Minimal Cell Structures for G-CW Complexes

UROP+ Final Paper, Summer 2016
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August 31, 2016

Abstract: In this paper, we consider the minimal cell structure problem for G-CW complexes. A CW complex is a nice approximation of general topological spaces, which is constructed by repeating attaching higher dimensional cells to lower ones. A G-CW complex is its generalized version to spaces with a group action. The structure theorem for ordinary CW complexes is well-studied, and we can know the minimal number of cells needed to describe the topology properties of a space completely. However, when the group action is involved, the structure becomes much more complicated. In this paper, we set up an algebraic model for G-CW complexes and simplify the minimal cell structure problem a lot. We also successfully get the minimal structure in the simple case $G = \mathbb{Z}/p\mathbb{Z}$.

1 Introduction

A CW complex is a space constructed by repeatedly attaching higher dimensional cells to lower ones. Compared to general spaces, the topology properties of a CW complex is easier to find and describe. In fact, each space can be approximated by a CW complex (up to weak equivalence, see [ref]), which means, we can study the topology properties of any space from its approximated complex. A natural question is, what is the minimal number of cells (for each dimension) we need to approximate a space. [2] gives several conditions with which we can get a finite answer. When the space is simply-connected, we can also get the exact minimal number:

Theorem 1.1. *([2], pp. 68) Suppose X is a finite simply-connected CW complex such that $H_i(X)$ has rank β_i and τ_i torsion coefficients, for each $i > 0$. Then X is (weak) equivalent to a CW complex Y with $\alpha_i = \beta_i + \tau_i + \tau_{i-1}$ i-cells for each i.*

More precisely, we can write each $H_i(X) = \mathbb{Z}^a \times \mathbb{Z}/s_1\mathbb{Z} \times \mathbb{Z}/s_2\mathbb{Z} \times ... \times \mathbb{Z}/s_u\mathbb{Z}$ and $H_{i-1}(X) = \mathbb{Z}^b \times \mathbb{Z}/t_1\mathbb{Z} \times \mathbb{Z}/t_2\mathbb{Z} \times ... \times \mathbb{Z}/t_v\mathbb{Z}$, such that $s_1|s_2|...|s_u$, $t_1|t_2|...|t_v$.

Then the number of i-cells in Y is $a + u + v$. According to the cellular calculation of the homology groups, this number is the minimal one to make these homology groups possible.

In this paper, we consider the same question for G-spaces, i.e., spaces with a group action. The concepts of CW complex, homotopy and homology theories have generalized version for G-spaces. As ordinary CW complexes, each G-space can be approximated by a G-CW complex ([1], pp. 17–18). We don't consider under which conditions a G-space can be approximated by a finite G-CW complex. Instead we assume that we have already got a finite G-CW complex, and we want to reduce its cells as much as possible.

More rigorously, suppose X is a finite G-CW complex, we want to find the minimal number (we will define this minimality later) of cells for a G-CW complex Y such that X is G-homotopy equivalent to Y (then the topology properties of X are inherited by Y).

In this paper, we will consider G as a finite discrete group. We haven't solved this question completely. In fact, we only get the answer for the most simple case $G = \mathbb{Z}/p\mathbb{Z}$ for some prime p (see section 5). But we set up an algebraic model and simplify the question a lot (see section 3,4), which might be quite helpful for the case of general groups.

The solution for the case $G = \mathbb{Z}/p\mathbb{Z}$ is as follows:

Theorem 1.2. *Let $G = \mathbb{Z}/p\mathbb{Z}$ for some prime p. Suppose X is a finite G-simply connected G-CW complex. Then it is G-equivalent to another G-CW complex Y, such that Y^G is a minimal ordinary CW complex (which means that we can use Theorem* 1.1 *to count the number of cells in it), and the number of i-cells in Y/Y^G is $p(\tau_G(X/X^G, i) + \tau_G(X/X^G, i-1) + gen(H_i^G(X/X^G; N_p)))$. Furthermore, all these numbers reach their minimal possible values.*

Here τ_G denotes the G-torsion, *gen* denotes the minimal number of generators of a G-module, N_p is a special coefficient system. All of their definitions can be found at the beginning of section 5.2. In fact, the information of all these numbers are contained in the Bredon homology groups of X (which means we don't even need to find out X^G and X/X^G).

In section 2 we introduce some basic definitions and properties of topology for G-spaces. In section 3, we introduce the concept of G-chain complexes and transfer the question into an algebraic version. In section 4, we discuss the properties for the algebraic model in section 3 and simplify the question. In section 5, we give a practical method to calculate the minimal structure from the G-version homology groups in the most simple case $G = \mathbb{Z}/p\mathbb{Z}$. The result of section 3 works for both abelian and non-abelian groups. But in the following sections we only consider the abelian groups.

2 Preliminaries

In this section, we will show the definitions of G-spaces and the corresponding topology concepts.

2.1 G-CW complexes

Definition 2.1. *A G-**space** X is a space with a group action of G on it, such that, for each $g \in G$, $g : X \to X$ is a continuous map.*

For each subgroup $H \leq G$, we use X^H to denote all invariant points under H, i.e., $X^H := \{x \in X : h \cdot x = x, \forall h \in H\}$.

In this paper, we consider this group action as a left action. For each point $x \in X$, we use $g \cdot x$ to denote its image under the action of element $g \in G$.

Definition 2.2. *A G-space is G-connected or G-simply connected if X^H is connected or simply connected for each $H \leq G$.*

Definition 2.3. *For each point x in a G-space X, $G_x := \{g \in G : g \cdot x = x\}$ is the **isotropy group** of x. For each subspace $Y \subset X$, $G_Y := \{g \in G : g \cdot y = y, \forall y \in Y\}$.*

Definition 2.4. *A G-**map** $f : X \to Y$ between the G-spaces X, Y is a continuous map such that $f(g \cdot x) = g \cdot f(x)$ for any $g \in G$ and $x \in X$.*

*For any two G-maps $f_0, f_1 : X \to Y$, they are G-**homotopic** if there exists a continuous G-map $F : X \times I \to Y$ (G acts trivially on I) such that $F(\cdot, 0) = f_0(\cdot)$, $F(\cdot, 1) = f_1(\cdot)$.*

Definition 2.5. *A G-**orbit** is the coset G/H with a subgroup $H \leq G$. The group action of G on it is defined naturally: $g \cdot g'H = gg'H$. The topology of G/H is induced by the topology of G.*

Definition 2.6. *A G-**cell** is the product of a cell with a G-orbit: $G/H \times D^n$. It can also be viewed as a G-space with the group action induced by the action on G/H.*

Definition 2.7. *A G-**CW complex** X is the union of sub G-spaces X^n such that X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching G-cells $G/H \times D^{n+1}$ along equivariant attaching maps $G/H \times S^n \to (X^n)^H$. ([1], pp. 16)*

On other words, a G-CW complex X is a G-space with a CW complex structure which is compatible with the group action of G, i.e., for each cell as a map $f : D^n \to X$, we can find a cell $f' : D^n \to X$ (it can be f itself), such that $g \cdot f(v) = f'(v)$, $\forall v \in D^n$.

In this paper, we only consider the G-simply connected G-CW complexes with base points, i.e., a 0-cell which is fixed under the group action.

Definition 2.8. *For two G-spaces X, Y, a G-**homotopy equivalence** is a G-map $f : X \to Y$ which has a G-homotopy inverse, i.e., a G-map $f' : Y \to X$ such that both $f \circ f'$ and $f' \circ f$ are G-homotopic to the identity map on the two spaces.*

Definition 2.9. *For two G-spaces X, Y, a **weak** G**-homotopy equivalence** is a G-map $f : X \to Y$ such that for each $H \leq G$, the induced map $f : X^H \to Y^H$ is a weak equivalence.*

As ordinary cases, a G-homotopy equivalence is also a weak one. The inverse is true when X, Y are G-CW complexes ([1], pp. 17). We don't need to distinguish them in this paper since we only consider the complexes. We use "equivalence" to denote a (weak) G-homotopy equivalence.

2.2 Bredon homology theory

The Bredon homology theory can be viewed as a generalization of the ordinary homology theory onto the G-spaces. The Bredon homology groups come from the corresponding chain complexes based on a coefficient system. It is also our main tool in this paper.

Definition 2.10. *The **orbit category** $\mathcal{O}(G)$ of G is a category with the G-orbits G/H as its objects and the G-maps among these orbits as its morphisms.*

It's not hard to describe the morphisms in $\mathcal{O}(G)$:

Lemma 2.11. *The morphisms in $\mathcal{O}(G)$ from G/K to G/H are determined by the image of K/K in G/H:*

$$(G/H)^K = \{[g] \in G/H : K \subseteq gHg^{-1}\}$$

Definition 2.12. *A **coefficient system** is a covariant functor from $\mathcal{O}(G)$ to the category of abelian groups.*

The chain complex for the Bredon homology is the tensor product of one such coefficient system and a fixed contravariant functor $\underline{C}_n(X)$:

Definition 2.13. *Let X be a G-CW complex. The **equivariant** n**-chains** on X, $\underline{C}_n(X)$, is the contravariant functor from $\mathcal{O}(G)$ to abelian groups formed by the relative homology groups*

$$\underline{C}_n(X)(G/H) := H_n((X^n)^H, (X^{n-1})^H)$$

as objects and morphisms defined naturally as follows:

Let f be a morphism $f : G/K \to G/H$ and g is an element of G such that $f(K) = gH$. For each element in $\underline{C}_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H)$ expressed as $e : D^n \to (X^n)^H$ with $e(\partial D^n) \subset (X^{n-1})^H$, its image under $\underline{C}_n(X)(f)$ is the element in $\underline{C}_n(X)(G/K) = H_n((X^n)^K, (X^{n-1})^K)$ expressed by $g \circ e : D^n \to (X^n)^K$.

We can also define the boundary map $d : \underline{C}_n(X)(G/H) \to \underline{C}_{n-1}(X)(G/H)$, which is induced naturally by the boundary map in $C_*(X^H)$.

We can also view these $\underline{C}_n(X), n = 0, 1, ...$ as one functor $\underline{C}(X)$ from the orbit category to chain complexes:

Each object $\underline{C}(X)(G/H)$ is the chain complex corresponding to the CW structure of X^H. The morphism is the chain complex homomorphism (i.e., ring homomorphisms in each dimension which commute with the boundary map) formed by the $\underline{C}_*(X)(f)$ in Definition 2.13 in each dimension.

Definition 2.14. *Let M be a contravariant functor and N be a covariant functor from $\mathcal{O}(G)$. The tensor product $M\hat{\otimes}N$ is the abelian group*

$$M\hat{\otimes}N := \left(\bigoplus_{K \leq G} M(G/K) \otimes_{\mathbb{Z}} N(G/K) \right) / \sim$$

with the relation generated by $M(f)(m) \otimes n \sim m \otimes N(f)(n)$ for all morphisms $f : G/K \to G/H$ and $m \in M(G/H)$, $n \in N(G/K)$.

Definition 2.15. *The Bredon chain complex for a chosen coefficient system N is $C_n^G(X; N) := \underline{C}_n(X)\hat{\otimes}N$, with the boundary map $\partial = d \otimes 1$. The Bredon homology groups are $H_n^G(X; N) = Ker\partial/Im\partial$ for $C_n^G(X; N)$.*

The following examples (with X a G-CW complex) show the generality of this definition.

Example 2.16. *$N(G/H) := \mathbb{Z}[G/H]$, and for each $f : G/K \to G/H$, $N(f) : \mathbb{Z}[G/K] \to \mathbb{Z}[G/H]$ is defined naturally. In this case, we have $C_n^G(X; N) \cong C_n^{CW}(X)$, and the Bredon homology groups are just the ordinary homology groups of X.*

Example 2.17. *Let N be the functor sending each orbit to $\mathbb{Z}$ and each morphism to the identify map. Then all cells in the same orbit are identified. So $C_n^G(X; N)$ forms the CW chain complex for the orbit space $H \setminus X$. Then $H_n^G(X; N)$ is also the corresponding homology groups.*

In the following examples, we will use $[H]$ to denote the conjugacy class of $H \leq G$: $[H] := \{g^{-1}Hg : g \in G\}$.

Example 2.18. *For a fixed $H \leq G$, consider N such that $N(G/K) = \mathbb{Z}[G/K]$ for all $K \in [H]$ and $N(G/K) = 0$ otherwise, with the morphisms defined naturally. Then only the cells with isotropy groups contained in $[H]$ are preserved. So we get the homology groups for the quotient space of a subspace. In the case that G is abelian, it's the space $X^H / \cup_{H<K} X^K$.*

Example 2.19. *For a fixed $H \leq G$, consider N_H with $N_H(G/K) = \mathbb{Z}[H \setminus G/K]$, with morphisms defined naturally. Then all cells in the same H-orbit are identified. So we get the homology groups for $H \setminus X$.*

The first two examples are special cases of this one, with $H = \{1\}$ or G.

As the ordinary relation between homotopy and homology, weak G-homotopy equivalences are also Bredon homology equivalences ([1], pp. 22). Now we can state the main question we will discuss about in this paper:

For G-CW complexes X, let $S(X,H)$ be the number of cells in X with $H<G$ as their isotropy group (the total number of such cells of all dimensions). We say that a G-CW complex Y is smaller than X if $S(Y,H) \leq S(X,H)$ for all $H<G$, and Y is said to be strictly smaller than X if additionally $S(Y,H) < S(X,H)$ for some H.

X is called **minimal** if there doesn't exists a smaller complex Y which is also equivalent to it.

For any G-CW complex X, it might be equivalent to multiple minimal G-CW complexes X''s (automatically these X''s are equivalent to each other). We will show in section 4 that, when G is abelian, $S(X',H)$ is fixed for all such minimal X''s. In Theorem 1.1 we use homology groups to calculate the minimal number of cells for ordinary CW complexes. Here the question is, can we use Bredon homology groups for X to calculate these minimal numbers $S(X',H)$.

3 G-chain complexes

For ordinary CW complexes, the algebraic structure for their cells only contains a chain of free abelian groups and a boundary map whose square vanishes. In the case of G-CW complexes, this structure becomes more complicated. There is an induced G-action. The boundary map also becomes a G-map with some additional restrictions. In this section, we introduce the G-chain complex, as an algebraic model, and its relation with G-CW structures. At the end of this section, we will use this new tool to transfer our main topological question into an algebraic one.

Definition 3.1. *A G-**chain complex** $\mathcal{A}$ consists of:*

(I) Free abelian groups $\{\mathcal{A}_i(H)\ |i=0,1,2,...,H \leq G\}$ with $\mathcal{A}_i(H) \subset \mathcal{A}_i(H')$ for any $H' \leq H \leq G$, such that there exist a G-subset S_i of each $\mathcal{A}_i$ satisfying

$$\mathcal{A}_i(H) = \mathbb{Z}(S_i^H)$$

for any i,H. Here S_i^H denotes the set of elements in S_i which are fixed under the action of $H \leq G$. Write $\mathcal{A}_i = \mathcal{A}_i(\{1\})$.

(II) An equivariant boundary map $\partial : \mathcal{A}_{i+1} \to \mathcal{A}_i$, satisfying $\partial^2 = 0$, and $\partial(A_{i+1}(H)) \subset \mathcal{A}_i(H)$ for each $H \leq G$.

We use $\mathcal{A}(H)$ to denote the sequence of free abelian groups $\mathcal{A}_0(H), \mathcal{A}_1(H), \ldots$. It's not necessarily a G-chain complex unless G is abelian.

Remark: For each $H \leq G$, by restriction, a G-chain complex $\mathcal{A}$ becomes a H-chain complex.

An equivalent definition of $\mathcal{A}$ is a special kind of contravariant functor from the orbit category to chain complexes (see Definition 2.13). Define the functor $\underline{\mathcal{A}}$ such that $\underline{\mathcal{A}}(G/H) = \mathcal{A}(H)$ and for each G-map $f : G/K \to G/H$ with $f(K) = gH$, $\underline{\mathcal{A}}(f) : \mathcal{A}(H) \to \mathcal{A}(K)$ is the multiplication of g in each dimension $\mathcal{A}_i(H) \to \mathcal{A}_i(K)$. It's easy to check that for different $\mathcal{A}$, the corresponding functors $\underline{\mathcal{A}}$ are also different.

The Bredon homology groups of $\mathcal{A}$ can be defined naturally with the functor $\underline{\mathcal{A}}$.

We can view the G-subset S_i as a "basis" of $\mathcal{A}_i$. Actually the choice of such "basis" is not part of the structure. We only need the abelian groups $\mathcal{A}_i(H)$ and the boundary map ∂ to form a G-chain complex.

Definition 3.2. *A G-chain complex $\mathcal{A}'$ is a* ***subcomplex*** *of another G-chain complex $\mathcal{A}$ if we can find permutation representations $S_0, S_1, ...$ inducing $\mathcal{A}$ and $S'_0 \subset S_0, S'_1 \subset S_1, ...$ inducing $\mathcal{A}'$. We write $\mathcal{A}' \subset \mathcal{A}$ to denote this relation.*

For $\mathcal{A}' \subset \mathcal{A}$, we define the ***quotient complex*** *$\mathcal{A}/\mathcal{A}'$ in the natural way.*

Definition 3.3. *A* ***homomorphism*** *between G-chain complexes $F : \mathcal{A} \to \mathcal{A}'$ is a chain of group homomorphisms $F_i : \mathcal{A}_i \to \mathcal{A}'_i$, which commutes with the G-action and the boundary maps, and sends each $\mathcal{A}_i(H)$ into $\mathcal{A}'_i(H)$. An* ***isomorphism*** *is a bijective homomorphism. We use $\mathcal{A} \cong \mathcal{A}'$ to denote it.*

F is called an ***equivalence*** *if the it induces isomorphisms $H_i(\mathcal{A}(H)) \cong H_i(\mathcal{A}'(H))$ for each $H \leq G$.*

Since such F induces maps on homology groups $H_i(\mathcal{A}(H)) \to H_i(\mathcal{A}'(H))$ for each $H \leq G$, it also induces maps for Bredon homology groups.

The idea of G-chain complex comes from the G-CW complexes. There is a direct relation between them, as follows:

Proposition 3.4. *For each G-simply connected G-CW complex X, the abelian groups $\underline{C}_i(X)(G/H)$ and the boundary map form a G-chain complex. We still use the functor $\underline{C}(X)$ to denote it. Conversely, for each G-chain complex $\mathcal{A}$, we can find a G-CW complex X such that $\underline{C}(X) \cong \mathcal{A}$.*

For each pair of $\mathcal{A}' \subset \mathcal{A}$, we can find G-CW complexes X corresponding to $\mathcal{A}$ and X' corresponding to $\mathcal{A}'$, such that X' is a subcomplex of X.

Each G-map between G-CW complexes $f : X \to Y$ induces a homomorphism between G-chain complexes $F : \underline{C}(X) \to \underline{C}(Y)$ (F may be not unique), such that f and F induce the same maps on the homology groups $H_i(X^H) \to H_i(Y^H)$.

Proof: We only show the idea for the last part. Each G-map $f : X \to Y$ is G-homotopic to a cellular map ([1], pp. 17). So we can assume f itself cellular. Then for each i-cell e in X, $f(e) \subset Y^n$ and $f(\partial e) \subset Y^{n-1}$. So $f(e)$ can be written as a linear combination of n-cells in Y up to homotopy (i.e., expressing the same element in $\pi_n(Y^n, Y^{n-1})$). Since f is a G-map, it sends each X^H into Y^H. So we get a homomorphism $F : \underline{C}(X) \to \underline{C}(Y)$ which induces the same homology maps as f. □

Proposition 3.4 shows that, each equivalence between G-CW complexes induces an equivalence between their corresponding G-chain complex. Since we can define the same minimality on G-chain complexes, we transfer our main question onto the algebraic chain complexes in one direction. The following proposition shows that we can also transfer it back:

Proposition 3.5. *Let X be a G-simply connected G-CW complex. $\mathcal{A}$ is another G-chain complex with an equivalence $F : \mathcal{A} \to \underline{C}(X)$. Then we can find another G-CW complex Y with an isomorphism $F' : \mathcal{A} \to \underline{C}(Y)$, and a skeletal G-homotopy equivalence $f : Y \to X$, such that $C_*(f) \circ F' = F$.*

Proof: We will use the same induction in [2] and a little adjustment to make it into a G-version.

Since X is G-simple connected, $\pi_1(X^H) = 0$ for all $H \leq G$. Therefore we can construct Y^1 and a G cellular map $f_1 : Y^1 \to X$ which can be factored as 1-connected maps $f_1(H) : (Y^1)^H \to X^H$ for all $H \leq G$ (we can map Y^1 to the base point of X).

Suppose we have constructed Y^r and a G cellular map $f_r : Y^r \to X$, which can be factored as r-connected maps $f_r(H) : (Y^r)^H \to X^H$ for all $H \leq G$, such that $\underline{C}(Y^r)$ is isomorphic to the first r dimension parts of $\mathcal{A}$. We want to attach $(r+1)$-cells to Y^r and make it into a new G-CW complex Y^{r+1}, satisfying:

(I) $\underline{C}(Y^{r+1})$ is isomorphic to the first $r+1$ dimensions of $\mathcal{A}$. Notice that $\mathcal{A}_{r+1} \cong \mathbb{Z}S_{r+1}$ for some permutation representation S_{r+1}. We only need to add one cell corresponding to each element of S_{r+1}. Then the G-action on S_{r+1} induces the G-action on these $(r+1)$-cells.

(II) We also need to extend $f_r : Y^r \to X$ to a G-map $f_{r+1} : Y^{r+1} \to X$ which factors as $f_{r+1}(H) : (Y^{r+1})^H \to X^H$ for all $H \leq G$. To finish this, we need to extend f_r to the $(r+1)$-cells compatibly with the G-action on them. And the $(r+1)$-cells should kill $\pi_{r+1}(f_r(H))$ (the homotopy group of the mapping cone of $f_r(H)$) for all $H \leq G$.

[2] shows that there is a natural epimorphism $\mathcal{A}_{r+1}(H) \to \pi_{r+1}(f_r(H))$. Since the homomorphism $\mathcal{A} \to \underline{C}(X)$ commutes with the G-action on each $\mathcal{A}_r$, the map $\mathcal{A}_{r+1}(H) \to \pi_{r+1}(f_r(H))$ also commutes with the G-action. Since these maps can be obtained naturally, for any $H \leq H' \leq G$, we have the following commuting diagram:

$$\begin{array}{ccc} A_{r+1}(H') & \longrightarrow & \pi_{r+1}(f_r(H')) \\ \downarrow & & \downarrow \\ A_{r+1}(H) & \longrightarrow & \pi_{r+1}(f_r(H)) \end{array}$$

The row maps are the natural maps mentioned before. The left column map is the injection and the right column map is induced by the injection.

Now we can attach the $(r+1)$-cells according to these natural G-epimorphisms. For each $(r+1)$-cell e corresponding to an element in S_{r+1}, choose the smallest subgroup H such that e is fixed under H. Then attach e to Y^r and extend f^r to it through the image of the corresponding element in S_{r+1} under the map $\mathcal{A}_{r+1}(H) \to \pi_{r+1}(f_r(H))$. Then the adding of e eliminate some elements in $\pi_{r+1}(f_r(H))$ naturally. For any H' such that e appears in $\mathcal{A}_{r+1}(H')$ (we have $H \leq H'$), the above commuting diagram tells us that the cell e also eliminate elements in $\pi_{r+1}(f_r(H'))$ naturally. In fact, we can add one cell in each orbit first, then add the rest of the orbit according to the G-action in Y^{r+1} and extend f^r according to the G-action in X. Then finally we can get a G-map $f_{r+1} : Y^{r+1} \to X$ and continue the induction.

Finally, according to the Whitehead Theorem, we get another G-CW complex Y with a G-homotopy equivalence $f : Y \to X$. As we attach the cells of Y through the homomorphism $\mathcal{A}_i \to \underline{C}_i(X)$ for each dimension, the induced map $C_*(f) : \underline{C}(Y) \to \underline{C}(X)$ agrees with F automatically (which gives the isomorphism F' we need). □

If X is not G-simply connected, we can get the same result but we need $\mathcal{A}$ to be the same as $\underline{C}(X)$ on the first two dimensions. In this case, it's hard to tell whether the number of cells in X is minimal or not.

G-simply connected G-CW complexes correspond to G-chain complexes with $H_1(\mathcal{A}(H)) = 0$ for all $H \leq G$. We also call this kind of G-chain complexes G-simply connected. Since the equivalence relation for G-CW complexes is invertible, for these G-simply connected G-chain complexes, the equivalence relation among them is also invertible.

Proposition 3.4 and 3.5 give us both directions to change our main question to an algebraic one:

For each G-chain complex induced by $S_0, S_1, ...$, let $S(\mathcal{A}, H)$ be the total number of elements in $S_0, S_1, ...$ with isotropy group $H \leq G$ (which doesn't depend on the choice of $S_0, S_1, ...$). We can define the same minimality for G-chain complexes. Then we only need to find out the minimal number of "cells" (more exactly, $S(\mathcal{A}', H)$ for an equivalent minimal G-chain complex $\mathcal{A}'$) for a G-simply connected G-chain complex from the information given by its Bredon homology groups.

4 More about minimal G-chain complexes

For the rest of this paper, we will discuss the algebraic version of our main question in the case that G is abelian. In this case, each $\mathcal{A}(H)$ is a subcomplex since the G-action is closed on it (which is not necessarily true for non-abelian G).

In the previous sections, we have shown that the equivalence of complexes induces isomorphisms on their Bredon homology groups. When G is abelian, the inverse is also true, since each $H_i(\mathcal{A}(H))$ can be expressed by a Bredon homology group $H_i^G(\mathcal{A}; N_{\geq H})$, with the coefficient system $N_{\geq H}$ defined as follows:

$N_{\geq H}(G/K) = \mathbb{Z}[G/K]$ if $H \leq K$, otherwise $N_{\geq H}(G/K) = 0$. For any two $K_1, K_2 \geq H$ and a G-map $f : G/K_1 \to G/K_2$, we must have $K_1 \leq K_2$ (in the non-abelian case, K_1 is conjugate to a subgroup of K_2) and f is induced by the multiplication of an element $g \in G$. The morphism $N_{\geq H}(f) : \mathbb{Z}[G/K_1] \to \mathbb{Z}[G/K_2]$ is induced by the same multiplication.

From the definition of Bredon homology for $\mathcal{A}$, we have the following immediate result:

Lemma 4.1. *$\mathcal{A}$ is G-simply connected if and only if all its 1-dimensional Bredon homology groups are trivial.*

In this section, we will first simplify the question to free complexes. A G-chain complex $\mathcal{A}$ is called **free** if each $\mathcal{A}_i$ is a free G-module. More exactly, for any choice of S_i (see Definition 3.1), the G-action on it is free.

4.1 Free parts of general complexes

Suppose that $\mathcal{A}$ is an arbitrary G-chain complex with abelian group G. For each $H < G$, we define the subcomplex $\mathcal{A}(> H)$ such that each $\mathcal{A}_i(> H)$ is the union of $\mathcal{A}_i(H')$ for all H' strictly larger than H (i.e., not equal H but contain H as a subgroup). Let $S_0, S_1, ...$ be a chain of permutation representations inducing $\mathcal{A}$. Define $S_i^{>H}$ to be the subrepresentation consisting of all elements with isotropy groups strictly larger than H. Then $S_0^{>H}, S_1^{>H}, ...$ induces $\mathcal{A}(> H)$ (it doesn't depend on the choice of $S_0, S_1, ...$).

Define the quotient complex $\mathcal{A}(H*) = \mathcal{A}(H)/\mathcal{A}(> H)$, which is also induced by the quotient representations $S_i^{H*} := S_i^H / S_i^{>H}$. Then each $\mathcal{A}(H*)$ is a free complex under group G/H. In fact, any homomorphism $\mathcal{A} \to \mathcal{A}'$ induces homomorphisms $\mathcal{A}(H*) \to \mathcal{A}'(H*)$.

Notice that each $\mathcal{A}_i$ can be viewed as the direct sum of $\mathcal{A}_i(H*)$ for all $H \leq G$, and each $\mathcal{A}(H*)$ is a free G/H-chain complex. So we divide $\mathcal{A}$ into several free parts.

In this section, we will show that, $\mathcal{A}$ is minimal if and only if all its free parts $\mathcal{A}(H*)$ are minimal. Furthermore, the information of Bredon homology groups for these free parts are also contained in the Bredon homology groups of $\mathcal{A}$ itself. Then automatically, the minimal complexes equivalent to the same $\mathcal{A}$ share the same total number of cells with isotropy H for all $H \leq G$, since they are also equivalent to each other (this makes the definition of minimal G-chain complexes at the end of section 2 clearer). And we only need to consider the minimal free complexes.

Proposition 4.2. *The Bredon homology groups of $\mathcal{A}(H*)$ under group G/H can be expressed as Bredon homology groups of $\mathcal{A}$ under group G. Therefore, any equivalence $\mathcal{A} \to \mathcal{A}'$ induces equivalences $\mathcal{A}(H*) \to \mathcal{A}'(H*)$ for all $H < G$.*

Proof: For each $H < G$, consider the following coefficient system N_{*H}:

$N_{H*}(G/K) = 0$ for $K \neq H$ and $N_{H*}(G/H) = \mathbb{Z}[H]$. Each G-map $f : G/H \to G/H$ is induced by a multiplication of element $g \in G$. Let $N_{H*}(f) : \mathbb{Z}[H] \to \mathbb{Z}[H]$ to be induced by the same multiplication.

For each coefficient system N for group G/H, it can be viewed as a coefficient system of G by sending each G/K to $N(G/H/K \cap H)$ and each G-map $f : G/K_1 \to G/K_2$ induced by the multiplication of g to $N(f')$, where $f' : G/H/K_1 \cap H \to G/H/K_2 \cap H$ is induced by the multiplication of $g/H \in G/H$.

Consider the tensor product $N_{H*} \otimes N$. We have $H_i^G(\mathcal{A}; N_{H*} \otimes N) \cong H_i^{G/H}(\mathcal{A}(H*); N)$ since $\mathcal{A}(H*)$ is free. □

Together with Lemma 4.2, if $\mathcal{A}$ is G-simply connected, so are all its free parts $\mathcal{A}(H*)$. We also have:

Corollary 4.3. *If $\mathcal{A}(H*)$ is minimal for all $H < G$, then $\mathcal{A}$ is minimal.*

Proof: Assume that $\mathcal{A}$ is not minimal in this case. Then we can find an equivalence $\mathcal{A}' \to \mathcal{A}$ such that $S(\mathcal{A}'(H*), H) = S(\mathcal{A}', H) < S(\mathcal{A}, H) = S(\mathcal{A}(H*), H)$ for some $H \leq G$. However, $\mathcal{A}'(H*) \to \mathcal{A}(H*)$ is also an equivalence, which contradicts with the minimality of $\mathcal{A}(H*)$. □

The converse is also true:

Proposition 4.4. *Each equivalence $F : \mathcal{A}' \to \mathcal{A}(H*)$ can be extended to an equivalence $\mathcal{A}'' \to \mathcal{A}$ such that $\mathcal{A}' = \mathcal{A}''(H*)$ and $S(\mathcal{A}, H') = S(\mathcal{A}'', H')$ for all $H' \neq H$. Hence if $\mathcal{A}$ is minimal, so are all its free parts.*

Proof: We extend the equivalence F with two steps. First we consider the case that $H = 1$. Then $\mathcal{A}(H*) = \mathcal{A}(H)/\mathcal{A}(> H) = \mathcal{A}/\mathcal{A}(> H)$. Let $S_0, S_1, ...$ induce $\mathcal{A}$ and $S_0', S_1', ...$ induce $\mathcal{A}'$. Construct $\mathcal{A}''$ by $S_i'' = S_i' \sqcup S_i^{>H}$, $i = 0, 1, ...$ First we define a homomorphism $\mathcal{A}'' \to \mathcal{A}$ and then define the boundary map compatibly.

Notice that S_i can be viewed as $S_i^{>H} \sqcup S_i^{H*}$, and F_i maps $\mathbb{Z}(S_i')$ into $\mathbb{Z}(S_i^{H*})$. Together with the identity map $\mathbb{Z}(S_i^{>H}) \to \mathbb{Z}(S_i^{>H})$. We can define $F_i' = F_i \oplus id : \mathbb{Z}(S_i'') = \mathbb{Z}(S_i') \oplus \mathbb{Z}(S_i^{>H}) \to \mathbb{Z}(S_i^{H*}) \oplus \mathbb{Z}(S_i^{>H}) = \mathbb{Z}(S_i)$.

Assume that ∂, ∂' are boundary maps for $\mathcal{A}$ and $\mathcal{A}'$. The boundary map ∂'' for $\mathcal{A}''$ is defined as follows:

Each element in $\mathcal{A}_i''$ can be written as $\alpha + \beta$ with $\alpha \in \mathbb{Z}(S_i')$ and $\beta \in \mathbb{Z}(S_i^{>H})$. Then α and β can also be viewed as elements in $\mathcal{A}_i'$ and $\mathcal{A}_i(> H) \subset \mathcal{A}_i$ respectively. Since F is a homomorphism, $\partial(F_i(\alpha)) = F_{i-1}(\partial'(\alpha)) + \beta'$ with $\beta' \in \mathcal{A}_{i-1}^{>H}$. Let $\partial''(\alpha + \beta) := \partial'(\alpha) + \beta' + \partial(\beta)$. Then ∂'' commutes with F''.

We also have $(\partial'')^2 = 0$: Since $(\partial')^2(\alpha) = 0$, we have $\partial''(\partial'(\alpha)) = \partial(F_{i-1}(\partial'(\alpha))$ and hence

$$\partial''(\alpha + \beta) = \partial''(\partial'(\alpha)) + \partial(\beta') + \partial^2(\beta) = \partial^2(F_i(\alpha) + \beta) = 0,$$

which makes $\mathcal{A}''$ a well-defined G-chain complex and F'' a homomorphism. Moreover, it's easy to check that, $\mathcal{A}''(H*) \cong \mathcal{A}'$ and the induced homomorphism

$\mathcal{A}''(H*) \to \mathcal{A}(H*)$ agrees with F. Notice that the induced homomorphisms $\mathcal{A}''(> H) \to \mathcal{A}(> H)$ is an isomorphism. So we have $S(\mathcal{A}.H') = S(\mathcal{A}'', H')$ for all $H' \neq H$.

Finally, F' is also an equivalence. Since we have $\mathcal{A}''(> H) \cong \mathcal{A}(> H)$, we only need to show that F' induces isomorphisms $H_i(\mathcal{A}'') \cong H_i(\mathcal{A})$. We have short exact sequences $\mathcal{A}_i(> H) \to \mathcal{A}_i \to \mathcal{A}_i(H*)$ and $\mathcal{A}''_i(> H) \to \mathcal{A}''_i \to \mathcal{A}''_i(H*)$ which induces long exact sequences and the following diagram:

$$\begin{array}{ccccccccc} ... & \longrightarrow & H_i(\mathcal{A}''(> H)) & \longrightarrow & H_i(\mathcal{A}'') & \longrightarrow & H_i(\mathcal{A}''(H*)) & \longrightarrow & ... \\ & & \downarrow & & \downarrow & & \downarrow & & \\ ... & \longrightarrow & H_i(\mathcal{A}(> H)) & \longrightarrow & H_i(\mathcal{A}) & \longrightarrow & H_i(\mathcal{A}(H*)) & \longrightarrow & ... \end{array}$$

All the column maps are induced by F'. Notice that all column maps with forms $H_i(\mathcal{A}''(H*)) \to H_i(\mathcal{A}(H*))$ (agrees with the maps induced by F) and $H_i(\mathcal{A}''(> H)) \to H_i(\mathcal{A}(> H))$ (induced by isomorphisms of chain complexes) are isomorphisms. According to the Five Lemma, $H_i(\mathcal{A}'') \to H_i(\mathcal{A})$ is also an isomorphism.

Now let's consider the general case. According to the previous argument, we can extend the equivalence to $\mathcal{A}(H)$ (since $\mathcal{A}(H*) = \mathcal{A}(H)/\mathcal{A}(> H)$). So we can assume that we have already got an equivalence $F : \mathcal{A}' \to \mathcal{A}(H)$ and we want to extend it to $\mathcal{A}$. We still choose $S_0, S_1, ...$ inducing $\mathcal{A}$ and $S'_0, S'_1, ...$ inducing $\mathcal{A}'$ and use ∂, ∂' to denote the boundary maps. Let $S_i^{/H} := S_i/S_i(H)$ and $\mathcal{A}_i(/H) := \mathcal{A}_i/\mathcal{A}_i(H)$. Then $S_i^{/H}$ induces $\mathcal{A}_i(/H)$ and we can still view S_i as $S_i^{/H} \sqcup S_i^H$. Construct $\mathcal{A}''$ by $S''_i = S'_i \sqcup S_i^{/H}$. The homomorphism $F' : \mathcal{A}'' \to \mathcal{A}$ and the boundary map ∂'' for $\mathcal{A}''$ are much more complicated in this case. We have to define them inductively.

First, F'_0 maps $\mathbb{Z}(S'_0)$ into $\mathbb{Z}(S_i^H)$. We can define $F'_0 = F_0 \oplus id : \mathbb{Z}(S''_0) = \mathbb{Z}(S'_0) \oplus \mathbb{Z}(S_0^{/H}) \to \mathbb{Z}(S_0^H) \oplus \mathbb{Z}(S_0^{/H}) = \mathbb{Z}(S_0)$. Automatically ∂'' on $\mathcal{A}''_0$ is the zero map.

Suppose that we have defined $F'_0, F'_1, ...F'_{i-1}$ and ∂'' on $\mathcal{A}''_0, ...\mathcal{A}''_{i-1}$ whose square is zero and commutes with F'. In addition, the induced map $\mathcal{A}''_j(H) \to \mathcal{A}_j(H)$ agrees with F_j and the induced map $\mathcal{A}''_j(/H) \to \mathcal{A}_j(/H)$ is an isomorphism. Now let's define F' and ∂'' on $\mathcal{A}''_i$. In fact, we only need to define them on each element of $S''_i = S'_i \sqcup S_i^{/H}$ and then extend them linearly.

For each $\alpha \in S'_i$, let $F'_i(\alpha) := F(\alpha)$ and $\partial''(\alpha) = \partial'(\alpha) \in \mathbb{Z}(S'_{i-1})$.

For each $\beta \in S_i^{/H}$, we choose $F'_i(\beta) := \beta + \alpha$ for some $\alpha \in S_i^H$. We need to choose α such that $\partial(\beta + \alpha)$ is contained in $F'_{i-1}(\mathcal{A}''_{i-1})$ and it has a pre-image under F' in $\mathcal{A}''_{i-1} \cap \ker \partial''$ in order to make F' a homomorphism and satisfy $(\partial'')^2 = 0$.

Write $\partial(\beta) = F'_{i-1}(\beta') + \alpha'$ for some $\beta' \in \mathbb{Z}(S_{i-1}^{/H})$ and $\alpha' \in \mathcal{A}_{i-1}^H$. Then $\partial''(\beta') \in \mathbb{Z}(S'_{i-2})$ and $\partial(\alpha') = -\partial(F'_{i-1}(\beta')) = F'_{i-2}(\partial''(\beta')) = F_{i-2}(\partial''(\beta'))$ is in the image of F. Since $\partial(\alpha')$ is also a boundary (contained in the image of ∂) and F is an equivalence, its pre-image $\partial''(\beta')$ is also a boundary in $\mathcal{A}'_{i-2}$. So we can find $\gamma \in \mathcal{A}'_{i-1} \cong \mathcal{A}''_{i-1}(H)$ such that $\partial''(\gamma) = \partial'(\gamma) = -\partial''(\beta')$.

Now $F'_{i-1}(\gamma) = F_{i-1}(\gamma)$ has the same boundary as α', and $F'_{i-1}(\gamma) - \alpha'$ becomes a cycle. Since F is an equivalence, we can find a cycle $\gamma' \in \mathcal{A}'_{i-1}$ and a boundary $\alpha'' \in \mathcal{A}^H_{i-1}$ such that $F_{i-1}(\gamma') = F'_{i-1}(\gamma) - \alpha' + \alpha''$. Choose $\alpha \in \mathcal{A}^H_i$ to be any pre-image of $-\alpha''$ under ∂. Then $\partial(\beta + \alpha) = F'_{i-1}(\beta') + \alpha' - \alpha'' = F'_{i-1}(\beta' + \gamma - \gamma')$, which is in the image of F'_{i-1} and the pre-image $\beta' + \gamma - \gamma'$ is a cycle. We only need to define $F'_i(\beta) = \beta + \alpha$ and $\partial''(\beta) = \beta' + \gamma - \gamma'$.

In order to agree with the G-action, we only need to do this process for one element of each orbit in $S_i^{/H}$.

Finally, we get the complex $\mathcal{A}''$ together with a homomorphism $F' : \mathcal{A}'' \to \mathcal{A}$, such that the induced map $\mathcal{A}''(H) \to \mathcal{A}(H)$ agrees with F and the map $\mathcal{A}''(/H) \to \mathcal{A}(/H)$ is an isomorphism. F' is also an equivalence according to the following diagram and the Five Lemma:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_i(\mathcal{A}''(H \cap H')) & \longrightarrow & H_i(\mathcal{A}''(H')) & \longrightarrow & H_i(\mathcal{A}''(H')/\mathcal{A}''(H \cap H')) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_i(\mathcal{A}(H \cap H')) & \longrightarrow & H_i(\mathcal{A}(H')) & \longrightarrow & H_i(\mathcal{A}(H')/\mathcal{A}(H \cap H')) & \longrightarrow & \cdots
\end{array}$$

for all subgroups $H' \leq G$. $\square$

Now we only need to consider the free complexes. For the rest of this section, we discuss some necessary conditions for a free complex to be minimal.

4.2 Free complexes

When $\mathcal{A}$ is free, we have $\mathcal{A}(\{1\}) = \mathcal{A}$ and $\mathcal{A}(H) = 0$ for any nontrivial subgroup H. The chain complex becomes a single chain of free abelian groups, more exactly, free G-modules ($\mathbb{Z}[G]$-modules), $\mathcal{A}_0, \mathcal{A}_1, \ldots$.

We say that a **basis** for $\mathcal{A}_i$ is a G-module basis. In fact, if $\{e_1, e_2, \ldots, e_r\}$ form a basis, then $\{g \cdot e_i \mid g \in G, i = 1, 2, \ldots, r\}$ form a free permutation representation which induces $\mathcal{A}_i$.

The following proposition tells which elements can be selected into a basis:

Proposition 4.5. *For $e \in \mathcal{A}_i$, if $\mathbb{Z}[G]e \cong \mathbb{Z}[G]$, and for any $e' \in \mathcal{A}_i$, $a \in \mathbb{Z}$, $0 \neq ae' \in \mathbb{Z}[G]e \Longrightarrow e' \in \mathbb{Z}[G]e$, then we can find a basis containing e.*

Proof: These conditions are equivalent to that, if we view $\mathcal{A}_i$ as a $\mathbb{Z}$-module, then $\mathbb{Z}[G]e \cong \mathbb{Z}^{|G|}$ and $\mathcal{A}_i/\mathbb{Z}[G]e$ is still a free $\mathbb{Z}$-module. In other words, $\{ge \mid g \in G\}$ can be extended into a $\mathbb{Z}$-module basis of $\mathcal{A}_i$.

We choose $e_1, e_2, \ldots, e_r$ as an arbitrary basis for $\mathcal{A}_i$ (as $\mathbb{Z}[G]$-module). Then $\{g \cdot e_i \mid g \in G, i = 1, 2, \ldots, r\}$ is a $\mathbb{Z}$-module basis. We consider the $\mathbb{Z}[G]$ and $\mathbb{Z}$ coordinates under these two basis separately.

Consider $\mathbb{Z}$-matrix M whose rows are the coordinates of $\{ge\ |g \in G\}$. Since $\{ge\ |g \in G\}$ can be extended into a basis, by adding other elements in this basis to M, we can extend M to a square matrix M', which can be transformed into the unit matrix by invertible column operations on $\mathbb{Z}$. Therefore we can use invertible column operations (in $\mathbb{Z}$) to change M into

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}$$

Now consider $\mathbb{Z}[G]$-matrix with only one row corresponding to the $\mathbb{Z}[G]$ coordinates of e. The previous operations of M induces invertible column operations in $\mathbb{Z}[G]$ which can change this 1-row matrix into

$$\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

Apply the inverse of these operations to the unit square $\mathbb{Z}[G]$-matrix, then the first row becomes e, and all rows form a $\mathbb{Z}[G]$-module basis for $\mathcal{A}_i$ which contains e. $\Box$

This proof can also be applied for a generalized version:

Corollary 4.6. *For $e_1, e_2, ..., e_l \in \mathcal{A}_i$, if $\mathbb{Z}[G][e_1, e_2, ..., e_l] \cong \mathbb{Z}[G]^l$, and for any $e' \in \mathcal{A}_i, a \in \mathbb{Z}$, $0 \neq ae' \in \mathbb{Z}[G][e_1, e_2, ..., e_l] \Longrightarrow e' \in \mathbb{Z}[G][e_1, e_2, ..., e_l]$, then we can extend $\{e_1, e_2, ..., e_l\}$ to a basis of $\mathcal{A}_i$.*

In other words, if $M_1 \subset M_2$ are free G-modules and M_2/M_1 is a free $\mathbb{Z}$-module, then M_2/M_1 is also a free G-module, and we can extend any basis of M_1 to a basis of M_2. We call M_1 a ***torsion free submodule*** *of M_2.*

We call all elements satisfying the conditions in Proposition 4.5 as **basic elements**. Then we can get necessary conditions for a minimal free complex.

Proposition 4.7. *For a free G-chain complex $\mathcal{A}$, if the image of boundary map contains at least one basic element, then $\mathcal{A}$ is equivalent to one of its subcomplex (which is not $\mathcal{A}$ itself) by the inclusion map.*

Proof: Assume that $e \in \mathcal{A}_i$ is a basic element in the image of the boundary map. Let $e' \in \mathcal{A}_{i+1}$ be a pre-image of e. First we show that e' is also a basic element:

Since $\partial e' = e$ and $\mathbb{Z}[G]e \cong \mathbb{Z}[G]$, we must have $\mathbb{Z}[G]e' \cong \mathbb{Z}[G]$ and $\partial : \mathbb{Z}[G]e' \to \mathbb{Z}[G]e$ is an isomorphism. For each $\alpha \in \mathcal{A}_{i+1}$ and $s \in \mathbb{Z} - \{0\}$, if $s\alpha \in \mathbb{Z}[G]e'$, then $s\partial(\alpha) = \partial(s\alpha) \in \mathbb{Z}[G]e$. So $\partial(\alpha) \in \mathbb{Z}[G]e$. Let β is the pre-image of $\partial(\alpha)$ in $\mathbb{Z}[G]e\prime$. We have $\partial((s\alpha - s\beta)e') = 0$, hence $\alpha = \beta \in \mathbb{Z}[G]e'$.

Extend e to a basis $\{e = e_0, e_1, ..., e_s\}$ of $\mathcal{A}_i$ and e' to $\{e' = e'_0, e'_1, ..., e'_t\}$ of $\mathcal{A}_{i+1}$ (as $\mathbb{Z}[G]$-modules). We can adjust $e'_1, e'_2, ...$ such that each $\partial e'_j$ can be written as $a_1e_1 + a_2e_2 + ... + a_se_s$ for $a_1, a_2, ... \in \mathbb{Z}[G]$. In fact, if $\partial e'_j = a_0e_0 + a_1e_1 + a_2e_2 + ... + a_se_s$, we only need to replace it by $e'_j - a_0e'$.

Now e, e' are "separated" from other basis elements. Let $\mathcal{A}'$ be a subcomplex of $\mathcal{A}$ such that $\mathcal{A}'_i = \mathbb{Z}[G][e_1, e_2, ..., e_s]$, $A'_{i+1} = \mathbb{Z}[G][e'_1, ..., e'_t]$, and all other dimensions are the same as $\mathcal{A}$. From the calculation of homology groups, we can show that the inclusion $\mathcal{A}' \hookrightarrow \mathcal{A}$ induces isomorphisms on all Bredon homology groups. $\square$

According to these two propositions, we get the following necessary condition for minimal complexes:

Corollary 4.8. *Suppose $\mathcal{A}$ is a minimal free G-chain complex. Then for each $e \in \mathcal{A}_i \cap Im\partial$, we can find an ideal $I < \mathbb{Z}[G]$ (not $\mathbb{Z}[G]$ itself), such that $e \in I\mathcal{A}_i$.*

Proof: If we cannot find such ideal I, it's easy to show that e will satisfy the condition in Proposition 4.5. Then Proposition 4.7 gives the contradiction. $\square$

For some special G, we can show that Corollary 4.8 is also a sufficient condition. In fact, it might be true for general G. In the next section, we show a practical method to calculate the number of cells for a minimal free complex for $G = \mathbb{Z}/p\mathbb{Z}$, which satisfies the inverse of Corollary 4.8. In fact, assume $\mathcal{A}$ is a free complex satisfies the condition in Corollary 4.8 (when $G = \mathbb{Z}/p\mathbb{Z}$), we can use its Bredon homology groups to calculate the dimensions of all $\mathcal{A}_i$. Hence automatically $\mathcal{A}$ is minimal and Corollary 4.8 must be true for $G = \mathbb{Z}/p\mathbb{Z}$.

5 $\mathbb{Z}/p\mathbb{Z}$-chain complexes

In this section, we discuss the special case that $G = \mathbb{Z}/p\mathbb{Z}$ for some prime p and show how to get the dimensions of any minimal G-chain complex from its Bredon homology groups.

We will mainly show the following result:

Proposition 5.1. *For a free G-chain complex $\mathcal{A}$ satisfying the conditions in Corollary* 4.8*, we can use its Bredon homology groups to calculate* $\dim(\mathcal{A}_i)$ *for each i.*

We will explain how to calculate that practically at the beginning of the subsection 5.2.

Following this Proposition, the inverse of Corollary 4.8 is automatically true, since if $\mathcal{A}$ is equivalent to another minimal complex, with the Bredon homology groups, we can get the same dimensions for that complex. Then according to the argument in the last section, we can simplify general G-chain complexes to free complexes for G and for $G/G = \{1\}$, where the latter is just the ordinary CW complex. So finally we can get a method to get dimensions for any $\mathbb{Z}/p\mathbb{Z}$-chain complex from its Bredon homology groups.

Now suppose that $\mathcal{A}$ is a free G-chain complex satisfying the condition in Corollary 4.8. Let $g \in G$ be a generator of G. We will find a basis for each $\mathcal{A}_i$ which makes the boundary map clear.

5.1 A clear basis of $\mathcal{A}_i$

Definition 5.2. *The* ***augmentation map*** $C : \mathbb{Z}[G] \to \mathbb{Z}$ *is a ring homomorphism which sends each* $g \in G$ *to* 1. *For each integer* $a \in \mathbb{Z}$, *define the ideal* $I_a := C^{-1}(a)$.

Consider the following chain of ideals: $Z[G] = J_{1,0} \supset I_p = J_{1,1} \supset J_{1,2} \supset .... \supset J_{2,1} \supset J_{2,2} \supset ... \supset \{0\}$, such that $J_{1,j} = \mathbb{Z}[G][1+g+g^2+...+g^{p-1}, (1-g)^j]$, $J_{2,j} = \mathbb{Z}[G][(1+g+g^2+...+g^{p-1})^l]$. Then we have:

Lemma 5.3. *For each ideal* I *in this chain, it has a successor* I' *(i.e., the ideal just after it). We also have* $I \cdot I_p \subset I'$.

For each finite or infinite set of ideals in this chain, the lower limit (or intersection) of all elements in this set is still an ideal in this chain.

Define an order among these ideals according to the inclusion relation ($\{0\}$ has the smallest order and $\mathbb{Z}[G]$ has the largest). Then an order is induced to the elements in $\mathcal{A}_i$ as follows: For each $\alpha \in \mathcal{A}_i$, let I be the smallest ideal in the chain such that $\partial\alpha \in I \cdot \mathcal{A}_{i-1}$ (I must exist since the chain is closed under lower limits). Then give α the same order as I. Define a map O which sends each α to such I with the same order.

Now we construct a basis $(e_1, e_2, ...e_r)$ for $\mathcal{A}_i$ satisfying:

(I) $e_1 \le e_2 \le e_3 \le ... \le e_r$;

(II) For each $\alpha = \sum a_j e_j \in \mathcal{A}_i$, if $C(a_j) \notin p\mathbb{Z}$ (or equivalently, $a_j \notin I_p$) for some j, then $\alpha \ge e_j$.

The basis is defined inductively according to the dimension of $\mathcal{A}_i$ as a finite dimensional free G-module. In fact, the order can be defined on any pair of (M, M'), where M is a finite dimensional free G-module, M' is an arbitrary finite dimensional G-module, together with a homomorphism $\partial : M \to M'$ (here we have $M = \mathcal{A}_i, M' = \mathcal{A}_{i-1}$).

It's easy to show that such basis exists when $\dim M = 1$. Assume the existence when $\dim M = r-1$, we consider the case of r:

Let I be the smallest ideal in the chain such that $\partial M \subset I \cdot M'$. Then we can find a basic element e_r in M such that the order of e_r is the same as I. Now replace M by $M_0 = M/\mathbb{Z}[G]e_r$ (which is still a free G-module) and M' by $M'_0 = M'/\mathbb{Z}[G]\partial e_r$. By induction, we can find such basis $(e_1^*, e_2^*, ..., e_{r-1}^*)$ for M_0. It can be extended to a basis $(e_1, e_2, ..., e_r)$ of M such that each e_j has the same order as e_j^*. Then automatically $e_1 \le e_2 \le ... \le e_r$. For each $\alpha \in M$, $\alpha = \sum a_l e_l$ such that $\alpha < e_j$ for some j, if $j < r$, then the image $\alpha^* = \sum a_l e_l^*$ of α in M_0 cannot have larger order than α itself. So we have $\alpha^* < e_j^*$ and hence $C(a_l) \in p\mathbb{Z}$ by induction. If $j = r$, assume that the order of $e_r, e_{r-1}, ..., e_{r-t}$ are the same but e_{r-t-1} is smaller. Then by the previous case, we know that $C(a_l) \in p\mathbb{Z}$ for $l = r-1, r-2, ..., r-t$. Assume that e_r has the same order as I and I' is the successor of I. Then $\partial\alpha$ and each $\partial(a_l e_l)$ are contained in $I'M'$ for $l = 1, 2, ..., r-1$. So we must have $\partial(a_r e_r) \in I'M'$ and hence $C(a_r) \in p\mathbb{Z}$.

Now we finish the construction of such basis. We divide $(e_1, e_2, ...)$ into 4 parts: $P_1 := \{e_1, e_2, ..., e_{t_1}\}$ have image $\{0\}$ under O; $P_2 := \{e_{t_1+1}, ..., e_{t_2}\}$ have image with the form $J_{2,j}$ for some j; $P_3 := \{e_{t_2+1}, ...e_{t_3}\}$ have image with the form $J_{1,j}$ for $j > 0$; $P_4 := \{e_{t_3+1}, e_{t_3+2}, ..., e_{\dim \mathcal{A}_i}\}$ have image $J_{1,0} = \mathbb{Z}[G]$. We write $\mathbb{Z}[G][P_j]$ to denote the (free) submodule generated by all elements in P_j.

From the condition (II) of this basis, we can get the following properties:

Lemma 5.4. *For any $\alpha \in \mathbb{Z}[G][P_4]$, if $\partial\alpha \in I_p\mathcal{A}_{i-1}$, then $\alpha \in I_p\mathcal{A}_i$.*

Corollary 5.5. *For each $\alpha \in J_{1,1}\mathcal{A}_i$ written in the form under basis $(e_1, e_2, ...)$, the coefficient of each element in P_4 is in I_p.*

Lemma 5.6. *For each $\alpha \in \mathbb{Z}[G][P_3 \cup P_4]$, if $\partial\alpha \in (1 + g + ... + g^{p-1})\mathcal{A}_{i-1} = J_{2,1}\mathcal{A}_{i-1}$, then $\alpha \in (1 + g + ... + g^{p-1})\mathcal{A}_i = J_{2,1}\mathcal{A}_i$.*

Corollary 5.7. *For each $\alpha \in J_{2,1}\mathcal{A}_i$, the coefficient of each element in $P_3 \cup P_4$ is in $\mathbb{Z}[1 + g + ... + g^{p-1}] = J_{2,1}$.*

Lemma 5.8. *For each $\alpha \in \ker\partial \cap \mathcal{A}_i$, the coefficient of each element in P_2 is in I_p.*

We still need to do some more adjustment for the elements in P_1 according to the image of the boundary map:

Consider the coefficient system N_p such that $N_p(G/G) = \mathbb{Z}[G]/I_p \cong \mathbb{Z}/p\mathbb{Z}$ and the morphisms are defined naturally. Then $\ker\partial$ for N_p is $\mathbb{Z}[G]/I_p[P_1 \cup P_2 \cup P_3]$ according to Corollary 5.5 and the choice of P_1, P_2, P_3. For each element $\alpha \in Im\partial$, Lemma 5.6 and 5.8 tell us that the coefficients for elements in $P_2 \cup P_3 \cup P_4$ are zero for N_p (since they are contained in I_p in ordinary case). Therefore $Im\partial \subset \mathbb{Z}[G]/I_p[P_1]$. Since $\mathbb{Z}[G]/I_p[P_1]$, $Im\partial$ are free $\mathbb{Z}/p\mathbb{Z}$(as a ring, not a group)-modules, we can select a $\mathbb{Z}/p\mathbb{Z}$-basis for $Im\partial$ and extend it to $\mathbb{Z}[G]/I_p[P_1]$. The new choice of basis of $\mathbb{Z}[G]/I_p[P_1]$ can be lifted into a basis of $\mathbb{Z}[G][P_1]$.

In conclusion, finally we can find proper $\{e_1, e_2, ..., e_{t_1}\}$ with some $0 \leq t \leq t_1$, such that, for any $\alpha \in Im\partial$, the coefficient of each e_j with $j > t$ are contained in I_p. Furthermore, for each $j = 1, 2, ..., t$, we can find $\alpha \in Im\partial$, such that only the coefficient of e_j is not in I_p.

5.2 Calculation of $\dim \mathcal{A}_i$

We state the result first:

We need the following coefficient systems: N_p, N_{orb}, N_g with $N_p(G/G) = \mathbb{Z}[G]/I_p$, $N_{orb}(G/G) = \mathbb{Z}[G]/I_0$, $N_g(G/G) = \mathbb{Z}[G]/\mathbb{Z}[1 + g + ... + g^{p-1}]$ and the morphisms are defined naturally. We also need a map $f : H_i^G(\mathcal{A}; N_g) \to H_i^G(\mathcal{A}; N_p)$ induced by the quotient may $\mathbb{Z}[G]/\mathbb{Z}[1 + g + ... + g^{p-1}] \to \mathbb{Z}[G]/I_p$.

For any G-module M, use $gen(M)$ to denote the minimal number of generators for M.

Define the **G-torsion** for dimension i, written as $\tau_G(i)$, to be the larger one of $gen(H_i^G(\mathcal{A};N_g)) - gen(Im(f))$ and the torsion coefficient of $H_i^G(\mathcal{A};N_{orb})$ mod p, i.e., we can write $H_i^G(\mathcal{A};N_{orb}) = \mathbb{Z}^a \times \mathbb{Z}/a_1\mathbb{Z} \times \mathbb{Z}/a_2\mathbb{Z}... \times \mathbb{Z}/a_s\mathbb{Z}$ such that each a_j divides a_{j+1}, and then consider the maximal j such that a_j is not a multiple of p.

Then we have

Proposition 5.9. $\dim \mathcal{A}_i = \tau_G(i) + \tau_G(i-1) + gen(H_i^G(\mathcal{A};N_p))$.

Recall that in the last subsection, we find a basis $\{e_1, e_2, ...\}$ for $\mathcal{A}_i$ and divide it into five parts (four parts P_1, P_2, P_3, P_4 and we divide P_1 into two parts $\{e_1, ..., e_t\}, \{e_{t+1}, ..., e_{t_1}\}$ at the end).

We only need to show that $\tau_G(i) = t$, $\tau_G(i-1) = \dim \mathcal{A}_i - t_3$, and $gen(H_i^G(\mathcal{A};N_p)) = t_3 - t$.

In fact, we can also give $\mathcal{A}_{i-1}$ such a basis and divide it into five parts. Then its first part (see the end of subsection 5.1) will have the same number of elements as P_4 (according to Lemma 5.4). Therefore, we only need to prove $\tau_G(i) = t$ and $gen(H_i^G(\mathcal{A};N_p)) = t_3 - t$.

We consider $gen(H_i^G(\mathcal{A};N_p))$ first. Notice that $H_i^G(\mathcal{A};N_p)$ can be written as a power of $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}[G]/I_p$ where G acts trivially. At the end of the last section, we have shown that under N_p, $Im\partial = \mathbb{Z}[G]/I_p[e_1, e_2, ..., e_t]$. The definition of P_1, P_2, P_3 and Corollary 5.5 tells us that $\ker \partial = \mathbb{Z}[G]/I_p[e_1, e_2, ..., e_{t_3}]$. Therefore, $gen(H_i^G(\mathcal{A};N_p)) = t_3 - t$.

Now we consider $\tau_G(i)$. There are two possible cases and we need two different methods to deal with them separately (that's why we define the G-torsion to be the larger one of two numbers).

Proposition 5.10. *For each $\alpha \in Im\partial\mathcal{A}_i$, use $I(\alpha)$ to denote the ideal of $\mathbb{Z}[G]$ generated by all coefficients of α (which doesn't depend on the choice of basis of $\mathcal{A}_i$). Then the following two cases cannot exist at the same time:*

(I) There exists $\alpha_1 \in Im\partial\mathcal{A}_i$ such that $I(\alpha_1) + \mathbb{Z}[1 + g + ... + g^{p-1}] = \mathbb{Z}[G]$;

(II) There exists $\alpha_2 \in Im\partial\mathcal{A}_i$ such that $I(\alpha_2)$ contains an element in $C^{-1}(1)$.

Proof: We only need to show that, if both α_1, α_2 exist in $Im\partial$, then we can find a basic element in $Im\partial$, which contradicts with our assumption of $\mathcal{A}$. Let $\alpha_1 = a_1e_1 + a_2e_2 + ...$, $\alpha_2 = b_1e_1 + b_2e_2 + ...$. Consider the following $\mathbb{Z}[G]$-matrix:

$$\begin{bmatrix} a_1 & a_2 & \dots & \dots \\ b_1 & b_2 & \dots & \dots \end{bmatrix}$$

We only need to apply row and column operations (not necessarily invertible) to it to get 1 as an entry. Since each row operation commutes with any column operations. We can do all the row operations first. Then each row still expresses an element in $Im\partial$. If we can apply column operations to one row in order to get 1, then that row expresses the basic element we want.

First we use column operations to get two special entries s_1, s_2 in two rows separately, such that s_1 can be written as $a(1 + g + ... + g^{p-1}) + 1$ for some $a \in \mathbb{Z}$, and $C(s_2) = 1$. We can assume that s_2 has already exist, since the condition of α_2 implies that $\gcd(C(b_1), C(b_2), ...) = 1$, and we can use invertible column operations to get such s_2. Then the ideal generated by the entries in the first row will not change (since the operations are inveritble). Assume that $c_1a_1 + c_2a_2 + ... = a(1 + g + ... + g^{p-1}) + 1$ and $s_2 = b_1$, we only need to use column operations to replace a_2 by s_1 and b_2 by $c_1b_1 + c_2b_2 + ...$ Now we only need to consider the first 2×2 submatrix:

$$\begin{bmatrix} a_1 & s_1 \\ s_2 & b_2 \end{bmatrix}$$

Replace (column 2) by (column 2)+$k(1 + g + ... + g^{p-1})$(column 1) for some $k \in \mathbb{Z}$. Then the two elements in column 2 becomes $d_1 = (a + kC(a_1))(1 + g + ... + g^{p-1}) + 1$ and d_2 with $C(d_2) = C(b_2) + pk$. Since $\gcd((a + kC(a_1))p + 1, C(b_2) + pk) = \gcd(ap + 1 - C(a_1)C(b_2), C(b_2) + pk)$, we can always find k to make $(a + kC(a_1))p + 1$ and $C(b_2) + pk$ relative prime to each other. Now we can write $d_1 = a'(1 + g + ... + g^{p-1}) + 1$ with $a'p + 1$ relative prime to $C(d_2)$. Then there exist $x, y \in \mathbb{Z}$ such that $x(a'p + 1) + a' = yC(d_2)$. Then $1 = x(1 + g + ... + g^{p-1})d_1 + d_1 - y(1 + g + ... + g^{p-1})d_2$ and we can use row operations to get it. □

We define $\tau_G(i)$ as the larger one of two numbers. We consider one of them first: $gen(H_i^G(\mathcal{A}; N_g)) - gen(Im(f))$.

Under N_g, Corollary 5.7 and the definition of P_1, P_2 tell us $\ker \partial = \mathbb{Z}[G]/\mathbb{Z}[1 + g + ... + g^{p-1}][P_1 \cup P_2]$. So $gen(H_i^G(\mathcal{A}; N_g)) \leq t_2$. Together with the structure of $H_i^G(\mathcal{A}; N_p)$, we have $Im(f) = \mathbb{Z}[G]/I_p[e_{t+1}, e_{t+2}, ..., e_{t_2}]$. So $gen(Im(f)) = t_2 - t$. When condition (I) in Proposition 5.10 doesn't happen, we have $gen(H_i^G(\mathcal{A}; N_g)) = t_2$. Otherwise, consider the quotient map $\mathbb{Z}[G][P_1 \cup P_2] \to \mathbb{Z}[G]/\mathbb{Z}[1 + g + ... + g^{p-1}][P_1 \cup P_2] \to H_i^G(\mathcal{A}; N_g)$. Since the generators of $H_i^G(\mathcal{A}; N_g)$ is less than the dimension of $\mathbb{Z}[G][P_1 \cup P_2]$ as a free G-module, we can find a basic element α with zero image. Then there is $\beta \in (1 + g + ... + g^{p-1})\mathcal{A}_i$ such that $\alpha + \beta$ is in $Im\partial$. Then condition (I) happens.

Now let's consider the other one, the torsion coefficient (we write as τ_p here) of $H_i^G(\mathcal{A}; N_{orb}) = H_i(G \setminus \mathcal{A})$ mod p. Write $H_i^G(\mathcal{A}; N_{orb}) = H_i(G \setminus \mathcal{A}) = \mathbb{Z}^a \times \mathbb{Z}/a_1\mathbb{Z} \times \mathbb{Z}/a_2\mathbb{Z} ... \times \mathbb{Z}/a_s\mathbb{Z}$ such that each $a_j | a_{j+1}$ and τ_p is the largest number with p not dividing a_{τ_p}.

From the definition of elements in P_1 at the end of subsection 5.1, $Im\partial / Im\partial \cap I_p\mathcal{A}_i$ can be generated by at most t elements. In fact, when we change into the orbit space, the result is more exact: $Im\partial / Im\partial \cap pG\mathcal{A}_i$ can be generated by exactly t elements. Consider the structure of $H_i^G(\mathcal{A}; N_{orb}) = \ker \partial / Im\partial$, we have τ_p no larger than $gen(Im\partial / Im\partial \cap pG\mathcal{A}_i)$. So we have $\tau_p \leq t$.

If the condition (II) in Proposition 5.10 doesn't happen, we get $\tau_p = t$. In this case, there are no basic elements in $G \setminus \mathcal{A}_i \cap Im\partial$. So we have $\ker \partial \cong \mathbb{Z}^{a+s}$. Hence $Im\partial = \mathbb{Z}[a_1 e_1', a_2 e_2', ... a_s e_s']$, where $\{e_1', ..., e_s'\}$ can be extended to a basis of $G \setminus \mathcal{A}_i$. Therefore $\tau_p = gen(Im\partial / Im\partial \cap pG\mathcal{A}_i) = t$.

Finally, in conclusion, we get $\tau_G(i) = t$, and hence finish the proof of Proposition 5.9.

6 Acknowledgments

I would like to thank my mentor Siddharth Venkatesh for his help and suggestions on this project. I would also like to thank Professor Haynes Miller for suggesting this project and all his valuable help. Finally, I'm grateful for MIT Math department's providing this precious research opportunity.

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