# ON THE NON NEGATIVITY OF A SPECIAL CLASS OF GENERALIZED LITTLEWOOD RICHARDSON COEFFICIENTS

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Abstract. In this paper we give a combinatorial proof of the non negativity of the generalized Littlewood Richardson coefficients $c^w_{u,v}$ in the case where $l(v) = 2$. The proof is based on the local structure of the Bruhat order of $S_n$ and on properties of generalized dual Schubert polynomials. We also reprove Monk's formula using the same technique.



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## 1. Introduction

Schubert and dual Schubert polynomials were initially introduced as objects in the field of algebraic geometry. The first systematic combinatorial study of Schubert polynomials is due to Lascoux and Schutzenberger [2]. One of the most important problems in the area of Schubert polynomials is to combinatorially prove the non negativity of the generalized Littlewood Richardson coefficients. These arise when one analyzes the product of two Schubert polynomials in the basis of Schubert polynomials. More precisely, the problem asks to give a combinatorial proof and equivalently find a way to calculate the numbers $c^w_{u,v}$ in the expression

$$S_u(x)S_v(x) = \sum c^w_{u,v} S_w(x).$$

Although this result is known to be true, the proof is based on algebraic geometry and only special cases have been proved combinatorially. The most general case known to the author which has been proved combinatorially is [4]. In that paper the non negativity of the numbers $c^w_{u,v}$ is proved for arbitrary $u$ and for $v$ being a Grassmannian permutation. In the present paper we give a proof of the non negativity of $c^w_{u,v}$ for general $u$ and for any $v$ of length 2.

The paper is organized in two main sections. In section 2 we present all of the background material needed for our proof of the above statement. More precisely, we begin by stating some basic knowledge about permutations and we define the Bruhat order. Next we introduce the usual Schubert polynomials, we prove that they are an integral basis of $\mathbb{C}[y_1, y_2, ...]$ and we define the generalized Littlewood Richardson coefficients. We also give a complete proof of Monk's formula. In the final subsection of section 2 we define the dual and generalized dual Schubert polynomials, which are the main tools for our proof in section 3.

In the first part of section 3 we give an alternative proof of Monk's formula using generalized dual Schubert polynomials and the structure of the Bruhat order. In the second part of section 3, we generalize the this method to prove the main result, namely that $0 \leq c^w_{u,v}$ for $l(v) = 2$.

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## 2. Preliminary definitions and results

Consider a permutation $w \in S_n$. We can write $w$ as a word $w(1)w(2)...w(n)$, meaning that number i is mapped to $w(i)$ by $w$. Denote $t_{ij}$ to be the transposition that interchanges $i$ and $j$ in a word, keeping all other elements fixed. In the special case where $j = i + 1$ write $t_{i,i+1} = s_i$.

For any permutation $w \in S_n$, we define the length $l(w)$ of $w$ to be the number of inversions of $w$, i.e. the number of pairs $(i, j)$ such that $i < j$ and $w(i) > w(j)$, denoted by $I(w)$. A string of numbers $a_1a_2...a_m$ such that $w = s_{a_1}s_{a_2}...s_{a_m}$ with m minimal is called a reduced word for $w$. It can be shown that $m = l(w)$. Finally, define $\mathbf{R}(w)$ to be the set of all reduced words of $w$.

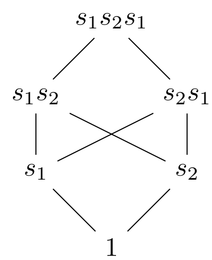



FIGURE 1. Bruhat Order of $S_3$

2.1. **The Bruhat order.** The (strong) Bruhat order is defined in the following way. We say that for $v, w \in S_n$, $v$ precedes $w$ if $l(w) = l(v) + 1$ and there exist integers $i < j$ such that $w = vt_{ij}$. We will write $v \lessdot w$ if $v$ precedes $w$ and more generally $v \leq w$ if there exists a chain of permutations connecting $v$ and $w$, each preceding the following. In figure 1 we explicitly give the Bruhat order of $S_3$.

**Lemma 2.1.** *Let $w = s_{i_1}...s_{i_l}$ be a reduced word for $w$. Then the set of inversions of $w$ is given by:*

$$I(w) = \{s_{i_l}...s_{i_{m+1}}(i_m, i_m + 1),\ 1 \leq m \leq l\}.$$

*Proof.* Consider a permutation v and an integer $m \in \mathbb{N}$ such that $l(vs_m) = l(v) + 1$. It is easy to verify that $I(vs_m) = s_m I(v) \cup \{(m, m+1)\}$. The conclusion follows by induction on the length $l(v)$ of the permutation $v$. $\square$

2.2. **Schubert polynomials.** We begin by defining the divided difference operators denoted by $\partial_i$, which act on polynomials of $n$ variables. They are defined for $1 \leq i < n$ by the following relation:

$$(\partial_i P)(x_1, x_2, ..., x_n) = \frac{P(..., x_i, x_{i+1}, ...) - P(..., x_{i+1}, x_i, ...)}{x_i - x_{i+1}}$$

Divided difference operators satisfy the relations

$$\begin{aligned} \partial_i^2 &= 0 \\ \partial_i\partial_j &= \partial_j\partial_i \ if\ |i-j| \geq 2 \\ \partial_i\partial_{i+1}\partial_i &= \partial_{i+1}\partial_i\partial_{i+1} \end{aligned}$$

Now for any permutation we define $\partial_w = \partial_{a_1}\partial_{a_2}...\partial_{a_m}$, where $a_1a_2...a_m$ is a reduced word for $w$. It can be shown, using the relations above, that $\partial_w$ does not depend on the reduced word and thus $\partial_w$ is well defined.

**Definition 2.2.** The simple Schubert polynomial indexed by permutation $w \in S_n$, is defined as $S_w = \partial_{w^{-1}w_0} x_1^{n-1} x_2^{n-2} ... x_n^0$ where $w_0$ is the permutation of maximal length.

It can be shown that the Schubert polynomials $S_w(x)$ are homogeneous polynomials of degree $l(w_0) - l(w)$. The relations of the divided difference operators combined with the definition of the Schubert polynomials immediately imply the following lemma:

**Lemma 2.3.**
$$\partial_u S_w = \begin{cases} S_{wu^{-1}}, & if\ l(wu^{-1}) = l(w) - l(u) \\ 0, & \text{otherwise.} \end{cases}$$

Now lets fix a reduced word $\mathbf{a} = a_1 a_2 ... a_m$ of $w$. We call a word $\mathbf{b} = b_1 b_2 ... b_m$ compatible with $\mathbf{a}$ if the following relations hold

$$b_1 \leq b_2 \leq ... \leq b_m$$
$$b_i \leq a_i,\ \forall i,$$
$$b_i < b_{i+1}\ if\ a_i < a_{i+1}.$$

Denote by $\mathbf{C}(a)$ the set of all words $\mathbf{b}$ compatible with $\mathbf{a}$.

**Theorem 2.4.** *For all permutations $w \in S_n$,*
$$S_w(x) = \sum_{\mathbf{a} \in \mathbf{R}(w)} \sum_{\mathbf{b} \in \mathbf{C}(a)} x_{b_1} x_{b_2} ... x_{b_l}.$$

A complete proof of this theorem can be found in [1].

Now, let $\mathcal{P}_n$ be the ring of polynomials in $n$ variables with integer coefficients and $\mathcal{H}_n$ the subgroup generated by the monomials $x^{\mathbf{a}}$ where $a_i \leq n - i\ \forall\ 1 \leq i \leq n$. It is easy to see that this basis contains $n!$ elements and that the Schubert polynomials corresponding to a permutation $w \in S_n$ belong to $\mathcal{H}_n$.

**Theorem 2.5.** *The Schubert polynomials $S_w(x)$ for $w \in S_n$ form an integral basis of $\mathcal{H}_n$.*

*Proof.* The number of permutations $w \in S_n$, and therefore of the corresponding Schubert polynomials is $n!$. Since the number of the basis elements of $\mathcal{H}_n$ is also $n!$, we only need to show that $S_w$ for $w \in S_n$ are linearly independent. For this, consider the relation $\sum_w s_w S_w(x) = 0$. Apply on this relation the operator $\partial_v$ with $v \in S_n$ fixed and consider only the terms of degree zero in the resulting expression. By lemma 2.3 and the fact that Schubert polynomials are homogeneous we get that the constant term must be equal to $s_v$. Thus $s_v = 0\ \forall\ v \in\ S_n$ since $v$ is chosen arbitrarily. Also, since divided difference operators send polynomials with integer coefficients to polynomials with integer coefficients, $S_w$ are polynomials with integer coefficients. Therefore, they must form a rational basis of $\mathcal{H}_n$. Now if $P(x) \in\ \mathcal{H}_n$ write it as $P(x) = \sum_w p_w S_w(x)$. Again apply $\partial_v$ to $P(x)$. Since the resulting polynomial has integer coefficients, as above we get that $p_v$ is an integer $\forall v$. Therefore the Schubert polynomials form an integral basis of $\mathcal{H}_n$. $\square$

It is an immediate consequence of the above theorem that $S_w$ for $w \in S_\infty$ form an integral basis of $\mathcal{P}_\infty$.

Now we are ready to define the generalized Littlewood Richardson coefficients.

For any $u, v \in S_\infty$, since $S_u(x), S_v(x) \in \mathcal{P}_\infty$, $S_u(x)S_v(x)$ is an element of $\mathcal{P}_\infty$ and thus we can write

$$S_u(x)S_v(x) = \sum_{w \in \ S_\infty} c_{u,v}^w S_w.$$

**Definition 2.6.** The integers $c_{u,v}^w$ are called the generalized Littlewood Richardson coefficients.

It is a result of intersection theory that these numbers are nonnegative. Proving this assertion combinatorially and providing a way to compute these numbers is an important problem in algebraic combinatorics. In section 4 of this paper we give a combinatorial proof of the non negativity of these numbers in the case $l(v) = 2$. One last thing that we are going to need in order to give a combinatorial description of the dual Schubert polynomials below, is a variation of theorem 2.7.

**Lemma 2.7.** *$S_{s_i}(x) = x_1 + x_2 + ... + x_i$, where $s_i$ is the transposition of $i$ and $i+1$.*

*Proof.* This is a result of theorem 2.3. Indeed the only word in $\mathbf{R}(s_i)$ is i. Therefore $S_{s_i}(x) = \sum_{j \leq i} x_j$ as wanted. □

**Theorem 2.8** (Monk's Formula)**.** *For any permutation $w \in S_\infty$, and for any $n \in \mathbb{N}$,*

$$S_w(x)S_{s_n}(x) = \sum_{\substack{j \leq n < k \\ l(wt_{jk}) = l(w)+1}} S_{wt_{jk}}.$$

*Proof.* First, it is easy to verify that the divided difference operator acts on the product of two polynomials in the following way:

$$\partial_i(A(x)B(x)) = (\partial_i A(x))s_i B(x) + A(x)\partial_i B(x).$$

Now consider a permutation $u$ such that $l(u) = l(w)+1$ and $v = s_{a_1}...s_{a_l}$ be a reduced decomposition of $v$. Since $S_{s_n}(x)$ is a degree one polynomial, it follows that

$$\partial_v(S_w S_{s_n}(x)) = \sum_{m=0}^{l} \partial_{a_1}...\partial_{a_{m-1}}\partial a_{m+1}...\partial_{a_l} S_w(x) \cdot \partial_{a_m}(s_{a_{m+1}}...s_{a_l} S_{s_n}(x)).$$

Since $l(w) = l(u) - 1$, the term $\partial_{a_1}...\partial_{a_{m-1}}\partial_{a_{m+1}}...\partial_{a_l} S_w$ is nonzero iff $a_1...a_{m-1}a_{m+1}...a_l \in \mathbf{R}(w)$. This implies that $w \lessdot v$ in the Bruhat order, so $v = wt_{jk}$. Lemma 2.2 implies that $\{j,k\} = s_{a_l}...s_{a_{m+1}}\{a_m, a_m+1\}$ which is equivalent to $\{a_m, a_m+1\} = s_{a_{m+1}}...s_{a_l}\{j,k\}$ since $s_i^2 = 1$.

Now the quantity $\partial_{a_m}(s_{a_{m+1}}...s_{a_l} S_{s_n}) = \partial_{a_m} \sum_{i=1}^{n} x_{s_{a_{m+1}}...s_{a_l}(i)} \neq 0$ iff exactly one of $\{x_{a_m}, x_{a_m+1}\}$ appears in the sum. Due to the above observation, $x_{a_m}$ appears when $i = j$ and $x_{a_{m+1}}$ when $i = k$. Therefore the expression is nonzero iff $j \leq n < k$ is which case it is equal to 1. This observation combined with the fact that Schubert polynomials are a basis of $\mathcal{P}_\infty$, implies that $S_w(x)S_{s_n}(x)$ is in fact equal to

$$\sum_{\substack{j \leq n < k \\ l(wt_{jk}) = l(w)+1}} S_{wt_{jk}}$$

as wanted. □

2.3. **Dual Schubert polynomials.** In order to define the dual Schubert polynomials we will need some algebra.

**Definition 2.9.** Given polynomials $P, Q \in \mathbb{C}[x_1, x_2, ..., x_n]$, we define the symmetric bilinear form

$$\langle P, Q\rangle_D = P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)Q(x_1, x_2, ..., x_n)\Big|_{x=0}.$$

We will call this form D-pairing of P and Q.

**Definition 2.10.** Assume that $I \subset \mathbb{C}[x_1, x_2, ..., x_n]$ is a graded ideal. Define $H_I$ to be the space of I-Harmonic polynomials i.e. the space of polynomials perpendicular to polynomials in I with respect to the D-pairing.

The fact that I is an ideal, implies that $H_I$ is the space of polynomials Q such that $P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)Q(x_1, x_2, ..., x_n) = 0 \ \forall \ P \in I$. Also call two bases $f_i$ and $g_i$ dual with respect to the D-pairing when the following holds:

$$\langle f_i, g_j\rangle_D = \delta_{ij}.$$

Finally,we will denote by $I_n$ the (graded) ideal generated by symmetric polynomials in n variables without constant terms.

**Lemma 2.11.** *Suppose that $\{\bar{f}_i\}$ is a graded basis of $\mathbb{C}[x_1, x_2, ..., x_n]/I$ and $\{g_i\}$ is a basis of $H_I$. The following are equivalent:*

1. *$\{\bar{f}_i\}$ is dual to $\{g_i\}$*
2. *$e^{x\cdot y} = \sum_i \bar{f}_i(x)g_i(y) \pmod{\hat{I}}$ where $x \cdot y = x_1y_1 + ... + x_ny_n$ and $\hat{I}$ is the extension of I to $\mathbb{C}[[x_1, x_2, ..., x_n]]$.*

The proof of this lemma can be found in [3].

**Lemma 2.12.** *The cosets $\bar{S}_w(x) = S_w(x) + I_n$ form a linear basis of $\mathbb{C}[x_1, x_2, ..., x_n]/I_n(= C_n)$.*

*Proof.* The proof is very similar to theorem 2.5. We leave the details to the reader. □

**Definition 2.13.** The Dual Schubert polynomials $D_w(y_1, y_2, ..., y_n) \in \mathbb{C}[y_1, y_2, ..., y_n]$ are defined as the basis of $H_{I_n}$, dual to the basis $\{\bar{S}_w(x)\}$ of $C_n$.

The definition implies that

$$e^{x\cdot y} = \sum_{w\in S_n} S_w(x)D_w(y) \pmod{I_n}.$$

Now since both $S_w$ and $D_w$ are stable under the inclusion $S_n \subset S_{n+1}$, $D_w$ for $w \in S_\infty$ form a basis of $\mathbb{C}[y_1, y_2, ...]$. This implies the identity

$$e^{x\cdot y} = \sum_{w\in S_\infty} S_w(x)D_w(y).$$

**Theorem 2.14.** $D_w(y + z) = \sum c^w_{uv} D_u(y)D_v(z)$.

*Proof.*

$$\begin{aligned}\sum_{u,v,w} c_{uv}^w S_w(x)D_u(y)D_v(z) &= \sum_{uv}(S_u(x)D_u(y))(S_v(x)D_v(z)) \\ &= e^{x\cdot y}e^{x\cdot z} = e^{x\cdot(y+z)} = \sum_w S_w(x)D_w(y+z).\end{aligned}$$

The result follows since $S_w(x)$ is a basis of $\mathcal{P}_\infty$. $\square$

**Lemma 2.15.** *For $w \in S_n$ we have*

$$(y_1x_1 + ... + y_nx_n)S_w(x) = \sum_{\substack{i<j \\ l(wt_{ij})=l(w)+1}} (y_i - y_j)S_{wt_{ij}}(x) \pmod{I_n}.$$

*Proof.* First observe that the following formula is equivalent to Monk's formula

$$x_mS_u(x) = \sum_{\substack{m<k \\ l(ut_{mk})=l(u)+1}} S_{ut_{mk}}(x) - \sum_{\substack{j<m \\ l(ut_{jm})=l(u)+1}} S_{ut_{jm}}(x) \pmod{I_n}$$

and therefore

$$(y_mx_m)S_u(x) = \sum_{\substack{m<k \\ l(ut_{mk})=l(u)+1}} y_mS_{ut_{mk}}(x) - \sum_{\substack{j<m \\ l(ut_{jm})=l(u)+1}} y_mS_{ut_{jm}}(x) \pmod{I_n}.$$

The result follows by adding up these relations for $1 \le m \le n$. $\square$

The following theorem allows us to express the Dual Schubert polynomials combinatorially in terms of the Bruhat order of $S_n$.

**Theorem 2.16.** *If $l(v) = l(w) + 1$ and $v = wt_{ij}$ or in other words $w$ precedes $v$ in the Bruhat order, attach to the edge connecting $w$ and $v$ the weight $y_i - y_j$. Now consider the Bruhat order of $S_n$ and an arbitrary $w \in S_n$. Denote by $P$ a path from id to $w$ in the Bruhat order and by w(P) the product of the weights of the edges of the path. Then we have*

$$D_w(y) = \frac{1}{l(w)!}\sum_P w(P),$$

*summed over all paths from id to $w$.*

*Proof.* By the above lemma repeatedly to $\frac{(y_1x_1+...+y_nx_n)^kS_{id}(x)}{k!}$ we obtain

$$\frac{(y_1x_1 + ... + y_nx_n)^k}{k!} = \sum_{l(w)=k} (\frac{1}{k!}\sum_P w(P))S_w(x) \pmod{I_n}.$$

Adding these relations we get that

$$e^{x\cdot y} = \sum_{w\in S_n} S_w(x)(\frac{1}{l(w)!}\sum_P w(P)) \pmod{I_n}.$$

This implies that $D_w(y) = \frac{1}{l(w)!}\sum_P w(P)$ as wanted. $\square$

**Definition 2.17.** We define the generalized dual Schubert polynomials for $u, w \in S_n$ such that $u \leq w$ in the Bruhat order in the following way:

$$D_{u,w}(y) = \frac{1}{(l(w) - l(u))!} \sum_P P(u \to w)(y)$$

where the sum is taken over all weighted paths in the Bruhat order of $S_n$ from $u$ to $w$.

**Lemma 2.18.**

$$e^{x \cdot y} S_u(x) = \sum_{u \leq w} D_{u,w}(y) S_w(x)$$

.

*Proof.* From lemma 2.15 for $w \in S_n$ we have that

$$(y_1 x_1 + \ldots + y_n x_n) S_w(x) = \sum_{\substack{i<j \\ l(wt_{ij}) = l(w)+1}} (y_i - y_j) S_{wt_{ij}}(x) \pmod{I_n}.$$

Using the definition of $D_{u,w}(y)$ it is easy to see that

$$\frac{1}{k!}(y_1 x_1 + \ldots + y_n x_n)^k S_u(x) = \sum_{\substack{w \in S_n \\ l(w) = l(u)+k}} S_w(x) D_{u,w}(y) \pmod{I_n}$$

. Adding these relations we obtain

$$e^{x \cdot y} S_u(x) = \sum_{\substack{u \leq w \\ w \in S_n}} D_{u,w}(y) S_w(x) \pmod{I_n}.$$

Now it is easy to see by its definition that $D_{u,w}(y)$ is stable under the inclusion $S_n \subset S_{n+1}$ and so is $S_w(x)$. Therefore we obtain the required identity by letting $n \to \infty$. $\square$

**Theorem 2.19.**

$$D_{u,w}(y) = \sum_v c_{uv}^w D_v(y).$$

*Proof.* We can write

$$\begin{aligned} e^{x \cdot y} S_u(x) &= \sum_{v \in S_\infty} (S_u(x) S_v(x)) D_v(y) \\ &= \sum_{v,w \in S_\infty} c_{uv}^w S_w(x) D_v(y) \end{aligned}$$

Now fix $w = w_1$ and consider the D-pairing

$$\begin{aligned} D_{u,w_1}(y) &= \langle D_{w_1}(x), \sum_{u \leq w} D_{u,w}(y) S_w(x) \rangle_D \\ &= \langle D_{w_1}(x), \sum_{v,w \in S_\infty} c_{uv}^w S_w(x) D_v(y) \rangle_D \\ &= \sum_v c_{uv}^{w_1} D_v(y) \end{aligned}$$

since the usual and dual Schubert polynomials are dual bases with respect to the D-pairing. Since $w_1$ was arbitrary the theorem is proved. $\square$

## 3. Combinatorial proof of the positivity of generalized Littlewood Richardson coefficients $c_{uv}^{w}$ for $l(v) = 2$

3.1. **An alternative proof of Monk's formula.** In this subsection we will use the generalized dual Schubert polynomials $D_{u,w}(y)$ with $l(w) = l(u)+1$ to reprove Monk's formula. In the following subsection we will generalize the argument in the proof to show the non negativity of $c_{u,v}^{w}$ for $l(v) = 2$.

Consider $D_{u,w}(y)$ as above. Then this polynomial is 0 unless $w = ut_{ij}$ for some $i, j$. In this case $D_{u,w}(y) = (y_i - y_j)$. Now, since all weights of segments with one vertex being the identity are of the form $(y_k - y_{k+1})$ we can write $D_{u,w}(y) = \sum_{i \leq m < j}(y_m - y_{m+1})$. Since the dual Schubert polynomials are a basis of $\mathbb{C}[y_1, y_2, ...]$, this decomposition is unique. This immediately implies that for $v = s_k$, $c_{u,v}^{w} = 0$ unless $w = ut_{ij}$ with $i \leq k < j$, in which case it is 1. Now $S_u(x)S_{s_k}(x) = \sum c_{u,s_k}^{w} S_w(x)$ with $l(w) = l(u) + 1$, so the above result immediately implies that

$$S_u(x)S_{s_k}(x) = \sum_{\substack{i \leq k < j \\ l(ut_{ij}) = l(u)+1}} S_{ut_{jk}}$$

as wanted.

3.2. **Proof of the main theorem.** In this subsection we present the proof of the following theorem.

**Theorem 3.1.** *(Main Theorem) The generalized Littlewood Richardson coefficients $c_{uv}^{w}$ with $l(v) = 2$ are non negative.*

The following lemma gives a full characterization of intervals of length 2 in the Bruhat order of $S_n$.

**Lemma 3.2.** *Consider the permutations u,v,w in the Bruhat order of $S_n$ such that $u \lessdot v \lessdot w$. There exists a unique permutation $x \neq v$ such that $u \lessdot x \lessdot w$.*

*Proof.* The $u, v$ and $w$ above satisfy $w = vt_{ij} = ut_{kl}t_{ij}$ such that $l(w) = l(v) + 1 = l(u) + 2$.

If all of $i, j, k, l$ are distinct then it holds that $t_{kl}t_{ij} = t_{ij}t_{kl}$. Clearly in this case the only possible $x$ is $x = ut_{ij}$. We can assume without loss of generality that $i < j, k < l$ and $j < l$. If $j < k$ or $k < i$, then it can be seen immediately that $l(x) = l(u)+1$. In the case $i < k < j < l$ we could have $u(i) < u(k) < u(j)$ in which case $l(x) = l(u)+2$. However this cannot happen. Since $l(v) = l(u)+1$ and $u(k) < u(j)$ it must hold that $u(i) < u(k) < u(l) < u(j)$. This would imply that $l(w) = l(v)+2$ which is absurd. Thus $x$ satisfies the required property.

Now assume that only three of i,j,k,l are distinct with $l$ being one of the other three. Without loss of generality, let $i < j < k$. Consider the word $u = ...u(i)...u(j)...u(k)....$ The only relations between $u(i), u(j)$ and $u(k)$ that can hold so that we can go to a $w$ such that $l(w) = l(v)+1 = l(u)+2$ by permuting only $i, j$ and $k$ are the following:

$$1) u(i) < u(j) < u(k)$$
$$2) u(j) < u(i) < u(k)$$
$$3) u(i) < u(k) < u(j)$$

In the first case there can be two possible rearrangements of $i, j$ and $k$ that can give a word $w$ with $l(w) = l(u) + 2$. One is $w = ut_{ij}t_{jk} = ut_{jk}t_{ik}$ giving $w = ...u(k)...u(i)...u(j)...$ with all other letters in the same order as in $u$. It can be seen that $l(v) = l(ut_{ij}) = l(u) + 1$ and that $x = ut_{jk}$ also satisfies $l(x) = l(ut_{jk}) = l(u) + 1$. It is easy to see that the only other way to obtain $w$ from $u$ by permuting $i, j$ and $k$ is $w = ut_{ik}t_{ij}$. However $l(ut_{ik}) = l(u) + 3$ absurd. Thus $x$ is unique. Similarly, the other is $w = ut_{ij}t_{ik} = ut_{jk}t_{ij}$ giving $w = ...u(k)...u(i)...u(j)...$ with all other letters in the same order as in $u$. $v = ut_{ij}$ and $x = ut_{jk}$ are the only two permutations such that $l(v) = l(x) = l(u) + 1$. The reasoning is the same as above. In the other two cases there is only one possible rearrangement of $i, j$ and $k$ that give a word w with $l(w) = l(u) + 2$.
In the second case the only $w$ that works is $w = ut_{jk}t_{ik} = ut_{ik}t_{ij}$ giving $w = ...u(k)...u(i)...u(j)...$ with all other letters in the same order as in $u$. $v = ut_{jk}$ and $x = ut_{ik}$. The reasoning that these two are the only permutations with this property is the same as above.
In the third case the only $w$ that works is $w = ut_{ij}t_{ik} = ut_{ik}t_{jk}$ giving $w = ...u(j)...u(k)...u(i)...$ with all other letters in the same order as in $u$. $v = ut_{ij}$ and $x = ut_{ik}$ are the unique two permutations with $l(v) = l(x) = l(u) + 1$. □

Figure 2 illustrates all of the above cases. From now on we attach on the edges of the Bruhat order of $S_n$ their weights as defined above. Consider the generalized dual Schubert polynomials $D_{u,w}$ such that $l(w) = l(u) + 2$. Since these intervals are of length two, they have one of the shapes in figure 2.

**Example 3.3.** For example, assuming that $i < j < k$, the generalized dual Schubert polynomial corresponding to shape ( B ) in figure 2 is $D_{u,w}(y) = \frac{1}{2}[(y_i - y_j)(y_j - y_k) + (y_j - y_k)(y_i - y_k)]$.

We wish to show that these polynomials can be written as sums of dual Schubert polynomials $D_v(y)$ with $l(y) = 2$. Since the dual Schubert polynomials are a basis of $\mathbb{C}[y_1, y_2, ...]$, this decomposition will be unique, and since we know that

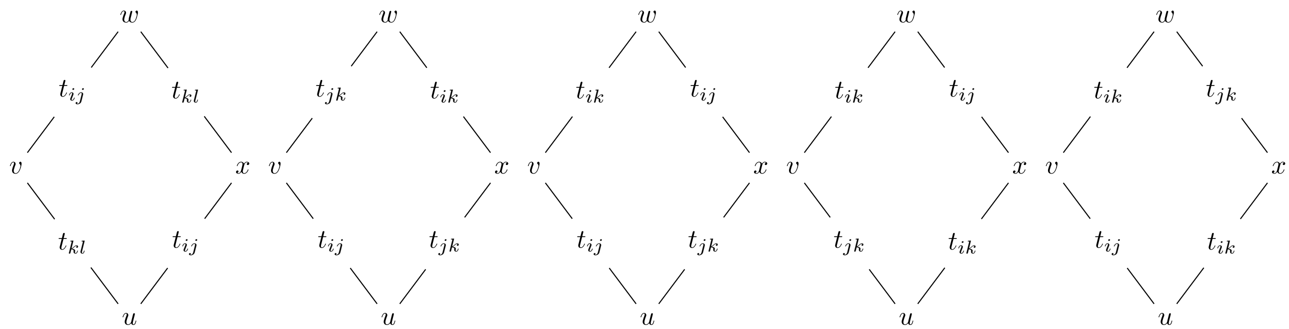
$$D_{u,w}(y) = \sum_v c_{uv}^w D_v(y),$$

this will prove that $0 \leq c_{uv}^w$ when $l(v) = 2$.

Now consider the edges of the Bruhat order of $S_n$ between id and any $w$ with $l(w) = 2$. The weights of the edges connecting id to a $u$ with $l(u) = 1$ can only have the form $(y_i - y_{i+1})$ for some $1 \leq i \leq n - 1$. The edges connecting a $u$ with $l(u) = 1$ to a $w$ with $l(w) = 2$ can either be of the above form or of the form $(y_i - y_{i+2})$. The latter case can only occur when the weight of the edge joining id with $u$ is either $(y_i - y_{i+1})$ or $(y_{i+1} - y_{i+2})$. Also for an edge connecting id to a $u$ with $l(u) = 1$ with weight $(y_i - y_{i+1})$, it is easy to see that there exist $w$ with $l(w) = 2$ such that the edge connecting $u$ to $w$ has weight $(y_j - y_{j+1})$ for any $j \neq i$, while $j = i$ cannot occur.

*Proof.* (Main Theorem) First we do the simplest case. For this consider the generalized dual Schubert polynomial corresponding to shape ( A ) in figure 2 with the further restriction $i < j < k < l$. We have that

$$D_{u,w}(y) = \frac{1}{2}[(y_i - y_j)(y_k - y_l) + (y_k - y_l)(y_i - y_j)] =$$
$$\sum_{\substack{i \leq m < j \\ k \leq n < l}} \frac{1}{2}[(y_m - y_{m+1})(y_n - y_{n+1}) + (y_n - y_{n+1})(y_m - y_{m+1})].$$




(A) $t_{kl}t_{ij} = t_{ij}t_{kl}$

(B) $t_{ij}t_{jk} = t_{jk}t_{ik}$

(C) $t_{ij}t_{ik} = t_{jk}t_{ij}$

(D) $t_{jk}t_{ik} = t_{ik}t_{ij}$

(E) $t_{ij}t_{ik} = t_{ik}t_{jk}$

FIGURE 2. All possible length two intervals

From the last observation in the previous paragraph, it can be seen that each summand of the sum corresponds to a dual Schubert polynomial. Therefore in this case our theorem is proved.

Now we do the case $i < k < j < l$. If we try to analyze the generalized dual polynomial as in the case above, terms of the form $(y_i - y_{i+1})^2$ with $k \le i < l$ would appear. However, we have ruled out this possibility above. Therefore we need to find a way of analyzing generalized dual Schubert polynomials, avoiding terms like this. To do this, we write

$$
\begin{aligned}
&2D_{u,w}(y) = [(y_i - y_j)(y_k - y_l) + (y_k - y_l)(y_i - y_j)] = \\
&\sum_{\substack{i\le m<k\\ k\le n<l}} (y_m - y_{m+1})(y_n - y_{n+1}) + \sum_{k\le s<j} (y_s - y_{s+1})\Big( \sum_{\substack{k\le n<l\\ n\ne\{s,s+1\}}} (y_n - y_{n+1}) + (y_s - y_{s+2})\Big) + \\
&\sum_{\substack{i\le m<j\\ j\le n<l}} (y_m - y_{m+1})(y_n - y_{n+1}) + \sum_{k\le s<j} \Big( \sum_{\substack{k\le n<l\\ n\ne\{s-1,s\}}} (y_n - y_{n+1}) + (y_{s-1} - y_{s+1})\Big)(y_s - y_{s+1}).
\end{aligned}
$$

Now we can rewrite this by pairing up terms in order to form dual Schubert polynomials in the following way:

$$
\begin{aligned}
2D_{u,w}(y) &= [(y_i - y_j)(y_k - y_l) + (y_k - y_l)(y_i - y_j)] = \\
&\sum_{\substack{i \le m < j \\ k \le n < l \\ m \notin \{n-1, n, n+1\}}} [(y_m - y_{m+1})(y_n - y_{n+1}) + (y_n - y_{n+1})(y_m - y_{m+1})] + \\
&\sum_{k \le s < j} [(y_{s-1} - y_s)(y_s - y_{s+1}) + (y_s - y_{s+1})(y_{s-1} - y_{s+1})] + \\
&\sum_{k < s \le j} [(y_s - y_{s+1})(y_{s-1} - y_s) + (y_{s-1} - y_s)(y_{s-1} - y_{s+1})].
\end{aligned}
$$

Finally, we do the case $k < i < j < l$. Again we are going to use a clever decomposition of $2D_{u,w}(y) = [(y_i - y_j)(y_k - y_l) + (y_k - y_l)(y_i - y_j)]$. Here the decomposition is trickier. For this reason we present the decompositions of $(y_i - y_j)(y_k - y_l)$ and $(y_k - y_l)(y_i - y_j)$ seperately. We have that:

$$
\begin{aligned}
(y_i - y_j)(y_k - y_l) = &\sum_{\substack{i \le n < j \\ \{k \le m < i-1\} \cup \{j < m < l\}}} (y_n - y_{n+1})(y_m - y_{m+1}) + \\
&(y_i - y_{i+1})\Big((y_{i-1} - y_{i+1}) + \sum_{i+2 \le m \le j} (y_m - y_{m+1})\Big) + \\
&\sum_{i < n < j-2} (y_n - y_{n+1})\Big((y_n - y_{n+2}) + \sum_{\{i-1 \le m < n\} \cup \{n+2 \le m \le j\}} (y_m - y_{m+1})\Big) + \\
&(y_{j-2} - y_{j-1})\Big(\sum_{i-1 \le m < j-2} (y_m - y_{m+1}) + (y_{j-3} - y_{j-1}) + (y_{j-1} - y_j)\Big) + \\
&(y_{j-1} - y_j)\Big(\sum_{i-1 \le m < j-1} (y_m - y_{m+1}) + (y_{j-1} - y_{j+1})\Big).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(y_k - y_l)(y_i - y_j) = &\sum_{\substack{\{k \le m < i\} \cup \{j \le m < l\} \\ i \le n < j}} (y_m - y_{m+1})(y_n - y_{n+1}) + \\
&(y_i - y_{i+1})\Big((y_i - y_{i+2}) + \sum_{i+2 \le n < j} (y_n - y_{n+1})\Big) + \\
&\sum_{\substack{i+1 \le m < j \\ m \ne j-2}} (y_m - y_{m+1})\Big((y_{m-1} - y_{m+1}) + \sum_{\{i \le n < m-1\} \cup \{m+1 \le n < j\}} (y_n - y_{n+1})\Big) + \\
&(y_{j-2} - y_{j-1})\Big((y_{j-2} - y_j) + \sum_{i \le n < j-2} (y_n - y_{n+1}).
\end{aligned}
$$

Now adding the above two relations and rearranging,we obtain the relation

$$
\begin{aligned}
&2D_{u,w}(y) = [(y_i - y_j)(y_k - y_l) + (y_k - y_l)(y_i - y_j)] = \\
&\sum_{\substack{i \le m < j \\ k \le n < l \\ m \notin \{n-1,n,n+1\}}} [(y_m - y_{m+1})(y_n - y_{n+1}) + (y_n - y_{n+1})(y_m - y_{m+1})] + \\
&[(y_{i-1} - y_i)(y_i - y_{i+1}) + (y_i - y_{i+1})(y_{i-1} - y_{i+1})] + \\
&[(y_{j-1} - y_j)(y_{j-1} - y_{j+1}) + (y_j - y_{j+1})(y_{j-1} - y_j)] + \\
&\sum_{i \le s < j-1} \Big( [(y_s - y_{s+1})(y_s - y_{s+2}) + (y_{s+1} - y_{s+2})(y_s - y_{s+1})] + \\
&[(y_s - y_{s+1})(y_{s+1} - y_{s+2}) + (y_{s+1} - y_{s+2})(y_s - y_{s+2})] \Big).
\end{aligned}
$$

It is clear that all terms in brackets divided by two are dual Schubert polynomials corresponding to a permutation $v$ with $l(v) = 2$.

Now in order to finish the proof, we need to decompose the generalized dual Schubert polynomials of the cases ( B ), ( C ), ( D ) and ( E ) of figure 2. However the polynomials corresponding to cases ( D ) and ( E ) are the same. Also we only need to do one of the cases ( B ) and ( C ) since the decompositions of the two are very similar.

In case ( B ) the generalized dual Schubert polynomial is $D_{u,w}(y) = \frac{1}{2}[(y_i - y_j)(y_j - y_k) + (y_j - y_k)(y_i - y_k)]$. Consider the following decomposition:

$$
\begin{aligned}
&2D_{u,w}(y) = [(y_i - y_j)(y_j - y_k) + (y_j - y_k)(y_i - y_k)] = \\
&\sum_{\substack{i \le m < j \\ j \le n < k}} (y_m - y_{m+1})(y_n - y_{n+1}) + \sum_{j \le m < k} (y_m - y_{m+1}) \Big( (y_{m-1} - y_{m+1}) + \sum_{\substack{i \le n < k \\ n \ne \{m-1,m\}}} (y_n - y_{n+1}) \Big) = \\
&\sum_{\substack{i \le m < k \\ j \le n < k \\ m \ne \{n-1,n,n+1\}}} [(y_m - y_{m+1})(y_n - y_{n+1}) + (y_n - y_{n+1})(y_m - y_{m+1})] + \\
&\sum_{j-1 \le s < k} [(y_s - y_{s+1})(y_{s+1} - y_{s+2}) + (y_{s+1} - y_{s+2})(y_s - y_{s+2})]
\end{aligned}
$$

Again it is clear that all terms in brackets divided by two are dual Schubert polynomials corresponding to a permutation $v$ with $l(v) = 2$.

Finally we decompose the generalized dual Schubert polynomial corresponding to case ( D ) in figure 2 above.

$$
\begin{aligned}
&2D_{u,w}(y) = [(y_i - y_k)(y_i - y_j) + (y_j - y_k)(y_i - y_k)] = \\
&\sum_{i \le m < j-1} (y_m - y_{m+1})\Big((y_m - y_{m+2}) + \sum_{\substack{i \le n < j \\ n \ne \{m, m+1\}}} (y_n - y_{n+1})\Big) + \\
&(y_{j-1} - y_j)\Big(\sum_{i \le n < j-1} (y_n - y_{n+1}) + (y_{j-1} - y_{j+1}) - (y_j - y_{j+1})\Big) + \\
&2 \sum_{\substack{j \le m < k \\ i \le n < j}} (y_m - y_{m+1})(y_n - y_{n+1}) + \sum_{j \le s < k} (y_s - y_{s+1})\Big((y_s - y_{s+2}) + \sum_{\substack{j \le n < k-1 \\ n \ne \{s, s+1\}}} (y_n - y_n + 1)\Big) + \\
&(y_{k-1} - y_k)\Big(\sum_{j \le n < k-2} (y_n - y_{n+1}) + (y_{k-2} - y_k)\Big).
\end{aligned}
$$

Although there is a negative term in the decomposition, it cancels out and the above decomposition can be rewritten in the following way:

$$
\begin{aligned}
&2D_{u,w}(y) = [(y_i - y_k)(y_i - y_j) + (y_j - y_k)(y_i - y_k)] = \\
&\sum_{\substack{j \le m < k \\ i \le n < j \\ m \ne \{n-1, n, n+1\}}} [(y_m - y_{m+1})(y_n - y_{n+1}) + (y_n - y_{n+1})(y_m - y_{m+1})] + \\
&\sum_{\substack{\{i \le m, n < j\} \cup \{j \le m, n < k\} \\ m \ne \{n-1, n, n+1\}}} [(y_m - y_{m+1})(y_n - y_{n+1}) + (y_n - y_{n+1})(y_m - y_{m+1})] + \\
&\sum_{i \le s < k} [(y_s - y_{s+1})(y_s - y_{s+2}) + (y_{s+1} - y_{s+2})(y_s - y_{s+1})] + \\
&[(y_{k-2} - y_{k-1})(y_{k-1} - y_k) + (y_{k-1} - y_k)(y_{k-2} - y_k)].
\end{aligned}
$$

□

It can be seen from the above decompositions that the following holds:

$$c_{uv}^{w} \in \{0, 1, 2\} \ for \ l(v) = 2.$$

As in the previous subsection, one can derive generalized Monk's formulas for multiplication of an arbitrary Schubert polynomial with a Schubert polynomial of the form $S_v(x)$ for $l(v) = 2$. The derivation is left as an exercise to the interested reader.

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