A Central Limit Theorem for Fluctuations of Internal Diffusion-Limited Aggregation with Multiple Sources

UROP+ Final Paper 2016

Eli Sadovnik

Mentor: Ricardo Grande Izquierdo

Project suggested by: Prof. David Jerison

August 31st, 2016

Abstract

Classical internal diffusion-limited aggregation (internal DLA) is a probabilistic lattice growth model in which an occupied set A_t is inductively defined at each step by starting a random walk on $\mathbb{Z}^d$ at the origin, and if x is the first point the walk visits which isn't already in A_t, we let $A_{t+1} = A_t \cup \{x\}$ (the base case is $A_0 = \emptyset$). Lawler, Bramson, and Griffeath showed that, after rescaling, A_t asymptotically converges to a ball, its deterministic shape. Subsequently, Jerison, Levine, and Sheffield proved a central limit theorem which shows that the fluctuations of the rescaled A_t away from the deterministic shape themselves converge weakly to a modified version of the Gaussian Free Field. Internal DLA, however, can also be modified to have multiple sources by starting the random walks at different points on the lattice according to a starting density function σ. Levine and Peres proved that, after rescaling, the occupied set for internal DLA with multiple sources converges to the solution of a certain obstacle problem. We show that the central limit theorem proved by Jerison, Levine, and Sheffield in the single point source case generalizes in part to the multiple source case in a natural way.

1 Introduction

Internal diffusion-limited aggregation (internal DLA) with a single point source is a random process $\{A_t | t \in \mathbb{Z}_{\geq 0}\}$ defined inductively, representing a growing cluster of particles. Here, A_t is called the "occupied set" and we define it to start empty, $A_0 = \emptyset$. Then, at each step $t \geq 1$, a particle is added at the origin, and moves along an independent random walk on $\mathbb{Z}^d$. To form A_t, we let x be the first point the random walk representing the t^{th} particle visits which isn't in A_{t-1}, and let $A_t = A_{t-1} \cup \{x\}$ (i.e. we stop the particle once it leaves the cluster, and add that lattice site to the cluster). This process was first proposed by Meakin and Deutch [1] in 1986, hoping the process would be useful to model chemical phenomena such as "electropolishing, corrosion, and etching." Internal DLA has been used in practice to model such processes as Copper electropolishing [2] and the high viscosity regime of diffusion of dense, water-immiscible liquids, such as certain oils, into water [3]. It is important to understand the surface regularity of such phenomena, and this can be investigated by studying internal DLA. Meakin and Deutch performed simulations of internal DLA (on cylindrical lattices, as opposed to the normal Euclidean lattices we will use) and obtained numerical evidence that, in two dimensions, the standard deviation of the height of the occupied set (on the cylindrical lattice, after rescaling) is $O(\sqrt{\log n})$ where n is the number of particles simulated, while in three dimensions, the standard deviation of the height of the occupied set actually seemed to stay constant as n grew, i.e. it was $O(1)$.

In 1992, Lawler, Bramson, and Griffeath [4] proved that, with probability one, the rescaled occupied set (for internal DLA on $\mathbb{Z}^d$ this time, i.e. on Euclidean lattices) converges to a ball centered at the origin, in the sense that it will eventually contain any smaller ball centered at the origin and be contained in any bigger ball centered at the origin. This was notable for being the first time a probabilistic lattice growth process was rigorously proved to have the ball as its deterministic shape; among the small number of such processes for which the deterministic shape had been characterized, all had been shown to have anisotropic growth.

This established the deterministic shape of the internal DLA cluster as a ball, and so said that with probability one, the fluctuations away from the ball will eventually be $o(n^{-d})$ where d is the dimension of the lattice and n is the number of points added to the cluster. However, this bound on the size of the maximum fluctuations is larger than what the numerical data predicts [5]. In 2012 and 2013, Jerison, Levine, and Sheffield proved, first in two dimensions [5], and then in higher dimensions [6], with probability one, the maximum fluctuations are eventually $O(\log n)$ in two dimensions and $O(\sqrt{\log n})$ in higher dimensions. This is in line with what numerical simulations expect the maximum fluctuations to be, and so it is thought that this bound is tight [5]. However, this bound is on the maximum fluctuations, and only gives an upper bound on what the standard deviation of the fluctuations could be, which is what Meakin and Deutch studied [1] (indeed, in both two and higher dimensions, the maximum fluctuation bounds proved by Jerison, Levine, and Sheffield are $\sqrt{\log n}$ times what Meakin and Deutch suggested the standard deviations should be).

In 2014, Jerison, Levine, and Sheffield [7] proved a central limit theorem establishing that the fluctuations weakly converge to the restriction of a modified version of the Gaussian free field (what they called the "augmented" Gaussian free field) to the boundary of the deterministic ball. This result does not directly say that the standard deviation of the fluctuations is $O(\sqrt{\log n})$ in two dimensions and $O(1)$ in higher dimensions, which is what Meakin and Deutch [1] predicted from numerical data in 1986, but it does heuristically suggest that this is true.

All of the above results concern the model of internal DLA with a single source of particles at the origin, but one can define the model to have multiple different sources, with varying intensities. In 1992, Diaconis and Fulton [8] defined a growth model that contained internal DLA as a rather specific sub-case. They were then able to prove a number of algebraic properties of their general model, most notably the fact that it is abelian. This allows for a definition of internal DLA with multiple sources. In particular, if σ is an integer-valued function on $\mathbb{Z}^d$ which is nonzero at only finitely many points, then the internal DLA cluster for the starting density σ is defined in a similar way as the normal single-source cluster above. Let $A_0 = \emptyset$, and let $\{x_i\}_{i=1}^n$ be some ordering of the points in the support of σ such that for each x in $\mathbb{Z}^d$, x is represented in the sequence exactly $\sigma(x)$ times (which implies that $n = \sum_{x \in \mathbb{Z}^d} \sigma(x)$). Then, we inductively define A_t by initiating an independent random walk on $\mathbb{Z}^d$ starting at x_t, and adjoining the first point the walk visits which isn't in A_{t-1} to A_{t-1}. The abelian property that Diaconis and Fulton proved extends to this case as well, and says specifically that the law of the last cluster, A_n, is independent of the order of the points $\{x_i\}_{i=1}^n$ (and so we were justified in arbitrarily choosing the sequence beforehand).

In 2009, Levine and Peres [9] showed that this model of multiple source internal DLA has a deterministic shape as well, which is the solution to a certain PDE free-boundary problem, specifically a certain obstacle problem. The proof utilized many of the techniques that were used for the single-source case proved by Lawler, Bramson, and Griffeath [4] in 1992. However, no further fluctuation bounds have been proven for the multiple source case. In this report, we aim to extend the central limit theorem proved by Jerison, Levine, and Sheffield [7] in 2014 to the multiple source case. Taking our cues from Levine and Peres, we model our proof on the proof in the single-source case due to Jerison, Levine, and Sheffield, making necessary changes to generalize to the multiple source case.

Multiple source internal DLA is of interest as it has the potential to provide new insights into the underlying geometry of the aggregation process. Jerison, Levine, and Sheffield proved that on the cylinder, the fluctuations of internal DLA weakly converge to the *regular* Gaussian free field [10], as opposed to the "augmented" Gaussian free field in the case of Euclidean lattices, as noted above. They conjectured that the need for the modification of the Gaussian free field to accomodate the Euclidean lattices was due, at least in part, to the nonzero mean curvature of the boundary of the deterministic shape (i.e. the ball) in the Euclidean case, while in the cylindrical case, the boundary is flat [7]. Understanding how to modify the Gaussian free field in order to accomodate the multiple source case could confirm and make precise these heuristic geometric arguments. Unfortunately, while we have been able to elucidate the underlying covariance structure of the fluctuations in this report, we have not found a way to modify the Gaussian free field to match this structure. Finding the proper way to modify the Gaussian free field to match the covariance structure we've found is the logical next step in this line of research.

After this Introduction, we establish the nomenclature and precise definitions of the objects of study in the Preliminaries section, along with stating theorems and lemmas proved in the literature described above which we will need in the course of our proof. The Results section that follows will consist of the proof of our main result and a proof of a lemma needed for our main result. Finally, in the Next Steps section, we discuss modifying the Gaussian free field to match the covariance structure we've found.

I would like to thank the MIT UROP+ program, organized by Slava Gerovitch, for providing resources for me to pursue this project, and specifically Ricardo Grande Izquierdo for his mentorship throughout. I would also like to especially thank Professor David Jerison for suggesting this project and helping me to interpret this research in a broader context.

2 Preliminaries

We define the multiple source internal DLA cluster A_t on a lattice with starting density σ (an integer-valued function on the lattice which is nonzero only finitely often) in the same way that Levine and Peres do [9], as described above. In particular, we label the points in the support of σ $\{x_1, \ldots, x_n\}$, where the multiplicity of a point in the sequence is the value of σ there (i.e. $\#\{i; x_i = x\} = \sigma(x)$ for all x in the lattice). We set $A_0 = \emptyset$, and define A_t recursively by starting an independent random walk on the lattice at x_t, and adding to A_{t-1} the first point in the walk which is not in A_{t-1}. At a first glance, it seems that A_t depends on the order chosen for $\{x_1, \ldots, x_n\}$. However, the law of A_n (i.e. the final cluster) does *not* actually depend on the order chosen, which was proved by Diaconis and Fulton [8]. All of the following notation and definitions correspond to what is in the paper of Levine and Peres [9].

First, if f is a function on $\frac{1}{m}\mathbb{Z}^d$, then we define $f^\square$ on $\mathbb{R}^d$ by

$$f^\square(x) = f\left(\left(x + \left(-\frac{1}{2m}, \frac{1}{2m}\right]^d\right) \cap \frac{1}{m}\mathbb{Z}^d\right).$$

We clearly have that for all x in $\mathbb{R}^d$, the set $\left(x + \left(-\frac{1}{2m}, \frac{1}{2m}\right]^d\right) \cap \frac{1}{m}\mathbb{Z}^d$ will consist of the single point in $\mathbb{Z}^d$ closest to x, rounding up if there's a conflict in some direction, and so it makes sense to speak of f applied to this singleton set. In general, we will abuse the notation throughout by considering a function applied to a set containing a single point to be the function evaluated at the point in the set.

Similarly, for a set $B \subset \frac{1}{m}\mathbb{Z}^d$, we let $B^\square = B + [-\frac{1}{2m}, \frac{1}{2m}]$ be the subset of $\mathbb{R}^d$ consisting of the lattice boxes surrounding the points in B.

Additionally, for any domain $U \subset \mathbb{R}^d$ and any $\epsilon > 0$, we define the inner and outer ϵ-neighborhoods of D to be

$$U_\epsilon = \{x \in U | B(x, \epsilon) \subset U\},$$
$$U^\epsilon = \{x \in \mathbb{R}^d | B(x, \epsilon) \not\subset U^c\},$$

where $B(x, \epsilon)$ refers to the open ball centered at x with radius ϵ.

Throughout this report, we will assume $\sigma : \mathbb{R}^d \to \mathbb{Z}_{\geq 0}$ and, for all positive integers m, $\sigma_m : \frac{1}{m}\mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ are compactly supported functions with the following properties (which are equations 27-31, 64-67 in the paper of Levine and Peres [9]). First, there must be a bound M such that $0 \leq \sigma \leq M$ and $0 \leq \sigma_m \leq M$ holds everywhere and for all m. Second, there must be a compact set Γ which simultaneously contains all the supports of σ and σ_m for all m. Additionally, σ must be continuous almost everywhere. Furthermore, for all x where σ is continuous, we must have

$$\sigma_m^\square(x) \to \sigma(x).$$

Also, we must have that for all x in $\mathbb{R}^d$, either $\sigma(x) \geq 1$ or $\sigma(x) = 0$. We define $\Omega = \{\sigma \geq 1\}^o$, and we require that $\{\sigma \geq 1\} = \overline{\Omega}$. Finally, we must assume that for all $\epsilon > 0$, there is a $W(\epsilon)$ such that both of the following conditions hold:

$$x \in \{\sigma \geq 1\}_\epsilon \Rightarrow \sigma_m(x) \geq 1 \text{ for all } m \geq W(\epsilon),$$
$$x \notin \{\sigma \geq 1\}^\epsilon \Rightarrow \sigma_m(x) = 0 \text{ for all } m \geq W(\epsilon).$$

These conditions on σ, σ_m, Ω and Γ are designed to be as general as possible. The idea is that σ_m is a sequence of functions on $\frac{1}{m}\mathbb{Z}^d$ such that they converge in the above sense to σ. If σ and Γ satisfy all the requirements above which don't involve σ_m, then setting σ_m to be

$$\sigma_m(x) = \left\lfloor \fint_{x+[-\frac{1}{2m}, \frac{1}{2m}]^d} \sigma(y)dy \right\rceil$$

guarantees the conditions involving σ_m to hold, where $\lfloor \cdot \rceil$ rounds real numbers to the nearest integer, and breaks ties upward.

Let $\{x_{m,i}\}_{i=1}^{n_m}$ be an ordering of the points in the support of σ_m according to multiplicity as above, and let $\{A_{m,t}\}_{t=1}^{n_m}$ be the multiple source internal DLA cluster on $\frac{1}{m}\mathbb{Z}^d$ (defined simply by scaling the corresponding cluster on $\mathbb{Z}^d$ by $\frac{1}{m}$) with initial density σ_m (and choosing the points in the order dictated by $\{x_{m,i}\}$). Thus, $n_m = \sum_{x \in \frac{1}{m}\mathbb{Z}^d} \sigma_m(x)$ is the index of the final cluster.

The last bit of nomenclature and conditions we have to get out of the way concerns the deterministic shape of the multiple source internal DLA. Let $g(x, y)$ be the Green's function on $\mathbb{R}^d \setminus \{0\}$, defined by

$$g(x, y) = \begin{cases} -\frac{1}{2\pi} \log |x - y| & d = 2 \\ \frac{1}{n(n-2)\,\mathrm{vol}(B_1^d)} |x - y|^{2-d} & d \geq 3 \end{cases}$$

where B_1^d is the unit ball in $\mathbb{R}^d$.

Then we define the "obstacle" γ in the obstacle problem defining the deterministic shape of the multiple source internal DLA by

$$\gamma(x) = -|x|^2 - \int_{\mathbb{R}^d} g(x, y)\sigma(y)dy.$$

We now let $s : \mathbb{R}^d \to \mathbb{R}$ be the least superharmonic majorant of γ, so

$$s(x) = \inf\{f(x) | f \text{ is continuous, superharmonic, and } f \geq \gamma\}.$$

Then the "odometer function" for the obstacle problem is $s - \gamma$, and the "noncoincidence set" for this obstacle problem is the set of points D where the odometer function is nonzero, i.e.

$$D = \{x \in \mathbb{R}^d | s(x) > \gamma(x)\}.$$

We then let $\widetilde{D} = D \cup \Omega$. The main relevant result in the paper of Levine and Peres [9] is that $\widetilde{D}$ is the deterministic shape for the multiple source internal DLA process, as we shall now formally state, in addition to three other previous results which we will need to prove our result.

Theorem 1 (Thm. 5.1 in [9]). *With the nomenclature above, given any $\epsilon > 0$, with probability one, we have for sufficiently large m,*

$$\widetilde{D}_\epsilon \cap \frac{1}{m}\mathbb{Z}^d \subset A_{m,n_m} \subset \widetilde{D}^\epsilon \cap \frac{1}{m}\mathbb{Z}^d.$$

The next lemma we will need concerns the roughness of the internal DLA cluster. It gives examples of events which have exponentially small probabilities, which we will be able to paste together to our advantage. The lemma has been modified from its original form to be on $\frac{1}{m}\mathbb{Z}^d$ instead of just $\mathbb{Z}^d$, which makes more sense for use in this context.

Lemma 1 (Lem. 5.12 in [9]). *Let $Q(z, \rho)$ be the cube centered at z with sidelength ρ. There are constants b_0, b_1, and b_2 depending only on the dimension d such that if ρ satisfies*

$$n_m - \#\{x; \sigma_m(x) \neq 0\} \leq b_0 m^d \rho^d$$

and if $Q(z, 3\rho)$ is disjoint from $\{x; \sigma_m(x) \neq 0\}$, then we have

$$\mathbb{P}(\{A_{m,n_m} \not\subset Q(z,\rho)^c\}) \leq b_1 e^{-b_2 m\rho}.$$

This is actually a somewhat weaker statement than lemma 5.12 in [9], but it will suffice to prove what we want.

The next thing we need to define is the final mass configuration for the divisible sandpile. The divisible sandpile is another lattice model, but which is deterministic instead of probabilistic, like internal DLA. We won't go into what exactly it is in too much detail, as it only serves to help us interpret our main result, so knowing its properties is sufficient for our purposes.

Theorem 2 (Thm. 3.9 and Eq. (6) in [9]). *Let ν_m be the final mass configuration for the divisible sandpile started on the density function σ_m. Then $0 \leq \nu_m \leq 1$, and for any $\epsilon > 0$ we have for sufficiently large m*

$$\widetilde{D}_\epsilon \cap \frac{1}{m}\mathbb{Z}^d \subset \{\nu_m = 1\} \subset \{\nu_m > 0\} \subset \widetilde{D}^\epsilon \cap \frac{1}{m}\mathbb{Z}^d.$$

Additionally, for any lattice harmonic function h, we have

$$\sum_{x \in \frac{1}{m}\mathbb{Z}^d} h(x)\nu_m(x) = \sum_{x \in \frac{1}{m}\mathbb{Z}^d} h(x)\sigma_m(x).$$

It should be noted that the sequence of inclusions above was proved by Levine and Peres without $\{\nu_m > 0\}$ in it, however the proof can be easily modified to include it as well.

A "lattice harmonic" function is a function h on $\frac{1}{m}\mathbb{Z}^d$ such that for all x in $\frac{1}{m}\mathbb{Z}^d$,

$$\frac{1}{2d}\sum_{y \sim x}(u(y) - u(x)) = 0.$$

Here, the sum is taken over the $2d$ lattice sites directly adjacent to x. The last result we need concerns lattice harmonic functions, and in particular how well they approximate harmonic (in the regular sense) polynomials on $\mathbb{R}^d$ as the lattice gets finer and finer. The following lemma is a synthesis of a few results from section 2.2 in the 2014 paper by Jerison, Levine, and Sheffield [7].

Lemma 2 (Sect. 2.2 in [7]). *For each positive integer m, there is a linear map from the space of harmonic polynomials on $\mathbb{R}^d$ to the space of lattice harmonic polynomials on $\frac{1}{m}\mathbb{Z}^d$ mapping $\psi \mapsto \psi_{(m)}$, which has the following properties. If ψ has degree k, then there is a constant $C(\psi)$ such that*

$$|\psi(x) - \psi_{(m)}(x)| \leq C(\psi)|x|^{k-2}m^{-2}.$$

In particular, for every bounded subset U of $\mathbb{R}^d$, there is a constant $C(U, \psi)$ such that for all x in U,

$$|\psi(x) - \psi_{(m)}(x)| \leq C(U, \psi)m^{-2}.$$

Finally, for every harmonic polynomial ψ on $\mathbb{R}^d$, we define the random quantity $\Phi_\sigma^m(\psi)$ by

$$\Phi_\sigma^m(\psi) = m^{-d/2}\left(\sum_{x\in A_{m,n_m}} \psi_{(m)}(x) - \sum_{x\in\frac{1}{m}\mathbb{Z}^d} \sigma_m(x)\psi_{(m)}(x)\right).$$

This sequence of random variables will be the main subject of study in our main result. We offer an interpretation of what this quantity represents using the divisible sandpile. Since $\psi_{(m)}$ is lattice harmonic, we have, by Theorem 2,

$$\sum_{x\in\frac{1}{m}\mathbb{Z}^d} \psi_{(m)}(x)\nu_m(x) = \sum_{x\in\frac{1}{m}\mathbb{Z}^d} \psi_{(m)}(x)\sigma_m(x).$$

Thus,

$$\begin{aligned}
\Phi_\sigma^m(\psi) &= m^{-d/2}\left(\sum_{x\in A_{m,n_m}} \psi_{(m)}(x) - \sum_{x\in\frac{1}{m}\mathbb{Z}^d} \sigma_m(x)\psi_{(m)}(x)\right) \\
&= m^{-d/2}\left(\sum_{x\in A_{m,n_m}} \psi_{(m)}(x) - \sum_{x\in\frac{1}{m}\mathbb{Z}^d} \psi_{(m)}(x)\nu_m(x)\right) \\
&= m^{-d/2}\sum_{x\in\frac{1}{m}\mathbb{Z}^d} \psi_{(m)}(x)(1_{A_{m,n_m}}(x) - \nu_m(x)).
\end{aligned}$$

For functions f and g on $\frac{1}{m}\mathbb{Z}^d$ we define the bilinear form

$$(f,g) = m^{-d}\sum_{x\in\frac{1}{m}\mathbb{Z}^d} f(x)g(x).$$

Then if we define

$$E_\sigma^m(x) = m^{d/2}\sum_{x\in\frac{1}{m}\mathbb{Z}^d}(1_{A_{m,n_m}}(x) - \nu_m(x)),$$

we get

$$\Phi_\sigma^m(\psi) = (E_\sigma^m, \psi).$$

We call E_σ^m the "discrepancy function" because it essentially measures the difference between A_{m,n_m} and $\widetilde{D}$ (by Theorem 2, ν_m becomes an arbitrarily good approximator for $1_{\widetilde{D}}$ as $m\to\infty$). Thus, if we can find the limiting distribution of $\Phi_\sigma^m(\psi)$ for all harmonic polynomials ψ (as we will in Theorem 3), we will have a result about the weak limit of E_σ^m as a distribution, which tells us about the fluctuations of A_{m,n_m} away from $\widetilde{D}$.

3 Results

Our main theorem is a generalization of theorem 1.4 from the 2014 paper by Jerison, Levine, and Sheffield [7]:

Theorem 3. *For any harmonic polynomials $\psi_1,\ldots,\psi_l$ with corresponding degrees $k_1,\ldots,k_l$ and starting density σ, we have that $(\Phi_\sigma^m(\psi_j))_{j=1}^l$ converges in law as $m\to\infty$ to a multivariate normal random vector $(N_j)_{j=1}^l$ with mean 0 and covariance matrix Σ given by*

$$\Sigma_{i,j} = \mathrm{Cov}(N_i, N_j) = \int_{\widetilde{D}} \psi_i\psi_j(1-\sigma).$$

Using the interpretation of $\Phi_\sigma^m(\psi)$ at the end of the Preliminaries section, we get the following corollary of Theorem 3, which tells us about the weak distributional limit of E_σ^m.

Corollary 1. *For any harmonic polynomials $\psi_1, \ldots, \psi_l$ with corresponding degrees $k_1, \ldots, k_l$ and starting density σ, we have that $[(E_\sigma^m, \psi_{(m)})]_{j=1}^l$ converges in law as $m \to \infty$ to a multivariate normal random vector $(N_j)_{j=1}^l$ with mean 0 and covariance matrix Σ given by*

$$\Sigma_{i,j} = \mathrm{Cov}(N_i, N_j) = \int_{\widetilde{D}} \psi_i \psi_j (1 - \sigma).$$

Before we prove Theorem 3, we must prove a lemma establishing some rather weak fluctuation bounds for internal DLA with multiple sources, utilizing Lemma 1, one of the lemmas due to Levine and Peres [9]. Recall that Γ is the compact set we required to exist which contains all the supports of the $\{\sigma_m\}$ and σ itself as well.

Lemma 3. *There exists a bounded set $B(\Gamma, \sigma) \subset \mathbb{R}^d$ such that*

$$m^a \mathbb{P}(\{A_{m,n_m} \not\subset B(\Gamma, \sigma)\}) \to 0 \text{ as } m \to \infty \text{ for all } a > 0.$$

Proof. Since $n_m = \sum_{x \in \frac{1}{m}\mathbb{Z}^d} \sigma_m(x)$, we have that there is a constant c_1 such that $n_m \le c_1 m^d$. Thus, we may choose ρ to be so large that $b_0 \rho^d > c_1$, in which case the first condition of Lemma 1 is satisfied. Let $R = \mathrm{diam}\,\Gamma + 3\frac{\sqrt{d}}{2}\rho + 1$. Let o be some point inside Γ, let T be the boundary of the ball of radius R centered at o, and let T_b be the closed ball of radius R centered at o. Clearly, we can cover T with a finite number of sets of the form $Q(z, \rho)$, where z is in T (this immediately follows from the compactness of T, for example), so we may let $\{z_1, \ldots, z_h\}$ be a set of points on T such that $\bigcup_{i=1}^h Q(z_i, \rho) \supset T$. Furthermore, since we defined R large enough, we have that $Q(z_i, 3\rho)$ is disjoint from Γ, and therefore $\{x; \sigma_m(x) \neq 0\}$, for each i in $\{1, \ldots, h\}$. Thus, by Lemma 1, we have for each i in $\{1, \ldots, h\}$,

$$\mathbb{P}(\{A_{m,n_m} \not\subset Q(z_i, \rho)^c\}) \le b_1 e^{-b_2 m \rho}.$$

Now, we have

$$\bigcup_{i=1}^h \{A_{m,n_m} \not\subset Q(z_i, \rho)^c\} = \left\{ A_{m,n_m} \not\subset \bigcap_{i=1}^h Q(z_i, \rho)^c \right\} = \left\{ A_{m,n_m} \not\subset \left(\bigcup_{i=1}^h Q(z_i, \rho) \right)^c \right\}.$$

Now, $\left(\bigcup_{i=1}^h Q(z_i, \rho)\right)^c$ has two connected components, one a subset of T_b, one disjoint from T_b. Since σ_m is supported within Γ, which is a subset of T_b, if $A_{t,m} \subset \left(\bigcup_{i=1}^h Q(z_i, \rho)\right)^c$, then we must have that $A_{t,m} \subset T_b$, as every point in $A_{t,m}$ must be connected by a path in $\frac{1}{m}\mathbb{Z}^d$ to a point in the support of σ_m. Thus, we have that

$$\bigcup_{i=1}^h \{A_{m,n_m} \not\subset Q(z_i, \rho)^c\} = \left\{ A_{m,n_m} \not\subset \left(\bigcup_{i=1}^h Q(z_i, \rho) \right)^c \right\} = \{A_{m,n_m} \not\subset T_b\}.$$

So,

$$\mathbb{P}(\{A_{m,n_m} \not\subset T_b\}) = \mathbb{P}\left(\bigcup_{i=1}^h \{A_{m,n_m} \not\subset Q(z_i, \rho)^c\} \right) \le h b_1 e^{-b_2 m \rho}.$$

Going back through the proof, the definitions of T_b and h only depended on R, which depended on Γ and ρ. The definition of ρ only depended on d (and constants which depend only on d), so the right hand side of the above inequality has no further dependence on m than what is apparent, and T_b is a valid candidate for $B(\Gamma, \sigma)$. In particular, the above inequality shows that:

$$m^a \mathbb{P}(\{A_{m,n_m} \not\subset T_b\}) \le h b_1 m^a e^{-b_2 m \rho} \to 0 \text{ as } m \to \infty \text{ for all } a > 0$$

Thus, setting $B(\Gamma, \sigma) = T_b$, we see that the lemma indeed holds. $\square$

Proof of Theorem 3. We start by proving the theorem for a single harmonic polynomial ψ with degree k, and then generalize to the multivariate case.

The proof relies on exploiting the martingale properties of the quantity

$$M_m(t) = m^{-d/2}\left(\sum_{x\in A_{m,t}} \psi_{(m)}(x) - \sum_{i=1}^{t} \psi_{(m)}(x_{m,i})\right).$$

We first note that $M_m(n_m) = \Phi_\sigma^m(\psi)$, so our goal is to show that $M_m(n_m)$ converges in law to a normal random variable with mean zero and variance given in the theorem statement. We'll also see that the "rows" of this quantity (i.e. keeping m fixed) form martingales up to time $t = n_m$. In order to make use of this, we define the following sigma algebras on which we'll form a martingale difference array from $M_m(t)$. For m in $\mathbb{Z}_{\geq 1}$ and t in $\{0, \dots, n_m\}$,

$$\mathscr{F}_{m,t} = \sigma\left(\{A_{m,i}\}_{i=0}^{t}\right).$$

Now, for all m in $\mathbb{Z}_{\geq 1}$ and t in $\{1, \dots, n_m\}$, we define $X_{m,t} = M_m(t) - M_m(t-1)$ for $t > 1$ and let $X_{m,1} = M_m(1)$. Written out, we see

$$X_{m,t} = m^{-d/2}\left(\psi_{(m)}(A_{m,t} \setminus A_{m,t-1}) - \psi_{(m)}(x_{m,t})\right).$$

We note that the one point in $A_{m,t} \setminus A_{m,t-1}$ is the location a random walk, starting at $x_{m,t}$ exits the set $A_{m,t-1}$. Thus, since ψ_m is harmonic on the lattice $\frac{1}{m}\mathbb{Z}^d$, we have that

$$\mathbb{E}[\psi_{(m)}(A_{m,t} \setminus A_{m,t-1}) | \sigma(\{A_{m,i}\}_{i=0}^{t-1})] = \psi_{(m)}(x_{m,t}).$$

(This follows from Theorem 1.4.5 in [11]). Thus, we have $\mathbb{E}[X_{m,t}|\mathscr{F}_{m,t-1}] = 0$, so X is indeed a zero-mean martingale difference array adapted to $\{\mathscr{F}_{m,t}\}$. We will use a version of the martingale central limit theorem to prove our desired result (see Theorem 3.2 in [12] and the subsequent Remarks). This version tells us that if the following conditions hold on the martingale difference array X, then we will have that $\sum_{t=1}^{n_m} X_{m,t}$ converges in law as $m \to \infty$ to a zero-mean normal random variable with variance $\int_{\widetilde{D}} \psi^2(1-\sigma)$:

$$\mathbb{E}\left[\max_{1\leq t\leq n_m} X_{m,t}^2\right] \quad \text{is bounded in } m, \tag{1}$$

$$\max_{1\leq t\leq n_m} |X_{m,t}| \to 0 \quad \text{in probability as } m, \to \infty \tag{2}$$

$$\sum_{t=1}^{n_m} X_{m,t}^2 \to \int_{\widetilde{D}} \psi^2(1-\sigma) \quad \text{in probability as } m \to \infty. \tag{3}$$

It's worth noting that condition 1 implies the array is square-integrable, so we don't have to prove that separately. Additionally, since $\sum_{t=1}^{n_m} X_{m,t} = M_m(n_m) = \Phi_\sigma^m(\psi)$, showing these properties hold indeed suffices to prove the theorem in the single variable case.

In order to prove conditions 1 and 2 at once, we will show that $\mathbb{E}\left[\max_{1\leq t\leq n_m} |X_{m,t}|^a\right] \to 0$ as $m \to \infty$ for $a \geq 1$. Condition 1 immediately follows from the case $a = 2$, and the $a = 1$ case implies that the mean of $\max_{1\leq t\leq n_m} |X_{m,t}|$ converges to zero, which implies that it converges to zero in the L^1 norm since it is nonnegative everywhere, which implies that it converges to zero in probability, giving condition 2.

We start by showing that for all $a \geq 0$, we have that

$$\mathbb{E}\left[\max_{1\leq t\leq n_m} |\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|^a\right]$$

is bounded independently of m. We let $\mathscr{E}_m = \{A_{m,n_m} \subset B(\Gamma,\sigma) \cap \frac{1}{m}\mathbb{Z}^d\}$, where $B(\Gamma,\sigma)$ is as in Lemma 3, which tells us that for sufficiently large m, we have that $\mathbb{P}(\mathscr{E}_m^c) \leq m^{-a(d-1)k}$. Then, by conditioning on $\mathscr{E}_m$, we have the expectation above is equal to

$$\mathbb{E}\left[\max_{1\leq t\leq n_m} |\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|^a \middle| \mathscr{E}_m\right]\mathbb{P}(\mathscr{E}_m) + \mathbb{E}\left[\max_{1\leq t\leq n_m} |\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|^a \middle| \mathscr{E}_m^c\right]\mathbb{P}(\mathscr{E}_m^c).$$

Now, on $\mathscr{E}_m$, we have that $A_{m,t} \subset B(\Gamma, \sigma)$, so

$$|\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|^a \leq 2^{a-1} \left(\sup_{x \in B(\Gamma,\sigma)} |\psi(x)|^a + C(B(\Gamma, \sigma), \psi)^a m^{-2a} \right).$$

Thus, we have that

$$\mathbb{E}\left[\max_{1 \leq t \leq n_m} |\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|^a \middle| \mathscr{E}_m \right] \mathbb{P}(\mathscr{E}_m) \leq 2^{a-1} \left(\sup_{x \in B(\Gamma,\sigma)} |\psi(x)|^a + C(B(\Gamma, \sigma), \psi)^a m^{-2a} \right).$$

So the first term above is bounded independently of m. Now we show the same of the second term. Let $R = \sup_{x \in \Gamma} |x|$. An upper bound on the norm of the point in $A_{m,t} \setminus A_{m,t-1}$ is $t/m + R$, as $A_{m,t}$ must be connected, there are only t points in it on the grid $\frac{1}{m}\mathbb{Z}^d$, and the origin is in Γ. Furthermore, $t \leq n_m$, and since $n_m = \sum_{x \in \frac{1}{m}\mathbb{Z}^d} \sigma_m(x)$, we have that there is a constant c_1 such that $n_m \leq c_1 m^d$. Thus, we have that

$$\max_{1 \leq t \leq n_m} |\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|^a \leq 2^{a-1} \left(C'(\psi)^a (c_1 m^{d-1} + R)^{ak} + C(\psi)^a m^{-2} (c_1 m^{d-1} + R)^{a(k-2)} \right).$$

(Here, $C'(\psi)$ is a constant such that $|\psi(x)| \leq C'(\psi)|x|^k$ everywhere but zero, guaranteed to exist since ψ has degree k. Also, $C(\psi)$ is given by Lemma 2.) However, Lemma 3 tells us that for sufficiently large m, $\mathbb{P}(\mathscr{E}_m^c) \leq m^{-a(d-1)k}$. This means

$$\begin{aligned}
&\mathbb{E}\left[\max_{1 \leq t \leq n_m} |\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|^a \middle| \mathscr{E}_m^c \right] \mathbb{P}(\mathscr{E}_m^c) \\
&\leq 2^{a-1} \left(C'(\psi)^a (c_1 + m^{-(d-1)} R)^{ak} + C(\psi)^a m^{-2-2a(d-1)} (c_1 + m^{-(d-1)} R)^{a(k-2)} \right).
\end{aligned}$$

Thus, since this is decreasing in m, it is bounded independent of m. Thus, both terms are bounded independent of m, meaning that our original quantity

$$\mathbb{E}\left[\max_{1 \leq t \leq n_m} |\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|^a \right] \leq K(a, \psi, \sigma, \Gamma)$$

is bounded by a constant $K(a, \psi, \sigma, \Gamma)$ independently of m (this constant K will be used later as well). Thus, we have

$$\begin{aligned}
\mathbb{E}\left[\max_{1 \leq t \leq n_m} |X_{m,t}|^a \right] &= m^{-ad/2} \mathbb{E}\left[\max_{1 \leq t \leq n_m} |\psi_{(m)}(A_{m,t} \setminus A_{m,t-1}) - \psi_{(m)}(x_{m,t})|^a \right] \\
&\leq 2^{a-1} m^{-ad/2} \left(\mathbb{E}\left[\max_{1 \leq t \leq n_m} |\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|^a \right] + \max_{1 \leq t \leq n_m} |\psi_{(m)}(x_{m,t})|^a \right).
\end{aligned}$$

Clearly if the quantity inside the parentheses is bounded independent of m, the whole expression will tend to zero as $m \to \infty$. We just showed that the first term in the parentheses is bounded independent of m, and it's not hard to see the second one is as well:

$$\max_{1 \leq t \leq n_m} |\psi_{(m)}(x_{m,t})|^a \leq 2^{a-1} \left(\max_{1 \leq t \leq n_m} |\psi(x_{m,t})|^a + C(\Gamma, \psi)^a m^{-2a} \right) \leq 2^{a-1} \left(\sup_{x \in \Gamma} |\psi(x)|^a + C(\Gamma, \psi)^a m^{-2a} \right).$$

Thus, we've shown that $\mathbb{E}\left[\max_{1 \leq t \leq n_m} |X_{m,t}|^a\right] \to 0$ for all $a \geq 1$, which, as noted before, implies that conditions 1 and 2 hold.

The final step is to show that condition 3 holds. We define the following random variables to help with this:

$$S_m(t) = \sum_{i=1}^{t} X_{m,t}^2,$$

$$Z_m(t) = m^{-d} \sum_{x \in A_{m,t}} \psi_{(m)}(x)^2 - m^{-d} \sum_{i=1}^{t} \psi_{(m)}(x_{m,i})^2,$$

$$N_m(t) = S_m(t) - Z_m(t).$$

We note that $S_m(t)$, $Z_m(t)$ and $N_m(t)$ are adapted to the filtration $\mathscr{F}_{m,t}$, since so are $X_{m,t}$ and $A_{m,t}$. We will show that $\mathbb{E}[N_m(n_m)^2] \to 0$ as $m \to \infty$, which implies that $N_m(n_m) = S_m(n_m) - Z_m(n_m)$ converges in probability to zero as $m \to \infty$. Then, if we can show that $Z_m(n_m)$ converges in probability to $\int_{\widetilde{D}} \psi^2(1-\sigma)$, we have that $S_m(n_m)$ converges in probability to this value as well, which is precisely what condition 3 requires.

So, we try to show that $\mathbb{E}[N_m(n_m)^2] \to 0$. We note that the increments of N_m have the martingale property. We have

$$\mathbb{E}\left[N_m(t) - N_m(t-1)|\mathscr{F}_{m,t-1}\right] = \mathbb{E}\left[X_{m,t}^2 - m^{-d}(\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})^2 - \psi_{(m)}(x_{m,t})^2)|\mathscr{F}_{m,t-1}\right].$$

This is equal to

$$\begin{aligned} m^{-d}&\mathbb{E}\left[(\psi_{(m)}(A_{m,t} \setminus A_{m,t-1}) - \psi_{(m)}(x_{m,t}))^2 - (\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})^2 - \psi_{(m)}(x_{m,t})^2)|\mathscr{F}_{m,t-1}\right] \\ &= m^{-d}\mathbb{E}\left[-2\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})\psi_{(m)}(x_{m,t}) + 2\psi_{(m)}(x_{m,t})^2|\mathscr{F}_{m,t-1}\right] \\ &= 2m^{-d}\left(\psi_{(m)}(x_{m,t})^2 - \psi_{(m)}(x_{m,t})\mathbb{E}[\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|\mathscr{F}_{m,t-1}]\right). \end{aligned}$$

We noted earlier that $\mathbb{E}[\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})|\mathscr{F}_{m,t-1}] = \psi_{(m)}(x_{m,t})$ since $\psi_{(m)}$ is discrete harmonic and $A_{m,t} \setminus A_{m,t-1}$ is the point where a random walk exits $A_{m,t-1}$ (so the result follows from Theorem 1.4.5 in [11]). Thus, we get that the expression above is equal to $2m^{-d}(\psi_{(m)}(x_{m,t})^2 - \psi_{(m)}(x_{m,t})^2) = 0$. Thus, the increments of M_m have the martingale property, which tells us that the increments also have zero covariance, as follows. Let $1 \le j < i \le n_m$. Then we have that

$$\mathbb{E}[(N_m(i) - N_m(i-1))(N_m(j) - N_m(j-1))] = \mathbb{E}[\mathbb{E}[(N_m(i) - N_m(i-1))(N_m(j) - N_m(j-1))|\mathscr{F}_{m,i-1}]].$$

Since $i > j$, $i-1 \ge j$ and $i-1 \ge j-1$, so $N_m(j)$ and $N_m(j-1)$ are $\mathscr{F}_{m,i-1}$-measurable. Thus, we have

$$\mathbb{E}[\mathbb{E}[(N_m(i)-N_m(i-1))(N_m(j)-N_m(j-1))|\mathscr{F}_{m,i-1}]] = \mathbb{E}[(N_m(j)-N_m(j-1))\mathbb{E}[N_m(i)-N_m(i-1)|\mathscr{F}_{m,i-1}]].$$

Since we just showed that the increments of N_m have the martingale property, the inner expectation in the second expression above is zero, so we have that the covariance of distinct increments of N_m must be zero. Finally, we note that $S_m(1) = 0$ and $Z_m(1) = 0$, both since the first point in the internal DLA cluster must be the same as the starting point for the first walk, since the cluster is empty, so the first walk's starting position must be empty. Thus,

$$\mathbb{E}[N_m(n_m)^2] = \mathbb{E}[(N_m(n_m) - N_m(1))^2] = \mathbb{E}\left[\left(\sum_{t=2}^{n_m} N_m(t) - N_m(t-1)\right)^2\right].$$

When we expand the square, the cross-terms will have expectation zero by what we just showed, so we have

$$\mathbb{E}[N_m(n_m)^2] = \sum_{t=2}^{n_m} \mathbb{E}\left[(N_m(t) - N_m(t-1))^2\right].$$

Now we'll estimate the increments of N_m. We have

$$\mathbb{E}[(N_m(t) - N_m(t-1))^2] \le 2\mathbb{E}[(S_m(t) - S_m(t-1))^2] + 2\mathbb{E}[(Z_m(t) - Z_m(t-1))^2].$$

To estimate the first term, we see

$$\mathbb{E}[(S_m(t) - S_m(t-1))^2] = \mathbb{E}[X_{m,t}^4] = m^{-2d}\mathbb{E}[\big(\psi_{(m)}(A_{m,t} \setminus A_{m,t-1}) - \psi_{(m)}(x_{m,t})\big)^4].$$

Splitting this up further gives

$$\begin{aligned} \mathbb{E}[(S_m(t) - S_m(t-1))^2] &\le 8m^{-2d}\mathbb{E}[\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})^4 + \psi_{(m)}(x_{m,t})^4] \\ &\le 8m^{-2d}\left(K(4,\psi,\sigma,\Gamma) + 8\left(\sup_{x\in\Gamma}|\psi(x)|^4 + C(\Gamma,\psi)^4 m^{-8}\right)\right). \end{aligned}$$

Here, we've reused the K from earlier. Thus, we've shown there is a constant $C_S(\psi, \sigma, \Gamma)$ such that

$$\mathbb{E}[(S_m(t) - S_m(t-1))^2] \leq C_S(\psi, \sigma, \Gamma) m^{-2d}.$$

Now we estimate the second term:

$$\begin{aligned}\mathbb{E}[(Z_m(t) - Z_m(t-1))^2] &= m^{-2d}\mathbb{E}[(\psi_{(m)}(A_{m,t} \setminus A_{m,t-1})^2 - \psi_{(m)}(x_{m,i})^2)^2] \\ &\leq 2m^{-2d}\left(K(4, \psi, \sigma, \Gamma) + 8\left(\sup_{x\in\Gamma}|\psi(x)|^4 + C(\Gamma, \psi)^4 m^{-8}\right)\right).\end{aligned}$$

Thus, there is also a constant $C_Z(\psi, \sigma, \Gamma)$ such that

$$\mathbb{E}[(Z_m(t) - Z_m(t-1))^2] \leq C_Z(\psi, \sigma, \Gamma) m^{-2d}.$$

Thus, we have that

$$\mathbb{E}[N_m(n_m)^2] \leq 2\sum_{t=2}^{n_m} m^{-2d}(C_Z(\psi, \sigma, \Gamma) + C_S(\psi, \sigma, \Gamma)) \leq 2(C_Z(\psi, \sigma, \Gamma) + C_S(\psi, \sigma, \Gamma)) n_m m^{-2d}.$$

However, we still have that $n_m \leq c_1 m^d$. Thus, there is a constant $C_N(\psi, \sigma, \Gamma)$ such that

$$\mathbb{E}[N_m(n_m)^2] \leq C_N(\psi, \sigma, \Gamma) m^{-d}.$$

Thus, $N_n(n_m) \to 0$ in the L^2 norm, and so converges in probability as well.

So to prove the theorem in the single variable case, it suffices to show that $Z_m(n_m) \to \int_{\widetilde{D}} \psi^2(1-\sigma)$ in probability. We have that

$$Z_m(n_m) = m^{-d}\left(\sum_{x\in A_{m,n_m}} \psi_{(m)}(x)^2 - \sum_{t=1}^{n_m} \psi_{(m)}(x_{m,i})^2\right) = m^{-d}\sum_{x\in A_{m,n_m}} \psi_{(m)}(x)^2(1 - \sigma_m(x)).$$

Now, by standard integration theory, we have

$$m^{-d}\sum_{x\in A_{m,n_m}} \psi_{(m)}(x)^2(1 - \sigma_m(x)) = \int_{A^\square_{m,n_m}} \psi^\square_{(m)}(x)^2(1 - \sigma^\square_m(x)).$$

We will show that this converges in probability to $\int_{\widetilde{D}} \psi^2(1-s)$ by showing that it is equal to $\int_{\widetilde{D}} \psi^2(1-s)$ minus three other random variables, each of which converges to zero almost surely. For each positive integer i, we let U_i be the event that

$$\widetilde{D}_{1/i} \cap \frac{1}{m}\mathbb{Z}^d \subset A_{m,n_m} \subset \widetilde{D}^{1/i} \cap \frac{1}{m}\mathbb{Z}^d$$

holds for sufficiently large m. Theorem 1 tells us that $\mathbb{P}(U_i) = 1$ for all i. Let $U = \bigcap_{i\geq 1} U_i$. Since U is a countable intersection of probability one events, it has probability one itself. We define

$$Y^1_m = \int_{\widetilde{D}} \psi^\square_{(m)}(x)^2(1-\sigma^\square_m(x)) - \int_{A^\square_{m,n_m}} \psi^\square_{(m)}(x)^2(1-\sigma^\square_m(x)) = \int_{\widetilde{D}\cup A^\square_{m,n_m}} \psi^\square_{(m)}(x)^2(1-\sigma^\square_m(x))(1^\square_{A_{m,n_m}} - 1_{\widetilde{D}}).$$

We clearly have that for $B_1, B_2 \subset \frac{1}{m}\mathbb{Z}^d$, if $B_1 \subset B_2$ then $B_1^\square \subset B_2^\square$. Thus, we have that for every outcome in U and every positive integer i, for sufficiently large m,

$$\left(\widetilde{D}_{1/i} \cap \frac{1}{m}\mathbb{Z}^d\right)^\square \subset A^\square_{m,n_m} \subset \left(\widetilde{D}^{1/i} \cap \frac{1}{m}\mathbb{Z}^d\right)^\square.$$

I claim that $\widetilde{D}_{1/i+\sqrt{d}/m} \subset \left(\widetilde{D}_{1/i} \cap \frac{1}{m}\mathbb{Z}^d\right)^\square$ and $\left(\widetilde{D}^{1/i} \cap \frac{1}{m}\mathbb{Z}^d\right)^\square \subset \widetilde{D}^{1/i+\sqrt{d}/m}$. If x is in $\widetilde{D}_{1/i+\sqrt{d}/m}$, then $B(x, 1/i + \sqrt{d}/m) \subset \widetilde{D}$, so by the triangle inequality, if $z_m(x)$ is the nearest lattice point to x, breaking ties upward, then $B(z_m(x), 1/i) \subset B(x, 1/i + \sqrt{d}/m) \subset \widetilde{D}$ (since $z_m(x)$ is at most a distance of $\frac{\sqrt{d}}{2m}$ away from x), so $z_m(x)$ is in $\widetilde{D}_{1/i}$. On the other hand, if x is in $\left(\widetilde{D}^{1/i} \cap \frac{1}{m}\mathbb{Z}^d\right)^{\square}$, then $z_m(x)$ is in $\widetilde{D}^{1/i}$, so $B(z_m(x), 1/i) \not\subset \widetilde{D}$. Since $z_m(x)$ is at most a distance $\frac{\sqrt{d}}{2m}$ away from x, by the triangle inequality, $B(x, 1/i + \sqrt{d}/m) \supset B(z_m(x), 1/i)$, so $B(x, 1/i + \sqrt{d}/m) \not\subset \widetilde{D}$ as well, so x is in $D^{1/i+\sqrt{d}/m}$. Thus, for every outcome in U and every positive integer i, for sufficiently large m,

$$\widetilde{D}_{1/i+\sqrt{d}/m} \subset A^{\square}_{m,n_m} \subset \widetilde{D}^{1/i+\sqrt{d}/m}.$$

Since we clearly also have that $\widetilde{D}_{1/i+\sqrt{d}/m} \subset \widetilde{D} \subset \widetilde{D}^{1/i+\sqrt{d}/m}$, we have that for all outcomes in U and positive integers i, $1_{A^{\square}_{m,n_m}} - 1_{\widetilde{D}}$ can only be supported on $\widetilde{D}^{1/i+\sqrt{d}/m} \setminus \widetilde{D}_{1/i+\sqrt{d}/m}$ for sufficiently large m. We first note that, for each outcome in U, for sufficiently large m, we have

$$\begin{aligned}\sup_{x \in \widetilde{D} \cup A^{\square}_{m,n_m}} |\psi^{\square}_{(m)}(x)^2(1 - \sigma^{\square}_m(x))| &\le 2(M+1)\left(\sup_{x \in \widetilde{D}^1} |\psi(x)|^2 + \sup_{x \in \widetilde{D}^1} |\psi(x) - \psi_m(x)|^2\right) \\ &\le 2(M+1)\left(\sup_{x \in \widetilde{D}^1} |\psi(x)|^2 + C(\widetilde{D}^1, \psi)m^{-2}\right).\end{aligned}$$

This is bounded independently of m, so we have that for each outcome in U

$$\sup_{x \in \widetilde{D} \cup A^{\square}_{m,n_m}} |\psi^{\square}_{(m)}(x)^2(1 - \sigma^{\square}_m(x))| \le K'(\psi, M, \widetilde{D})$$

is bounded independently of m (here, K' is actually a random variable as it depends on the outcome chosen in U). Thus, for each outcome in U and each positive integer i, we can choose m so large that

$$|Y^1_m| \le \int_{\widetilde{D} \cup A^{\square}_{m,n_m}} |\psi^{\square}_{(m)}(x)^2(1 - \sigma^{\square}_m(x))(1^{\square}_{A_{m,n_m}} - 1_{\widetilde{D}})| \le K'(\psi, M, \widetilde{D})\mathcal{L}\left(\widetilde{D}^{1/i+\sqrt{d}/m} \setminus \widetilde{D}_{1/i+\sqrt{d}/m}\right).$$

Here, $\mathcal{L}$ denotes the Lebesgue measure. Now, given any positive integer j, by setting $i = 2j$ and requiring that m also be large enough that $\frac{\sqrt{d}}{m} \le \frac{1}{2j}$, we can get that for each outcome in U and each positive integer j, we can choose m so large that

$$|Y^1_m| \le K'(\psi, M, \widetilde{D})\mathcal{L}(\widetilde{D}^{1/j} \setminus \widetilde{D}_{1/j}).$$

Now, clearly $\{\widetilde{D}^{1/j} \setminus \widetilde{D}_{1/j}\}$ is a decreasing sequence ordered by inclusion, and $\bigcap_{j \ge 1} \widetilde{D}^{1/j} \setminus \widetilde{D}_{1/j} = \partial\widetilde{D}$. Thus, by the monotonicity properties of measures, since $\mathcal{L}(\partial\widetilde{D}) = 0$ (this is precisely Proposition 2.12(i) in the paper of Levine and Peres [9]), we must have that $\mathcal{L}\left(\widetilde{D}^{1/j} \setminus \widetilde{D}_{1/j}\right) \to 0$. Thus, given an outcome in U, and an $\epsilon > 0$, we can find a j so large such that

$$\mathcal{L}\left(\widetilde{D}^{1/j} \setminus \widetilde{D}_{1/j}\right) < \frac{\epsilon}{K'(\psi, M, \widetilde{D})}.$$

Then we can choose m so large that, for this outcome, we have

$$|Y^1_m| \le K'(\psi, M, \widetilde{D})\mathcal{L}\left(\widetilde{D}^{1/j} \setminus \widetilde{D}_{1/j}\right) < \epsilon.$$

Thus, we've shown that $Y^1_m \to 0$ on U, which is a set of probability one, so $Y^1_m \to 0$ almost surely.

Now we define Y^2_m by

$$Y^2_m = \int_{\widetilde{D}} \psi^{\square}_{(m)}(x)^2(1 - \sigma(x)) - \int_{\widetilde{D}} \psi^{\square}_{(m)}(x)^2(1 - \sigma^{\square}_m(x)) = \int_{\widetilde{D}} \psi^{\square}_{(m)}(x)^2(\sigma^{\square}_m(x) - \sigma(x)).$$

As above, for each outcome in U, we can choose m so large that

$$\sup_{x\in\widetilde{D}} |\psi^{\square}_{(m)}(x)|^2 \leq 2\left(\sup_{x\in\widetilde{D}^1} |\psi(x)|^2 + \sup_{x\in\widetilde{D}^1} |\psi(x)-\psi_m(x)|^2\right)$$
$$\leq 2\left(\sup_{x\in\widetilde{D}^1} |\psi(x)|^2 + C(\widetilde{D}^1,\psi)m^{-2}\right).$$

Thus, $\sup_{x\in\widetilde{D}} |\psi^{\square}_{(m)}(x)|^2$ is bounded independent of m (but not independently of the outcome in U), say by a constant $K''(\psi,\widetilde{D})$. So for every outcome in U, we have

$$|Y_m^2| \leq \int_{\widetilde{D}} \left|\psi^{\square}_{(m)}(x)^2(\sigma^{\square}_m(x)-\sigma(x))\right| \leq K''(\psi,\widetilde{D})\int_{\widetilde{D}} |\sigma^{\square}_m - \sigma|$$

We have that for all m, $|\sigma^{\square}_m| \leq M1_\Gamma$, and the latter is Lebesgue integrable. Additionally, since we required that we have $\sigma^{\square}_m \to \sigma$ everywhere σ is continuous, and we required σ to be continuous almost everywhere, we have that $\sigma^{\square}_m \to \sigma$ almost everywhere. Thus, by dominated convergence, we get that $\int_{\widetilde{D}} |\sigma^{\square}_m - \sigma| \to 0$. This gives that, on U, $Y_m^2 \to 0$, and since U has probability one, $Y_m^2 \to 0$ almost surely.

Finally, we define Y_m^3 by

$$Y_m^3 = \int_{\widetilde{D}} \psi(x)^2(1-\sigma(x)) - \int_{\widetilde{D}} \psi^{\square}_{(m)}(x)^2(1-\sigma(x)) = \int_{\widetilde{D}} (\psi(x)^2 - \psi^{\square}_{(m)}(x)^2)(1-\sigma(x)).$$

Thus, we have

$$|Y_m^3| \leq (M+1)\int_{\widetilde{D}} \left|\psi + \psi^{\square}_{(m)}\right| \left|\psi - \psi^{\square}_{(m)}\right|.$$

So,

$$|Y_m^3| \leq (M+1)\left(\sup_{x\in\widetilde{D}} |\psi(x)| + K''(\psi,\widetilde{D})^{1/2}\right)\int_{\widetilde{D}} \left|\psi - \psi^{\square}_{(m)}\right|.$$

Now, fix x in $\widetilde{D}$ and $\epsilon > 0$. Since ψ is continuous, there is a $\delta > 0$ such that for any y, $|x-y| < \delta$ implies that $|\psi(x)-\psi(y)| < \epsilon/2$. We can choose m so large that

$$m^{-2} \leq \frac{\epsilon}{2C(\widetilde{D},\psi)}$$

and $\frac{\sqrt{d}}{2m} < \min(\delta, d(x,\partial\widetilde{D}))$. Then, letting $z_m(x)$ be the closest lattice point to x as above, we will have

$$|\psi(x) - \psi^{\square}_{(m)}(x)| = |\psi(x) - \psi_{(m)}(z_m(x))| \leq |\psi(x) - \psi(z_m(x))| + |\psi(z_m(x)) - \psi_{(m)}(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $\psi^{\square}_{(m)} \to \psi$ pointwise, and since, on $\widetilde{D}$, $|\psi^{\square}_{(m)}| \leq C(\widetilde{D}^2,\psi)m^{-2} \leq C(\widetilde{D}^2,\psi)$, dominated convergence gives us that $\int_{\widetilde{D}} \left|\psi - \psi^{\square}_{(m)}\right| \to 0$. Thus, for each outcome in U, $Y_m^3 \to 0$, so $Y_m^3 \to 0$ almost surely. Thus, we have that

$$Z_m(n_m) + Y_m^1 + Y_m^2 + Y_m^3 = \int_{\widetilde{D}} \psi^2(1-\sigma).$$

Since Y_m^1, Y_m^2, and Y_m^3 all converge to zero almost surely, they converge to zero in probability, so $Z_m(n_m)$ converges in probability to $\int_{\widetilde{D}} \psi^2(1-\sigma)$ as desired.

Now, going back to the multivariate case, we prove that the characteristic function converges to the characteristic function of the multivariate normal distribution with the desired mean and covariance matrix. Let $\Phi_m = (\Phi^m_\sigma(\psi_i))_{j=1}^l$, and let $t = (t_1,\ldots,t_l)$ be an arbitrary vector in $\mathbb{R}^l$. We want to calculate $\mathbb{E}[\exp(i\langle t,\Phi_m\rangle)]$. However, since the map $\psi \mapsto \psi_{(m)}$ is linear (by Lemma 2), the map $\psi \mapsto \Phi^m_\sigma(\psi)$ is linear (as it was defined as a linear function of $\psi_{(m)}$). Thus, $\langle t,\Phi_m\rangle = \Phi^m_\sigma\left(\sum_{i=1}^l t_i\psi_i\right)$. Since we just showed that, for each individual ψ harmonic, $\Phi_\sigma^m(\psi)$ converges in law as $m \to \infty$ to a normal random variable with variance $\int_{\widetilde{D}} \psi^2(1-\sigma)$, by Lévy's continuity theorem, we have

$$\mathbb{E}[\exp(i\langle t, \Phi_m\rangle)] = \mathbb{E}\left[\exp\left(i\Phi_\sigma^m\left(\sum_{i=1}^{l} t_i\psi_i\right)\right)\right] \to \exp\left(-\frac{1}{2}\int_{\widetilde{D}}\left(\sum_{i=1}^{l} t_i\psi_i\right)^2(1-\sigma)\right).$$

Now, we also have

$$\exp\left(-\frac{1}{2}\int_{\widetilde{D}}\left(\sum_{i=1}^{l} t_i\psi_i\right)^2(1-\sigma)\right) = \exp\left(-\frac{1}{2}\int_{\widetilde{D}}\sum_{i,j=1}^{l} t_it_j\psi_i\psi_j(1-\sigma)\right) = \exp\left(-\frac{1}{2}t^T\Sigma t\right).$$

Here, Σ is the matrix given in the statement of the theorem. Thus, we have that $\mathbb{E}[\exp(i\langle t, \Phi_m\rangle)] \to \exp\left(-\frac{1}{2}t^T\Sigma t\right)$ for each t in $\mathbb{R}^l$, so the random vector $(\Phi_\sigma^m(\psi_i))_{i=1}^l$ converges in law to the normal random vector given in the theorem statement, again by Lévy's continuity theorem, as desired.

□

4 Next Steps

In this report, we've partially generalized theorem 1.4 from [7] to the multiple source case. However, one of the features of the original theorem is that the weak limit, which they termed the "augmented" Gaussian free field, was able to be interpreted as a Gaussian Hilbert space isomorphic to a certain closed subspace of $H_0^1(\mathbb{R}^d)$ with a norm different, but equivalent to the Dirichlet norm on the subspace. On the other hand, the normal Gaussian free field is a Gaussian Hilbert space isomorphic to $H_0^1(\mathbb{R}^n)$ with the Dirichlet norm. Expressing the result as a Gaussian Hilbert space allowed for a much better understanding of the geometry of the problem, but its counterpart in the multiple sources case we expect is significantly more complicated. In the single source case, using a heuristic symmetry argument, Jerison, Levine, and Sheffield reasonably conjectured that since the smoothing and dampening effects on the fluctuations should be rotationally invariant, they should act independently on each spherical Fourier mode. This motivated them to construct, at least in two dimensions, the augmented Gaussian free field as a Gaussian Hilbert space isomorphic to a subspace of H_0^1 with the norm:

$$\|\eta\|_{nr}^2 = \sum_{0<|k|<\infty} 2\pi\int_0^\infty \left(|r\partial_r\eta_k(r)|^2 + (|k|+1)^2|\eta_k(r)|^2\right)\frac{dr}{r}$$

where

$$\eta_k(r) = \frac{1}{2\pi}\int_0^{2\pi}\eta(re^{i\theta})e^{-ik\theta}d\theta$$

(we've implicitly made use of the identification of $\mathbb{R}^2$ with $\mathbb{C}$). The subspace here is just the orthogonal (in the sense of the Dirichlet norm) complement of the null space of the norm above. For reference, the normal Dirichlet norm, which corresponds to the normal Gaussian free field, can be written as

$$\|\eta\|_\nabla^2 = \int_{\mathbb{R}^2}|\nabla\eta|^2 = \sum_{0\le|k|<\infty} 2\pi\int_0^\infty \left(|r\partial_r\eta_k(r)|^2 + |k|^2|\eta_k(r)|^2\right)\frac{dr}{r}$$

which only really differs from the previous norm in that the previous norm uses $(|k|+1)^2$ instead of $|k|^2$ in the second term (hence the term "augmented"). The inclusion or exclusion of the $k=0$ term in the sums above corresponds to whether or not the particles in the cluster are started at Poisson intervals or simple integer intervals (respectively); Jerison, Levine, and Sheffield opt for the former choice, while we've chosen the latter, but the choice is not really salient mathematically, it's more a stylistic choice. We suspect modifying the norm to generate a Gaussian Hilbert space to match the covariance structure we've found for the multiple source case will be harder, as we've lost the rotational symmetry which suggested looking at the Fourier modes in the first place. Nevertheless, based on our work above, we derive below a condition which the desired norm must satisfy.

To be more specific, we want to find a closed subspace $H \subset H_0^1(\mathbb{R}^d)$ with its own, different inner product $(\cdot,\cdot)_H$ (which is still equivalent to the Dirichlet norm on the subspace, however) inducing a norm $\|\cdot\|_H$ such that for all harmonic polynomials ψ, we have

$$\int_{\widetilde{D}} \psi^2(1-\sigma) = \sup_{\|\eta\|_H \leq 1} \left| \int_{\partial \widetilde{D}} \psi\eta \right|^2.$$

Why this is desireable is as follows. For every measure ϕ in H^{-1}, let $\Psi_\phi : H_0^1 \to \mathbb{R}$ be the continuous linear functional which pairs elements of H_0^1 with ϕ, i.e.

$$\Psi_\phi(\eta) = \int \eta d\phi = (\eta, \phi).$$

Let $\lambda : H' \to H$ be the Hilbert space isomorphism between H (with the inner product $(\cdot,\cdot)_H$) and its continuous dual which is guaranteed by the Riesz representation theorem, so $\lambda((\eta,\cdot)_H) = \eta$. Since Ψ_ϕ is a continuous linear functional on H_0^1, it's a continuous linear functional on H, so we have that $\lambda(\Psi_\phi)$ is in H. Thus, we have

$$(\eta, \lambda(\Psi_\phi))_H = \Psi_\phi(\eta) = \int \eta d\phi = (\eta, \phi).$$

Now, let g_H be a Gaussian Hilbert space isomorphic to H, i.e. for every η in H, $(g_H, \eta)_H$ is a zero mean normal random variable with variance $\|\eta\|_H^2$, and $(g_H, \cdot)_H$ is linear in its argument. Then are justified by the above equation in defining (g_H, ϕ) for each ϕ in H^{-1} by

$$(g_H, \phi) = (g_H, \lambda(\Psi_\phi))_H.$$

Then we would have

$$\mathrm{Var}(g_H, \phi) = \mathrm{Var}(g_H, \lambda(\Psi_\phi))_H = \|\lambda(\Psi_\phi)\|_H^2 = \|\Psi_\phi\|_{H'}^2 = \sup_{\|\eta\|_H \leq 1} |\Psi_\phi(\eta)|^2 = \sup_{\|\eta\|_H \leq 1} \left| \int \eta d\phi \right|^2.$$

Here we've used the fact that λ is a Hilbert space isomorphism. Thus, if we can show that the original equation above holds, then (letting $s_{\partial \widetilde{D}}$ be the surface measure on $\partial \widetilde{D}$) we have

$$\mathrm{Var}(g_H, \psi s_{\partial \widetilde{D}}) = \sup_{\|\eta\|_H \leq 1} \left| \int \eta d\left(\psi s_{\partial \widetilde{D}}\right) \right|^2 = \sup_{\|\eta\|_H \leq 1} \left| \int_{\partial \widetilde{D}} \psi\eta \right|^2 = \int_{\widetilde{D}} \psi^2(1-\sigma).$$

Then, Corollary 1 could be re-expressed as saying that $(E_\sigma^m, \psi_{(m)})$ converges in law to $(g_H, \psi s_{\partial \widetilde{D}})$, which tells us that E_σ^m converges in distribution in this rather weak sense (weaker even than in the usual sense since we have to use $\psi_{(m)}$ instead of ψ itself) to g_H restricted to $\partial \widetilde{D}$, which is what we want to be able to say. Thus, the natural next step in this line of research is to search for a subspace $H \subset H_0^1$ with a different inner-product-induced norm $\|\cdot\|_H$ such that

$$\int_{\widetilde{D}} \psi^2(1-\sigma) = \sup_{\|\eta\|_H \leq 1} \left| \int_{\partial \widetilde{D}} \psi\eta \right|^2.$$

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