An introduction to equivariant cohomology and the equivariant first Chern class

UROP+ Final Paper, Summer 2017

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September 1, 2017

Abstract

The goal of this paper to provide a relatively accessible and integrated introduction to the algebraic topology of spaces with Lie group actions, in both the smooth and the holomorphic category. We present a detailed treatment of the basic constructions in equivariant de Rham theory and Dolbeault theory. We also discuss equivariant connections and curvature on vector bundles equipped with infinitesimal lifts, as well as the equivariant first Chern class of line bundles with infinitesimal lifts. A novel feature of this presentation is the definition of infinitesimal lift via differential operators on the vector bundle.

1 Introduction

Many topological and geometric theories admit *equivariant* analogs, which incorporate extra structure associated to the action of a Lie group on the underlying space. In addition to their intrinsic attraction, such equivariant analogs are important technical tools in the study of objects with symmetries and the construction of quotient objects.

In this paper we give an account of the very basics of equivariant de Rham and Dolbeault cohomology and the equivariant first Chern class, which lies at the foundation of interesting modern research in geometric invariant theory, equivariant index theory, and Hamiltonian group actions. Discussion of this material in the available literature is hard to find all in one place, and often assumes a fair amount of background knowledge, as well as the ability to navigate through concise notation that differs from author to author. Here, we aim to provide a fairly unified, accessible and annotated treatment.

In section 2, we review the construction of the equivariant de Rham cohomology. Section 3 provides a detailed account of the equivariant Dolbeault cohomology. This construction is fairly well-known ([Tel00, section 7], [Lil03, section 5.1]), but an elementary treatment is not available at the moment to the awareness of the author.

In section 4, we introduce the notion of an *infinitesimal lift* of a group action on a space to a vector bundle over it. Vector bundles with such infinitesimal lifts are generalizations of the better-known *equivariant vector bundles* (see, e.g., [BGV03, section 1.5]). Here we define an infinitesimal lift to be a certain collection of differential operators on the vector bundle. This definition is new, to the knowledge of the author, but should coincide with the definition of infinitesimal lifts via vector fields in [MiR01], [HY76], and [Kos70]. Then we discuss equivariant connections and curvatures on a vector bundle equipped with such an infinitesimal lift. We also define the equivariant first Chern class of a complex line bundle with such an infinitesimal lift, following the construction of the equivariant first Chern class in [BGV03, section 7.1]. This definition is also hard to find in the literature as presented in the infinitesimal setting, although it is probably well-known.

1.1 Notation and conventions

In this paper, we assume familiarity with smooth manifolds and actions of Lie groups on manifolds (as in [Lee13]), as well as some knowledge of complex geometry, in particular complex line bundles and Kähler manifolds (as in [CdS01] and part of [Huy05]). We will use the Einstein summation convention throughout.

Let G be a compact, connected Lie group. Denote its Lie algebra by $\mathfrak{g}$ and the dual of the Lie algebra by $\mathfrak{g}^*$. Let M be a compact, connected smooth manifold. Unless specified, we mean by p a general point in the underlying set of M, g an element of G, and X an element of $\mathfrak{g}$.

Suppose that M admits a smooth G-action

$$\rho : G \times M \to M,\ (g,p) \mapsto g \cdot p.$$

This induces a G-action on the space of smooth funtions $C^\infty(M,\mathbb{C}) = \mathscr{A}^0(M)$ via the pullback

$$(g \cdot f)(p) = f(g^{-1} \cdot p)$$

and thus an action on the graded algebra $\mathscr{A}^*(M)$ of complex-valued smooth differential forms on M. Each $X \in \mathfrak{g}$ generates a vector field $X^\sharp$ on M, called the *fundamental vector field* of X, by

$$X^\sharp_p := \frac{d}{dt}\Big|_{t=0} (\exp(tX) \cdot p).$$

2 Equivariant de Rham cohomology

In this section, we present a construction of the equivariant de Rham complex based on discussions in [Lib07] and [GS13].

To start with, we want to extend the notion of a differential form to the equivariant setting. The intuitive approach is to directly apply the definition of an *G-equivariant map*, which is a map that intertwines with the action of G.

Definition 2.1. A *G-equivariant differential form* is a smooth polynomial $\alpha : \mathfrak{g} \to \mathscr{A}^*(M)$ that is G-equivariant, i.e., for all $g \in G$ and $X \in \mathfrak{g}$

$$(g \cdot \alpha)(X) = \alpha(\mathrm{Ad}_g \cdot X),$$

where $g \mapsto \mathrm{Ad}_{g^{-1}}$ is the adjoint representation of G on $\mathfrak{g}$.

One advantage of this definition is that $\alpha(X) \in \mathscr{A}^*(M)$ is a linear combination of ordinary differential forms. However, this notation can be rather cumbersome and confusing as the theory develops.

Alternatively, we use the identification of the symmetric algebra $S(\mathfrak{g}^*)$ with $\mathbb{C}[\mathfrak{g}]$. Note that $\mathscr{A}^*(G) \otimes S(\mathfrak{g}^*)$ is naturally equipped with a G-action

$$(g \cdot \beta)(X) = g \cdot \beta(\mathrm{Ad}_{g^{-1}} \cdot X),\ \forall \beta \in \mathscr{A}^*(M) \otimes S(\mathfrak{g}^*).$$

One can easily check that with respect to this action, $\alpha \in \mathscr{A}^*(G) \otimes S(\mathfrak{g}^*)$ is G-invariant if and only if the corresponding polynomial is G-equivariant in the sense of Definition 2.1. Hence we obtain the following equivalent definition:

Definition 2.2. A *G-equivariant differential form* on M is a G-invariant element

$$\alpha \in \mathscr{A}^*_G(M) := (\mathscr{A}^*(M) \otimes S(\mathfrak{g}^*))^G.$$

In order to make $\mathscr{A}^*_G(M)$ into a differential graded algebra, we need a well-defined notion of degree for its elements and a operator that serves as the differential.

Note that $\mathscr{A}^*(M) \otimes S(\mathfrak{g}^*)$ is spanned by the forms $\omega \otimes f$ with $\omega \in \mathscr{A}^p(M)$ and $f \in S^q(\mathfrak{g}^*)$. Assigning $\omega \otimes f$ the degree $p + 2q$ yields a $\mathbb{Z}$-grading

$$\mathscr{A}^*_G(M) = \bigoplus \mathscr{A}^k_G(M),$$

where
$$\mathscr{A}_G^k(M) = \bigoplus_{p+2q=k} (\mathscr{A}^p(M)\otimes S^q(\mathfrak{g}^*))^G.$$

Lemma 2.3. *There exists a unique operator $d_G : \mathscr{A}_G^*(M) \to \mathscr{A}_G^*(M)$, called the* equivariant differential*, such that*
$$(d_G\alpha)(X) = d(\alpha(X)) - \iota_{X^\sharp}(\alpha(X))$$
for all $\alpha \in \mathscr{A}_G^*(M)$, $X \in \mathfrak{k}$. Moreover $d_G : \mathscr{A}_G^k(M) \to \mathscr{A}_G^{k+1}(M)$ for all k.

Proof. We define d_G by linearly extending to $\mathscr{A}_G^*(M) \to \mathscr{A}_G^*(M)$ the map
$$d_G(\omega\otimes f)(X) = d\omega\otimes f(X) - \iota_{X_i^\sharp}\omega\otimes u^i f(X),$$
where d is the exterior derivative, $\iota_{X^\sharp}$ the interior product with $X^\sharp$, u^i a basis for $\mathfrak{k}^*$ with dual basis ξ_i for $\mathfrak{g}$, and $X_i = u^i(X)\xi_i$. Note that $\deg(d\omega\otimes f) = (p+1)+2q$ and $\deg(\iota_{X_i^\sharp}\omega\otimes u^i f) = p-1+2(q+1)$. On the other hand, d commutes with pullbacks, so
$$g\cdot d(\alpha(X)) = d(g\cdot\alpha(X)) = d(\alpha(\mathrm{Ad}_g\cdot X)),$$
while $\iota_{\mathrm{Ad}_g\cdot X} = g\cdot\iota_X\cdot g^{-1}$. Hence d_G preserves G-equivariance and restricts to a map $\mathscr{A}_G^k(M) \to \mathscr{A}_G^{k+1}(M)$ for all k. ☐

Proposition 2.4. *The equivariant differential satisfies $d_G\circ d_G = 0$, thus making $(\mathscr{A}_G^*(M),\, d_G)$ a differential graded algebra.*

Proof. Using Cartan's formula and the G-invariance of α, we have
$$\begin{aligned}
(d_G)^2(\alpha)(X) =& d^2(\alpha(X)) - (d(\iota_{X^\sharp}\alpha(X)) + \iota_{X^\sharp}(d(\alpha(X))) + \iota_{X^\sharp}(\iota_{X^\sharp}(\alpha(X))\\
=& -\mathscr{L}_{X^\sharp}(\alpha(X))\\
=& -\frac{d}{dt}\Big|_{t=0}(\exp(tX)\cdot\alpha(X))\\
=& \frac{d}{dt}\Big|_{t=0}\alpha(\mathrm{Ad}_{\exp(tX)}\cdot X))\\
=& \frac{d}{dt}\Big|_{t=0}\alpha([tX,\, X]) = \ 0.
\end{aligned}$$
☐

As a consequence, we can define the *G-equivariant de Rham cohomology* of M to be the cohomology of the chain complex $(\mathscr{A}_G^*(M);\, d_G)$ and the p-th *equivariant cohomology group* to be the quotient
$$H^p(\mathscr{A}_G^*(M)) := \frac{\{\ker d_G : \mathscr{A}_G^p(M) \to \mathscr{A}_G^{p+1}(M)\}}{\{\mathrm{Im}\, d_G : \mathscr{A}_G^{p-1}(M) \to \mathscr{A}_G^p(M)\}}.$$

Remark 2.5. The study of equivariant cohomology combines the ordinary cohomology theory with Lie group actions and gives rise to many profound results. To start with, a remarkable theorem of Cartan shows that the de Rham construction coincides with the algebro-topological construction via universal bundles. Since it is quite beyond the scope of this paper, we will not go into details here. Instead, we refer the reader to [Lib07] for an introductory account of the subject and [AB84], [GS13] for a detailed discussion in a more general setting.

Naturally, we would want to give an equivariant version of the Poincaré lemma, which says that a closed differential form is locally exact. We say that $\alpha \in \mathscr{A}_G^*(M)$ is *equivariantly closed* if $d_G\alpha = 0$, *equivariantly exact* if $\alpha = d_G\beta$ for some $\beta \in \mathscr{A}_G^*(M)$.

It turns out that "bad" things happen at the fixed points of the G-action. For instance, if p_0 is fixed by G, then around p_0 equivariantly closed 2-forms are equivariantly exact only up to some constants in the center of $\mathfrak{g}$. But on the subset $M - \{p_0 \in M \mid g\cdot p_0 = p_0$ for some $g \in G - \{e\}\}$, we do have the following generalization, which is the starting point of several important localization theorems. (cf. [BGV03, section 7.2])

Lemma 2.6 (Equivariant Poincaré lemma). *If $\alpha \in \mathscr{A}_G^*(M)$ is equivariantly closed, then it is locally equivariantly exact away from the zero locus of the G-action.*

3 Equivariant Dolbeault cohomology

Suppose that $M = (M,J)$ is a compact complex manifold and G acts on M holomorphically. In this section, we present an equivariant version of the Dolbeault cohomology on M following the outline given in [Lil03, Theorem 5.1].

Recall from complex geometry that the complexification of the cotangent bundle $T^*M \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ yields a splitting of the exterior differential into $d = \partial + \bar{\partial}$. The del and delbar operators ∂, $\bar{\partial}$ restrict to maps $\partial : \mathscr{A}^{p,q}(M) \to \mathscr{A}^{p+1,q}(M)$, $\bar{\partial} : \mathscr{A}^{p,q}(M) \to \mathscr{A}^{p,q+1}(M)$ on $\mathscr{A}^{p,q}(M) = \Gamma(M, \bigwedge^p T^{1,0} \otimes \bigwedge^q T^{0,1})$. Accordingly, a vector field X splits into its holomorphic and antiholomorphic part, i.e., $X = Z + \bar{Z}$ with

$$Z = \frac{1}{2}(X - iJX) \in T^{1,0}, \ \bar{Z} = \frac{1}{2}(X + iJX) \in T^{0,1}.$$

On the other hand, we can complexify $S(\mathfrak{g}^*)$ and write $S(\mathfrak{g}^*)_\mathbb{C} := S(\mathfrak{g}^*) \otimes \mathbb{C}$. Let u_i be a basis of $S(\mathfrak{g}^*)_\mathbb{C}$ and assign each u_i the bidegree $(1,1)$. Thus we get a bigrading on the symmetric algebra $S^*(\mathfrak{g}^*)_\mathbb{C}$.

Assigning $\omega \otimes f \in \mathscr{A}^{p,q}(M) \otimes S^r(\mathfrak{g}^*)_\mathbb{C}$ the bidegree $(p+r, q+r)$ yields a bigrading on $\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C}$, given by

$$(\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^{m,n} := \bigoplus_{p+r=m, \ q+r=n} \mathscr{A}^{p,q}(M) \otimes S^r(\mathfrak{g}^*)_\mathbb{C}.$$

Now we define the equivariant del and delbar operators on $\mathscr{A}_G^{*,*}(M) = (\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^G$. For $X \in \mathfrak{g}$ and tensor $\alpha = \omega \otimes f \in (\mathscr{A}^{p,q}(M) \otimes S^r(\mathfrak{g}^*))^G$, set

$$(\partial_G \alpha)(X) = (\partial \omega \otimes f - \iota_{\bar{Z}_i^\sharp} \omega \otimes u^i f)(X),$$
$$(\bar{\partial}_G \alpha)(X) = (\bar{\partial} \omega \otimes f - \iota_{Z_i^\sharp} \omega \otimes u^i f)(X).$$

Extending the above linearly to $\mathscr{A}_G^{*,*}(M)$ yields two operators $\partial_G, \bar{\partial}_G : \mathscr{A}_G^{*,*}(M) \to \mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C}$. It is clear that $d_G = \partial_G + \bar{\partial}_G$. Next, we need to check that ∂_G and $\bar{\partial}_G$ are indeed differentials on $\mathscr{A}_G^{*,*}(M)$.

Lemma 3.1. *For all (p,q), the equivariant del and delbar operators restrict to maps*

$$\partial_G : (\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^{m,n} \to (\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^{m+1,n},$$
$$\bar{\partial}_G : (\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^{m,n} \to (\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^{m,n+1}.$$

Proof. Given $\alpha = \omega \otimes f \in \mathscr{A}^{p,q}(M) \otimes S^r(\mathfrak{g}^*)_\mathbb{C} \subset (\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^{p+r,q+r}$, the components of $\partial_G \alpha$ are $\partial \omega \otimes f$ with bidegree $(p+1,q) + (r,r)$ and $\iota_{\bar{Z}_i^\sharp} \omega \otimes u^i f$ with bidegree $(p, q-1) + (r+1, r+1)$. Hence $\partial_G \alpha \in (\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^{p+r+1,q+r}$ and ∂_G defines a map

$$\partial_G : (\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^{p+r,q+r} \to (\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C})^{p+r+1,q+r}$$

as desired. Likewise for $\bar{\partial}_G$. □

Lemma 3.2. *The del and delbar operators preserve the subspace $\mathscr{A}_G^{*,*}(M) \subset \mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_\mathbb{C}$ and restrict to operators $\partial_G : \mathscr{A}_G^{*,*}(M) \to \mathscr{A}_G^{*+1,*}(M)$ and $\bar{\partial}_G : \mathscr{A}_G^{*,*}(M) \to \mathscr{A}_G^{*,*+1}(M)$ on $\mathscr{A}_G^{*,*}(M)$.*

Proof. Recall that d_G preserves G-equivariance, i.e., for all $\alpha \in \mathscr{A}_G^*(M)$, $g \cdot (d_G \alpha(X)) = (d_G \alpha)(\mathrm{Ad}_g \cdot X)$. The splitting d_G yields

$$\underbrace{g \cdot (\partial_G \alpha)(X)}_{(p+1,q)} + \underbrace{g \cdot (\bar{\partial}_G \alpha)(X)}_{(p,q+1)} = \underbrace{(\partial_G \alpha)(\mathrm{Ad}_g \cdot X)}_{(p+1,q)} + \underbrace{(\bar{\partial}_G \alpha)(\mathrm{Ad}_g \cdot X)}_{(p,q+1)}.$$

Comparing terms of the same degree, we deduce that

$$g \cdot (\partial_G \alpha(X)) = (\partial_G \alpha)(\mathrm{Ad}_g \cdot X), \ g \cdot (\bar{\partial}_G \alpha(X)) = (\bar{\partial}_G \alpha)(\mathrm{Ad}_g \cdot X),$$

i.e., ∂_G and $\bar{\partial}_G$ preserves G-equivariance. □

Proposition 3.3. *The equivariant Dolbeault operators satisfy*

$$\partial_G \circ \partial_G = \bar{\partial}_G \circ \bar{\partial}_G = 0,\ \partial_G \circ \bar{\partial}_G + \bar{\partial}_G \circ \partial_G = 0.$$

Proof. It follows from the identity $d_G \circ d_G = 0$ that for all $\alpha \in \mathscr{A}_G^{p,q}(M)$,

$$d_G d_G \alpha = \underbrace{\partial_G \partial_G \alpha}_{(p+2,q)} + \underbrace{(\partial_G \bar{\partial}_G + \bar{\partial}_G \partial_G)\alpha}_{(p+1,q+1)} + \underbrace{\bar{\partial}_G \bar{\partial}_G \alpha}_{(p,q+2)} = 0.$$

For degree reasons, all three terms vanish as desired. □

It follows that $((\mathscr{A}^{*,*}(M) \otimes S(\mathfrak{g}^*)_{\mathbb{C}})^G; \partial_G, \bar{\partial}_G)$ is a differential bigraded algebra. Hence we can define the *equivariant Dolbeault cohomology* on M as in the non-equivariant case.

Proposition 3.4. *If a pure form $\alpha \in \mathscr{A}_G^{p,q}(M)$ is d_G-closed, then $\partial_G \alpha = \bar{\partial}_G \alpha = 0$.*

Proof. Note that the $(p+1,q)$-component of $d_G\alpha$ is $\partial_G\alpha$ and the $(p,q+1)$-component is $\bar{\partial}_G\alpha$. Since $d_G\alpha = 0$, we deduce that $\partial_G\alpha = \bar{\partial}_G\alpha = 0$. □

A fundamental fact in Kähler geometry is the $\partial\bar{\partial}$-lemma, which says that if M is a compact Kähler manifold, then d-closed pure forms on M are locally $\partial\bar{\partial}$-exact. (See, e.g., [Huy05, Lemma 3.12].) This is related to the fact that compact Kähler manifolds are *formal*. For an equivariant analogue of this lemma to hold, we need to impose analogous conditions on the pair M and the G-action.

Definition 3.5. A topological manifold M with a G-action is *equivariantly formal* if the Serre spectral sequence of the fibration $M \to EG \times_G M \to BG$ degenerates at the E_2 term, where EG is the universal bundle and BG the classifying space of G.

An exposition on the various terminologies involved in this definition is way beyond the scope of this paper. We refer the interested reader to [GS13, chapter 6] for an overview. For our purpose, it suffices to note that M is equivariantly formal if it is compact and admits an equivariant symplectic form, i.e. the G-action is Hamiltonian with respect to this symplectic form. (cf. [GS13, note 6.9.4].)

Lemma 3.6. ($\partial_G\bar{\partial}_G$-Lemma; [Lil03, Theorem 5.1]) *Let M be a compact Kähler manifold with a holomorphic action by a compact, connected Lie group G. Suppose that M is equivariantly formal with respect to the G-action. If a pure form $\alpha \in \mathscr{A}_G^{p,q}(M)$ is equivariantly closed, then α is equivariantly exact if and only if*

$$\alpha = i\partial_G\bar{\partial}_G\beta$$

for some $\beta \in \mathscr{A}_G^{p-1,q-1}(M)$.

4 The equivariant first Chern class

In this section, we examine the equivariant connection and curvature on vector bundles equipped with an infinitesimal lift of the G-action on the base manifold to the total space, which is a generalization of equivariant vector bundles. We also extend the classical properties of the first Chern class on holomorphic line bundles to the equivariant case. In each subsection, we briefly go over the non-equivariant constructions before defining their equivariant analogs.

Let $\pi : E \to M$ be a complex vector bundle over M. A *twisted differential* p-form is a p-form with values in E, or equivalently, a section of the twisted form bundle $\bigwedge^p M \otimes E$ over M. Let $\{s^i\}$ be a trivialization of E over U, then a section $\sigma \in \mathscr{A}^p(M,E) = \Gamma(U, \mathscr{A}^p(M) \otimes E)$ can be written as $\sigma = \alpha_i \otimes s^i$ with $\alpha_i \in \mathscr{A}^p(M)$. Denote by $\mathscr{A}^*(E) = \mathscr{A}^*(M,E)$ the collection of twisted differential forms.

4.1 Lifts of an action to vector bundles

Suppose that G acts smoothly on M. An *infinitesimal lift* of the G action on M to E is a Lie algebra homomorphism

$$\Phi : \mathfrak{g} \to \{\text{first order differential operators on } \Gamma(M,E)\},\ X \mapsto \Phi_X$$

such that for all $s \in \Gamma(M,E),\ f \in C^\infty(M)$, we have

$$\Phi_X(f \cdot s) = (\mathscr{L}_{X^\sharp} f) \cdot s + f \cdot \Phi_X(s).$$

Heuristically, an infinitesimal lift is a "lift" of the action of the Lie algebra $\mathfrak{g}$ on $\mathscr{A}^*(M)$ to $\mathscr{A}^*(E)$. This concept should be equivalent to the notion of infinitesimal lifts in [Kos70], [HY76], and more recently [MiR01] defined via vector fields on the total space of E, although we won't prove the equivalence here.

In some cases, an infinitesimal lift comes from a lift of the G-action on the base space.

Definition 4.1. A *lift* of the G-action on M to E is an action of G on the total space of E covering the action on M such that the associated map $\hat{g} : E_p \to E_{g \cdot p}$ is linear for all $p \in M$ and $g \in G$.

A vector bundle $E \to M$ equipped with a lift of the G-action on M to E is called a *G-equivariant vector bundle*.

A lift induces an infinitesimal lift as follows (cf. [BGV03, section 1.1]): Given a lift of the G-action on M to E, there is an induced G-action on the space of sections $\mathscr{A}^0(E) = \Gamma(M,E)$ given by

$$(g \cdot s)(p) = \hat{g} \cdot s(g^{-1} \cdot p),\ \forall s \in \Gamma(M,E), p \in M.$$

This induces a $\mathfrak{g}$-parametrized family of first-order differential operators $\mathscr{L}_X^E$ on $\Gamma(M,E)$ given by

$$\mathscr{L}_X^E(s) := \frac{d}{dt}\Big|_{t=0} (\exp(tX) \cdot s).$$

One can check that for all $f \in C^\infty(M),\ s \in \Gamma(M,E)$,

$$\mathscr{L}_X^E(f \cdot s) = \mathscr{L}_{X^\sharp} f \cdot s + f \cdot \mathscr{L}_X^E(s).$$

Hence $\mathscr{L}^E$ is an infinitesimal lift.

Remark 4.2. In general, an infinitesimal lift does not necessarily come from a lift of the group action on the base manifold. On complex line bundles, conditions for an infinitesimal lift to correspond to a lift are examined in [HY76] and [MiR01].

From here on, we fix an infinitesimal lift Φ of the G-action on M to E. We would like to define an action of the Lie algebra $\mathfrak{g}$ on $\alpha \in \mathscr{A}^*(E) \otimes S(\mathfrak{g}^*)$. However, it is difficult to give an explicit characterization since taking the derivative of the component in $S(\mathfrak{g}^*)$ generates more terms as the degree of the associated polynomial grows. For our purposes, it suffices to consider two cases where the degree of the associated polynomial is small.

Proposition 4.3. *i) There exists a unique Lie algebra homomorphism*

$$\mathscr{L}^\Phi : \mathfrak{g} \to \{\text{first order differential operators on } \mathscr{A}^*(E)\}$$

such that for all $\omega \in \mathscr{A}^k(M), \sigma \in \Gamma(M,E)$, we have

$$\mathscr{L}_X^\Phi(\omega \otimes \sigma) = \mathscr{L}_{X^\sharp} \omega \otimes \sigma + \omega \otimes \Phi_X(\sigma).$$

ii) There exists a unique Lie algebra homomorphism

$$D^\Phi : \mathfrak{g} \to \{\text{first order differential operators on } \mathscr{A}^0(E) \otimes \mathfrak{g}^*\}$$

such that for all $\alpha \in \mathscr{A}^0(E) \otimes \mathfrak{g}^*, Y \in \mathfrak{g}$, we have

$$D_X^\Phi(\alpha)(Y) = \Phi_X(\alpha(Y)) - \alpha(\mathrm{ad}_X Y).$$

Note that the operator Φ induces an associated map $\hat{\Phi} : \mathscr{A}^0(E) \to \mathscr{A}^0(E) \otimes \mathfrak{g}^*$ defined as follows: for any $\sigma \in \mathscr{A}^0(E)$, the tensor $\hat{\Phi}(\sigma) \in \mathscr{A}^0(E) \otimes \mathfrak{g}^*$ is the twisted form associated to the map $X \mapsto \Phi_X(\sigma)$ from $\mathfrak{g}$ to $\mathscr{A}^0(E)$.

Lemma 4.4. *The map $\hat{\Phi}$ commutes with the $\mathfrak{g}$-action as defined in Proposition 4.3, i.e., for all $\sigma \in \Gamma(M,E)$ and $X \in \mathfrak{g}$,*

$$\hat{\Phi}(\Phi_X(\sigma)) = D_X^{\Phi}(\hat{\Phi}(\sigma)).$$

Proof. This is just unraveling the definition given above. For all $Y \in \mathfrak{g}$, we have

$$D_X^{\Phi}(\hat{\Phi}(\sigma))(Y) = \Phi_X(\hat{\Phi}(\sigma)(Y)) - \hat{\Phi}(\sigma)([X,Y]) = \Phi_X(\Phi_Y(\sigma)) - \Phi_{[X,Y]}(\sigma),$$

$$\hat{\Phi}(\Phi_X(\sigma))(Y) = \Phi_Y(\Phi_X(\sigma)).$$

Then the statement follows from the fact that D^{Φ} is a Lie algebra homomorphism. □

4.2 Connections and curvature

4.2.1 The nonequivariant case

Definition 4.5. A *connection* on E is an operator $\nabla : \mathscr{A}^0(E) \to \mathscr{A}^1(E)$ that satisfies the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for all $s \in \mathscr{A}^0(E)$ and $f \in C^\infty(M)$. This extends uniquely to an operator $d^\nabla : \mathscr{A}^*(E) \to \mathscr{A}^{*+1}(E)$ with

$$d^\nabla(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^p \alpha \wedge d^\nabla(\sigma)$$

for all $\sigma \in \mathscr{A}^*(E)$ and $\alpha \in \mathscr{A}^p(M)$.

Definition 4.6. The *curvature* of ∇ is the composition $F^\nabla = d^\nabla \circ d^\nabla : \mathscr{A}^*(E) \to \mathscr{A}^{*+2}(E)$.

In particular, with respect to a local trivialization $\{s^i\}$, any connection ∇ can be written as $\nabla = d + \theta$, where $\theta \in \mathscr{A}^1(M, \mathrm{End}(E))$ is a matrix of 1-forms. We call θ the *connection form* associated to $\{s^i\}$, noting that θ depends on the choice of trivialization. (cf. [Huy05, remark 4.2.5]) In comparison, the curvature $F_\nabla = d\theta + \theta \wedge \theta$ is independent of the choice of trivialization.

One can see that if $E = L$ is a line bundle, then $F^\nabla = d\theta$ is a closed differential two-form. This is a special case of the *second Bianchi identity*, which says that $d^\nabla(F^\nabla) = 0$ for any connection ∇ on a vector bundle E.

Remark 4.7. [BGV03, section 1.1] presents a more motivated definition of connection forms on vector bundles (or fiber bundles in general). Here we give a (very) rough sketch of the main idea.

A tangent vector V on $\pi : E \to M$ is *vertical* if it is tangent to the fibers, i.e., $V(\pi^* f) = 0$ for all smooth function f on M and $X \in \mathfrak{g}$. Heuristically, a vertical vector points to the direction along the fibers and hence is "vertical" to M.

Denote by VE the bundle of vertical vectors and TE the tangent bundle of E. A *connection one-form* θ is an element of $\mathscr{A}^1(E, VE)$ such that $\iota_{\mathscr{V}}\theta = \mathscr{V}$ for all sections $\mathscr{V}$ of VE. Roughly speaking, θ measures the projection of a vector field on E along the "vertical directions". The kernel of θ, dubbed the *horizontal bundle*, is isomorphic to $\pi^*(TM)$. This agrees with our previous definition, as the exterior derivative d measures the change along the "horizontal directions".

Now we are ready to examine the equivariant analog of the above notions.

4.2.2 Equivariant connections

Fix an infinitesimal lift Φ of the G-action on M to E. We would like to define an equivariant connection on $(\mathscr{A}^*(E) \otimes \mathfrak{g}^*)^{\Phi}$. Imitating the construction of equivariant exterior derivative $d_G = d - \iota_{X^\sharp}$, an equivariant connection would be of the form $d^\nabla - \iota_{X^\sharp}$ for some connection ∇ on E. Here $\iota_{X^\sharp}$ acts on $\mathscr{A}^*(E)$ by

$$\iota_{X^\sharp}(\alpha \otimes \sigma) = (\iota_{X^\sharp}\alpha) \otimes \sigma,\ \forall \alpha \in \mathscr{A}^*(M),\ \sigma \in \Gamma(M,E).$$

We would like such a map to preserve the space of twisted forms that are Φ-invariant, so we require ∇ to be a Φ-invariant connection.

Definition 4.8. A connection ∇ on a vector bundle E with an infinitesimal lift Φ of the G-action is Φ-*invariant* if for all $X \in \mathfrak{g}$, $\sigma \in \mathscr{A}^0(E)$ and smooth vector fields $Y \in \Gamma(M,TM)$,

$$\mathscr{L}_X^{\Phi}(\nabla_Y \sigma) = \nabla_{[X^\sharp,Y]}(\sigma) + \nabla_Y(\Phi_X(\sigma)).$$

Therefore, we have the following construction:

Proposition 4.9. *Let ∇ be a Φ-invariant connection on E. Then there exists a unique differential operator d_G^{∇} associated with a ∇ is a differential operator $d_G^{\nabla} : \mathscr{A}^*(E) \otimes \mathfrak{g}^* \to \mathscr{A}^*(E) \otimes \mathfrak{g}^*$ with the property that*

$$(d_G^{\nabla}\alpha)(X) := d^{\nabla}(\alpha(X)) - \iota_{X^\sharp}(\alpha(X))$$

for all $\alpha \in \mathscr{A}^*(E) \otimes \mathfrak{g}^*$, $X \in \mathfrak{g}$.

We call this differential operator d_G^{∇} the Φ-*equivariant connection* associated to the Φ-invariant connection ∇ on E.

Lemma 4.10. *The operators $d_G^{\nabla} : \mathscr{A}^0(E) \to \mathscr{A}^1(E)$, $d_G^{\nabla} : \mathscr{A}^1(E) \to \mathscr{A}^2(E) \oplus \mathscr{A}^0(E) \otimes (g)^*$ commute with the $\mathfrak{g}$-actions constructed in Proposition 4.3. That is,*

1. *For all $\sigma \in \mathscr{A}^0(E)$, $X \in \mathfrak{g}$,*
$$d_G^{\nabla}(\Phi_X(\sigma)) = \mathscr{L}_X^{\Phi}(d_G^{\nabla}\sigma).$$
2. *For all $\alpha \in \mathscr{A}^1(E)$, $X \in \mathfrak{g}$,*
$$d_G^{\nabla}(\mathscr{L}_X^{\Phi}(\alpha)) = (\mathscr{L}_X^{\Phi}, D_X^{\Phi})(d_G^{\nabla}\alpha).$$

Proof. The first equation is just a reformulation of Definition 4.8.

Write $\alpha = \omega \otimes \sigma$ with $\omega \in \mathscr{A}^1(M)$ and $\sigma \in \Gamma(M,E)$. Then

$$\begin{aligned} &d_G^{\nabla}(\mathscr{L}_X^{\Phi}(\alpha))(Y) = d_G^{\nabla}(\mathscr{L}_{X^\sharp}\omega \otimes \sigma + \omega \otimes \Phi_X(\sigma))(Y) \\ =&d_G(\mathscr{L}_{X^\sharp}\omega)(Y) \otimes \sigma - (\mathscr{L}_{X^\sharp}\omega)(Y) \otimes \nabla_{Y^\sharp}(\sigma) + d_G(\omega)(Y) \otimes \Phi_X(\sigma) - \omega(Y) \otimes d_G^{\nabla}(\Phi_X(\sigma))(Y) \\ =&(\mathscr{L}_{X^\sharp}(d_G\omega)(Y) \otimes \sigma - \iota_{[X^\sharp,Y^\sharp]}\omega \otimes \sigma) + d_G(\omega)(Y) \otimes \Phi_X(\sigma) - (\mathscr{L}_{X^\sharp}\omega)(Y) \otimes \nabla_{Y^\sharp}(\sigma) - \omega(Y) \otimes \mathscr{L}_X^{\Phi}(d_G^{\nabla}\sigma)(Y). \end{aligned}$$

One can check that the last line is exactly the expansion of $(\mathscr{L}_X^{\Phi} + D_X^{\Phi})(d_G^{\nabla}\alpha)$. $\square$

4.2.3 Equivariant curvature

As is in the non-equivariant case, there is an *equivariant curvature* associated to d_G^{∇}.

Definition 4.11. The *equivariant curvature* of an equivariant connection d_G^{∇} is given by the map $F_G^{\nabla} : \mathscr{A}^0(E) \to \mathscr{A}^2(E) \oplus \mathscr{A}^0(E) \otimes \mathfrak{g}^*$,

$$(F_G^{\nabla}\sigma)(X) = ((d_G^{\nabla})^2\sigma)(X) + \Phi_X(\sigma) = F^{\nabla}\sigma - \nabla_{X^\sharp}\sigma + \Phi_X\sigma.$$

Note that by Lemma 4.4 and Lemma 4.10, the map F_G^{∇} commutes with the $\mathfrak{g}$-actions constructed in Proposition 4.3. That is, for all $\sigma \in \mathscr{A}^0(E)$, $X \in \mathfrak{g}$,

$$F_G^{\nabla}(\Phi_X(\sigma)) = (\mathscr{L}_X^{\Phi}, D_X^{\Phi})(F_G^{\nabla}\sigma).$$

Lemma 4.12. *The operator F_G^{∇} is tensorial.*

Proof. Note that F^{∇} is tensorial and for all $f \in C^\infty(M)$ and sections $\sigma \in \Gamma(M,E)$,

$$\begin{aligned} (\Phi_X - \nabla_{X^\sharp})(f \cdot \sigma) =&(\mathscr{L}_{X^\sharp}f) \cdot \sigma + f \cdot \Phi_X(\sigma) - (f \cdot (\nabla_{X^\sharp}\sigma) + (\nabla_{X^\sharp}f) \cdot \sigma) \\ =&f \cdot \Phi_X(\sigma) - f \cdot (\nabla_{X^\sharp}\sigma). \end{aligned}$$

$\square$

Hence F_G^{∇} defines an element in $\mathscr{A}^2(\mathrm{End}(E)) \oplus (\mathscr{A}^0(\mathrm{End}(E)) \otimes \mathfrak{g}^*)$. Here an element in $\mathscr{A}_{\Phi}^*(\mathrm{End}(E))$ acts on $\sigma \in \mathscr{A}^0(E)$ by wedge product on the form part and evaluation on the section part, producing an element of $\mathscr{A}^2(E) \oplus \mathscr{A}^0(E) \otimes \mathfrak{g}^*$. Moreover the tensor F_G^{∇} is Φ-invariant, i.e., invariant with respect to the $\mathfrak{g}$-action on $\mathscr{A}^2(\mathrm{End}(E)) \oplus (\mathscr{A}^0(\mathrm{End}(E)) \otimes \mathfrak{g}^*)$ induced by those constructed in Proposition 4.3.

Proposition 4.13 (Equivariant second Bianchi identity)**.** *The operator*

$$d_G^\nabla : A^2(\mathrm{End}(E)) \oplus (\mathscr{A}^0(\mathrm{End}(E)) \otimes \mathfrak{g}^*) \to A^3(\mathrm{End}(E)) \oplus (\mathscr{A}^1(\mathrm{End}(E)) \otimes \mathfrak{g}^*)$$

annihilates F_G^∇.

Proof. $d_G^\nabla(F_G^\nabla) = d^\nabla(F^\nabla) + (-\iota_{X^\sharp} F^\nabla + d^\nabla(\Phi_X - \nabla_{X^\sharp}))$. Note that $d^\nabla(F^\nabla) = 0$ by the second Bianchi identity. Hence for all $\sigma \in \Gamma(M, E)$ and smooth vector fields Y over M, we have

$$\begin{aligned}
((d_G^\nabla F_G^\nabla)\sigma)(Y) &= -F^\nabla(X^\sharp, Y)\sigma + \nabla_Y(\Phi_X - \nabla_{X^\sharp}))\sigma \\
&= -F^\nabla(X^\sharp, Y)\sigma + \nabla_Y(\Phi_X(\sigma) - \nabla_{X^\sharp}\sigma) - (\Phi_X - \nabla_{X^\sharp})(\nabla_Y \sigma) \\
&= -F^\nabla(X^\sharp, Y)\sigma + [\nabla_Y, \Phi_X]\sigma - [\nabla_Y, \nabla_{X^\sharp}]\sigma \\
&= -F^\nabla(X^\sharp, Y)\sigma + (\nabla_{[Y, X^\sharp]} - [\nabla_Y, \nabla_{X^\sharp}])\sigma = 0.
\end{aligned}$$

The last step makes use of the classic definition of the curvature tensor in Riemannian geometry. See, e.g., [Huy05, section 4.A]. □

In particular, if $E = L$ is a complex line bundle, then the space of linear endomorphisms $\mathbb{C} \to \mathbb{C}$ is isomorphic to $\mathbb{C}$. Hence we can identify $A^2(\mathrm{End}(E)) \oplus (\mathscr{A}^0(\mathrm{End}(E)) \otimes \mathfrak{g}^*)$ with $A^2(M) \oplus (\mathscr{A}^0(M) \otimes \mathfrak{g}^*)$, and the Φ-invariant subspace of the former with the G-invariant subspace of the latter, namely, $\mathscr{A}_G^2(M)$, Since $d_G^\nabla = d_G$ on $\mathscr{A}_G^*(M)$, the following is immediate from the equivariant Bianchi identity.

Corollary 4.14. *Suppose that* ∇ *is a* Φ*-invariant connection on a line bundle* L *with an infinitesimal lift* Φ*. The equivariant curvature* $F_G^\nabla \in \mathscr{A}_G^2(M)$ *is an equivariantly closed two-form.*

Set $\mu(X) = \Phi_X - \nabla_{X^\sharp} = F_G^\nabla(X) - F^\nabla$, so μ defines an equivariant map $\mathfrak{g} \to \Gamma(M, \mathrm{End}(E))$. It follows from the equivariant Bianchi identity that

$$\nabla(\mu(X)) = \iota_{X^\sharp} F^\nabla(X).$$

We call μ the *moment map* of the action Φ due to its resemblance to the moment map in symplectic geometry.

4.3 First Chern class on complex line bundles

For the sake of completeness, we briefly discuss in this section the equivariant first Chern class of a complex line bundle L with an infinitesimal lift Φ.

4.3.1 The non-equivariant case

Suppose that L is a complex line bundle. The *first Chern class* of L is a special case of the classic construction of Chern classes on any complex vector bundle. (c.f. [Huy05, 4.4.8].) Explicitly, it is the cohomology class $c_1(L) = \frac{i}{2\pi}[F^\nabla] \in H^2(M, \mathbb{R})$ for any connection ∇ on L. In particular, the first Chern class is independent of the choice of the connection ∇.

4.3.2 The equivariant first Chern class

Suppose that G acts smoothly on M and $L \to M$ is a complex line bundle equipped with an infinitesimal lift Φ, which we denote by the pair (L, Φ). Suppose further that L admits a Φ-invariant connection ∇. Note that such a connection exists by averaging if Φ exponentiates to a lift of the G-action on L, but the author does not know whether (L, Φ) necessarily admits such a connection in general.

Lemma 4.15. *The cohomology class* $\frac{i}{2\pi}[F_G^\nabla] \in H_G^2(M, \mathbb{R})$ *is independent of choice of the* Φ*-invariant connection* ∇.

Proof. Let ∇' be another Φ-invariant connection on L. Note that locally $\nabla' - \nabla = \eta \in \mathscr{A}^1(\mathrm{End}(E))$ is Φ-invariant and $F^{\nabla'} - F^{\nabla} = d\eta$. (cf. [Huy05, 4.2.3, 4.3.4]) Hence

$$(F_G^{\nabla'} - F_G^{\nabla})(X) = (F^{\nabla'} - F^{\nabla})(X) - (\nabla'_{X^\sharp} - \nabla_{X^\sharp})(X) = (d\eta)(X) - \iota_{X^\sharp}(\eta)(X) = (d_G\eta)(X)$$

is equivariantly exact, concluding the proof. □

As a result, we can define the *equivariant first Chern class* of (L, Φ) to be the cohomology class

$$c_1^G(L,\Phi) := \frac{i}{2\pi}[F_G^{\nabla}] \in H_G^2(M,\mathbb{R}).$$

4.4 First Chern class on holomorphic vector bundles

4.4.1 The non-equivariant case

To start with, we quickly summarize the construction of the first Chern class on a holomorphic line bundle L (cf. [Fin, chapter 2], [Huy05, chapter 4]).

If M is a complex manifold and E is a holomorphic vector bundle, then a connection $\nabla : \mathscr{A}^0(E) \to \mathscr{A}^1(E)$ splits as $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where $\nabla^{1,0}, \nabla^{0,1}$ are respectively the compositions of ∇ with the canonical projections of $\mathscr{A}^1(E)$ onto $\mathscr{A}^{1,0}(E)$, $\mathscr{A}^{1,0}(E)$. The Leibniz rule implies that

$$\nabla^{0,1}(f \cdot s) = \bar{\partial} f \otimes s + f \cdot \nabla^{0,1}(s).$$

On the other hand, the delbar operator $\bar{\partial}$ can be extended to an operator $\bar{\partial}_E$ on $\mathscr{A}^{*,*}(E)$. Explicitly, let $\{s^i\}$ be a holomorphic local trivialization of L. Then $\bar{\partial}$ acts on $\sigma = \alpha_i \otimes s^i$ by $\bar{\partial}\sigma := \bar{\partial}(\alpha_i) \otimes s^i$. One can check that this is independent of the choice of local trivialization.

Definition 4.16. A connection ∇ on E is *compatible with the holomorphic structure* if $\nabla^{0,1} = \bar{\partial}_E$.

In fact, such a connection always exists. Here we only concern ourselves with a holomorphic line bundle $E = L$. Note that a hermitian structure h on L is locally a real-valued positive smooth function. We say that a connection ∇ is *compatible with the hermitian structure* h if

$$dh(\sigma_1, \sigma_2) = h(\nabla\sigma_1, \sigma_2) + h(\sigma_1, \nabla\sigma_2).$$

Given h, there exists a unique connection ∇^h, the *Chern connection*, that is compatible with both the holomorphic structure and the hermitian structure h. Locally, the Chern connection can be written as

$$\nabla^h = d + \partial \log h$$

and the curvature F^h of ∇^h is given by the closed $(1,1)$-form $F^h = \bar{\partial}\partial \log h$.

The cohomology class $c_1(L) = \frac{i}{2\pi}[F^h]$ of L is independent of choice of the hermitian structure h on L. Moreover, the converse is also true, provided that M is a compact Kähler manifold.

Lemma 4.17. ([Fin, lemma 2.10]) *Let $L \to M$ be a holomorphic line bundle over a compact Kähler manifold. Given a real $(1,1)$-form $\omega \in c_1(L)$, there is a hermitian metric h, unique up to scaling by a constant, such that $\frac{i}{2\pi}F^h = \omega$.*

Proof. Pick any hermitian metric h_0 as reference, which produces a closed real $(1,1)$-form $\omega_0 = \frac{i}{2\pi}\bar{\partial}\partial \log h$ with cohomology class $c_1(L)$. Given any closed real $(1,1)$-form ω in the same cohomology class, we have $d(\omega - \omega_0) = 0$. The $\partial\bar{\partial}$-lemma says that $\omega - \omega_0 = i\partial\bar{\partial} f$ for some smooth real function f, which is unique up to some constant. One can check that $h = e^{-2\pi f}h_0$ is a hermitian metric on L with $\frac{i}{2\pi}F^h = \omega$. □

Now we shall consider the analogue of the above for a holomorphic line bundle L with an infinitesimal lift of the G-action.

4.4.2 The equivariant Chern class on holomorphic line bundles with infinitesimal lifts

Let M be a complex manifold equipped with a holomorphic G-action and L a holomorphic line bundle over M.

Definition 4.18. An infinitesimal lift Φ to a vector bundle E is *holomorphic* if, for all $X \in \mathfrak{g}$ and smooth vector fields Y on M,

$$[\Phi_X, \iota_Y \bar{\partial}_E] = \iota_{[X^\sharp, Y]} \bar{\partial}_E.$$

In the rest of this subsection, let Φ be a holomorphic infinitesimal lift on L.

Definition 4.19. A hermitian metric on L is Φ-*invariant* if for all $\sigma \in \Gamma(M, E)$

$$X^\sharp(h(\sigma,\sigma)) = h(\Phi_X \sigma, \sigma) + h(\sigma, \Phi_X \sigma).$$

In particular, if Φ exponentiates to a lift of the G-action on L, then a Φ-invariant hermitian metric always exists by averaging. Now we suppose that L admits a Φ-invariant hermitian metric h.

Lemma 4.20. *The Chern connection ∇^h associated with a Φ-invariant hermitian metric h on L is Φ-invariant in the sense of Definition 4.8.*

Proof. Fix a non-vanishing local section σ. Since Φ covers the G-action on M, locally we can write $\Phi_X(f \cdot \sigma) = X^\sharp f + f \cdot \phi(\sigma)$ for some function $\phi \in C^\infty(L)$. One can check that Φ is holomorphic implies that ϕ is holomorphic.

The Φ-invariance of the hermitian metric h implies that

$$X^\sharp(h(\sigma,\sigma)) = h(\phi\sigma,\sigma) + h(\sigma,\phi\sigma) = (\phi + \overline{\phi})h(\sigma,\sigma),$$

i.e., $X^\sharp \log h = \phi + \overline{\phi}$. Note that ϕ is holomorphic, so $\bar{\partial}\phi = 0$ and $\partial\overline{\phi} = 0$. Denote by $Y^{1,0}$ the holomorphic part of a vector field Y on M, i.e., $\iota_Y \partial = \iota_{Y^{1,0}} d$, then we have

$$\begin{aligned}
[\Phi_X, \nabla_Y] - \nabla_{[X^\sharp, Y]} =& [X^\sharp + \phi, \iota_Y(d + \partial \log h)] - \iota_{[X^\sharp,Y]}(d + \partial \log h) \\
=& ([X^\sharp, Y] + [X^\sharp, Y^{1,0} \log h] - [\iota_Y d, \phi]) - ([X^\sharp, Y] + \iota_{[X^\sharp,Y]} \partial \log h) \\
=& ((X^\sharp Y^{1,0}) \log h + Y^{1,0}(X^\sharp \log h) - \iota_Y(d\phi)) - (X^\sharp Y^{1,0}) \log h \\
=& \iota_{Y^{1,0}} d(\phi + \overline{\phi}) - \iota_Y(\partial\phi) \\
=& \iota_Y(\partial\phi + \partial\overline{\phi}) - \iota_Y(\partial\phi) \\
=& 0.
\end{aligned}$$

$\square$

Hence we obtain the *equivariant Chern connection* ∇^h_G associated with a Φ-invariant hermitian metric h on L. Locally $\nabla^h_G = d_G + \partial \log h$ and its curvature $F^h_G(X) = F^h(X) + (\Phi_X - \nabla^h_{X^\sharp}(X))$ is an equivariantly closed imaginary $(1,1)$-form.

It follows from Lemma 4.15 that every Φ-invariant hermitian metric h corresponds to an element $\frac{i}{2\pi} F^h_G$ in the cohomology class $c^G_1(L,\Phi)$. Again, we want to know if every element in $c^G_1(L,\Phi)$ comes from a Φ-invariant hermitian metric. Recall that there is a canonical decomposition $\mathscr{A}^2_G(M) = (\mathscr{A}^2(M) \otimes S^0(\mathfrak{g}^*))^G \oplus (\mathscr{A}^0(M) \otimes S^1(\mathfrak{g}^*))^G$ and the subspace of equivariantly closed real $(1,1)$-forms is

$$Z^{1,1}_G(M) = \{(\omega,\ \mu) \in \mathscr{A}^{1,1}(M)^G \oplus C^\infty(M,\mathfrak{g}^*)^G \mid d\omega = 0,\ d(\mu, X) = \iota_X \omega,\ \forall X \in \mathfrak{g}\}.$$

In order to adapt the argument in the proof of Lemma 4.17, which makes use of the $\partial\bar{\partial}$-lemma, we require that M be equivariantly formal.

Theorem 4.21. *Let M be a compact Kähler manifold with a holomorphic G-action such that M is equivariantly formal with respect to G. Let L be a holomorphic line bundle over M with an infinitesimal lift Φ. Suppose that L admits a Φ-invariant hermitian metric. Then given a real $(1,1)$-form $\alpha \in c^G_1(L,\Phi)$, there is a Φ-invariant hermitian metric h, unique up to scaling by a constant, such that $\frac{i}{2\pi} F^h_G = \alpha$.*

Proof. Fix a Φ-invariant hermitian metric h_0 on L and write its curvature as $\frac{i}{2\pi}F_G^{h_0} = (\omega_0,\ \mu_0) \in Z_G^{1,1}(M)$. Given any $[(\omega,\ \mu)] \in c_1^G(L,\Phi)$, the equivariant two-form $(\omega - \omega_0,\ \mu - \mu_0)$ is equivariantly exact. Since M is equivariantly formal, the $\partial_G\bar{\partial}_G$-lemma (Lemma 3.6) ensures the existence of a G-invariant real-valued smooth function $f : M \to \mathfrak{g}$ such that $(\omega - \omega_0,\ \mu - \mu_0) = i\partial_G\bar{\partial}_G f$.

Set $h = e^{-2\pi f}h_0$. Since $\nabla^h - \nabla^{h_0} = -2\pi\partial f$, it follows from the proof of Lemma 4.15 that $F_G^h - F_G^{h_0} = d_G(-2\pi\partial f)$. Note that $(\bar{\partial}_G\bar{\partial}_G f)(X) = ((\bar{\partial} - \iota_{Z^\sharp})(\partial - \iota_{\overline{Z}^\sharp})f)(X) = (\bar{\partial}\partial f - \iota_{Z^\sharp}\partial f)(X)$. Hence

$$\begin{aligned}\frac{i}{2\pi}F_G^h =& \frac{i}{2\pi}F_G^{h_0} - i\cdot d_G(\partial f) = (\omega_0,\mu_0) - i(\bar{\partial}\partial f - \iota_{X^\sharp}\partial f)\\ =&(\omega_0,\mu_0) - i\bar{\partial}_G\partial_G f = (\omega_0,\mu_0) + i\partial_G\bar{\partial}_G f = (\omega,\mu).\end{aligned}$$

Moreover h is Φ-invariant since $X^\sharp f = 0$ for all $X \in \mathfrak{g}$. This concludes the proof. $\square$

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