As Hom is left exact.} \\ 0 \to & M^\vee \otimes N \to & R^{\oplus m} \otimes N \to & R^{\oplus n} \otimes N & \text{As } N \text{ is flat.} \end{array}$$

Similarly we get:

$$\begin{array}{rrrrr} R^{\oplus n} \to & R^{\oplus m} \to & M \to & 0 & \text{for integers } n \text{ and } m. \\ 0 \to & \mathrm{Hom}(M,N) \to & \mathrm{Hom}(R^{\oplus m},N) \to & \mathrm{Hom}(R^{\oplus n},N) & \text{As Hom is left exact.} \end{array}$$

We can combine these results to get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^\vee \otimes N & \longrightarrow & N^{\oplus m} & \longrightarrow & N^{\oplus n} \\ & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ 0 & \longrightarrow & \mathrm{Hom}(M,N) & \longrightarrow & N^{\oplus m} & \longrightarrow & N^{\oplus n} \end{array}$$

Then we can prove that $M^\vee \otimes N \simeq \mathrm{Hom}(M,N)$, with a diagram chase.

□

Theorem 6.1. *For a finitely presented module M and any module N, we have the following isomorphism in the derived category:*

$$LHom(M,N) \simeq M^\vee \overset{L}{\otimes} N.$$

Proof. Let P_N be a projective resolution $P_N = \cdots \to P_2 \to P_1 \to P_0$ of N. Then

$$\begin{aligned} \mathrm{L}_i\mathrm{Hom}(M,N) &\simeq H_i(\mathrm{Hom}(M,P_N)) && \text{definition of Left Derived Functor.} \\ &\simeq H_i(M^\vee \otimes P_N) && \text{By Lemma 6.1.} \\ &\simeq \mathrm{Tor}_i(M^\vee, N) && P_N \text{ is projective resolution of N.} \end{aligned}$$

So we know that the homology groups agree and as we have the natural map ψ of Lemma 6.1 we can see that $\mathrm{LHom}(M,N)$ is homotopically equivalent to $M^\vee \overset{L}{\otimes} N$ and so we have the following equivalence in the derived category

$$\mathrm{LHom}(M,N) \simeq M^\vee \overset{L}{\otimes} N.$$

□

We now present the following lemma found in Cartan-Eilenberg "Homological Algebra".

Proposition 6.1 ([3], Propositions 5.2 and 5.3)**.** *For modules A, B, C the natural map $\sigma : Hom(B,C) \otimes A \to Hom(Hom(A,B),C)$ is an isomorphism if any of the two following conditions is satisfied.*

(a) A is a finitely generated projective module.
(b) The ring R is noetherian, A is finitely generated and C is an injective module.

Lemma 6.2. *For modules A, B, C, the natural map $\sigma : Hom(B,C) \otimes A \to Hom(Hom(A,B),C)$ is an isomorphism if A is finitely presented and C is an injective module.*

Proof. We first prove that the functors $\mathrm{Hom}(B,C) \otimes -$ and $\mathrm{Hom}(\mathrm{Hom}(-,B),C)$ are right exact. To see this we notice that for the first functor we see this property comes from the right exactness of tensor product. For the other functor we notice that if we apply $\mathrm{Hom}(-,B)$ to an exact sequence

$$X \to Y \to Z \to 0$$

we get the sequence

$$0 \to \mathrm{Hom}(Z,B) \to \mathrm{Hom}(Y,B) \to \mathrm{Hom}(X,B)$$

right exact and applying $\mathrm{Hom}(-,C)$ which is exact, as C is injective, we get

$$0 \to \mathrm{Hom}(\mathrm{Hom}(X,B),C) \to \mathrm{Hom}(\mathrm{Hom}(Y,B),C) \to \mathrm{Hom}(\mathrm{Hom}(Z,B),C)$$

and so it is a right exact functor as desired.

Now, if A is finitely presented module we have an exact sequence $F_1 \to F_0 \to A \to 0$ and applying the two functors described in the paragraph above, combined with part (a) of proposition 6.1 we get the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}(B,C) \otimes F_1 & \longrightarrow & \mathrm{Hom}(B,C) \otimes F_0 & \longrightarrow & \mathrm{Hom}(B,C) \otimes A & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \\
\mathrm{Hom}(\mathrm{Hom}(F_1,B),C) & \longrightarrow & \mathrm{Hom}(\mathrm{Hom}(F_0,B),C) & \longrightarrow & \mathrm{Hom}(\mathrm{Hom}(A,B),C) & \longrightarrow & 0
\end{array}$$

And from a diagram chase we can conclude that $\psi : \mathrm{Hom}(B,C) \otimes A \to \mathrm{Hom}(\mathrm{Hom}(A,B),C)$ is a natural isomorphism.

□

Corollary 6.1. *For modules M and N over a ring R the map*

$$M \otimes N \xrightarrow{\psi} Hom(M^\vee, N)$$

is a natural map of bifunctors.
Furthermore, ψ is an isomorphism if M is finitely presented and N is an injective module.

Proof. The result follows from the naturality shown in 6.1 and using Lemma 6.2 with $A = M$, $B = R$ and $C = N$ for the isomorphism.

□

Theorem 6.2. *For a finitely presented module N and any module M, we have the following isomorphism in the derived category:*

$$M \overset{R}{\otimes} N \simeq RHom(N^\vee, M).$$

Proof. Let I_M be an injective resolution $I_M = 0 \to I_0 \to I_1 \to I_2 \to \cdots$ of M. Then

$$\begin{aligned} M\overset{R^i}{\otimes}N &\simeq H^i(I_M \otimes N) && \text{Right Derived Functor.} \\ &\simeq H^i(\mathrm{Hom}(N^\vee, I_M)) && \text{By Corollary 6.1.} \\ &\simeq \mathrm{Ext}(N^\vee, M) && I_M \text{ is a resolution of M.} \end{aligned}$$

So we know that the homology groups coincide and as we have the natural map ψ of Corollary 6.1 we can see that $M\overset{R}{\otimes}N$ is homotopically equivalent to $\mathrm{RHom}(N^\vee, M)$ and so we have the following equivalence in the derived category

$$M\overset{R}{\otimes}N \simeq \mathrm{RHom}(N^\vee, M).$$

□

Finally, using theorem 6.1 and theorem 6.2, we can prove that for a finitely presented module M the functors LHom and $\overset{R}{\otimes}$ are a pair of adjoint functors in the derived category of R.

Corollary 6.2. *For a finitely presented module M in R-mod, the functors $LHom(M, -)$ and $-\overset{R}{\otimes}M$ satisfy*

$$Hom(LHom(M, A), B) \simeq Hom(A, B\overset{R}{\otimes}M)$$

Proof. We have from theorems 6.1 and 6.2 that $\mathrm{LHom}(M, A) \simeq \mathrm{Tor}(M^\vee, A) \simeq \mathrm{Tor}(A, M^\vee)$ and $B\overset{R}{\otimes}M \simeq \mathrm{Ext}(M^\vee, B)$ so that

$$\begin{aligned} \mathrm{Hom}(\mathrm{LHom}(M, A), B) &\simeq \mathrm{Hom}(A, B\overset{R}{\otimes}M) \\ \mathrm{Hom}(\mathrm{Tor}(A, M^\vee), B) &\simeq \mathrm{Hom}(A, \mathrm{Ext}(M^\vee, B)) \end{aligned}$$

and the result follows from the fact that $\mathrm{Tor}(-, M)$ and $\mathrm{Ext}(M, -)$ are an adjoint pair.

□

7 Spectral Sequences

We will now describe spectral sequences that converge to Ext and Tor but that in one of their E_2 pages we have LHom, *LHom or $\overset{R}{\otimes}$. We will describe in detail how to get the spectral sequence for LHom and in the other case only give the E_2 pages, but the proofs follow the same idea as that for LHom.

Proposition 7.1. *Let M, N be two R modules. Then the spectral sequence that comes from a sign change of the double complex $Hom(P_M, Q_N)$ where $P_M := \cdots P_2 \to P_1 \to P_0 \to 0$ and $Q_N := \cdots Q_2 \to Q_1 \to Q_0 \to 0$ are projective resolutions of M and N respectively, converges to $E^\infty_{p,0} = Ext^p(M, N)$ under one of the filtrations, and under the other filtration, we get the following E_2 page:*

$$
\begin{array}{cccc}
L_0 Hom(M,N) & L_0 Ext^1(M,N) & L_0 Ext^2(M,N) & \cdots \\
L_1 Hom(M,N) & L_1 Ext^1(M,N) & L_1 Ext^2(M,N) & \cdots \\
L_2 Hom(M,N) & L_2 Ext^1(M,N) & L_2 Ext^2(M,N) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

Furthermore, if M is finitely generated and R is noetherian we can use that $Hom(M,N) \simeq M^\vee \otimes N$ and get the E_2 page:

$$
\begin{array}{cccc}
Tor_0(Ext^0(M,R),N) & Tor_0(Ext^1(M,R),N) & Tor_0(Ext^2(M,R),N) & \cdots \\
Tor_1(Ext^0(M,R),N) & Tor_1(Ext^1(M,R),N) & Tor_1(Ext^2(M,R),N) & \cdots \\
Tor_2(Ext^0(M,R),N) & Tor_2(Ext^1(M,R),N) & Tor_2(Ext^2(M,R),N) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

Proof. We first include a diagram of the spectral sequence:

$$
\begin{array}{ccccccc}
\mathrm{Hom}(P_0,Q_0) & \longrightarrow & \mathrm{Hom}(P_1,Q_0) & \longrightarrow & \mathrm{Hom}(P_2,Q_0) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}(P_0,Q_1) & \longrightarrow & \mathrm{Hom}(P_1,Q_1) & \longrightarrow & \mathrm{Hom}(P_2,Q_1) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}(P_0,Q_2) & \longrightarrow & \mathrm{Hom}(P_1,Q_2) & \longrightarrow & \mathrm{Hom}(P_2,Q_2) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \ddots
\end{array}
$$

If we first consider the filtration going through the vertical direction we will get that as $\mathrm{Hom}(P_i,-)$ is an exact functor, that the sequence collapses to the x-axis as the line

$$\mathrm{Hom}(P_0,N) \to \mathrm{Hom}(P_1,N) \to \mathrm{Hom}(P_2,N) \to \cdots$$

which then is straight forward to see gives us the E_2 page:

$$\mathrm{Ext}^0(M,N) \qquad \mathrm{Ext}^1(M,N) \qquad \mathrm{Ext}^2(M,N) \qquad \cdots$$

If we considered the other filtration then it is straightforward to see that we get the following E_1 page:

$$\begin{array}{cccc}
\mathrm{Hom}(M,Q_0) & \mathrm{Ext}^1(M,Q_0) & \mathrm{Ext}^2(M,Q_0) & \cdots \\
\uparrow & \uparrow & \uparrow & \\
\mathrm{Hom}(M,Q_1) & \mathrm{Ext}^1(M,Q_1) & \mathrm{Ext}^2(M,Q_1) & \cdots \\
\uparrow & \uparrow & \uparrow & \\
\mathrm{Hom}(M,Q_2) & \mathrm{Ext}^1(M,Q_2) & \mathrm{Ext}^2(M,Q_2) & \cdots \\
\uparrow & \uparrow & \uparrow & \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

From this page we can calculate the E_2 page which would get us the following:

$$\begin{array}{cccc}
\mathrm{LHom}^0(M,N) & L_0\mathrm{Ext}^1(M,N) & L_0\mathrm{Ext}^2(M,N) & \cdots \\
\mathrm{LHom}^1(M,N) & L_1\mathrm{Ext}^1(M,N) & L_1\mathrm{Ext}^2(M,N) & \cdots \\
\mathrm{LHom}^2(M,N) & L_2\mathrm{Ext}^1(M,N) & L_2\mathrm{Ext}^2(M,N) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

To prove the second part of the theorem we notice that as M is finitely generated and R is noetherian then all the P_i are finitely generated projective modules. Then by Lemma 6.1 we can rewrite the E_0 page as:

$$\begin{array}{ccccccc}
P_0^\vee \otimes Q_0 & \longrightarrow & P_1^\vee \otimes Q_0 & \longrightarrow & P_2^\vee \otimes Q_0 & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
P_0^\vee \otimes Q_1 & \longrightarrow & P_1^\vee \otimes Q_1 & \longrightarrow & P_2^\vee \otimes Q_1 & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
P_0^\vee \otimes Q_2 & \longrightarrow & P_1^\vee \otimes Q_2 & \longrightarrow & P_2^\vee \otimes Q_2 & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

Then, we notice that $- \otimes Q_j$ is exact for all j and so

$$\begin{aligned}
H^i(\mathrm{Hom}(P_M, R) \otimes Q_j) &\simeq H^i(\mathrm{Hom}(P_M, R)) \otimes Q_j \\
&\simeq \mathrm{Ext}^i(M, R) \otimes Q_j
\end{aligned}$$

we get the E_1 page:

$$\begin{array}{cccc}
\mathrm{Ext}^0(M,R) \otimes Q_0 & \mathrm{Ext}^1(M,R) \otimes Q_0 & \mathrm{Ext}^2(M,R) \otimes Q_0 & \cdots \\
\uparrow & \uparrow & \uparrow & \\
\mathrm{Ext}^0(M,R) \otimes Q_1 & \mathrm{Ext}^1(M,R) \otimes Q_1 & \mathrm{Ext}^2(M,R) \otimes Q_1 & \cdots \\
\uparrow & \uparrow & \uparrow & \\
\mathrm{Ext}^0(M,R) \otimes Q_2 & \mathrm{Ext}^1(M,R) \otimes Q_2 & \mathrm{Ext}^2(M,R) \otimes Q_2 & \cdots \\
\uparrow & \uparrow & \uparrow & \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Then it is straightforward to get the E_2 page:

$$\begin{array}{cccc}
\mathrm{Tor}_0(\mathrm{Ext}^0(M,R),N) & \mathrm{Tor}_0(\mathrm{Ext}^1(M,R),N) & \mathrm{Tor}_0(\mathrm{Ext}^2(M,R),N) & \cdots \\
 & \nwarrow & \nwarrow & \\
\mathrm{Tor}_1(\mathrm{Ext}^0(M,R),N) & \mathrm{Tor}_1(\mathrm{Ext}^1(M,R),N) & \mathrm{Tor}_1(\mathrm{Ext}^2(M,R),N) & \cdots \\
 & \nwarrow & \nwarrow & \\
\mathrm{Tor}_2(\mathrm{Ext}^0(M,R),N) & \mathrm{Tor}_2(\mathrm{Ext}^1(M,R),N) & \mathrm{Tor}_2(\mathrm{Ext}^2(M,R),N) & \cdots \\
 & & & \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

$\square$

The second remark can actually be completed as a way to prove theorem 6.1 in the Noetherian case. This suggests that this spectral sequence is a good way to approximate and get information about the functor LHom.

Similarly we can get the following results:

Proposition 7.2. *Let M, N be two R modules. Then the spectral sequence that comes from a sign change of the double complex $I_M \otimes P_N$ where $I_M := 0 \to I_0 \to I_1 \to I_2 \to \cdots$ and $P_N := \cdots P_2 \to P_1 \to P_0 \to 0$ are injective resolution of M and projective of N respectively, converges to $E^{\infty}_{p,0} = Tor_q(M,N)$ under one of the filtrations and under the other filtration we get the following E_2 page:*

$$\begin{array}{cccc}
M\overset{R}{\otimes}N & R^0\,Tor_1(M,N) & R^0\,Tor_2(M,N) & \cdots \\
M\overset{R^1}{\otimes}N & R^1\,Tor_1(M,N) & R^1\,Tor_2(M,N) & \cdots \\
M\overset{R^2}{\otimes}N & R^2\,Tor_1(M,N) & R^2\,Tor_2(M,N) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Furthermore, if N is finitely generated and R is noetherian we can use that $M \otimes N \simeq Hom(N^{\vee}, M)$ and get the E_2 page:

$$\begin{array}{cccc}
Ext^0(Ext^0(N,R),M) & Ext^0(Ext^1(N,R),M) & Ext^0(Ext^2(N,R),M) & \cdots \\
Ext^1(Ext^0(N,R),M) & Ext^1(Ext^1(N,R),M) & Ext^1(Ext^2(N,R),M) & \cdots \\
Ext^2(Ext^0(N,R),M) & Ext^2(Ext^1(N,R),M) & Ext^2(Ext^2(N,R),M) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Proposition 7.3. *Let M, N be two R modules. Then the spectral sequence that comes from a sign change of the double complex $Hom(I_M, J_N)$ where $I_M := 0 \to I_0 \to I_1 \to I_2 \to \cdots$ and $Q_N := 0 \to J_0 \to J_1 \to J_2 \to \cdots$ are injective resolutions of M and N respectively, converges to $E^{\infty}_{0,q} = Ext^q(M,N)$ under one of the filtrations and under the other filtration we get the following E_2 page:*

$$\begin{array}{cccc}
{}^{*}LHom^{0}(M,N) & {}^{*}LHom^{1}(M,N) & LHom^{2}(M,N) & \cdots \\
L_0Ext^{1}(M,N) & L_1Ext^{1}(M,N) & L_2Ext^{1}(M,N) & \cdots \\
L_0Ext^{2}(M,N) & L_1Ext^{2}(M,N) & L_2Ext^{2}(M,N) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

To end this section we would like to say that even though we found similarities between the *wrong* derived functors studied in this paper and the classical ones, there is an important difference to note. Although we can resolve any of the two inputs in the classical derived functors, in these *wrong* derived functors resolving a different input gives a totally different functor. As an example it is a straightforward calculation that $\mathrm{LHom}(\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z}$, but ${}^{*}\mathrm{LHom}(\mathbb{Z},\mathbb{Z}) \simeq 0$.

8 Injective Cogenerator

In this section we will give some information about the functor ${}^{*}\mathrm{LHom}$ which we will relate with an injective cogenerator E and with its endomorphism ring $S = \mathrm{End}(E)$. We will have some important conditions that this module E must satisfy, it must be a noetherian ring or be a Morita duality (contravariant equivalence) module as described in [5], that means a balanced injective cogenerator for R and S. To begin the discussion of this section we introduce the concept of an injective cogenerator–the dual of a projective generator.

Definition 8.1 (Injective Cogenerator)**.** An Injective Cogenerator E of a category $\mathcal{C}$ is an injective object that satisfies the following condition: for every object A in $\mathcal{C}$ can be embedded into a product of E i.e. There is an exact sequence $0 \to A \to \prod_I E$ for some set I.

An equivalent definition of an injective cogenerator is that for every nonzero object A in $\mathcal{C}$ there is a nonzero map $f : A \to E$.

An important property of an injective cogenerator is that the functor $\mathrm{Hom}(-, E)$ is a faithfully exact functor. We won't prove this property as we won't use it on this paper, however it motivates many of the ideas used. An injective cogenerator as R^{+}, the character module of R, could be a good module E to consider. However, there can be other choices that could be used, as the minimal injective cogenerator which is the injective hull of the direct sum of all simple modules in R-mod.

Proposition 8.1. *If E is a Morita duality module or S is noetherian, we have that for an injective R module I the S module I^+ is a flat module.*

Proof. If E us a Morita duality module then the module I^+ would be a prjective module and thus it is flat. Otherwise if S is noetherian we see that. For E an injective cogenerator there is an embedding
$0 \to I \to \prod E$ for some (possibly infinite) product. But we know that if an injective module is a submodule of another module then it is a direct summand. So that we can write $\prod E \simeq I \oplus J$ for some module J. If we then apply the functor $\mathrm{Hom}(-, E)$ we see that $\prod S \simeq I^+ \oplus J^+$. As S is noetherian any product of copies of the ring S is a flat module and so then I^+ that is a direct summand of a flat module is itself flat, as desired. □

Lemma 8.1. *For any modules M and N over a ring R. We have a natural map*

$$M^+ \underset{End(E)}{\otimes} N^e \xrightarrow{\psi} Hom(M, N)$$

Furthermore if E is a Morita duality module or S is noetherian, ψ is an isomorphism when N is finitely copresented by E and M is an injective module.

Proof. We consider the maps $\varphi : M \to M'$ and $\phi : N \to N'$ and to prove naturality we do a diagram chase in the following diagrams:

$$\begin{array}{ccc} \underset{(f\circ\varphi)\otimes n)}{M^+ \otimes N^e} & \longrightarrow & \underset{(g(f(\varphi)))}{\mathrm{Hom}(M,N)} \\ \uparrow & & \uparrow \\ \underset{(f\otimes g)}{(M')^+ \otimes N^e} & \longrightarrow & \underset{(g\circ f)}{\mathrm{Hom}(M',N)} \end{array} \qquad \begin{array}{ccc} \underset{(f\otimes g)}{M^+ \otimes N^e} & \longrightarrow & \underset{(g\circ f)}{\mathrm{Hom}(M,N)} \\ \downarrow & & \downarrow \\ \underset{(f\otimes\phi(g))}{M^+ \otimes (N')^e} & \longrightarrow & \underset{(\phi(g(f)))}{\mathrm{Hom}(M,N')} \end{array}$$

To prove that ψ is an isomorphism when N is finitely copresented by E and M is injective we see the following diagrams:

$$\begin{array}{lrrrl} 0 \to & N \to & E^{\oplus m} \to & E^{\oplus n} & \text{for integers } n \text{ and } m. \\ 0 \to & \mathrm{Hom}(E,N) \to & \mathrm{Hom}(E,E^{\oplus m}) \to & \mathrm{Hom}(E,E^{\oplus n}) & \text{As Hom is left exact.} \\ 0 \to & M^+ \underset{\mathrm{End(E)}}{\otimes} N^e \to & M^+ \underset{\mathrm{End(E)}}{\otimes} \mathrm{End}(E)^{\oplus m} \to & M^+ \underset{\mathrm{End(E)}}{\otimes} \mathrm{End}(E)^{\oplus n} & \text{As } M^+ \text{ is flat.} \end{array}$$

Similarly,

$$\begin{array}{lrrrl} 0 \to & N \to & E^{\oplus m} \to & E^{\oplus n} & \text{for integers } n \text{ and } m. \\ 0 \to & \mathrm{Hom}(M,N) \to & \mathrm{Hom}(M,E^{\oplus m}) \to & \mathrm{Hom}(M,E^{\oplus n}) & \text{As Hom is left exact.} \end{array}$$

These two results yield the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^+ \underset{\mathrm{End(E)}}{\otimes} N^e & \longrightarrow & (M^+)^{\oplus m} & \longrightarrow & (M^+)^{\oplus n} \\ & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ 0 & \longrightarrow & \mathrm{Hom}(M,N) & \longrightarrow & (M^+)^{\oplus m} & \longrightarrow & (M^+)^{\oplus n} \end{array}$$

Where all the down arrows are the natural isomorphisms and so we conclude $M^+ \underset{\mathrm{End}(E)}{\otimes} N^e \to \mathrm{Hom}(M, N)$ is a natural isomorphism as desired.

$\square$

Theorem 8.1. *For any module M and N finitely copresented by E, the injective cogenerator, we have the following isomorphism:*

$$^*LHom(M, N) \simeq M^+ \overset{L}{\underset{S}{\otimes}} N^e.$$

Proof. Let Q be a projective resolution $P_0 \to P_1 \to P_2 \to \cdots$ that is a resolution of M. Then we have that

$$\begin{aligned}
^*\mathrm{LHom}(M, N) &\simeq H(\mathrm{Hom}(Q, N)) && \text{Left Derived Functor.} \\
&\simeq H(Q^+ \underset{S}{\otimes} N^e) && \text{By Lemma 8.1.} \\
&\simeq \mathrm{Tor}_S(M^+, N^e) && Q^+ \text{ Flat Resolution of } M^+.
\end{aligned}$$

So we know that the homology groups coincide and as we have the natural map ψ of Lemma 8.1 we can see that $^*\mathrm{LHom}(M, N)$ is homotopically equivalent to $M^+ \overset{L}{\underset{S}{\otimes}} N^e$ and so we have the following equivalence in the derived category

$$^*\mathrm{LHom}(M, N) \simeq M^+ \overset{L}{\underset{S}{\otimes}} N^e.$$

$\square$

In the next sections we will use the three theorems that we have proved to understand the properties of the derived functors that we defined in this paper.

9 Rings of Dimension 2

In this section we will show a vanishing theorem for these rings. We will see that $\mathrm{L}_i\mathrm{Hom}$ $^*\mathrm{L}_i\mathrm{Hom}$ and $\overset{R^i}{\otimes}$ are zero whenever $i \geq 1$. To see this, we first show the following two results.

Lemma 9.1. *In a ring of Projective Dimension less than or equal to 2 we have that for any module M the module $M^\vee$ is projective.*

Proof. For M we have a presentation

$$F_1 \to F_0 \to M \to 0,$$

where the F_1, F_0 are finitely generated free modules. By taking duals we get an exact sequence $0 \to M^\vee \to P_0 \to P_1$ where P_i are projective modules, this happens as the dual module of a projective module is projective. Then taking the cokernel C of the map between P_0 and P_1 we have the exact sequence

$$0 \to M^\vee \to P_0 \to P_1 \to C \to 0.$$

Then by a dimension shift argument we have that

$$Ext^i(M^\vee, N) \simeq Ext^{i+2}(C, N)$$

for $i \geq 1$. As the projective dimension of the ring is 2 and $i+2 > 2$ then

$$Ext^{i+2}(C, N) \simeq 0$$

for all modules N. From this we get that $\mathrm{Ext}^1(M^\vee, N) \simeq 0$ for all modules N; and we conclude that $M^\vee$ is projective. □

We can see that if R is a ring of injective dimension less than or equal to 2 and E is a Morita duality module, then $\mathrm{End}(E)$ is a ring with projective dimension less than or equal to 2. We will use this fact to prove the following lemma.

Lemma 9.2. *In a ring of injective dimension less than or equal to* 2 *we have that any finitely copresented by a Morita duality module* E *module* M *we have that the module* M^e *is projective.*

Proof. For M we have a finite copresentation using the injective generator E,

$$0 \to M \to E_0 \to E_1$$

where the E_i are finite direct sums of E and C is the cokernel. Then, by applying the left exact functor $\mathrm{Hom}(E, -)$ in Mod-End(E) we get an exact sequence

$$0 \to M^e \to F_0 \to F_1 \to C \to 0$$

where F_0, F_1 are finitely generated free S modules and C is the cokernel of the map between them. Then we have that M^e is a projective S module by a dimension shifting we argument as in the previous lemma. □

Corollary 9.1. *For any module* M *and* N *that satisfy the properties of Theorems 6.1, 6.2 or 8.1 respectively, will have that* $L_iHom(M, N)$, $^*L_iHom(M, N)$ *or* $M \overset{R^i}{\otimes} N$ *are isomorphic to* 0 *for* $i \geq 1$.

Using Corollary 9.1 we can see that in many common rings as are $\mathbb{Z}$, any Dedekind domain or even polynomial rings of 1 or 2 variables, the functors $\mathrm{L}_i\mathrm{Hom}$, $^*\mathrm{L}_i\mathrm{Hom}$ and $\overset{R^i}{\otimes}$ vanish for all values of i except possibly for $i = 0$.

10 Nonvanishing Examples: Regular Local Rings

In this section we will provide some example of non-vanishing LHom and $\overset{R}{\otimes}$ to showcase that these functors are not identically 0 and that they may offer useful information for some modules. To show these we will do it over regular local rings, proving this result for this type of rings suggests that it is common over rings of higher dimensions that these functors do not vanish.

A local ring is a ring that has a unique maximal ideal, a way of generating these rings is by localizing any ring R over a maximal ideal $\mathfrak{m}$. There is a special kind of local rings with some nice properties which we describe right now. We use the definition given in [6].

Definition 10.1 (Regular Local Ring)**.** A regular local ring is a noetherian local ring R of dimension d with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$ that satisfies any of the following conditions:
(i) $dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$
(ii) $\mathfrak{m}$ can be generated by d elements.

Condition (ii) is related to the following important property of regular local rings.

Definition 10.2 (Regular Sequence)**.** A regular sequence in a regular local ring R is made of elements $f_1, f_2, ..., f_d$ such that f_i is not a zero divisor in $R/(f_1, f_2, ..., f_{i-1})$ for all i.

It is known that for all regular local rings there are regular sequences; however, we won't prove this result.

Lemma 10.1. *If $f_1, f_2, ..., f_d$ is a regular sequence and $M = R/(f_1, ..., f_n)$ then*

$$Ext^i(M, R) \simeq \begin{cases} 0 & \text{if } i \neq n \\ M & \text{if } i = n \end{cases}$$

Proof. We will show by induction that there is a free resolution

$$0 \to R^{\oplus\binom{n}{n}} \to ... \to R^{\oplus\binom{n}{0}} \to R/(f_1, ..., f_n) \to 0.$$

For $n = 1$ we have

$$0 \to R \xrightarrow{\times f_1} R \to R/(f_1) \to 0.$$

For the inductive step we take the free resolution of $R/(f_1, ..., f_n)$

$$0 \to R^{\oplus\binom{n}{n}} \to ... \to R^{\oplus\binom{n}{0}} \to 0$$

and take the tensor product with the resolution of $R/(f_{n+1})$

$$0 \to R \xrightarrow{\times f_1} R \to 0$$

doing this we get a double complex

$$\begin{array}{ccc}
R^{\oplus\binom{n}{0}} & \xrightarrow{\times f_d} & R^{\oplus\binom{n}{0}} \\
\uparrow & & \uparrow \\
R^{\oplus\binom{n}{1}} & \xrightarrow{\times f_d} & R^{\oplus\binom{n}{1}} \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
R^{\oplus\binom{n}{n}} & \xrightarrow{\times f_d} & R^{\oplus\binom{n}{n}}
\end{array}$$

We would like to calculate the homology of the total complex

$$R^{\oplus\binom{n}{n}} \to R^{\oplus(\binom{n}{n-1}+\binom{n}{n})} \to ... \to R^{\oplus(\binom{n}{1}+\binom{n}{0})} \to R^{\binom{n}{0}} \to 0$$

which is equivalent to

$$R^{\oplus\binom{n+1}{n+1}} \to R^{\oplus\binom{n+1}{n}} \to ... \to R^{\oplus\binom{n+1}{1}} \to R^{\binom{n+1}{0}} \to 0$$

which we would like to prove that is a free resolution of $R/(f_1, f_2, ..., f_{n+1})$.

To prove this we will make the commutative diagram above into a double complex by multiplying the maps on the right column by -1. Then we can calculate the homology of the total complex by looking at the spectral sequence.

For the first page we look at the cohomology in the columns, which we know gives us $R/(f_1, ..., f_n)$ as they are resolutions. Then the page E_1 looks like

$$R/(f_1, ..., f_n) \xrightarrow{\times f_{n+1}} R/(f_1, ..., f_n)$$

and as f_{n+1} is not a zero divisor then we get that the page E_2 is

$$0 \qquad R/(f_1, ..., f_{n+1}).$$

This means that the cohomology of the total complex is $R/(f_1, ..., f_n)$ at 0 and so indeed

$$R^{\oplus\binom{n+1}{n+1}} \to R^{\oplus\binom{n+1}{n}} \to ... \to R^{\oplus\binom{n+1}{1}} \to R^{\binom{n+1}{0}} \to R/(f_1, ..., f_{n+1}) \to 0$$

is a free resolution of $R/(f_1, ..., f_{n+1})$.

With this result we can now see that $\mathrm{Ext}(R/(f_1, ..., f_n), R)$ is the cohomology of

$$(R^\vee)^{\oplus\binom{n+1}{n+1}} \leftarrow (R^\vee)^{\oplus\binom{n+1}{n}} \leftarrow ... \leftarrow (R^\vee)^{\oplus\binom{n+1}{1}} \leftarrow (R^\vee)^{\binom{n+1}{0}} \leftarrow 0$$

, we can see that this complex agrees with the resolution of $R/(f_1, ..., f_n)$ and so the only non vanishing Ext is $\mathrm{Ext}^n(R/(f_1, ..., f_n), R) \simeq R/(f_1, ..., f_n)$. (We need to see that the maps are actually the same, but this can be seen by induction as we have linear maps. We just need to see their transposes and this is straightforward to do.)

$\square$

We will now apply the spectral sequence found in Björk [7, pg. 58-61] in the setting of a regular local ring and a module $M = R/(f_1, f_2, ..., f_n)$ where for $n \leq d$, $f_1, f_2, ..., f_d$ is a regular sequence.

Theorem 10.1. *Let R be a regular local ring of dimension $d \geq 3$ and maximal ideal $\mathfrak{m}$, let $k := R/\mathfrak{m}$ be its residue field. Then for any integer $1 \leq \ell \leq d-2$ there exists a module N such that:*

$$N \overset{R}{\otimes} R \simeq k[-\ell]$$

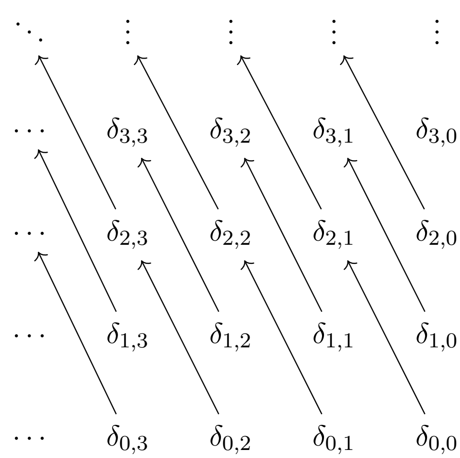
in the derived category.

Proposition 10.1. *[7, Proposition 4.15] For every module M, there is a spectral sequence that converges to M in its $(0, n)$ position, where we consider the position $(0,0)$ to be the bottom left corner. And for which one of the E_2 pages of its filtration are the following:*

$$\begin{array}{cccc}
Ext^n(Ext^n(M,R),R) & Ext^n(Ext^{n-1}(M,R)) & \cdots & Ext^n(Ext^0(M,R)) \\
Ext^{n-1}(Ext^n(M,R),R) & Ext^{n-1}(Ext^{n-1}(M,R),R) & \cdots & Ext^{n-1}(Ext^0(M,R),R) \\
\vdots & \vdots & \vdots & \vdots \\
Ext^0(Ext^n(M,R),R) & Ext^0(Ext^{n-1}(M,R),R) & \cdots & Ext^0(Ext^0(M,R),R)
\end{array}$$

Page E_2 of the spectral sequence

We give a better picture of the spectral sequences, showing the differentials, by using the shorthand notation $\delta_{a,b} = \mathrm{Ext}^a(\mathrm{Ext}^b(M,R),R)$. We then can picture the E_2 page as following:



Proof of Theorem 10.1. We use Proposition 10.1 with $M = R/(f_1, f_2, ..., f_n) = k$ the residue field of our regular local ring. It is straightforward to see from 10.1 that only

$\text{Ext}^n(\text{Ext}^n(M,R),R) \simeq k$ is a non-zero module in the E_2 page. Having this in mind we would like to make a dimension shifting argument to have this k away from the diagonal of $\delta_{i,i}$, and the way to do so is to consider the module K such that

$$K \to F_\ell \to ... \to F_1 \to M$$

with $\ell > 2$. With this K we have that that $\text{Ext}^i(K,N) \simeq \text{Ext}^{\ell+i}(k,N)$ for $i \geq 1$.

Now if we look at this spectral sequence with $M = K$, the page E_2 we can see that only $\text{Hom}(\text{Hom}(K,R),R)$ can be non-empty in the diagonal and as this diagonal gives a filtration of K. We can also see that the only modules that cannot be 0 are $\text{Ext}^n(\text{Ext}^{n-\ell}(K,R),R)$, and those that are $\text{Ext}^i(Hom(K,R),R)$.

Then by looking at the differentials going from $\text{Ext}^0(\text{Ext}^0(K,R),R)$ through all pages it will go to the positions where we had $\text{Ext}^v(\text{Ext}^{v-1}(K,R),R) \simeq 0$ and so as $\ell > 2$, we can conclude that $K^{\vee\vee} \simeq K$.

Finally, considering the module $N = K^\vee$ we have

$$\text{Ext}^{n-\ell}(\text{Hom}(N,R),R) \simeq M.$$

These are precisely the modules we were looking for. □

Keeping with the discussion of the proof, looking at the module N we found that $\text{Ext}(N,R) \not\simeq 0$. In particular as N is finitely presented, then by Lemma 6.1

$$R \overset{R}{\otimes} N \simeq \text{Ext}^i(N^\vee,R) \simeq M$$

for $i = n-\ell$. As M is finitely presented we also have that N is flat iff it is projective and so it is not flat and we would also find some A for which

$$\text{L}_i\text{Hom}(A,N) \simeq \text{Tor}^i(N^\vee,A) \not\simeq 0.$$

So, even though for L_iHom we did not give a concrete example, we know that for N we should be able to find a nonzero L_iHom.

One of the reasons why it is useful to prove the non-vanishing for local rings is how they naturally arise in many contexts; and they are also helpful in describing other rings, because after localizing at a prime ideal we get a local ring; furthermore, if Ext is non-vanishing in the localization then it must also be non-vanishing in the original ring.

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