INVESTIGATING DETERMINANTS OF SOME COMBINATORIAL HARD LEFSCHETZ ISOMORPHISMS

UROP+ FINAL PAPER, SUMMER 2017
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AUGUST 31, 2017

Abstract. We investigate the determinants of two different isomorphisms. The first type arises from simplicial polytopes. In particular, the hard Lefschetz theorem gives us an isomorphism between cohomology rings of a projective toric variety that corresponds to the simplicial polytope and a result of Stanley tells us that these cohomology groups are isomorphic to Stanley-Reisner rings modulo some linear relations. We can choose a basis for this isomorphism via a line shelling and we look explicitly at these isomorphisms for specific instances of simplicial polytopes. The second type of isomorphism is between vector spaces spanned by ranks in the poset determined by the strong Bruhat order on S_n. It is known that this poset has the strong Sperner property so such an isomorphism must exist. We investigate the determinant of an explicit example of such an isomorphism.

1. Introduction

Our work is motivated, in part, by questions and conjectures posed in [5], in which Stanley tries to prove that W_n, the poset structure of the weak Bruhat order of S_n, has the strong Sperner property. He gives an order-raising operator on W_n and conjectures its determinant up to sign. Part of this paper is dedicated to considering an analogous operator for the strong order, also called S_n, and investigating its determinant. However, this is not necessary to prove that S_n has the strong Sperner property; in [6] Stanley proves existence of a viable order-raising operator. This is because there is a connection between S_n and a complete flag variety $\mathcal{F}_n$ identifying ranks of the former to cohomology rings of the latter. Since the hard Lefschetz theorem on $\mathcal{F}_n$ implies an isomorphism between cohomology rings, it also implies the existence of an appropriate isomorphism on S_n [5].

This isomorphism between cohomology rings of varieties inspires the other part of our paper. Considering a specific projective toric variety $X_{\mathcal{P}}$ that corresponds to a particular simplicial polytope, we form an isomorphism between the cohomology rings of $X_{\mathcal{P}}$ and the *Stanley-Reisner ring* of the underlying polytope modulo some linear relations so the hard Lefschetz theorem implies an isomorphism in this new ring. We can procedurally select bases for this isomorphism via a shelling of the polytope but in general there is no canonical choice of basis or shelling. Furthermore, the linear relations that we mod out of the Stanley-Reisner ring are not only dependent on the combinatorial structure of the polytope, but also its location in Euclidean space, making the isomorphism and its determinant difficult to analyze generally. We will look at some specific instances of polytopes and the possible determinants the isomorphism can take on.

In section 2 we give some preliminary definitions and results. In section 3, we examine the hard Lefschetz isomorphisms for some specific instances of simplicial polytopes, including some n-simplices and regular cross-polytopes in n dimensions. In section 4, we investigate the determinant for an order-raising operator in S_n as well as list some explicit calculations in the appendix.

2. Preliminaries

2.1. **Simplicial Polytope Preliminaries.** Given an integral convex polytope $\mathcal{P}$ embedded in $\mathbb{Z}^d$, we let $X_{\mathcal{P}}$ be a projective toric variety defined as

$$X_{\mathcal{P}} = \mathrm{cl}\{x^{\alpha^1}y, ..., x^{\alpha^n}y : x_1, ..., x_d, y \in \mathbb{C}^*\} \subset \mathbb{P}^{n-1}_{\mathbb{C}},$$

where the α^i are the vertices of $\mathcal{P}$ and $x^{\alpha^i} = x_1^{\alpha_1^i} \cdots x_d^{\alpha_d^i}$ [4].

Definition 2.1. A polytope $\mathcal{P}$ is simplicial if each of its faces is a simplex [7].

Definition 2.2. Given Δ, the boundary complex of a simplicial polytope $\mathcal{P}$ with vertices $x_1, ..., x_n$, define $\mathbb{R}[\Delta] = \mathbb{R}[x_1, ..., x_n]/I_\Delta$, with

$$I_\Delta = \{x_{i_1}x_{i_2}\cdots x_{i_r} : i_1 < i_2 < \cdots < i_r, \{x_{i_1}, ..., x_{i_r}\} \notin \Delta\}$$

[4]. That is $\mathbb{R}[\Delta]$ is the polynomial ring in n variables modulo the ideal of monomials not in Δ. This ring is called the *Stanley-Reisner ring* (or *face ring*).

We now have our first couple theorems from [4]:

Theorem 2.3. *([4]) Let $\mathcal{P}$ be an integral simplicial d-polytope in $\mathbb{R}^d$, let $X_{\mathcal{P}}$ be its corresponding toric variety, and let $\mathbb{R}[\Delta]$ be its Stanley-Reisner ring. Then there is an isomorphism*

$$\varphi : H^*(X_{\mathcal{P}}; \mathbb{R}) \to \mathbb{R}[\Delta]/(\theta_1, ..., \theta_d)$$

for some choice of linear relations $\theta_1, ..., \theta_d \in \mathbb{R}[\Delta]$. This isomorphism halves degree so $\varphi(H^{2i}(X_{\mathcal{P}})) = \mathbb{R}[\Delta]_i$ and $\varphi(H^{2i+1}(X_{\mathcal{P}})) = 0$.

Let the θ_i take the form $\theta = \sum_i \alpha_i x_i$ for $\alpha_i \in \mathbb{R}$ and x_i in the vertex set of $\mathcal{P}$. The θ_i must be chosen so that the following two conditions are met: for each facet $F \in \mathcal{P}$ the restrictions of the θ_i, $\theta_i|F = \sum_{x_i \in F} \alpha_i x_i$ span a d-dimensional vector space. Secondly, each θ_i must arise from a linear function on the underlying Euclidean space, so $\theta = \sum_i p_i(x_i)x_i$ for some $p_i : \mathbb{R}^d \to \mathbb{R}$.

Note that in the previous definition each p_i must be strictly linear (and not affinely linear). This places the restriction that $p_i(0) = 0$. This implies that the rings $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_d)$ for polytopes that differ only in position in Euclidean space may not (and in fact are likely not) isomorphic.

Theorem 2.4. *([4]) There exists an element $\omega \in H^2(X_{\mathcal{P}})$ (in the Stanley-Reisner ring, this element is canonically $\omega = x_1 + \cdots + x_d$, the sum of the vertices) such* *that for all $0 \leq i \leq \lfloor d/2 \rfloor$, the map $\omega^{d-2i} : H^{2i}(X_{\mathcal{P}}) \to H^{2d-2i}(X_{\mathcal{P}})$, which in the Stanley-Reisner ring is given by multiplication by ω^{d-i}, is an isomorphism.*

We can see that this is the (or one of the) isomorphisms assured by the hard Lefschetz theorem.

Definition 2.5. ([4]) Again, let Δ be the boundary complex of a simplicial polytope. We say that Δ is *shellable* if there exists an ordering of the facets $F_1, F_2, ..., F_s$ such that for each subcomplex generated by the first i facets, i.e. $2^{F_1} \cup 2^{F_2} \cup \cdots \cup 2^{F_i}$, where 2^{F_i} denotes the facet F_i and all the faces of Δ that it contains, there exists a unique minimal face (with respect to dimension) among the faces which do not belong to the subcomplex generated by the first $i-1$ facets. In other words, if Δ_i denotes the subcomplex generated by the first i facets, then $\Delta_i \setminus \Delta_{i-1}$ contains a unique minimal element, called $r(F_i)$. This ordering of facets is called a *shelling order* or *shelling* of Δ.

Now, we have a theorem from [4] regarding how to choose a basis for our isomorphism:

Theorem 2.6. *([4]) Let $F_1, F_2, ..., F_s$ be the shelling order for Δ and let $\theta_1, ..., \theta_d$ be an appropriate set of linear relations as outlined in Theorem 2.3. Then,*

$$B = \{x^{r(F_i)} : 1 \leq i \leq s\},$$

with $r(F_i)$ as defined in Definition 2.5 and x^F is the face monomial $x^F = \prod_{x_i \in F} x_i$, is a basis for $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_d)$.

Definition 2.7. ([7]) For the boundary complex Δ of a $d-$dimensional simplicial polytope, define its h-vector to be $h(\Delta) = (h_0, h_1, ..., h_d)$ with

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$$

where f_k is the number of $k-$dimensional faces contained in the polytope.

Proposition 2.8. *([4]) Let $F_1, ..., F_s$ be a shelling order of Δ and let $h(\Delta) = (h_0, h_1, ..., h_n)$ be its h-vector. Then, we have*

$$h_k = \#\{j : |r(F_j) = i\}$$

where $r(F_j)$ is as defined in Definition 2.5.

2.2. **Strong Bruhat Order Preliminaries.** We start with a review of poset terminology, mainly from [5]. For a more detailed review of posets, see [3]. Let P be a finite graded rank-symmetric poset of rank m, so we can write $P = P_0 \cup P_1 \cup \cdots \cup P_m$ with $p_k = p_{m-k}$ if $p_k = |P_k|$.

Definition 2.9. Let U be a linear operator on P, $U : \mathbb{Q}P \to \mathbb{Q}P$, where $\mathbb{Q}P$ denotes the $\mathbb{Q}$-vector space with basis P. We call U *order-raising* if for every $t \in P$, we have $U(t) \in \mathbb{Q}C^+(t)$ where $C^+(t)$ is the set of elements that cover t.

Definition 2.10. ([5]) A poset P is *strong Sperner* if for all $r \geq 1$, the largest subset S of P that does not contain an $(r+1)$ element chain has the same number of elements as the largest union of r levels in P.

Theorem 2.11. *([6],[5]) Suppose there exists an order-raising operator $U : \mathbb{Q}P \to \mathbb{Q}P$ such that if $0 \le k < m/2$ then the linear transformation $U^{m-2k} : \mathbb{Q}P_k \to \mathbb{Q}P_{m-k}$ is a bijection. Then P is strongly Sperner.*

Definition 2.12. The *weak order* W_n of S_n is a poset so that a permutation w has rank k in W_n if w has k inversions. Furthermore, v covers u in W_n if and only if $v = us_i$ for some i, where s_i is the transposition $(i, i+1)$, and if $\ell(v) = \ell(u) + 1$, where $\ell(w)$ is the length, or number of inversions, of w.

In [5], Stanley tries to find such an order-raising operator to prove that W_n has the strong Sperner property. He considers the operator

$$U(u) = \sum_{i:\ell(us_i)=1+\ell(u)} i \cdot us_i.$$

He is able to deduce that

$$U^j(u) = \left(\binom{n}{2} - 2k\right)! \sum_v \nu_{vu^{-1}} v$$

where v ranges over permutations satisfying $\ell(v) = \ell(u) + j$ and $v > u$ in weak order, and where ν_w is defined as $\nu_w = \mathfrak{S}_w(1, 1, ..., 1)$; that is the corresponding Schubert polynomial evaluated at $(1, 1, ..., 1)$. Following the conventions set in [5], we let $D(n, k)$ be the matrix of the linear transformation $U^{\binom{n}{2}-2k} : \mathbb{Q}(W_n)_k \to \mathbb{Q}(W_n)_{\binom{n}{2}-k}$ with bases $(W_n)_k$ and $(W_n)_{\binom{n}{2}-k}$ in some order. Then the determinant of $D(n, k)$ is, up to sign, given by

$$\det D(n, k) = \left(\binom{n}{2} - 2k\right)!^{\#(W_n)_k} \det \tilde{D}(n, k)$$

where $\tilde{D}(n, k)_{uv}$ is given by $\nu_{vu^{-1}}$ if $u \le v$ in W_n and 0 otherwise. Stanley was able to state the following conjecture and prove it for $k = 1$:

Conjecture 2.13. ([5]) We have

$$\det \tilde{D}(n, k) = \pm \prod_{i=0}^{k-1} \left(\frac{\binom{n}{2} - (k+i)}{k-i} \right)^{\#(W_n)_i}.$$

A natural extension would be to consider a similar operator for the strong order S_n. Stanley suggests in [5] to consider the operator

$$V(u) = \sum_{\substack{1 \le i < j \le n \\ \ell(ut_{ij})=1+\ell(u)}} (j - i) \cdot ut_{ij}$$

where t_{ij} is the transposition between i and j. This choice of operator gives us some nice properties, including the fact due to [2] that $V^{\binom{n}{2}}(\mathrm{id}) = \binom{n}{2}! w_0$ where id is the identity permutation and $w_0 = n, n-1, ..., 1$ is the identity permutation reversed. We now define a type of polynomial as introduced in [2] that is closely related to S_n.

Definition 2.14. For S_n, the *degree polynomial* $\mathfrak{D}_{u,w} \in \mathbb{Q}[y_1, ..., y_n]$ is defined as

$$\mathfrak{D}_{u,w} = \frac{1}{(\ell(w) - \ell(u))!} \sum_C m_C(y)$$

where the summation runs over all saturated chains from u to w and $m_C(y)$ is the weight of the chain, which is the product of the weights of each individual edge in the chain. The weight of the edge $v < vs_{ij}$ is defined to be $y_i - y_j$. We let $\mathfrak{D}_{id,w} = \mathfrak{D}_w$.

From [2], we also have an alternate way of calculating these degree polynomials iteratively, which is as follows:

$$\mathfrak{D}_{w_0} = \frac{1}{1!2!\cdots(n-1)!} \prod_{1\le i<j\le n} (y_i - y_j)$$

$$\mathfrak{D}_{u,w} = \mathfrak{S}_u(\partial/\partial y_1, ..., \partial/\partial y_n)\mathfrak{S}_{w_0 w}(\partial/\partial y_1, ..., \partial/\partial y_n)\mathfrak{D}_{w_0},$$

where we again have that $\mathfrak{S}_u$ is the Schubert polynomial corresponding to the permutation u.

3. Hard Lefschetz Determinants

Here, we will examine some specific instances of simplicial polytopes and calculate the possible determinant of the bijection given by multiplication by ω^k where $\omega = x_1 + x_2 + \cdots + x_n \in \mathbb{R}[\Delta]/(\theta_1, ..., \theta_d)$.

3.1. **n-Simplices Centered at Origin.** Let $\mathcal{P}$ be any n-simplex whose centroid is the origin. It is well-known that the centroid of an n-simplex divides the medians in the ratio $1 : n$ (see [1] for instance). With this we can see that we can make the following choice of θ_i:

$$\begin{aligned}
\theta_1 &= x_1 + x_2 + \cdots + x_n - nx_{n+1} \\
\theta_2 &= x_1 + x_2 + \cdots + x_{n-1} + x_{n+1} - nx_n \\
&\vdots \\
\theta_n &= x_1 + x_3 + x_4 \cdots + x_{n+1} - nx_2.
\end{aligned}$$

We can easily check to see that these θ_i satisfy both conditions established in Theorem 2.3. Using these θ_i, we are now able to calculate all possible determinants for ω^k.

Proposition 3.1. *For any n-simplex whose centroid is located at the origin, we have* $\det \omega^k = (n+1)^k$.

Proof. From θ_1 we can see $x_1 = nx_{n+1} - x_2 - x_3 - \cdots - x_n$ in the ring $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_n)$. Substituting this into the rest of the θ_i, we can see $x_1 = x_2 = \cdots = x_{n+1}$. From this, we can see that $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_n) \cong \mathbb{R}[x]$ so in particular, all monomials of the same degree and coefficient are equivalent in this ring.

Since an n-simplex has $\binom{n+1}{k+1}$ k-faces, we can use Stanley's trick to calculate that the h-vector of an n-simplex is $(1, 1, ..., 1)$ [7]. This is because the difference table is a rotated version of Pascal's triangle and the bottom row from which the h-vector is read off is just a diagonal containing all 1's in Pascal's triangle. From Proposition 2.8, we can see any basis for $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_n)$ contains exactly one element of degree i for $0 \le i \le n$. Since the elements of this basis are face monomials, and we have already seen that all monomials of the same degree are equivalent in $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_n)$, we conclude that this basis is dependent only on n.

In particular, the basis is independent of shelling order. This also implies that the determinant for multiplication by ω^k is unique. Without loss of generality, let the basis be $B = \{1, x_1, x_1^2, ..., x_1^n\}$. Since $x_1 = x_2 = \cdots = x_{n+1}$, we have $\omega = (n+1)x_1$ so ω^k takes x_1^{i-k} to $(n+1)^k x_1^i$ so $\det \omega^k = (n+1)^k$. $\square$

3.2. **Cross-Polytope.** Let $\mathcal{P}$ be the standard n-dimensional cross-polytope with $2n$ vertices at $(\pm 1, 0, 0, ..., 0)$ and its permutations. Label its vertices such that x_{2i} and x_{2i-1} are opposite of each other for $1 \leq i \leq n$. Therefore, we can make the following choice of θ_i:

$$\begin{aligned} \theta_1 &= x_1 - x_2 \\ \theta_2 &= x_3 - x_4 \\ &\vdots \\ \theta_n &= x_{2n-1} - x_{2n} \end{aligned}$$

and we can see that they satisfy the necessary conditions. We first consider the determinant for ω^n (note that the exponent is n and not k).

Proposition 3.2. *For a standard n-dimensional cross-polytope, we have* $\det \omega^n = 2^n n!$.

Proof. Since $x_{2i-1}x_{2i}$ is killed when modding out I_Δ when forming the Stanley-Reisner ring, we can see that $x_1^2, x_2^2, ..., x_{2n}^2$ are all equivalent to 0 in the ring $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_n)$ since $x_{2i} = x_{2i-1} \Rightarrow x_{2i}^2 = x_{2i}x_{2i-1} = 0$. Let us consider ω^n. In its expansion, we can consider only the squarefree terms since all squares are 0. Since $x_{2i-1}x_{2i}$ is killed, all nonzero terms in the expansion of ω^n must contain exactly one factor from each of the pairs (x_{2i-1}, x_{2i}). There are 2^n such terms. Since these terms can be in any order, we have a total of $2^n n!$ nonzero terms in the expansion of ω^n. Also note that each of these terms are equal since $x_{2i-1} = x_{2i}$ in $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_n)$ so we have $\det \omega^n = 2^n n!$ independent of basis. $\square$

Now we consider the more general case of looking at the determinant for ω^k.

Proposition 3.3. *For a standard d-dimensional cross-polytope, if $k = d - 2i$ we have* $\det \omega^k = (2^k k!)^{\binom{d}{i}} \det D$, *where D is a certain 0/1 matrix, which is unique up to reordering the rows/columns.*

Proof. As in the previous case, we have that all squares are 0 in $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_d)$. Also recall that the h-vector for the cross-simplex is the nth row of Pascal's triangle so $h_j = \binom{d}{j}$. Therefore, by Proposition 2.8, the basis must have $\binom{d}{i}$ degree i and degree $d - i$ terms. Therefore, the matrix for the linear transformation ω^k is $\binom{d}{i} \times \binom{d}{i}$. Now let us examine how multiplication by ω^k affects degree i monomials. We know that any degree i monomial must exactly one factor from i of the pairs (x_{2i-1}, x_{2i}). Therefore, if this term is multiplied by any term in any of these pairs, it will be killed. Therefore, k of the remaining $d-i$ pairs must be chosen to select exactly one factor from. By the same logic as the previous example, there are $2^k k!$ ways to do this for each unique term and there are $\binom{d-i}{d-2i} = \binom{d-i}{k}$ unique degree $d - i$ terms that appear when a single degree i term is multiplied by ω^k. This means that the matrix for this linear transformation has entries that are all either

$2^k k!$ or 0. Since this is a $\binom{d}{i} \times \binom{d}{i}$ matrix, we have $\det \omega^k = (2^k k!)^{\binom{d}{i}} \det D$, where D is the 0/1 matrix that is the result when we factor out $(2^k k!)$. $\square$

Let us say a little more about D. We can think of D as follows: label the rows of D with the size i subsets of $[n]$ and label the rows of D with the size $d - i$ subsets of $[n]$. An entry in D is 1 if and only if the row is a subset of the column.

3.3. **Off-Centered Simplices.** To demonstrate that the determinant is not always unique and that the combinatorial structure alone is not enough to classify the determinant of ω^k, we will consider the case when $\mathcal{P}$ is the n-simplex with vertices $(1, 1, ..., 1)$ and permutations of $(2, 1, 1, ..., 1)$. This is just the standard simplex shifted by the vector $(1, 1, ..., 1)$. Label the vertex at $(1, 1, ..., 1)$ x_1 and the rest $x_2, x_3, ..., x_{n+1}$. We can choose for our θ_i

$$\begin{aligned}
\theta_1 &= x_1 + 2x_2 + x_3 + \cdots + x_{n+1} \\
\theta_2 &= x_1 + x_2 + 2x_3 + \cdots + x_{n+1} \\
&\vdots \\
\theta_n &= x_1 + x_2 + x_3 + \cdots + 2x_{n+1}
\end{aligned}$$

where each variable has the coefficient 2 in exactly one equation except for x_1. We can again verify that these θ_i satisfy our conditions.

Proposition 3.4. *For this off-centered n-simplex, $\det \omega^k$ is not independent of the basis. More precisely, $\det \omega^k$ can take the values $(-1)^k$ and $\frac{(-1)^{k-1}}{n+1}$.*

Proof. Note that by θ_1, we have $x_1 = -2x_2 - x_3 - \cdots - x_{n+1}$ in $\mathbb{R}[\Delta]/(\theta_1, ..., \theta_d)$. By substituting for x_1 in each of the other θ_i, we can see that $x_2 = x_3 = \cdots = x_{n+1}$ and $x_1 = -(n+1)x_2$. By summing all of the θ_i, we can also see that $(n+1)\omega - x_1 = 0$ so $\omega = \frac{1}{n+1}x_1 = -x_2 = -x_3 = ... = -x_{n+1}$. Let us consider ω^k which takes the space of $d - 2k$ degree polynomials to the space of $d - k$ degree polynomials. Let the basis element monomial for the former be A and the basis element monomial for the latter be B. If neither A nor B contain x_1, or if they both contain x_1, we can see that $\omega A = (-1)^k B$ so $\det \omega^k = (-1)^k$. If A does not contain x_1 but B does, then we can see that $\omega A = \frac{(-1)^{k-1}}{n+1} B$ so $\det \omega^k = \frac{(-1)^{k-1}}{n+1}$ and it is clear that the determinant is not independent of the basis. $\square$

This example shows why trying to find these determinants is in general difficult; simple translations are enough to change the determinant and force a dependence on the basis, and therefore the shelling behind it.

4. Strong Bruhat Order Determinants

In the strong order S_n, let us consider the operator $V : \mathbb{Q}S_n \to \mathbb{Q}S_n$ defined as

$$V(u) = \sum_{\substack{1 \le i < j \le n \\ \ell(ut_{ij}) = 1 + \ell(u)}} (j - i) \cdot ut_{ij}.$$

As in [5], let $E(n, k)$ be the matrix of $V^{n-2k} : \mathbb{Q}(S_n)_k \to \mathbb{Q}(S_n)_{\binom{n}{2}-k}$ with bases some ordering of$(S_n)_k$ and $(S_n)_{\binom{n}{2}-k}$. We also let $e(n, k)$ denote the determinant of $E(n,k)$, which is unique up to sign. With the aid of some computer computations, we have found $e(n,k)$ for several values of n and k (these calculations can be found in the appendix). We can see that the largest prime factors of $e(n,k)$ grow very quickly even for relatively small choices for n and k.

Note that by the definition of $\mathfrak{D}_{u,v}$, we have

$$E(n,k)_{u,v} = (\ell(v) - \ell(u))!\mathfrak{D}_{u,v}(\mathbf{y}) = \left(\binom{n}{2} - 2k\right)!\mathfrak{D}_{u,v}(\mathbf{y}),$$

where $\mathbf{y} = (-1,-2,...,-n)$ is the vector that gives us the weights that correspond to the operator V. This equality holds since the operator V sums all weights of edges between an element and the elements in the next rank so V^{n-2k} applied to all the elements in $(S_n)_k$ sums the weights of saturated chains between rank k and rank $\binom{n}{2} - k$, which is very similar to the definition of $\mathfrak{D}_{u,v}$. Also, since $u \in (S_n)_k$ and $v \in (S_n)_{\binom{n}{2}-k}$, we always have $(\ell(v) - \ell(u))! = (\binom{n}{2} - 2k)!$. This means we can factor out $(\binom{n}{2} - 2k)!$ from each element in the matrix $E(n,k)$ to obtain the matrix $\tilde{E}(n,k)$ so we can write the determinant as

$$e(n,k) = \left(\binom{n}{2} - 2k\right)!^{\#(W_n)_k} \det \tilde{E}(n,k),$$

where $\#(W_n)_k = \#(S_n)_k$ is the rank size. Note we have

$$\tilde{E}(n,k)_{u,v} = \mathfrak{D}_{u,v}(-1,-2,...,-n).$$

This formula is quite similar to a formula Stanley gives in [5] for the weak order analog; his states

$$d(n,k) = \left(\binom{n}{2} - 2k\right)!^{\#(W_n)_k} \det \tilde{D}(n,k),$$

where $d(n,k)$ is the determinant of the matrix $D(n,k)$ from [5] and $\tilde{D}(n,k)$ is the matrix that results from dividing each entry in $D(n,k)$ by $(\binom{n}{2} - 2k)!$ and, as mentioned in [5], has explicit form

$$\tilde{D}(n,k)_{u,v} = \mathfrak{S}_{uv^{-1}}(1,1,...,1).$$

This gives us two analogous formulas for the weak and strong order; furthermore, Schubert polynomials and degree polynomials are dual as described in [2] so the matrices $\tilde{D}(n,k)$ and $\tilde{E}(n,k)$ are also related despite the fact that $\tilde{D}(n,k)$ is an integer matrix while $\tilde{E}(n,k)$ is only guaranteed to have rational entries.

Now, we examine the $k = 1$ case more closely. Recall from our definitions that

$$\mathfrak{D}_{w_0} = \frac{1}{1!2!\cdots(n-1)!}\prod_{1\le i<j\le n}(y_i - y_j)$$
$$\mathfrak{D}_{u,w} = \mathfrak{S}_u(\partial/\partial y_1,...,\partial/\partial y_n)\mathfrak{S}_{w_0w}(\partial/\partial y_1,...,\partial/\partial y_n)\mathfrak{D}_{w_0},$$

and when $k = 1$, we have $\ell(u) = 1$, $\ell(w) = \binom{n}{2} - 1$, and $\ell(w_0w) = 1$. Therefore, we can say that $u = s_i$ and $w_0w = s_j$ for some i,j where s_i is the adjacent transposition $(i,i+1)$. It is known that $\mathfrak{S}_{s_i} = x_1 + x_2 + ... + x_i$ [5], so $\mathfrak{D}_{u,w}$ is the result of two degree one differential operators applied to $\mathfrak{D}_{w_0}$. Furthermore, we can index the entries of $\tilde{E}(n,1)$ so that

$$\tilde{E}(n,1)_{i,j} = (\partial/\partial y_1 + \cdots + \partial/\partial y_i)(\partial/\partial y_1 + \cdots + \partial/\partial y_j)\mathfrak{D}_{w_0}(-1,-2,...,-n).$$

We wish to evaluate $\tilde{E}(n,1)_{i,j}$ explicitly, so let us calculate all second order partial derivatives of $\mathfrak{D}_{w_0}$. First, let us evaluate partial derivatives of the form $\partial^2/\partial y_a^2 \mathfrak{D}_{w_0}$. We have

$$\begin{aligned}
\frac{\partial^2}{\partial y_a^2}\mathfrak{D}_{w_0} &= \frac{1}{1!\cdots(n-1)!}\prod_{\substack{1\le i<j\le n\\ i,j\neq a}}(y_i-y_j)\cdot\frac{\partial^2}{\partial y_a^2}(-1)^{a-1}\prod_{\substack{i=1\\ i\neq a}}^{n}(y_a-y_i)\\
&= \frac{(-1)^{a-1}}{1!\cdots(n-1)!}\prod_{\substack{1\le i<j\le n\\ i,j\neq a}}(y_i-y_j)\cdot\frac{\partial}{\partial y_a}\left(\sum_{\substack{k=1\\ k\neq a}}^{n}\prod_{\substack{m=1\\ m\neq k}}^{n}(y_a-y_m)\right)\\
&= \frac{(-1)^{a-1}}{1!\cdots(n-1)!}\prod_{\substack{1\le i<j\le n\\ i,j\neq a}}(y_i-y_j)\cdot 2\left(\sum_{\substack{1\le j<k\le n\\ j,k\neq a}}\prod_{\substack{m=1\\ m\neq j,k}}^{n}(y_a-y_m)\right).
\end{aligned}$$

Plugging in $y_i=-i$ yields

$$\begin{aligned}
\frac{\partial^2}{\partial y_a^2}\mathfrak{D}_{w_0}(\mathbf{y}) &= \frac{2}{1!\cdots(n-1)!}\cdot\frac{1!\cdots(n-1)!}{(a-1)!(n-a)!}\cdot\sum_{\substack{i=-a+1\\ i\neq 0}}^{n-a}\sum_{\substack{j=i+1\\ j\neq 0}}^{n-a}\frac{(a-1)!(n-a)!}{ij}\\
&= 2\sum_{\substack{i=-a+1\\ i\neq 0}}^{n-a}\sum_{\substack{j=i+1\\ j\neq 0}}^{n-a}\frac{1}{ij}\\
&= 2\left(\sum_{i=1}^{n-a}\sum_{j=i+1}^{n-a}\frac{1}{ij}+\sum_{i=-a+1}^{-1}\sum_{j=1}^{n-a}\frac{1}{ij}+\sum_{i=-a+1}^{-1}\sum_{j=i+1}^{-1}\frac{1}{ij}\right)\\
&= H_{n-a}^2 - H_{n-a,2} - 2H_{a-1}H_{n-a} + H_{a-1}^2 - H_{a-1,2},
\end{aligned}$$

where H_n is the nth harmonic number and $H_{n,2}$ is the nth generalized harmonic number of order 2; that is $H_n = \sum_{i=1}^{n} 1/i$ and $H_{n,2} = \sum_{i=1}^{n} 1/i^2$. Now let us consider the mixed partial derivatives. Let us evaluate $\partial^2/\partial y_b\partial y_a\mathfrak{D}_{w_0}$ and without loss of generality, let $a<b$. We can do this since $\partial^2/\partial y_b\partial y_a\mathfrak{D}_{w_0} = \partial^2/\partial y_a\partial y_b\mathfrak{D}_{w_0}$

because $\mathfrak{D}_{w_0}$ is a polynomial. We have

$$\frac{\partial^2}{\partial y_b \partial y_a}\mathfrak{D}_{w_0} = \frac{(-1)^{a-1}}{1!\cdots(n-1)!}\cdot\frac{\partial}{\partial y_b}\prod_{\substack{1\le i<j\le n\\ i,j\neq a}}(y_i-y_j)\left(\sum_{\substack{k=1\\k\neq a}}^{n}\prod_{\substack{m=1\\m\neq k}}^{n}(y_a-y_m)\right)$$

$$= \frac{(-1)^{a-b-3}}{1!\cdots(n-1)!}\prod_{\substack{1\le i<j\le n\\ i,j\neq a,b}}(y_i-y_j)\cdot\frac{\partial}{\partial y_b}\prod_{\substack{i=1\\i\neq a,b}}^{n}(y_b-y_i)\left(\sum_{\substack{k=1\\k\neq a}}^{n}\prod_{\substack{m=1\\m\neq k}}^{n}(y_a-y_m)\right)$$

$$= \frac{(-1)^{a-b-3}}{1!\cdots(n-1)!}\prod_{\substack{1\le i<j\le n\\ i,j\neq a,b}}(y_i-y_j)\left(\prod_{\substack{i=1\\i\neq a,b}}^{n}(y_b-y_i)\cdot(-1)\left(\sum_{\substack{k=1\\k\neq a,b}}^{n}\prod_{\substack{m=1\\m\neq k}}^{n}(y_a-y_m)\right)\right.$$

$$\left.+\left(\sum_{\substack{k=1\\k\neq a}}^{n}\prod_{\substack{m=1\\m\neq k}}^{n}(y_a-y_m)\right)\left(\sum_{\substack{k=1\\k\neq a,b}}^{n}\prod_{\substack{m=1\\m\neq k}}^{n}(y_b-y_m)\right)\right).$$

Plugging in $y_i=-i$ yields

$$\frac{\partial^2}{\partial y_b \partial y_a}\mathfrak{D}_{w_0}(\mathbf{y}) = \frac{(b-a)(-1)^{a-b-3}}{(a-1)!(n-a)!(b-1)!(n-b)!}\left(\frac{(-1)^{b-1}(b-1)!(n-b)!}{b-a}\right.$$

$$\left.\cdot\left(\sum_{\substack{k=1\\k\neq a,b}}^{n}\prod_{\substack{m=1\\m\neq k}}^{n}(y_a-y_m)\right)+\sum_{\substack{i=-a+1\\i\neq 0}}^{n-a}\frac{(-1)^{a-1}(a-1)!(n-a)!}{i}\cdot\sum_{\substack{k=1\\k\neq a,b}}^{n}\prod_{\substack{m=1\\m\neq k}}^{n}(y_b-y_m)\right)$$

$$= \frac{-1}{(a-1)!(n-a)!}\sum_{\substack{i=-a+1\\i\neq 0,b-a}}^{n-a}\frac{(a-1)!(n-a)!}{(b-a)i}$$

$$+\frac{b-a}{(b-1)!(n-b)!}\sum_{\substack{i=-a+1\\i\neq 0}}^{n-a}\frac{1}{i}\sum_{\substack{i=-b+1\\i\neq 0,b-a}}^{n-b}\frac{(b-1)!(n-b)!}{(b-a)i}$$

$$= -\sum_{\substack{i=-a+1\\i\neq 0,b-a}}^{n-a}\frac{1}{(b-a)i}+\sum_{\substack{i=-a+1\\i\neq 0}}^{n-a}\frac{1}{i}\sum_{\substack{i=-b+1\\i\neq 0,b-a}}^{n-b}\frac{1}{i}$$

$$= (H_{n-a}-H_{a-1})(H_{n-b}-H_{b-1}+\frac{1}{b-a})-\frac{1}{b-a}(H_{n-a}-H_{a-1}-\frac{1}{b-a}).$$

From this, we are able to construct any entry of $\tilde{E}(n,1)$; we have

$$\tilde{E}(n,1)_{i,j}=\sum_{a=1}^{i}\sum_{b=1}^{j}d_{ab}$$

where

$$d_{ab}=(H_{n-a}-H_{a-1})(H_{n-b}-H_{b-1}+\frac{1}{b-a})-\frac{1}{b-a}(H_{n-a}-H_{a-1}-\frac{1}{b-a})$$

if $a \neq b$ and

$$d_{aa} = H_{n-a}^2 - H_{n-a,2} - 2H_{a-1}H_{n-a} + H_{a-1}^2 - H_{a-1,2}.$$

By performing determinant-preserving row and column operations on $\tilde{E}(n,1)$, we can transform it into a matrix whose entry at (i,j) is d_{ij}. Therefore, we have

$$\det \tilde{E}(n,1) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} d_{i,\sigma(i)} \right)$$

and

$$e(n,1) = \left(\binom{n}{2} - 2 \right)^{n-1} \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} d_{i,\sigma(i)} \right).$$

Performing similar calculations on higher values of k seems less tractable as not only are the differential operators of a higher degree, their form, the the form of Schubert polynomials, are less regular and harder to work with. Furthermore, there would be several more types of mixed partial derivatives that would have to be calculated as well.

Appendix

Here, we list the results of calculating $e(n,k)$, many of which are not found in [5].

$e(4,1) = \pm 2^{7} \cdot 3 \cdot 5^{2} \cdot 19$

$e(4,2) = \pm 2^{6} \cdot 3 \cdot 29$

$e(5,1) = \pm 2^{22} \cdot 3^{6} \cdot 5^{5} \cdot 7^{4} \cdot 59 \cdot 89$

$e(5,2) = \pm 2^{23} \cdot 3^{11} \cdot 5^{12} \cdot 7^{4} \cdot 3361 \cdot 15817$

$e(5,3) = \pm 2^{25} \cdot 3^{10} \cdot 5^{17} \cdot 7 \cdot 127 \cdot 100151$

$e(5,4) = \pm 2^{15} \cdot 3^{6} \cdot 5^{7} \cdot 7^{2} \cdot 23 \cdot 59^{2}$

$e(6,1) = \pm 2^{41} \cdot 3^{20} \cdot 5^{8} \cdot 7^{7} \cdot 11^{5} \cdot 13^{5} \cdot 89 \cdot 173 \cdot 593$

$e(6,2) = \pm 2^{83} \cdot 3^{45} \cdot 5^{23} \cdot 7^{17} \cdot 11^{14} \cdot 13 \cdot 41 \cdot 2333 \cdot 12959 \cdot 37061 \cdot 92921 \cdot 390421$

$$\begin{aligned} e(6,3) = \pm 2^{136} \cdot 3^{83} \cdot 5^{27} \cdot 7^{30} \cdot 11^{5} \cdot 1657 \cdot 416579 \cdot 385403867 \cdot 66133454765140163 \\ \cdot 917967507536691554972359 3 \end{aligned}$$

$$\begin{aligned} e(6,4) = \pm 2^{168} \cdot 3^{72} \cdot 5^{22} \cdot 7^{53} \cdot 11 \cdot 24509 \cdot 2333732060086702024182044088101918 6187791 \\ \cdot 8811097997134706609839794213310412 7560769587 \end{aligned}$$

$$\begin{aligned} e(6,5) = \pm 2^{170} \cdot 3^{64} \cdot 5^{34} \cdot 7^{35} \cdot 17 \cdot 103 \cdot 1091 \cdot 4229 \cdot 8513 \cdot 107071 \cdot 209789 \cdot 602689 \\ \cdot 1198429681 \cdot 43379003403667 \cdot 5777805032770261 \cdot 119840876574655575019921 \end{aligned}$$

$$\begin{aligned} e(6,6) = \pm 2^{130} \cdot 3^{55} \cdot 5^{14} \cdot 7^{20} \cdot 19 \cdot 503 \cdot 541 \cdot 1153 \cdot 4409 \cdot 8059 \cdot 15937 \cdot 26839 \cdot 31799 \\ \cdot 45530819 \cdot 457896551011 \cdot 2519669695836271 63 \end{aligned}$$

$e(6,7) = \pm 2^{45} \cdot 3^{17} \cdot 5^{3} \cdot 7^{6} \cdot 13^{2} \cdot 43 \cdot 67 \cdot 20058992573 \cdot 24392276537$

$e(7,1) = \pm 2^{88} \cdot 3^{42} \cdot 5^{15} \cdot 7^{13} \cdot 11^{6} \cdot 13^{6} \cdot 17^{6} \cdot 19^{6} \cdot 29 \cdot 2333 \cdot 21341$

$$\begin{aligned} e(7,2) = \pm 2^{257} \cdot 3^{100} \cdot 5^{40} \cdot 7^{46} \cdot 11^{20} \cdot 13^{21} \cdot 17^{20} \cdot 19 \cdot 23 \cdot 1291 \cdot 309157 \cdot 312283 \\ \cdot 13377977218381 \cdot 12726512910740791 \end{aligned}$$

$$\begin{aligned} e(7,3) = \pm 2^{475} \cdot 3^{205} \cdot 5^{89} \cdot 7^{100} \cdot 11^{50} \cdot 13^{50} \cdot 17^{6} \cdot 7841 \cdot 13641238333 \\ \cdot 416008126418773345958608944763 \cdot 1098305273474733510354837760337 \\ \cdot 197967222248696261969575980295340892358680370139285167560542167029 7009 \end{aligned}$$

$e(8,1) = \pm 2^{149} \cdot 3^{67} \cdot 5^{36} \cdot 7^{21} \cdot 11^{14} \cdot 13^{14} \cdot 17^{7} \cdot 19^{7} \cdot 23^{7} \cdot 353 \cdot 701 \cdot 1777 \cdot 4987$

$$\begin{aligned} e(8,2) = \pm 2^{520} \cdot 3^{228} \cdot 5^{86} \cdot 7^{73} \cdot 11^{54} \cdot 13^{28} \cdot 17^{28} \cdot 19^{27} \cdot 23^{28} \cdot 127 \cdot 48406488656797 \cdot \\ 155343364730691858887 \cdot 546128540037304304 4959 \cdot 730073109984372602451748 7 \end{aligned}$$

$$\begin{aligned} e(9,1) = \pm 2^{239} \cdot 3^{116} \cdot 5^{51} \cdot 7^{29} \cdot 11^{24} \cdot 13^{16} \cdot 17^{17} \cdot 19^{8} \cdot 23^{8} \cdot 29^{8} \cdot 31^{8} \cdot 43 \\ \cdot 577 \cdot 279269 \cdot 352523 \end{aligned}$$

$$\begin{aligned} e(10,1) = 2^{325} \cdot 3^{159} \cdot 5^{72} \cdot 7^{48} \cdot 11^{29} \cdot 13^{27} \cdot 17^{18} \cdot 19^{18} \cdot 23^{9} \cdot 29^{9} \cdot 31^{9} \cdot 37^{9} \cdot 41^{9} \cdot 43^{9} \cdot 97 \cdot 181 \\ \cdot 34361 \cdot 956929 \cdot 1214929640347. \end{aligned}$$

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