Universal Homotopy Theories
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Abstract

The following paper is aimed to be a self-contained expository paper about Daniel Dugger's construction of a universal homotopy theory assigned to a small category [4].

Contents

0 Introduction

The following paper is aimed to be a self-contained expository paper about Daniel Dugger's construction of a universal homotopy theory assigned to a small category [4]. But what does this mean? Throughout the time I have been working on this project, I have felt an intense need to share what I am working on with my friends and family. The problem, as any mathematician would know, is that it is hard to explain one's work to non mathematicians, and even to mathematicians in a different field. As a result of many conversations in which I tried to explain my work, I have decided to write the following fictional short story that I hope serves as a good metaphor for what I have been working on. As any fictional story, it has as a starting point something real: physicists in the search of a theory of everything; however, this is just a starting point and most of it is just delirium from the author; which I hope you enjoy.

> Imagine that physicists in the search of a physical theory that encompasses all physical phenomenon in our universe, have decided that good approach is to first understand what a "theory of physics" means. To accomplish this, they first need to construct a definition of an abstract universe upon which they can endow a physical theory. What properties should a physicist expect such a definition to have? At the very least, a good definition should have as a property that our universe, which we will call the standard universe; and our physical theory, which we will call the standard physical theory, are special cases of these more general notions. We will call a universe endowed with a physical theory a physical universe, and our universe endowed with our physical theory will be called the standard physical universe. The philosophy of this approach is that by comparing the standard physical universe with other simpler physical universes, one should be able to obtain a great deal of useful information that would be to hard to notice otherwise.
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> When presented with a new framework, any scientist should ask: how can I use this framework to understand the objects that I care about? In particular, a physicist, when presented with the framework presented above, should ask: how can I use this framework to better understand the standard physical universe? The following would be a satisfying answer: imagine that physicists are able to assign to each universe a physical theory, and moreover, they are able to assign a simplest physical theory. For example, lets assume for a moment that physicists have decided that a physical theory should have a notion of positively and negatively charged particles, but the strength of their interactions is not specified. In particular, the strength of their interactions could be zero. It is then reasonable to assume that the simplest physical theory assigned to the standard universe says that the interactions between positively and negatively charged particles is zero.
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> Imagine furthermore, that the simplest physical theory assigned to a universe comes equipped with a way to compare it to every other physical theory assigned to the universe. Following the example above, the simplest physical theory assigned to the standard universe comes equipped with the following way of comparing it to the standard physical theory: if two particles, one positively charged and one negatively charged, are getting closer in the simplest physical theory, then these particles will also get closer in the standard physical theory. If more interesting ways of comparing these two physical theories exist, then they would give physicists the ability to reduce many interesting problems

about the standard physical universe to a simpler setting.

We will now explain this metaphor in the context of this paper. A 'universe' is a small category, and a 'physical theory' is a model structure. Model categories (a category with a model structure) first introduced by Quillen [13], form the foundation of modern homotopy theory. As in the short story above, where an abstract physical theory is an abstraction of the physical theory we see on our universe, a model structure is an abstraction of the standard homotopy theory in the category of topological spaces. The goal of this paper is to answer the following question: given a category, is it possible to assign a model structure to it? And if so, is it possible to assign a 'simplest' model structure? The answer is a positive. Moreover, as in the short story above, the 'simplest' model structure on the category $\mathcal{C}$ comes equipped with a way of comparing it with every other model structure assign to $\mathcal{C}$. This turns out to be a very powerful computational tool.

For the more knowledgable reader, I present the following short discussion regarding the material of the paper: our goal is to be able to assign a model category to any small category $\mathcal{C}$, such that this model category enjoys a certain universal property. To make this precise, let $\mathcal{C}$ be a small category, and let $\mathcal{M}$ be a model category. The universal model category model category assign to $\mathcal{C}$, will be denoted by $U\mathcal{C}$, this category is the category of simplicial presheaves $sPre(\mathcal{C})$ endowed with the projective model structure. The model category $U\mathcal{C}$ is universal in the following sense: for every map $\gamma : \mathcal{C} \to \mathcal{M}$ we obtain the following diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{Y}"] \arrow[dr, "\gamma"'] & U\mathcal{C} \arrow[d, "Re"] \arrow[dl, phantom, "\Swarrow\eta"] \\ & \mathcal{M} \end{tikzcd} \tag{1}$$

where the functor $\mathcal{Y} : \mathcal{C} \to U\mathcal{C}$ is the composition of the Yoneda embedding $\mathcal{C} \to Pre(\mathcal{C})$ and the canonical inclusion $Pre(\mathcal{C}) \to U\mathcal{C}$. And $\eta : Re \circ \mathcal{Y} \to \gamma$ is a natural transformation that makes the diagram commute. Moreover, the functor $Re : U\mathcal{C} \to \mathcal{M}$ is a left Quillen functor.

Organization of the Paper

As mentioned before, this paper is aimed to be a self-contained exposition of Daniel Dugger's construction of a universal homotopy theory assigned to a small category. Although the author has done his best to include everything needed to understand this paper within its pages, there are a few prerequisites to read before continuing with this paper. In particular, the author has assumed that the reader is familiar with basic results in category theory, like the Yoneda lemma. Any further prerequisites will be discussed at the beginning of each section. This paper is organized as follows:

- *Chapter 1: Motivation* A discussion about the category of presheaves of a small category $\mathcal{C}$ is presented. There are two main results in this chapter. The first result is that every presheaf is a colimit of representable presheaves, and the second is that $Pre(\mathcal{C})$ enjoys a certain universal property. I have called this chapter "Motivation" because the universal property of $U\mathcal{C}$ is the homotopical analog of the universal property of $Pre(\mathcal{C})$. A discussion on simplicial sets is also provided. I present them as a very important special case of a presheaf category.

- *Chapter 2: A Model Structure for Simplicial Sets* The category of Simplicial Sets is central to modern homotopy theory, because it has basically been designed to be the fiber of what mathematicians have decided 'homotopy theory' means. In this chapter, the existence of a model structure for Simplicial Sets will be proven. This approach is somewhat non traditional, as instead of proving it directly, some machinery is developed and the analysis proceeds from there. This approach has been decided upon because this machinery will be needed again later in the paper.

- *Chapter 3: A Model Structure on Simplicial Presheaves* The category of Simplicial Presheaves endowed with a certain model structure are the main object of study of this paper. In this chapter, it is proven that we can endow the category $sPre(\mathcal{C})$ with this model structure, however its universal property will not be proven until later. We will then proceed to discuss the Reedy model structure. This model structure will be essential in proving the universal property of the $U\mathcal{C}$ model category.

- *Chapter 4: Homotopy Colimits* In this section we provide our first homotopical analog of a concept in category theory–an homotopy colimit is the homotopical analog of a colimit. I begin this section with a discussion about what an homotopy colimit should intuitively look like. It should be noted that when presented with the actual definition, the connection might not be clear. However, at the end of the chapter it is proven that under a certain mild hypothesis the homotopy colimit does coincide with the one described in the introduction.

- *Chapter 5: Applications* In this chapter, it will finally be proven the homotopical analogs of the results presented in Chapter 1; namely, that every simplicial presheaf is an homotopy colimit of representables, and that the model category $U\mathcal{C}$ enjoys a certain universal property.

- *Chapter 6: Looking Forward* In this section we aim to present an informal discussion regarding some applications of the machinery developed in this paper.

I would also like to point out that I have taken advantage of the introductions of each chapter and presented an informal discussion of other material that is related to the one in the chapter, but which it is not covered. We have done this in the hope that it motivates the reader to learn the theory presented in the chapter, which at times can seem overly technical.

Acknowledgments

I would like to extend my gratitude to Pablo Boixeda, Joseph Hirsh, and Marc Hoyois, for many helpful discussions on the subject of this paper. To Veronica Wilson, for her emotional support and for carefully reading through this paper to help me fix many grammar mistakes. And to the MIT UROP+ program for the opportunity to pursue this project.

1 Motivation

Algebraic topologists, despite what we are usually led to believe by the name of the profession, are really not interested in the category of all topological spaces. The general notion of topological spaces is much too broad; in particular, it admits many pathological objects that the machinery of algebraic topology was simply not designed to deal with. However, there are various subcategories, like the category of CW-complexes, in which they are highly interested.

The category of manifolds, is one of these interesting subcategories. Unlike the category of CW complexes, the category of manifolds ends up being too small for homotopy theorists. Homotopy theorists would like to be able to glue manifolds together by taking colimits in the same way it is possible in the category of CW-complexes, but gluing manifolds might lead one to an object that is no longer a manifold. The homotopy theorist must somehow enlarge the category of manifolds such that the enlarged category is closed under small colimits in a way that pathologies like the ones in the category of all topological spaces are overruled.

Other settings exist where one would like to do something similar. Instead of studying CW-complexes, an homotopy theorist might want to study something like schemes. If there is any hope of applying the methods of homotopy theory to these subjects, the theorist again finds them self needing a bigger category. This enlarged category needs to be rich enough to permit us all the constructions wed like, but also coarse enough so that the geometric structure on our original spaces doesnt get lost. For instance, embedding the category of varieties into that of all topological spaces solves the first problem, but not the second since it makes us lose sight of the algebraic structure.

The goal of this chapter is to describe a general method for enlarging categories, such that the enlarged category is closed under colimits. What we plan to do, is to introduce a purely categorical construction which formalizes the notion of objects built from the elements of a category $\mathcal{C}$. For example, there will be a category of creatures built from schemes. The goal of this section is to make how we can build these new creatures precise. The author learned this motivation for presheaves from [2].

Throughout this chapter we suppose given a category $\mathcal{C}$ is somehow 'deficient' in colimits. Our goal is to produce a category $\hat{\mathcal{C}}$ which is not deficient and which admits a fully faithful embedding $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$ and which is as close as possible to $\mathcal{C}$, in the sense that it enjoys a certain universal property. 'Deficient' will sometimes mean that certain colimits just do not exist, but it may also mean that they do exists yet do not have the desired properties. The formal procedure will be to add colimits to $\mathcal{C}$ in as 'free' as possible. This is accomplished by the theory of presheaves, which is the topic of the first section. We will then present an example of a presheaf category, namely the category of simplicial sets, which is central to abstract homotopy theory.

1.1 Presheaves

Given a small category $\mathcal{C}$, our goal in this chapter is to develop a convenient way to formally add colimits to the category $\mathcal{C}$. The idea is the following: given some diagram $D : I \to \mathcal{C}$, we can formally add an object Ω to $\mathcal{C}$, and we require the morphisms to be:

(i) $\mathrm{Hom}_{\hat{\mathcal{C}}}(\Omega, Z) = \lim_{\alpha\in I} \mathrm{Hom}_{\mathcal{C}}(D_\alpha, Z)$

(ii) $\mathrm{Hom}_{\hat{\mathcal{C}}}(Z, \Omega) = \mathrm{colim}_{\alpha\in I} \mathrm{Hom}_{\mathcal{C}}(Z, D_\alpha)$

We would like to point out that this procedure destroys most of the colimits that already exist in the category $\mathcal{C}$. Too see this, consider the diagram $D : I \to \mathcal{C}$, such that $\mathcal{C}$ has a colimit for that diagram, lets call it K. Note that although the object K will satisfy (i), by the representability property of colimits, it is not required for K to satisfy (ii) for it to be the colimit of $D : I \to \mathcal{C}$. In case that (ii) is not satisfied, it will follow that K and Ω are not isomorphic, and in particular, K is no longer a colimit of the diagram in $\hat{\mathcal{C}}$. There are procedures through which we can add colimits, and at the same time preserve certain colimits that are considered important, this is attained by the theory of sheaves, but we will postpone this discussion until later.

Using the procedure described above, one could theoretically go about adding all colimits to a category. The difficulty that arises, is that diagrams that appear very different may have the same colimits, and we would not want to add the same colimit twice. As a simple example of two diagrams with the same colimit, we consider the following two diagrams,

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & & X \arrow[dl, "g"'] \arrow[dr, "h"] & \\ W & & Y & & Z \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\ W & Y & Z \end{tikzcd} \tag{2}$$

It is clear that these two diagrams will have the same colimit in any category. Therefore, a certain amount of bookkeeping will be needed to keep track of all such equivalent diagrams. A very elegant solution is provided by the theory of presheaves. As we will see below, there is a functorial procedure to assign a diagram to any presheaf, such that said presheaf is the colimit of the diagram.

I first learned about presheaves from Joseph Hirsh, at the beginning of the Summer of 2016. He challenged me to prove Theorem (1.1.7) by myself, and I am very grateful for it. I would also like to thank Pablo Boixeda, for helping me complete the proof of Theorem (1.1.7), and for helping me realize that I should have a better understanding of basic categorical concepts (like colimits and adjunctions) before going further in this project.

Theorem (1.1.7) made it clear to me why the definition of a simplicial set (a particular example of a presheaf) was the natural definition to have. For this reason, I have presented the theory of presheaves before simplicial sets.

The author learned this material from Joseph Hirsh and the following resources:

- Dugger - Sheaves and Homotopy Theory [2]
- Riehl - Category Theory in Context [16]

Definition 1.1.1. A *presheaf* on a (small) category $\mathcal{C}$ is a contravariant functor $\mathcal{C}^{op} \to Set$; morphisms of presheaves are just natural transformations of functors. The category of presheaves on $\mathcal{C}$ will be denoted $Pre(\mathcal{C})$.

Remark 1.1.2. Any object $X \in \mathcal{C}$ determines a presheaf $\mathrm{Hom}_{\mathcal{C}}(-, X) = rX$. This embedding $\mathcal{C} \hookrightarrow Pre(\mathcal{C})$ is known as the Yoneda Embedding. Particularly this means that it is a fully faithful embedding. More generally, by the Yoneda lemma we obtain the useful fact that $\mathrm{Hom}_{Pre(\mathcal{C})}(rX, F) = F(X)$ for any presheaf F.

Definition 1.1.3. A category is *complete* if it is closed under (small) limits. Dually, a category is *cocomplete* if it is closed under (small) colimits.

Lemma 1.1.4. *Let $\mathcal{C}$ be a (small) category. Then the category $Pre(\mathcal{C})$ is complete and cocomplete.*

Proof. Completeness and cocompleteness are inherited from the category Set as we will show now. Since the proof of completeness and cocompleteness are dual to each other, we will only show that $Pre(\mathcal{C})$ is cocomplete. Let $D : J \to Pre(\mathcal{C})$ be a diagram in the category $Pre(\mathcal{C})$. We will first construct a presheaf F from this diagram, and then we will note that it is actually the colimit. For every $x \in \mathcal{C}$ define

$$F(x) = \mathrm{colim}_J \, D_j(x) \tag{3}$$

this is equivalent as saying that $F(x)$ is the coequalizer of the following diagram

$$\textstyle\coprod_{g \in \mathrm{mor}\, J} D_{\mathrm{dom}(g)}(x) \overset{c}{\underset{d}{\rightrightarrows}} \coprod_{j \in J} D_j(x) \longrightarrow F(x) \tag{4}$$

where c is the identity map $c : D_{\mathrm{dom}(g)} \to D_j(x)$ for $j = \mathrm{dom}(g)$, and $d = D(g) : D_{\mathrm{dom}(g)} \to D_j$ for $j = \mathrm{cod}(g)$. With this description of the action of F on the objects of $\mathcal{C}$, we will show that there is a canonically induced action on the morphisms of $\mathcal{C}$. For any morphism $y \to x$ we have

$$\begin{tikzcd}
\coprod_{g \in \mathrm{mor}\, J} D_{\mathrm{dom}(g)}(x) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_{j \in J} D_j(x) \arrow[r] \arrow[d] & F(x) \arrow[d, dashed] \\
\coprod_{g \in \mathrm{mor}\, J} D_{\mathrm{dom}(g)}(y) \arrow[r, shift left] \arrow[r, shift right] & \coprod_{j \in J} D_j(y) \arrow[r] & F(y)
\end{tikzcd} \tag{5}$$

By the universal property of the coequalizer there is an induced map $F(x) \to F(y)$. It is easy to see that F is indeed the colimit of the diagram $J \to Pre(\mathcal{C})$. $\square$

Definition 1.1.5. The *category of elements* of a presheaf F has:

- as objects, pairs (c, x), where $c \in \mathcal{C}$ and $x \in F(c)$.
- a morphisms $(c, x) \to (c', x')$ is a morphism $f : c \to c'$ in $\mathcal{C}$ such that $Ff : x' \to x$.

we will denote this category by $el(F)$.

Lemma 1.1.6 (Density Theorem)**.** *Every presheaf F is a colimit of a $el(F)$ shape diagram of representables. And $el : Pre(\mathcal{C}) \to Cat$ is a faithful functor.*

Proof. There exists a canonical forgetful functor $U_F : el(F) \to Pre(\mathcal{C})$ that maps $(c, x) \to \mathrm{hom}(-, x)$. By the Yoneda lemma it follows that

$$\mathrm{Cone}_{Pre(\mathcal{C})}(U, G) \cong \mathrm{hom}_{Pre(\mathcal{C})}(F, G) \tag{6}$$

therefore $F = \mathrm{colim}_{el(F)} \, U_F(c, x)$. We have shown that every presheaf is a colimit of a canonical diagram of representables. By construction it is clear that el is functorial. Too see that el is a faithful functor, it suffices to see that the functor $el(h) : el(F) \to el(G)$ induced a morphism $\mathrm{colim}_{el(F)} \, U_F(c, x) \to \mathrm{colim}_{el(G)} \, U_G(c', x')$ which is precisely $h : F \to G$. $\square$

Theorem 1.1.7 (Universal Property of $Pre(\mathcal{C})$)**.** *Let $\mathcal{C}$ be a small category, $\mathcal{D}$ be any category, such that $\mathcal{D}$ is cocomplete, and let $H : \mathcal{C} \to \mathcal{D}$ be a functor. Then there exists a left adjoint functor $Re : Pre(\mathcal{C}) \to \mathcal{D}$ that makes the following diagram commute:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{Y}"] \arrow[dr, "\Gamma"'] & Pre(\mathcal{C}) \arrow[d, dashed, "Re"] \\ & \mathcal{D} \end{tikzcd} \tag{7}$$

in particular Re is a colimit preserving functor. Moreover, for any two such functors there is a unique natural isomorphism between them. We will denote the right adjoint functor of Re by $Sing : \mathcal{D} \to Pre(\mathcal{C})$.

Proof. For each object $F \in Pre(\mathcal{C})$ we have a induced diagram $el(F) \to \mathcal{C}$, such that $\mathcal{Y}$ and Γ induce $el(F)$ shaped diagrams in $Pre(\mathcal{C})$ and $\mathcal{D}$ respectively. Since we want Re to be colimit preserving, it follows that $Re(F)$ is determined up to isomorphism by the colimit of the $el(F)$ shaped diagram in $\mathcal{D}$. Note that the reason that Re is not unique, is because colimits are only unique up to a unique isomorphisms, and for any two colimit preserving functors $Re_1, Re_2 : Pre(\mathcal{C}) \to \mathcal{D}$ this induces a unique isomorphism between them.

Next, we will show that Re is a left adjoint to the functor $Sing : \mathcal{D} \to Pre(\mathcal{C})$ that maps $d \to \hom_{\mathcal{D}}(\Gamma-, d)$. We need to show that there is a bijection

$$\hom_{Pre(\mathcal{C})}(F, Sing(d)) \cong \hom_{\mathcal{D}}(Re(F), d) \tag{8}$$

For this consider the following identity

$$\hom_{Pre(\mathcal{C})}(F, Sing(d)) \cong \lim_{el(F)} \hom_{Pre(\mathcal{C})}(U_F(j), \hom_{\mathcal{D}}(\Gamma-, d)) \tag{9}$$

Since $U_F(j)$ is a representable presheaf, by the Yoneda lemma we obtain

$$\cong \lim_{el(F)} \hom_{\mathcal{D}}(\Gamma U_F(j), d) \cong \hom_{\mathcal{D}}(ReF, d_0) \tag{10}$$

It follows that Re is left adjoint, and therefore it is colimit preserving. $\square$

Although the definition we have for Re is very nice conceptually, it would be useful to have a more hands on description of Re so that we can understand examples of the adjunction $Pre(\mathcal{C}) \rightleftarrows \mathcal{D}$ in a better way. Our goal is to be able to describe the action of Re on the objects of $Pre(\mathcal{C})$ as a coequalizer of a simple diagram. For this we will introduce the following definitions.

For any set S and object $d \in \mathcal{D}$, the *copower* or *tensor* of d by S, denoted $S \otimes d$ is simply the coproduct $\amalg_S d$ of copies of d indexed by S. In particular if F is a presheaf, we may form copowers

$$F(c) \otimes \Gamma(c') \tag{11}$$

for any c, c' in $\mathcal{C}$. A morphism $f : c' \to c$ of $\mathcal{C}$ induces a map

$$f_* : F(c) \otimes \Gamma(c') \longrightarrow F(c) \otimes \Gamma(c) \tag{12}$$

which applies Γf to the copy of $\Gamma(c')$ in the component corresponding to $x \in F(c)$ and includes it in the component corresponding to x in $F(c) \otimes \Gamma(c)$, and also a map

$$f^* : F(c) \otimes \Gamma(c') \longrightarrow F(c') \otimes \Gamma(c') \tag{13}$$

which maps the component corresponding to $x \in F(c)$ to the component corresponding to $f^*x \in F(c')$. With some thought we can see that $Re(F)$ is the coequalizer of the following diagram

$$\coprod_{f \in \operatorname{mor} \mathcal{C}} F(c) \otimes \Gamma(c') \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{c \in \mathcal{C}} F(c) \otimes \Gamma(c) \dashrightarrow Re(F) \tag{14}$$

This is indeed a very nice description of the functor $Re : Pre(\mathcal{C}) \to \mathcal{D}$. And recall it has a right adjoint, the functor $Sing : \mathcal{D} \to Pre(\mathcal{C})$ that maps $d \to \hom_{\mathcal{D}}(\Gamma -, d)$. This constructions will be used later, when we proof a certain universal property of the category of simplicial presheaves.

1.2 Simplicial Sets

Let Δ be a category with the following description. Its objects are categories, which we will denote by $[n]$. The category $[n]$ is generated by the following objects and morphisms:

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n \tag{15}$$

Of course, since $[n]$ is a category, its morphisms are closed under composition. To get a better geometric picture, we present the following complete description of [2],

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 \arrow[rr] & & 2 \end{tikzcd} \tag{16}$$

For simplicity, it is standard to describe $[n]$ by just specifying its objects and 'generating morphisms', we have adopted this in Definition (1.2.1). The problem I see with Definition (1.2.1) is that in a first exposure to simplicial sets, you may not notice the geometry behind it. With the complete description of $[n]$ in mind, it is easier to see that $[n]$ is in some sense the categorification of the standard topological n-simplex. The category of simplicial sets is just $Pre(\Delta)$. Based on this discussion of the section on Presheaves, it is easy to see that the category $sSet$ are just the object built from $[n]$. Following the analogy that $[n]$ is the categorification of the topological n-simplex, we can extend this analogy and say that $sSet$ is the categorification of the category of CW-complexes.

The morphisms in Δ are "order preserving functions". More precisely, the morphisms are the possible functors between the categories $[n]$. A good geometric interpretation of the morphisms in Δ, is that a morphism $[n] \to [m]$ way in which you can 'paste' $[n]$ to $[m]$. This interpretation is particularly useful to understand the geometry of the category of simplicial sets, which we will denote by $sSet$. Extending this analogy, we can see that the values that a simplicial set X takes at $[n]$, are just the set of 'ways' in which one can paste $[n]$ to X. The benefits of dealing with $sSet$ rather than CW, is that the morphisms in $sSet$ are much simpler. As we know, continuous maps can be very badly behaved; and in particular for homotopy theory, where we only care about maps up to homotopy, the category $sSet$ is much easier to work with.

The main object of this section is to introduce the notion of simplicial sets and present some examples. Simplicial sets will be a central subject of this paper, so we have included several examples so that the reader can gain some familiarity with them, and at the same time get a glimpse of its importance in algebraic topology.

The material presented in this section can be found in the following sources:

- Riehl - A Leisurely Introduction to Simplicial Sets [14]
- Goerss and Jardine - Simplicial Homotopy Theory [8]

Definition 1.2.1. Let Δ be the category whose objects are finite, non-empty, totally ordered sets

$$[n] = \{0, 1, ..., n\} \tag{17}$$

and morphisms are order preserving functions. Equivalently, Δ is the full subcategory of Cat whose objects are the posets defining finite, non-empty ordinals, regarded as categories in the usual manner.

Definition 1.2.2. A *simplicial set* is a functor $\Delta^{op} \to Set$. This form a category where the morphisms are the natural transformations between functors, we denote the category of simplicial sets by $sSet$. More generally, for any category $\mathcal{C}$, a *simplicial object* in $\mathcal{C}$ is a functor $X : \Delta^{op} \to \mathcal{C}$. This again form a category which we denote by $s\mathcal{C}$.

By definition we have that the category $sSet$ is actually the presheaf category $Pre(\Delta)$. For each $[n] \in \Delta$, we denote its image under the Yoneda embedding

$$\mathcal{Y} : \Delta \to Pre(\Delta) = sSet \quad \text{by} \quad \Delta^n := \hom_\Delta(-, [n]) \tag{18}$$

This is the simplicial set representing the standard n-simplex. From the definition, $\Delta^n_k = \hom_\Delta([k], [n])$, i.e., k-simplices in Δ^n are maps $[k] \to [n]$ in Δ. Among all the morphisms $[m] \to [n]$ in Δ there exists special ones, namely

$$\begin{aligned} &d^i : [n-1] \to [n] \quad 0 \leq i \leq n \quad \text{(cofaces)} \\ &s^j : [n+1] \to [n] \quad 0 \leq j \leq n \quad \text{(codegeneracies)} \end{aligned} \tag{19}$$

where, by definition,

$$d^i(0 \to 1 \to \cdots \to n-1) = (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n) \tag{20}$$

(i.e. compose $i-1 \to i \to i+1$, giving a string of arrows of length $n-1$ in $[n]$), and

$$s^j(0 \to 1 \to \cdots \to n+1) = (0 \to 1 \to \cdots \to j \to j \to \cdots \to n) \tag{21}$$

(insert the identity Id_j in the j^{th} place, giving a string of length $n+1$ in $[n]$). It is easy to see that these functors satisfy a list of identities as follows, called the *cosimplicial identities*

$$\begin{cases} d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j \end{cases} \tag{22}$$

The maps d^j, s^i and these relations can be viewed as a set of generators and relations Δ. This in order to define a simplicial set Y it suffices to write down sets Y_n, $n \geq 0$ (sets of n-simplices) together with maps

$$\begin{aligned} &d_i : Y_n \to Y_{n-1} \quad 0 \geq i \geq n \quad \text{(faces)} \\ &s_j : Y_n \to Y_{n+1} \quad 0 \geq j \geq n \quad \text{(degeneracies)} \end{aligned} \tag{23}$$

satisfying the *simplicial identities*:

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j \end{cases} \tag{24}$$

This is the classical way to write down data for a simplicial set Y.

Next, we will present two classical examples of simplicial sets. The first one, the nerve of a category, is a way of assigning a (small) category $\mathcal{C}$ a simplicial set. The second example, the total singular complex of a space, assigns a simplicial set to a topological space, in a way that is closely related to the singular homology groups as we will show. We will construct this examples in a non-standard way, we will use the machinery develop in the Section on Presheaves to describe them. The author believes this approach gives a better conceptual understanding of the examples than the standard descriptions.

Example 1.2.3 (Nerve of a category)**.** Let Cat denote the category of small categories. Consider the canonical inclusion $\Delta \hookrightarrow Cat$, since Cat is a cocomplete category we consider the following diagram

$$\begin{tikzcd} \Delta \arrow[r, "\mathcal{Y}"] \arrow[dr, hook] & sSet \arrow[d, "Re"] \\ & Cat \end{tikzcd} \tag{25}$$

Such that $Re : sSet \rightleftarrows Cat : Sing$ is an adjunction, as shown in Theorem (1.1.7). The *nerve* of a category $\mathcal{C}$ is precisely $Sing(\mathcal{C})$. It is standard to denote the nerve of the category $\mathcal{C}$ by $N\mathcal{C}$. Concretely we have that

$$N\mathcal{C}_n = \hom_{Cat}([n], \mathcal{C}) \tag{26}$$

The face and degeneracy are induced by the functoriality of $\hom_{Cat}(-, \mathcal{C})$. This shows that categories can be studied through simplicial sets, this is the main object of the book Higher Topos Theory by Jacob Lurie [10]

Example 1.2.4 (Total singular complex of a space)**.** We define a functor $\Delta \to Top$ that sends $[n]$ to the standard topological n-simplex

$$\Delta_n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | x_0 + \cdots + x_n = 1, x_i \geq 0\} \tag{27}$$

The morphisms $d^i : \Delta_{n-1} \to \Delta_n$ insert a zero in the ith coordinate, while the morphisms $s^i : \Delta_{n+1} \to \Delta_n$ add the x_i and x_{i+1} coordinates. Geometrically, d^i inserts Δ_{n-1} as the ith face of Δ_n and s^i projects the $n+1$ simplex Δ_{n+1} onto the n-simplex that is orthogonal to its ith face.

Since Top is a cocomplete category we obtain the following diagram

$$\begin{tikzcd} \Delta \arrow[r, "\mathcal{Y}"] \arrow[dr] & sSet \arrow[d, "Re"] \\ & Top \end{tikzcd} \tag{28}$$

Such that $Re : sSet \rightleftarrows Top : Sing$ is an adjunction, as shown in Theorem (1.1.7). By this adjunction we can assign a simplicial set to a topologial space Y. Concretely we have that

$$SingY_n = \hom_{Top}(\Delta_n, Y) \tag{29}$$

which is the set of continuous maps from the standard topological n-simplex to Y . Elements of this set are called n-simplices of Y in algebraic topology, which coincides with our terminology. The morphisms are canonically induced by the functoriality of $\hom_{Top}(-, Y)$.

The functor $Re : sSet \to Top$ is also interesting in its own right. This functor captures the intuitive notion that simplicial sets are in some sense the categorification of CW-complexes. Form the discusion at the end of the section of Presheaves, we have a good simple description of how the functor Re acts on the simplicial sets. Recall that for a simplicial set X, its image under the functor Re is

$$Re(X) = \mathrm{coeq}\left[\coprod_{f:[n]\to[m]} X_m \otimes \Delta_n \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{[n]} X_n \otimes \Delta_n\right] \tag{30}$$

With some thought we can see that $Re(X)$ is a CW-complex. It is standard to denote this functor by $|-|: sSet \to Top$, and it is called the *geometric realization functor*.

We conclude our example by making a connection with the singular homology groups as we promised. From a simplicial set X, one may construct a simplicial abelian group $\mathbb{Z}X$ (i.e. a contravariant functor $\Delta^{op} \to Ab$), with $\mathbb{Z}X_n$ set equal to the free abelian group on X_n. $\mathbb{Z}X$ has associated to it a chain complex, called its *Moore complex* and also written $\mathbb{Z}X$, with

$$\cdots \xrightarrow{\partial} \mathbb{Z}X_2 \xrightarrow{\partial} \mathbb{Z}X_1 \xrightarrow{\partial} \mathbb{Z}X_0 \tag{31}$$

and

$$\partial = \sum_{i=0}^{n} (-1)^i d_i \tag{32}$$

in degree n. Recall that the integral singular homology groups $H_*(Y, \mathbb{Z})$ of the space Y are defined to be the homology groups of the chain complex $\mathbb{Z}Sing(Y)$.

We will conclude this section by describing two fundamental examples of simplicial sets which are central to modern homotopy theory, this are the simplicial n-sphere $\partial\Delta^n$ and the simplicial horn Λ^n_k. There are many descriptions of these simplicial sets, but the author believes that Emily Riehl's [14] description as subsets of Δ is the most natural. But for this, we need to understand what it means for a simplicial Y to be a subset of X.

Definition 1.2.5. We say that a simplicial set Y is a *subset* of a simplicial set X if there is a monomorphism $Y \to X$, i.e., if $Y_n \subset X_n$ for all $[n] \in \Delta$ and if

$$Xf|_{Y_n} = Yf \tag{33}$$

for all $f : [m] \to [n]$ in Δ. This second condition says that the subsets Y_n are closed under the right action by the face and degeneracy operations and furthermore that these operations agree with their definition for X.

In the presence of a simplicial set X, we often specify a simplicial subset by giving a set of *generators*, which will typically form a subset $S \subset X_n$ for some n.

Definition 1.2.6. The simplicial set *generated* by S is then the smallest simplicial subset of X that contains S. Its k-simplices will be the union of those k-simplices of X that are in the image of S under the right action by some $f : [k] \to [n]$ in Δ.

Example 1.2.7. The ith face $\partial_i\Delta^n$ of Δ^n is the simplicial subset generated by the image of

$$\Delta^{n-1}_k \xrightarrow{d^i} \Delta^n_k \tag{34}$$

for all k.

Example 1.2.8. The *simplicial n-sphere* $\partial\Delta^n$ is the simplicial subset of Δ^n given by the union of the faces $\partial_0\Delta^n, \dots, \partial_n\Delta^n$. The sphere $\partial\Delta^n$ has the property that $(\partial\Delta^n)_k = \Delta^n_k$ for all $k < n$; all higher simplices of $\partial\Delta_n$ are degenerate. In other words, $\partial\Delta^n$ is the (n-1)-*skeleton* of Δ^n.

Example 1.2.9. The *simplicial horn* Λ^n_k is the simplicial subset of Δ^n given by the union of the faces $\partial_0\Delta^n, \dots, \partial_{k-1}\Delta^n, \partial_{k+1}\Delta^n \dots \partial_n\Delta^n$. Equivalently, is the union of all the faces of Δ^n except for the kth face. The horn Λ^n_k has the property that $(\Lambda^n_k)_j = \Delta^n_j$ for $j < n-1$ and $(\Lambda^n_k)_{n-1} = \Delta_{n-1} \setminus \partial_k\Delta^n$, with higher simplices again being degenerate.

Remark 1.2.10. For each of these simplicial sets, their geometric realization is the topological object suggested by their name; $|\partial_i\Delta^n|$ is the ith face of the standard topological n-simplex $\Delta_n = |\Delta^n|$, $|\partial\Delta^n|$ is its boundary, and $|\Lambda^n_k|$ is the union of all faces but the kth.

2 A Model Structure for Simplicial Sets

Model categories, first introduced by Quillen in [13], have been an incredibly successful axiomatization of homotopy theory. The basic problem that model categories solve is the following: given a certain category, one often has certain maps (weak equivalences) that are not isomorphisms, but one would like to consider them to be isomorphisms. For example, in the category of topological spaces, we would like to consider the homotopy equivalences as isomorphisms. One can always formally invert the weak equivalences, but in this case one loses control of the morphisms in this new category. But, if the weak equivalences are part of a model structure, then the morphisms of the new category can be easily understood.

In these paper we will follow the following definition of a model category:

Definition 2.0.1. A *model category* is a category $\mathcal{C}$ which is equipped with three distinguised classes of morphisms in $\mathcal{C}$, called *cofibrations, fibrations*, and *weak equivalences*, in which the following axioms are satisfies:

(1) The category $\mathcal{C}$ admits all (small) limits and colimits.

(2) Given a composable pair of maps $f : X \to Y$ and $g : Y \to Z$, if any two of $g \circ f$, f and g are weak equivalences, then so is the third.

(3) Suppose $f : X \to Y$ is a retract of $g : X' \to Y'$: that is, suppose there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & X' \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "i'"] & Y' \arrow[r, "r'"] & Y \end{tikzcd} \tag{35}$$

where $r \circ i = \mathrm{Id}_X$ and $r' \circ i' = \mathrm{Id}_Y$. Then

(i) If g is a fibration, so is f.

(ii) If g is a cofibration, then so is f.

(iii) If g is a weak equivalence, then so is f.

(4) Given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{36}$$

A dotted arrow can be found rendering the diagram commutative if either

(i) The map i is a cofibration, and the map p is both a fibration and a weak equivalence.

(ii) The map i is both a cofibration and a weak equivalence, and the map p is a fibration.

(5) Any map $X \to Z$ in $\mathcal{C}$ admits a two types of functorial factorizations

(i) $f : X \to Y$ and $g : Y \to Z$ where f is a cofibration, and g is a fibration and a weak equivalence

(ii) $f' : X \to Y$ and $g' : Y \to Z$ where f' is a cofibration and a weak equivalence, and g is a fibration.

A map f in a model category is called a *trivial cofibration* if it is both a cofibration and a weak equivalence; similarly, f is called a *trivial fibration* if its both a fibration and a weak equivalence. By axiom (1), any model category $\mathcal{C}$ has a initial object $\emptyset$ and a final object $*$. An object $X \in \mathcal{C}$ is said to be *fibrant* if the unique map $X \to *$ is a fibration and *cofibrant* if the unique map $\emptyset \to X$ is a cofibration.

Definition 2.0.2. Suppose $\mathcal{C}$ is a model category, and denote the weak equivalences by W. Define the *homotopy category* $\operatorname{Ho}\mathcal{C}$ as follows. Form the free category $F(\mathcal{C}, W^{-1})$ on the arrows of $\mathcal{C}$ and the reversal of the arrows of W. An object of $F(\mathcal{C}, W^{-1})$ is an object if $\mathcal{C}$, and a morphism if a finite string of composable arrows $(f_1, f_2, \ldots, f_n)$ where f_i is either an arrow of $\mathcal{C}$ or the reversals w_i^{-1} of an arrow w_i of W. The empty string at a particular object is the identity at that object, and composition is defined by concatenation of strings. Now, define $\operatorname{Ho}\mathcal{C}$ to be the quotient category of $F(\mathcal{C}, W^{-1})$ by the relations $\mathrm{Id}_A = (\mathrm{Id}_A)$ for all objects A, $(f, g) = (g \circ f)$ for all composable arrows f, g of $\mathcal{C}$, and $\mathrm{Id}_{\operatorname{dom} w} = (w, w^{-1})$ and $\mathrm{Id}_{\operatorname{cod} w} = (w^{-1}, w)$ for all $w \in W$.

Note that although neither the fibrations or cofibrations appear in the construction of $\operatorname{Ho}\mathcal{C}$, they are essential to prove that $\operatorname{Ho}\mathcal{C}$ has only a set worth of morphisms between two objects. It is also possible to define the homotopy category by identifying homotopic morphisms. This definition yields category which is categorically equivalent to $\operatorname{Ho}\mathcal{C}$. But this will require us to take a detour from the main object of this paper. We refer the reader to the following resources from which the author learned the material:

- Hovey - Model Categories, Chapter 1 [9]
- Dwyer and Spalinski - Homotopy Theory and Model Categories [7]

Our goal in this chapter is to prove that the category of simplicial sets admits a model structure, we will call this model structure the *Kan model structure*. Moreover, it is well known that this model structure is equivalent in a strong sense to the standard model structure on topological spaces. For more details on the standard model structure on topological spaces we refer the reader to [9]. And for more details of its relation to the Kan Model structure we refer the reader to [8].

The chapter is organized as follows: The first section on "Presentable and Accessible Categories" is very technical and can be ignored on a first reading. As the title suggests, we will present a introduction to the theory of presentable and accessible categories. The results here will be used to prove the existence of certain model structures, but it serves no purpose besides that. In the second section, which is named "Combinatorial Model Categories", we will introduce a very important class of model categories, namely, combinatorial model categories. In particular, the category of simplicial sets admits a combinatorial model structure, and this will be the topic of the section "Kan model structure".

2.1 Presentable and Accessible Categories

As mentioned in the introduction, this section is very technical and may my skipped on a first reading. The results in this section will be used in the rest of the chapter to prove the existence of a very important class of model structures. All of this material can be found in:

- Lurie - Higer Topos Theory, A.1.1 [10]
- Lurie - Higer Topos Theory, A.2.6 [10]

A more detailed treatment of the material, which also describes connections of presentable and accessible categories to other branches of mathematics can be found in:

- Adamek and Rosicky - Locally Presentable and Accessible Subcategories [1]

It is worth noting that in the literature, what we call presentable categories is usually called *locally presentable categories.*

Definition 2.1.1. A partially ordered set J is κ-filtered if, for any subset $J_0 \subset J$ having certain cardinality $< \kappa$, there exists an upper bound for $J_0 \subset J$.

Let $\mathcal{C}$ be a category which admits (small) colimits and let X be an object of $\mathcal{C}$. Suppose we are given a κ-filtered partially ordered set J and a diagram $\{Y_\alpha\}_{\alpha \in J}$ in $\mathcal{C}$ indexed by J. Let Y denote a colimit of this diagram. Then there is an associated map of sets

$$\psi : \operatorname{colim}_J \operatorname{Hom}_{\mathcal{C}}(X, Y_\alpha) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(X, Y) \tag{37}$$

We say that X is κ-compact if ψ is bijective for every κ-filtered partially ordered set J and every diagram $\{Y_\alpha\}$ indexed by J. We say that X is small if it is κ-compact for some (small) regular cardinal κ. In this case, X is κ-compact for all sufficiently large regular cardinals κ.

Definition 2.1.2. A category $\mathcal{C}$ is presentable if it satisfies the following conditions:

(1) The category $\mathcal{C}$ admits all (small) colimits.

(2) There exists a (small) set S of objects of $\mathcal{C}$ which generates $\mathcal{C}$ under colimits; in other words, every object of $\mathcal{C}$ may be obtained as the colimit of a (small) diagram taking values in S.

(3) Every object in $\mathcal{C}$ is small. (Assuming (2), this is equivalent to the assertion that every object which belongs to S is small.)

(4) For any pair of objects $X, Y \in \mathcal{C}$, the set $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is small.

Example 2.1.3. Given a small category $\mathcal{C}$ we will show that $Pre(\mathcal{C})$ is a presentable category. Let S be the set of representable objects. Then conditions (1) and (2) follow directly from the discussion on the sections about Presheaves. For condition (3) let rX be a representable presheaf, and abusing the notation, denote by $el(Y)$ the diagram induced by the canonical functor $el(Y) \to Pre(\mathcal{C})$. Since $el(Y)$ is a faithful functor, it follows that

$$\psi : \operatorname{colim}_{el(Y)} \operatorname{Hom}_{Pre(\mathcal{C})}(rX, Y_\alpha) \longrightarrow \operatorname{Hom}_{Pre(\mathcal{C})}(rX, Y) \tag{38}$$

is surjective. Injectivity of ψ follows from the universal property of colimits. The general case for an arbitrary $X \in Pre(\mathcal{C})$ follows from the universal property of colimits and the fact that X is a colimit of representables. Condition (4) follows form the fact that $el : Pre(\mathcal{C}) \to Cat$ is a faithful functor and Cat is a locally small category.

Definition 2.1.4. Let $\mathcal{C}$ be a presentable category and let κ a regular cardinal. We will say that a full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ is a κ-accessible subcategory of $\mathcal{C}$ if the following conditions are satisfied:

(1) The full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ is stable under κ-filtered colimits.

(2) There exists a (small) subcategory $\mathcal{C}_0$ which generates $\mathcal{C}_0$ under κ-filtered colimits.

We will say that $\mathcal{C}_0 \subset \mathcal{C}$ is an accessible subcategory if $\mathcal{C}_0$ is κ-accessible subcategory of $\mathcal{C}$ for some regular cardinal κ.

Condition (2) of Definition (2.1.4) can be hard to use in practice. The following proposition is will provide a reformulation of condition (2) that may be easier to use. We will not provide a proof of Proposition (2.1.5) because it requires techniques that are beyond the scope of this paper. We refer the reader to [10] Proposition A.2.6.3

Proposition 2.1.5. *Let κ be a regular cardinal, $\mathcal{C}$ a presentable category, and $\mathcal{C}_0 \subset \mathcal{C}$ a full subcategory which is stable under κ-filtered colimits. Then $\mathcal{C}$ satisfies condition (2) of Definition (2.1.4) if and only if the following condition is satisfies for sufficiently large cardinals $\tau \gg \kappa$:*

(2_τ) *Let A be a τ-filtered partially ordered set, and $\{X_\alpha\}_{\alpha \in A}$ a diagram of τ-compact objects of $\mathcal{C}$ indexed by A. For every κ-filtered subset $B \subset A$ we let X_B denote the (κ-filtered) colimit of the diagram $\{X_\alpha\}_{\alpha \in B}$. Suppose that X_A belongs to $\mathcal{C}_0$. Then for every τ-small subset $C \subset A$ there exists a τ-small κ-filtered subset $B \subset A$ which contains C, such that X_B belongs to $\mathcal{C}_0$*

Corollary 2.1.6. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between presentable categories which preserves κ-filtered colimits and let $\mathcal{D}_0 \subset \mathcal{D}$ be a κ-accessible subcategory. Then $f^{-1}\mathcal{D}_0 \subset \mathcal{C}$ is a κ-accessible subcategory.*

Proof. First, suppose that there exists a κ-filtered diagram $\{X_\alpha\}_{\alpha \in A}$ in $f^{-1}\mathcal{D}_0$, such that $X_A \notin f^{-1}\mathcal{D}_0$. Since f preserves κ-filtered colimits, it follows that $f(X_A) \notin \mathcal{D}_0$, contradicting the hypothesis that $\mathcal{D}_0$ is stable under κ-filtered colimits. Let S be a subcategory of $\mathcal{D}_0$ that generates $\mathcal{D}_0$ under κ-filtered colimits. To proof condition (2) of Definition (2.1.4) we will invoke Proposition (2.1.5) and show that if $\mathcal{D}_0$ satisfies condition (2_τ) then so does $f^{-1}\mathcal{D}_0$. This follows by a similar argument to the one used for condition (1). $\square$

2.2 Combinatorial Model Categories

Among model categories, combinatorial model categories form a very important class of model categories. The reason is that it is possible to give a complete description of the model structure with just a small amount of data. This can be thought of as the homotopical analog of presentable categories. A combinatorial model category is just a presentable model category, which was a set of 'generating' cofibrations and trivial cofibrations. The goal of this section is to prove Theorem (2.2.4). This Theorem is a very powerful tool – it allows us to assign model structures to an important kind of category with very little work.

In this section, and for the rest of the paper, we will be using terms like "weakly saturated" and "small object argument". We will not present a treatment of this material in this paper, but we refer the reader to:

- Lurie - Higher Topos Theory, A.1.2 [10]

We strongly suggest to the reader who has never heard of these terms to read the section of [10] referenced above. It will be of central importance in the rest of this chapter and the following chapter. The author learned this material from

- Lurie - Higher Topos Theory, A.2.6 [10]

Definition 2.2.1 (Smith)**.** Let $\mathbf{A}$ be a model category. We say that $\mathbf{A}$ is combinatorial if the following conditions are satisfied:

(1) The category $\mathbf{A}$ is presentable

(2) There exists a set I of generating cofibrations such that the collection of all cofibrations in $\mathbf{A}$ is the smallest weakly saturated class of morphisms containing I.

(3) There exists a set J of generating trivial cofibrations such that the collection of all trivial cofibrations in $\mathbf{A}$ is the smallest weakly saturated class of morphism containing J.

Proposition 2.2.2 (Smith)**.** *Let $\mathbf{A}$ be a combinatorial model category, let $\mathbf{A}^{[1]}$ be the category of morphisms in $\mathbf{A}$, let $W \subset \mathbf{A}^{[1]}$ be the full subcategory spanned by the weak equivalences, and let $F \subset \mathbf{A}^{[1]}$ be the full subcategory spanned by the fibrations. Then F, W and $F \cap W$ are accessible subcategories of $\mathbf{A}^{[1]}$.*

Proof. For every morphisms $i : A \to B$, let $F_i : \mathbf{A}^{[1]} \to Set^{[1]}$ be the canonical functor that carries a morphism $f : X \to Y$ to the induced map of sets

$$\mathrm{Hom}_{\mathbf{A}}(B, X) \longrightarrow \mathrm{Hom}_{\mathbf{A}}(B, Y) \times_{\mathrm{Hom}_{\mathbf{A}}(A,Y)} \mathrm{Hom}_{\mathbf{A}}(A, X) \tag{39}$$

Note that a map $i : A \to B$ is a fibration (resp. trivial fibration) if and only if the induced functor $F_i : \mathbf{A}^{[1]} \to Set^{[1]}$ sends all trivial cofibration (resp. cofibration) to surjective maps. Also note that if A and B are κ-compact objects of $\mathbf{A}$, then F_i preserves κ-filtered colimits. This follows because (small) filtered colimits commute with finite limits, and a pull back square is a finite limit.

Let $\mathcal{C}_0$ be the full subcategory of $Set^{[1]}$ spanned by the collection of surjective maps between between sets. It is easy to see that $\mathcal{C}_0$ is an accessible subcategory of $Set^{[1]}$. It follows by Corollary (2.1.6) that the full subcategories $R(i) = F_i^{-1}\mathcal{C}_0 \subset \mathbf{A}^{[1]}$ are accessible subcategories of $\mathbf{A}^{[1]}$.

Let I be the sat of generating cofibrations for $\mathbf{A}$ and let J be the set of generating trivial cofibrations. We claim that the following subcategories

$$F = \bigcap_{j \in J} R(j) \qquad\qquad W \cap F = \bigcap_{i \in I} R(i) \tag{40}$$

are accessible subcategories of $\mathbf{A}^{[1]}$. We need to check the conditions of definition Definition (2.1.4). Condition (1) is clear and condition (2) is a direct consequence of Proposition (2.1.5). As noted above F and $W \cap F$ are the full subcategories of fibrations and trivial fibrations respectively.

By the small object argument we deduce that there exists a pair of functors $T', T'' : \mathbf{A}^{[1]} \to \mathbf{A}^{[1]}$, which carry arbitrary morphisms $f : X \to Y$ to a factorization

$$X \xrightarrow{T'(f)} Y \xrightarrow{T''(f)} Z \tag{41}$$

where $T'(f)$ is a trivial cofibration and $T''(f)$ is a fibration. Moreover, the functor T'' can be chosen to commute with the κ-filtered colimits for a sufficiently large regular cardinal κ. We now observe that W is the inverse image of $F \cap W$ under the functor $T'' : \mathbf{A}^{[1]} \to \mathbf{A}^{[1]}$ and therefore an accessible subcategory of $\mathbf{A}^{[1]}$ by Corollary (2.1.6). $\square$

Our next goal is to proof a converse to Proposition (2.2.2), which will allow us to construct examples of combinatorial model categories. First, we need the following preliminary result. We will not give a proof for this result, to proof it we will need techniques that will force us to take a big detour, which the author is not willing to make. For a complete proof we reference the reader to [10] Lemma A.2.6.7.

Lemma 2.2.3. *Let $\mathbf{A}$ be a presentable category. Suppose W and C are collections of morphisms of $\mathbf{A}$ with the following properties*

(1) The collection C is a weakly saturated class of morphisms of $\mathbf{A}$, and there exists a (small) subset $C_0 \subset C$ which generates C as a weakly saturated class of morphisms

(2) The intersection $C \cap W$ is a weakly saturated class of morphisms of $\mathbf{A}$.

(3) The full subcategory $W \subset \mathbf{A}^{[1]}$ is an accessible subcategory of $\mathbf{A}^{[1]}$.

(4) The class W has the two-out-of-three property.

Then $C \cap W$ is generated, as a weakly saturated class of morphisms, by a (small) subset $S \subset C \cap W$.

Theorem 2.2.4. *Let $\mathbf{A}$ be a presentable category and let W and C be classes of morphisms in $\mathbf{A}$ with the following properties:*

(1) The collection C is a weakly saturated class of morphisms of $\mathbf{A}$, and there exists a (small) subset $C_0 \subset C$ which generates C as a weakly saturated class of morphisms

(2) The intersection $C \cap W$ is a weakly saturated class of morphisms of $\mathbf{A}$.

(3) The full subcategory $W \subset \mathbf{A}^{[1]}$ is an accessible subcategory of $\mathbf{A}^{[1]}$.

(4) The class W has the two-out-of-three property.

(5) *If f is a morphisms in $\mathbf{A}$ which has the right lifting property with respect to each element of C, then $f \in W$*

Then $\mathbf{A}$ admits a combinatorial model structure which may be described as follows:

(C) *The cofibrations in $\mathbf{A}$ are the elements of C*

(W) *The weak equivalences in $\mathbf{A}$ are the element of W.*

(F) *A morphism in $\mathbf{A}$ is a fibration if it has the right lifting property with respect to every morphism in $C \cap W$.*

Proof. The category $\mathbf{A}$ has all (small) limits and colimits since it is presentable. The two-out-of-three property for W is among our assumptions, and the stability of W under retracts follows from the accessibility of $W \subset \mathbf{A}^{[1]}$. Indeed, let Y be a retract of X, then there exist maps $i : Y \to X$ and $r : X \to Y$ such that $r \circ i = \mathrm{Id}_Y$. This implies that there exists a map $p = i \circ r : X \to X$ such that $p \circ p = p$, and $\mathrm{colim}(p : X \to X) = Y$. We can conclude that if W is an accessible subcategory, then it is closed under retracts.

The class of cofibrations is closed under retracts by condition (1), and it is easy to see that the class of fibrations is closed under retracts since it is defined as the maps that have the right lifting property with respect to trivial cofibrations.

We next establish the factorization axioms. By the small object argument, any morphisms $X \to Z$ admits a factorization

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{42}$$

where $f \in C$ and g has the right lifting property with respect to every morphisms in C. In particular, g has the right lifting property with respect to every morphisms $W \cap C$, so that g is a fibration; and condition (5) then implies that g is a trivial fibration. We have shown that there exists a functorial factorization consisting of a cofibration followed by a trivial fibration. Similarly, using Lemma (2.2.3) we know that $C \cap W$ is generated as a weakly saturated class of morphisms by a (small) subset $S \subset C \cap W$. Using S instead of C we invoke the small object argument as above to obtain a functorial factorization consisting of a trivial cofibration followed by a fibration.

To complete the proof, it suffices to show that the cofibrations have the left lifting property with respect to trivial fibrations, and the trivial cofibrations have the left lifting property with respect to fibrations. The second of these statements is clear, since it is precisely the definition of the fibrations. For the first statement, let us consider an arbitrary trivial fibration $p : X \to Z$. By the small object argument, there exists a factorization of p

$$X \xrightarrow{q} Y \xrightarrow{r} Z \tag{43}$$

where q is a cofibration and r has the right lifting property with respect to all cofibrations. Then r is a weak equivalence by (5), so that q is a weak equivalence by the two-out-of-three property. Considering the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "q"'] & X \arrow[d, "p"] \\ Y \arrow[r, "r"'] \arrow[ur, dashed] & Z \end{tikzcd} \tag{44}$$

we deduce the existence of the dotted arrow from the fact that p is a fibration and q is a trivial cofibration. It follows that p is a retract of r, and therefore p also has the right lifting property with respect to all cofibrations. This completes the proof that $\mathbf{A}$ is a model category. The assertion that $\mathbf{A}$ is combinatorial follows immediately from (1) and from Lemma (2.2.3). $\square$

Corollary 2.2.5. *Let $\mathbf{A}$ be a presentable category equipped with a model structure. Suppose that there exists a (small) set which generates the collection of cofibrations in $\mathbf{A}$ (as a weakly saturated class of morphisms). Then the following are equivalent:*

(1) The model category $\mathbf{A}$ is combinatorial; in other words, there exists a (small) set which generates the collection of trivial cofibrations in $\mathbf{A}$ (as a weakly saturated class of morphisms).

(2) The collection of weak equivalences in $\mathbf{A}$ determines an accessible subcategory of $\mathbf{A}^{[1]}$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition (2.2.2), and the reverse implication follows from Theorem (2.2.4). $\square$

2.3 Kan Model Structure

The Kan model structure on simplicial sets is probably the most important model structure in this paper, and one of the most important ones in homotopy theory. The proof of the Kan model structure we provide here is a consequence of the general framework we have been developing in the previous section and in this section. This approach has a downside. Unlike the standard approach, we are unable to completely characterize the trivial cofibrations in the Kan model structure. This is not a big problem, since we are not going to need this fact in the rest of the paper. But we refer the reader to

- Hovey - Model Categories, Chapter 3 [9]

for a proof of a complete characterization of the trivial cofibrations. The author learned most of the material from

- Lurie - Higher Topos Theory, A.2.6 [10]
- Lurie - Higher Topos Theory, A.2.7 [10]

We would also like to thank [9] for a proof of a complete characterization of the cofibrations in the Kan model structure, a fact that we will use in the rest of the paper. And [14] for the discussion on $(\infty, 1)$ categories at the end of the section.

Definition 2.3.1. Let $\mathbf{A}$ be a presentable category. A class W of morphisms in $\mathcal{C}$ is *perfect* if it satisfies the following conditions:

(1) Every isomorphism belongs to W.

(2) Given a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$, if any two of the morphisms f, g, and $g \circ f$ belong to W, then so does the third.

(3) The class W is stable under filtered colimits. More precisely, suppose we are given a family of morphisms $\{f_\alpha : X_\alpha \to Y_\alpha\}$ which is indexed by a filtered partially ordered set. Let X denote a colimit of $\{X_\alpha\}$, Y a colimit of $\{Y_\alpha\}$, and $f : X \to Y$ the induced map. If each f_α belongs to W, then so does f.

(4) There exists a (small) subset $W_0 \subset W$ such that every morphism belonging to W can be obtained as a filtered colimit of morphisms belonging to W_0.

Corollary 2.3.2. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between presentable categories which preserves filtered colimits and let $W_{\mathcal{C}'}$ be a perfect class of morphisms in $\mathcal{C}$. Then $W_{\mathcal{C}} = F^{-1}W_{\mathcal{C}'}$ is a perfect class of morphisms in $\mathcal{C}$.*

Proof. Condition (1) and (2) are immediate. And conditions (3) and (4) are a direct consequence of Corollary (2.1.6). □

Theorem 2.3.3. *Let $\mathbf{A}$ be a presentable category. Suppose we are given a class W of morphisms of $\mathbf{A}$, which we will call weak equivalences, and a (small) set C_0 of morphisms of $\mathbf{A}$, which we will call generating cofibrations. Suppose furthermore that the following assumptions are satisfied:*

(1) The class W of weak equivalences is perfect.

(2) For any diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[d, "g"'] & Y' \arrow[d, "g'"] \\ X'' \arrow[r] & Y'' \end{tikzcd} \tag{45}$$

in which both squares are pushout squares, f belongs to C_0, and g belongs to W, the map g also belongs to W.

(3) If $g : X \to Y$ is a morphism in $\mathbf{A}$ which has the right lifting property with respect to every morphism in C_0, then g belongs to W.

Then there exists a left proper combinatorial model structure on $\mathbf{A}$ which may be described as follows:

(C) A morphism $f : X \to Y$ in $\mathbf{A}$ is a cofibration if it belongs to the weakly saturated class of morphisms generated by C_0.

(W) A morphism $f : X \to Y$ in $\mathbf{A}$ is a weak equivalence if it belongs to W.

(F) A morphism $f : X \to Y$ in $\mathbf{A}$ is a fibration if it has the right lifting property with respect to every map which is both a cofibration and a weak equivalence.

Proof. We first show that the class of weak equivalences is stable under pushouts by cofibrations. Let P denote the collection of all morphisms f in $\mathbf{A}$ with the following property: for a diagram composed of two pushout squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[d, "g"'] & Y' \arrow[d, "g'"] \\ X'' \arrow[r] & Y'' \end{tikzcd} \tag{46}$$

where g belongs to W, the map g' also belongs to W. By condition (2) we have that $C_0 \subset P$, it suffices to show that P is weakly saturated since then $C \subset P$ and the result follows. The only non trivial point is that P is that P is closed under transfinite composition, but this follows by the stability of W under filtered colimits.

It remains to show that $\mathbf{A}$ is a model category. We would like to invoke Theorem (2.2.4), given the hypothesis it suffices to show that $C \cap W$ is a weakly saturated class of morphisms. First, note that $C \cap W$ is closed under retracts. This follows from the fact that C is closed under retracts since it is a weakly saturated class of morphisms, and W is closed under retracts by condition (3). We now check that $C \cap W$ is stable under transfinite composition, this follows because W is stable under transfinite composition because it is stable under filtered colimits and finite composition, and C is a weakly saturated class of morphisms.

It remains to show that $C \cap W$ is stable under pushouts. Concretely, this means that given a pushout diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X'' \arrow[d, "f''"] \\ Y \arrow[r] & Y'' \end{tikzcd} \tag{47}$$

in which f belongs to $C \cap W$; we wish to show that f'' also belongs to $C \cap W$. Since C is weakly saturated, it will suffice to show that f'' belongs to W. Using the small object argument, we can factor the top horizontal map to produce two pushout squares

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[r, "h"] \arrow[d, "f'"] & X'' \arrow[d, "f''"] \\ Y \arrow[r] & Y' \arrow[r, "h'"] & Y'' \end{tikzcd} \tag{48}$$

in which g is a cofibration and h has the right lifting property with respect to all the morphisms in C_0. Since W is stable under the formation of pushouts by cofibrations, we deduce that f' belongs to W. Moreover, by assumption (3), h belongs to W. Since h' is a pushout of h by the cofibration f', we deduce that h' belongs to W as well. Applying the two-out-of-three property (twice), we deduce that f'' belongs to W. Now we can invoke Theorem (2.2.4) and the result follows. ☐

Our main goal in this section is that the category of simplicial sets has a model structure. This result is central to the rest of the paper, since all the following proofs regarding model categories will be based on the existence of a combinatorial left model structure on simplicial sets.

Proposition 2.3.4 (Kan Model Structure)**.** *The category of simplicial sets has a combinatorial left proper model structure. It may be described as follows:*

- *(C) A map of simplicial sets $f : X \to Y$ is a cofibration if it belongs to the weakly saturated class of morphisms generated by the canonical inclusion $\partial\Delta^n \hookrightarrow \Delta^n$*
- *(W) A map of simplicial sets $f : X \to Y$ is a weak equivalence if the induced map of geometric realizations $|X| \to |Y|$ is an homotopy equivalence of topological spaces*
- *(F) A map of simplicial sets $f : X \to Y$ is a fibration if it has the right lifting property with respect to every map which is both a cofibration and a weak equivalence.*

Proof. We would like to invoke Theorem (2.3.3), we only need to check that the class of morphisms W is perfect. The only non trivial point here is that there exists a subset $W_0 \subset W$ such that every morphism belonging to W can be obtained as a filtered colimit of morphisms belonging to W_0. This follows by Proposition (2.1.5) ☐

The Kan model structure has a more complete description in which the cofibrations and fibrations can be described explicitly as follows

- A map of simplicial sets $f : X \to Y$ is a *cofibration* if it is a monomorphism; that is, if the induced map $X_n \to Y_n$ is injective for all $n \geq 0$.
- A map if simplicial sets $F : X \to Y$ is a *fibration* if it is a Kan fibration: that ism if for any diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{49}$$

It is possible to supply the dotted arrow rendering the diagram commutative.

- A map of simplicial sets $f : X \to Y$ is a *weak equivalence* if the induced map of geometric realizations $|X| \to |Y|$ is a homotopy equivalence of topological spaces.

We will not be able to give a complete proof that the Kan model structure described on Proposition (2.3.4) is equivalent to this more explicit description. The reason is, that it will take us to much of a detour from the main topic of this paper. For a complete proof of this fact we refer the reader to [9] or [8]. But for the sake of completeness we will provide a proof of the fact that the weakly saturated category of maps generated by $\partial\Delta^n \hookrightarrow \Delta^n$ are indeed all the monomorphisms.

Proposition 2.3.5. *A map $f : X \to Y$ in sSet is a cofibration if and only if it is injective. In particular, every simplicial set is cofibrant.*

Proof. Denote the set of generating cofibrations $\partial\Delta^n \hookrightarrow \Delta^n$ by C_0. Certainly the maps in are injective. Since the class of injection is a weakly saturated class, it follows that every cofibration is injective. Conversely, suppose that $f : X \to Y$ is injective. We will give an explicit construction of f to show that it is an element of C. Define $K_0 = X$. Having defined $K_n \to Y$ that is an isomorphisms on simplices of dimension less than n, let S_n denote the set of n-simplices of Y not in the image of K_n. Each such simplex s is necessarily non degenerate, and corresponds to a map $\Delta^n \to Y$. The restriction of s to $\partial\Delta^n$ factors uniquely through K_n. Define K_{n+1} as the pushout in the diagram below.

$$\begin{tikzcd} \coprod_S \partial\Delta^n \arrow[r] \arrow[d] & K_n \arrow[d] \\ \coprod_S \Delta^n \arrow[r] & K_{n+1} \end{tikzcd} \tag{50}$$

Then the inclusion $X_n \to L$ extends to a map $X_{n+1} \to L$. This extension is surjective on simplices of dimension $\leq n$, by construction. It is also injective, since we are only adding non-degenerate simplices. The map $f : X \to Y$ is a composition of the sequence of maps $K_0 \to K_1 \to \cdots$, so f is an element of the weakly saturated class of C_0. $\square$

Although for the purpose of this paper, we do not need to know that fibrations in the Kan model structure are the maps that have the right lifting property with respect to $\Lambda^n_k \hookrightarrow \Delta^n$ for all k and n, this "horn filling" condition is central in the theory of higher categories. We will proceed to give a informal description about the theory of $(\infty, 1)$ categories as described by [10].

Definition 2.3.6. A *Kan complex* is a simplicial set X such that every horn has a filler (which is not assumed to be unique). This means that for each horn $\Lambda^n_k \to X$ in X there exists an extension along the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ as shown

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{51}$$

By the Yoneda lemma, the map $\Delta^n \to X$ identifies an n-simplex in X whose faces agree with those specified by the horn.

Lemma 2.3.7. *If X is a topological space, then $Sing(X)$ is a Kan complex*

Proof. By the adjunction $Re : sSet \rightleftarrows Top : Sing$, the following diagrams are equivalent

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & SX \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} |\Lambda^n_k| \arrow[r] \arrow[d] & X \\ |\Delta^n| \arrow[ur, dashed] & \end{tikzcd} \tag{52}$$

A topological Λ^n_k horn is a deformation retract of the standard n-simplex $\Delta_n = |\Delta^n|$ so the lift n the right hand side exists. The adjunct to this map gives us the desires lift on the left. $\square$

As we know, the study of fibrant and cofibrant objects are central in the study the the homotopy category of a given model structure. And in the Kan model structure we have that every simplicial set is cofibrant, and the Kan complexes are the fibrant objects. Moreover Lemma (2.3.7) is central in the proof that the Kan model structure on simplicial sets is Quillen equivalent to the standard model structure on the category of topological spaces. That is, there exists an equivalence of categories in their homotopy categories.

Definition 2.3.8. A $(\infty, 1)$ category is a simplicial set X such that every *inner horn*, i.e., horn Λ^n_k with $0 < k < n$, has a filler.

Example 2.3.9. For any category $\mathcal{C}$, its nerve $N\mathcal{C}$ is a $(\infty, 1)$ category. In fact, it is a $(\infty, 1)$ category with the special property that every inner horn has an unique filler. Conversely, any $(\infty, 1)$ category such that every inner horn has a unique filler is isomorphic to the nerve of a category.

We wont give formal proofs of these facts here, which can be found in [10] as Proposition 1.1.2.2, but we will at least provide some intuition for why the nerve of a category has a unique filler for horns $\Lambda^2_1 \to N\mathcal{C}$. This horn is often represented by the following picture:

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] & & x_2 \end{tikzcd} \qquad \begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & x_2 \end{tikzcd} \tag{53}$$

Here $f, g \in N\mathcal{C}_1$ are morphisms in $\mathcal{C}$ and $x_0, x_1, x_2 \in N\mathcal{C}_0$ are objects in $\mathcal{C}$. $f \circ d_1 = x_0$ and $f \circ d_0 = x_1$, colloquially, x_0 is the domain of f and x_1 is its codomain, and similarly for g. The essential point that this picture communicates is that if f and g are the generating 1-simplices of a horn $\Lambda^2_1 \to N\mathcal{C}$, then f and g are a composable pair of arrows in $\mathcal{C}$. The statement that this horn can be filled then simply expresses the fact that this pair necessarily has a composite $g \circ f$. Composition is unique in a category, so this horn can be filled uniquely. However in a $(\infty, 1)$ category, composition need not be unique. This lack of uniqueness, reflects some of the philosophy of homotopy theory, which is that morphisms are "equal" only up to homotopy.

3 A Model Structure for Simplicial Presheaves

In this chapter we prove the existence of the main object of study of this paper – a certain model structure on the category of simplicial presheaves. The category of simplicial presheaves is precisely the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, sSet)$, which we will denote by $sPre(\mathcal{C})$. As you may expect, this is the homotopical analog of the presheaf category $Pre(\mathcal{C})$. From the previous chapter we know that there exists a model structure on simplicial sets. It would then be reasonable to assume that we can assign a model structure to $sPre(\mathcal{C})$ by defining the weak equivalences to be morphisms $F \to G$ that are weak equivalences when evaluated at each U of $\mathcal{C}$. This is in fact correct, but we still need to specify the fibrations and cofibrations, and prove that these determine a model structure. This will be the goal of the first section.

Instead of describing why simplicial presheaves are important to this paper, I have decided that it might be better to describe other applications of simplicial presheaves. In particular, I am going to describe the application that got me interested in the subject. Simplicial presheaves, and more importantly simplicial sheaves, are essential in assigning a homotopy theory to the category of algebraic varieties, or more generally, to schemes. This homotopy theory, which is called *motivic homotopy theory* is due to Fabien Morel and Vladimir Voevodsky [12]. The underlying idea is that it should be possible to develop a purely algebraic approach to homotopy theory by replacing the unit interval $[0, 1]$, which is not an algebraic variety, with the affine line $\mathbb{A}^1$.

To explain how our study of simplicial presheaves fits into the study of motivic homotopy theory, we must first describe the model structure assigned to this homotopy theory. For this, we must first describe what a sheaf is. We start by presenting a discussion about sheaves and hoping that everything else falls into place. The following discussion on sheaves is due to Daniel Dugger [2].

The presheaf functor gives a way of embedding any category into one that is cocomplete. But if we apply this to the category of manifolds, for instance, what happens is that we lose the underlying geometry which made manifolds interesting in the first place. The point is that the process of formally adding all colimits also destroys most colimits we might have already had. The reason for this was described in the introduction on the section on presheaves. Here we present the solution to this problem, as promised before.

As an example, consider the category of manifolds and let M_1 and M_2 be two objects. These objects already have a coproduct in our original category, namely the disjoint union $M_1 \amalg M_2$. But if we embed everything in the presheaf category then rM_1 and rM_2 (i.e. the new copies of M_1 and M_2) now have a formal coproduct $rM_1 \amalg rM_2$, and this is not the same as $r(M_1 \amalg M_2)$. To get a sense of the difference, lets compute the set of maps from S^0 into both objects. Mapping S^0 into $M_1 \amalg M_2$ is equivalent to just picking two points on $M_1 \amalg M_2$: they may both be on M_1, both on M_2, or one on each. On the other hand, we may compute that

$$\mathrm{Hom}_{Pre(Man)}(rS^0, rM_1 \amalg rM_2) = \mathrm{Hom}_{Pre(Man)}(rS^0, rM_1) \amalg \mathrm{Hom}_{Pre(Man)}(rS^0, rM_2) \tag{54}$$

$$= \mathrm{Hom}_{Man}(S^0, M_1) \amalg \mathrm{Hom}_{Man}(S^0, M_2) \tag{55}$$

Thus, mapping S^0 into $rM_1 \amalg rM_2$ is equivalent to giving either two points on M_1 or two points on M_2. The difference is clear. It is clear that the original $M_1 \amalg M_2$ is the right coproduct, it is the coproduct which geometry gives us. However, in passing to the presheaf category we have exchanged our interesting coproduct $M_1 \amalg M_2$ for a formal and uninteresting one $rM_1 \amalg rM_2$. The theory of sheaves presents an elegant solution to this problem. This was developed by Grothendieck. The idea is that we give ourselves a collection of cones $\{D_\alpha \to X\}$ in $\mathcal{C}$ (where by cone we mean a diagram with a terminal vertex) which we want to become colimit cones in our expanded category $\hat{\mathcal{C}}$. For instance, the above example said that when we expand the category of manifolds we would still like the following cone to be a colimit $M_1 \to M_1 \amalg M_2 \leftarrow M_2$.

Example 3.0.1. Consider again the category of manifolds. If $\{U_\alpha\}$ is an open cover of a manifold M, then geometric considerations show that M can be built by gluing together all the U_α's along their intersections. In other words, the following is a coequalizer diagram

$$\coprod_{\beta,\gamma} U_\beta \cap U_\gamma \rightrightarrows \coprod_\alpha U_\alpha \longrightarrow M \tag{56}$$

where the two parallel arrows are induced by the inclusion of $U_\beta \cap U_\gamma$ into U_β and U_γ. The collection of cones of the above form turns out to be sufficient for encoding the essential geometry in our category of manifolds. Grothendieck realized that by generalizing the notion of 'cover' one could utilize this basic method to produce a sufficient collection of cones in other categories.

Definition 3.0.2. A *Grothendieck topology* on a category $\mathcal{C}$ is an assignment $\tau : ob(\mathcal{C}) \to Set$ such that every element $\tau(X)$ is a subset of $ob(C_{/X})$. In other words, to each object X we associate a family of *covers* $\{U_\alpha \to X\}$. We require the following properties:

(i) If $f : Y \to X$ is an isomorphism then $\{Y \to X\}$ is a cover of X.

(ii) If $\{U_\alpha \to X\}$ is a cover of X and $\{V_{\alpha\beta} \to U_\alpha\}$ are covers of each U_α, then the collection of composites $V_{\alpha\beta} \to X$ is a cover of X.

(iii) If $f : Y \to X$ and $\{U_\alpha \to X\}$ is a cover, then each $Y \times_X U_\alpha$ exists and $\{Y \times_X U_\alpha \to Y\}$ is a cover.

A *Grothendieck site* is a small category equipped with a Grothendieck topology.

The main thing to keep in mind from this definition is that each cover $\{U_\alpha \to X\}$ in a Grothendieck topology gives rise to a cone

$$\coprod_{\beta,\gamma} U_\beta \times_X U_\gamma \rightrightarrows \coprod_\alpha U_\alpha \longrightarrow X \tag{57}$$

and that these are the cones which we will want to become colimits. Denote the category of sheaves on a Grothendieck site $\mathcal{C}$ by $Shv(\mathcal{C})$. Before giving the definition of a sheaf, we introduce the following terminology: given a cone $U_\alpha \to X$ in a category $\mathcal{C}$, and object $Z \in \mathcal{C}$ *sees* $\{A_\alpha \to X\}$ *as a colimit* if $\mathrm{Hom}_{\mathcal{C}}(X, Z) = \lim_\alpha \mathrm{Hom}_{\mathcal{C}}(A_\alpha, Z)$. Note that $\{A_\alpha \to X\}$ is an actual colimit precisely when every object Z sees $\{A_\alpha \to X\}$ as a colimit.

Definition 3.0.3. When $\mathcal{C}$ is a category with a Grothendieck topology, a *sheaf* on $\mathcal{C}$ is a presheaf $F \in Pre(\mathcal{C})$ which sees all the distinguished cones as colimits. This means that for every cover $\{U_\alpha \to X\}$ the following is an equalizer diagram:

$$\mathrm{Hom}_{Pre(\mathcal{C})}(rX, F) \longrightarrow \prod_\alpha \mathrm{Hom}_{Pre(\mathcal{C})}(rU_\alpha, F) \rightrightarrows \prod_{\beta,\gamma} \mathrm{Hom}_{Pre(\mathcal{C})}(r(U_\beta \times_X U_\gamma), F) \tag{58}$$

$Shv(\mathcal{C})$ is the full subcategory of $Pre(\mathcal{C})$ whose objects are sheaves.

An usual way to phrase the sheaf condition is to use the identification $\mathrm{Hom}_{Pre(\mathcal{C})}(rX, F) = F(X)$ given by the Yoneda lemma, so that the diagram above becomes the following equalizer diagram

$$F(X) \longrightarrow \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\beta,\gamma} F(U_\beta \times_X U_\gamma) \tag{59}$$

But we think that the definition we have is a better fit for the discussion presented before. We finalize our discussion on sheaves by presenting the following result

Proposition 3.0.4. *Let $\mathcal{C}$ be a Grothendieck site. Then there exists a cocomplete category $Shv(\mathcal{C})$ and a functor $r : C \hookrightarrow Shv(\mathcal{C})$ such that r takes the distinguished cones of $\mathcal{C}$ to colimit cones in $Shv(\mathcal{C})$. Moreover, $Shv(\mathcal{C})$ has the following universal property: If $\mathcal{D}$ is a cocomplete category and $\gamma : \mathcal{C} \to calD$ a map taking distinguished cones to colimits, then γ admits a colimit-preserving factorization*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & Shv(\mathcal{C}) \arrow[d, dashed] \\ & \mathcal{D} \end{tikzcd} \tag{60}$$

Any two such factorizations admit a unique isomorphism between them.

We hope this discussion illustrates why it is more natural to study categories that have geometric meaning, such as the category of manifolds, by using sheaves instead of presheaves. Although this has been an interesting discussion, we still have not answered our question: why are simplicial presheaves interesting? We are getting closer to answering this question.

We say that a simplicial presheaf is a simplicial sheaf if it satisfies the sheaf condition when evaluated at every $[n]$. Morel and Voevodsky where able to assign an homotopy theory to the category of smooth schemes (of finite type), over a base scheme S, endowed with the Nisnevick topology. They achieved this by defining a model structure on $sShv(Sm/S)$. The underlying idea is that it should be possible to develop a purely algebraic approach to homotopy theory by replacing the unit interval $[0, 1]$, which is not an algebraic variety, with the affine line $\mathbb{A}^1$. We denote by $sShv(Sm/S)_{\mathbb{A}^1}$ the category $sShv(Sm/S)$ endowed with this model structure. We call this model structure the motivic model structure. The motivic model structure included among the weak equivalences the projection maps $X \times \mathbb{A}^1 \to X$, such that $\mathbb{A}^1$ could replace the unit interval $[0, 1]$.

Finally, we can provide at least a partial answer to the question. To understand the homotopy category of $sShv(Sm/S)_{\mathbb{A}^1}$ it is essential that one understands its fibrant and cofibrant objects in a the motivic model structure. Unfortunately, in $sShv(Sm/S)_{\mathbb{A}^1}$, the fibrations are described as maps that satisfy the right lifting property (as we will see later on) with respect to the trivial cofibrations. On the other hand, there exists a model structure on $sPre(Sm/S)$, in which we can give a complete description of the fibrant objects, and which is Quillen equivalent to $sShv(Sm/S)_{\mathbb{A}^1}$. This allows us obtain a better understanding of the homotopy category of $sShv(Sm/S)_{\mathbb{A}^1}$. Later in the paper we will provide a discussion about how this Quillen equivalence is achieved.

As it is standard, we will provide a brief overview of the material covered in this chapter. In the first section, we aim to prove the existence of a model structure on $sPre(\mathcal{C})$. There are actually two such model structures, but one enjoys a certain universal property, which we will discuss later on. The goal of the next two sections is to prove the existence of a model structure on $\mathrm{Fun}(\Delta, \mathcal{M})$, where $\mathcal{M}$ is a model category. This model structure, called the Reedy model structure, will prove very valuable for computations on homotopy colimits.

3.1 Diagram Categories

In this section, our goal is to prove that we can equip $sPre(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, sSet)$ with a model structure. Indeed, there are two model structures we can assign to $sPre(\mathcal{C})$, the projective and injective model structure. The projective model structure on $sPre(\mathcal{C})$ is the main object of study of this paper, as we will see later on, this model structure enjoys a certain universal property. Hence the name of this paper.

The author learned this material from:

- Lurie - Higher Topos Theory, A.2.8 [10]

Definition 3.1.1. Let $\mathcal{C}$ be a small category and let $\mathbf{A}$ be a model category. We well say that a natural transformation $\alpha : F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is:

- an *injective cofibration* if the induced map $F(C) \to G(C)$ is a cofibration in $\mathbf{A}$ for each $C \in \mathcal{C}$.
- a *projective fibration* if the induced map $F(C) \to G(C)$ is a fibration in $\mathbf{A}$ for each $C \in \mathcal{C}$.
- a *weak equivalence* if the induced map $F(C) \to G(C)$ is a weak equivalence in $\mathbf{A}$ for each $C \in \mathcal{C}$.
- an *injective fibration* if it has the right lifting property with respect to every morphism β in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ which is simultaneously a weak equivalence and an injective cofibration.
- a *projective cofibration* if it has the left lifting property with respect to every morphism β in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ which is simultaneously a weak equivalence and a projective fibration.

Theorem 3.1.2. *Let $\mathbf{A}$ be a combinatorial model category and let $\mathcal{C}$ be a small category. Then there exist two combinatorial model structures on* $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$*:*

- *The projective model structure determined by the projective cofibrations, weak equivalences, and projective fibrations.*
- *The injective model structure determined by the injective cofibrations, weak equivalences, and injective fibrations.*

We will need the following lemma, which is a key step in the proof of Theorem (3.1.2). Unfortunately, we will not provide the proof, we reference the reader to [10] Lemma A.3.3.3. We will only need this result to proof that $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ admits the injective model structure, since we will proof that $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ admits the projective model structure by hand.

Lemma 3.1.3. *Let $\mathbf{A}$ be a presentable category and let $\mathcal{C}$ be a small category. Let S_0 be a (small) set of morphisms of $\mathbf{A}$ and let $\overline{S}_0$ be the weakly saturated class of morphisms generated by S_0. Let $\widetilde{S}$ be the collection of all morphisms $F \to G$ in* $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ *with the following property: for every $C \in \mathcal{C}$, the map $F(C) \to G(C)$ belongs to S_0. Then there exists a (small) set of morphisms S of* $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ *which generates $\widetilde{S}$ as a weakly saturated class of morphisms.*

Proof of Theorem (3.1.2). We will first proof the case of the projective model structure. For each object $C \in \mathcal{C}$ and each $A \in \mathbf{A}$, we define

$$\mathcal{F}_A^C : \mathcal{C} \longrightarrow \mathbf{A} \tag{61}$$

by the formula

$$\mathcal{F}_A^C(C') = \coprod_{\alpha \in \mathrm{Map}_{\mathcal{C}}(C,C')} A \tag{62}$$

We note that if $i : A \to A'$ is a (trivial) cofibration in $\mathbf{A}$, then the induced map $\mathcal{F}_A^C \to \mathcal{F}_{A'}^C$ is a projective (trivial) cofibration in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$.

Let I_0 be the set of generating cofibrations $i : A \to B$ for $\mathbf{A}$ and let I be the set of all induced maps $\mathcal{F}_A^C \to \mathcal{F}_B^C$ (where C ranges over $\mathcal{C}$). Let J_0 be a set of generating trivial cofibrations for $\mathbf{A}$ and define J likewise. It follows immediately from the definitions that a morphism in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is a projective fibration if and only if it has the right lifting property with respect to every morphism in J, and a projective trivial fibration if and only if it has the right lifting property with respect to every morphism in I. Let $\overline{I}$ and $\overline{J}$ be the weakly saturated classes of morphisms of $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ generated by I and J, respectively. Using the small object argument, we deduce the following:

(i) Every morphism $f : X \to Z$ in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ admits a factorization

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z \tag{63}$$

where $f' \in \overline{I}$ and f'' is a projective trivial fibration.

(ii) Every morphisms $f : X \to Z$ in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ admits a factorization

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z \tag{64}$$

where $f' \in \overline{J}$ and f'' is a projective fibration.

(iii) The class $\overline{I}$ coincides with the class of projective cofibrations in $\mathbf{A}$.

Furthermore, since the class of trivial projective cofibrations in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is weakly saturated and contains J, it contains $\overline{J}$. This proves the existence of a functorial factorization in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$.

The category $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ inherits completeness and cocompleteness form from $\mathbf{A}$ as in the case of the presheaf category. It is clear that weak equivalences, fibrations and cofibrations are closed under retracts. The two-out-of-three property for weak equivalences follows trivially from the definition. The thing left to check is that $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ satisfies the lifting axioms. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ C \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{65}$$

in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$, where i is a projective cofibration and p is a projective fibration. We wish to show that there exists a dotted arrow as indicated provided that either i or p is a weak equivalence. If p is a weak equivalence, then this follows immediately from the definition of a projective fibration. Suppose instead that i is a trivial projective cofibration. We wish to show that i has the left lifting property with respect to every projective fibration. It will suffice to show that every trivial projective fibration belongs to $\overline{J}$ (this will also imply that J is a set of generating trivial projective cofibrations for $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$, which shows that the projective model structure on $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is combinatorial, given that $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is presentable). Suppose then that i is a trivial projective cofibration and choose a factorization

$$A \xrightarrow{i} B \xrightarrow{i''} C \tag{66}$$

where $i' \in \overline{J}$ and i'' is a projectove fibration. Then i' is a weak equivalence, so that i'' is a weak equivalence by the two-out-of-three property. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "i'"] \arrow[d, "i"] & B \arrow[d, "i''"] \\ C \arrow[r, equal] \arrow[ur, dashed] & C \end{tikzcd} \tag{67}$$

Since i is a cofibration, there exists a dotted arrow as indicated. This proves that i is a retract of i' and therefore belongs to $\overline{J}$, as desired. Finally, we only need to proof that this projective model structure on $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is combinatorial, for this, it suffices to show that $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is presentable. Let $\{A_\alpha\}$ be the set of generators of $\mathbf{A}$. We define

$$\mathcal{G}^C_{A_\alpha} : \mathcal{C} \longrightarrow \mathbf{A} \tag{68}$$

by the formula

$$\mathcal{G}^C_{A_\alpha}(C') = \begin{cases} A_\alpha & \text{if } C = C' \\ \emptyset & \text{otherwise} \end{cases} \tag{69}$$

The collection of this functors $\{\mathcal{G}^C_{A_\alpha}\}$ forms a set, and they generate $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ under small colimits. The rest of the properties of a presentable category are directly inherited from the presentability of A.

We now prove the existence of the injective model structure on $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$. Here it is difficult to proceed directly, so we will instead apply Theorem (2.2.4). It will suffice to check each of the hypotheses in turn:

(1) The collection of injective cofibrations in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is generated as a weakly saturated class by some small set of morphisms. This follows from Lemma (3.1.3)

(2) The collection of trivial injective cofibrations in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is weakly saturated: this follows immediately from the fact that the class of trivial cofibrations is weakly saturated in $\mathbf{A}$ by hypothesis, and from Lemma (3.1.3).

(3) The collection of weak equivalences in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is an accessible subcategory of $\mathrm{Fun}(\mathcal{C}, \mathbf{A})^{[1]}$: this follows from the proof of Proposition 5.4.4.3 in [10] since the collection of weak equivalences in A form an accessible subcategory of $\mathbf{A}^{[1]}$. We will not include the proof of this Proposition since it requires techniques beyond the scope of this paper.

(4) The collection of weak equivalences in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ satisfy the two-out-of-three property: this follows immediately from the fact that the weak equivalences in $\mathbf{A}$ satisfy the two-out-of-three property.

(5) Let $f : X \to Y$ be a morphism in $\mathbf{A}$ which has the right lifting property with respect to every injective cofibration. In particular, f has the right lifting property with respect to each of the morphisms in the class I defined above, so that f is a trivial projective fibration and, in particular, a weak equivalence.

It follows that $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ admits a combinatorial model structure defined as the injective model structure as above. $\square$

Remark 3.1.4. It follows from the proof of Theorem (3.1.2) that the class of projective cofibrations is generated (as a weakly saturated class of morphisms) by the maps $j : \mathcal{F}_A^C \to \mathcal{F}_{A'}^C$, where $C \in \mathcal{C}$ and $A \to A'$ is a cofibration in $\mathbf{A}$. We observe that j is an injective cofibration. It follows that every projective cofibration is an injective cofibration; dually, every injective fibration is a projective fibration.

Remark 3.1.5. In the situation of Theorem (3.1.2), if $\mathbf{A}$ is assumed to be right or left proper, then $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is likewise right or left proper (with respect to either the projective or the injective model structures). To see why this is true, in the injective model structure it is clear that if $\mathbf{A}$ is left proper, then $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is left proper; and this also holds for the projective model structure because by the previous remark, every projective cofibration is an injective cofibration. The claim follows similarly for the case of right properness.

Remark 3.1.6. The construction of Theorem (3.1.2) is functorial in the following sense: given a Quillen adjunction of combinatorial model categories $F : \mathbf{A} \rightleftarrows \mathbf{B} : G$ and a small category $\mathcal{C}$, composition with F and G determines a Quillen adjunction

$$F^{\mathcal{C}} : \mathrm{Fun}(\mathcal{C}, \mathbf{A}) \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathbf{B}) : G^{\mathcal{C}} \tag{70}$$

(with respect to either the injective or the projective model structures). Moreover, if (F, G) is a Quillen equivalence, then so is $(F^{\mathcal{C}}, G^{\mathcal{C}})$.

Moreover, because the projective and injective model structure on $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ have the same weak equivalence between then, the identity functor $\mathrm{Id}_{\mathrm{Fun}(\mathcal{C},\mathbf{A})}$ is a Quillen equivalence between them.

Example 3.1.7. Our main example of a diagram category will be the *simplicial presheaf* category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, sSet)$, which we will denote by $sPre(\mathcal{C})$. By Theorem (3.1.2) this category admits the projective and injective model structure. This category with the projective model structure is the main object of study of this paper, since it enjoys a certain universal property as we will see below.

3.2 Directed Categories

The following section serves mostly as a preface to the subsequent section, in which we prove the existence of a model structure on $\mathrm{Fun}(\Delta, \mathcal{M})$, where $\mathcal{M}$ is a model category. The main goal of this section is to provide a proof of Theorem (3.2.4).

The author learned most of the material, all but Lemma (3.2.6), in this section from

- Hovey - Model Categories, 5.1 [9]

Lemma (3.2.6) can be found in [10], in section $A.2.9$.

Recall that an ordinal is defined inductively as the totally ordered set of all smaller ordinals. If λ is an ordinal, we often think of λ as a category where there is one map from α to β if and only if $\alpha \leq \beta$.

Definition 3.2.1. Suppose $\mathcal{J}$ is a small category and λ is an ordinal.

(1) A functor $\mathcal{J} \to \lambda$ is called a *linear extension* if the image of a nonidentity map is a nonidentity map. We then refer to $f(j)$ as the *degree* of j. Note that all nonidentity maps raise the degree.

(2) The small category $\mathcal{J}$ is a *directed category* if there is a linear extension $\mathcal{C} \to \lambda$ for some ordinal λ.

(3) Dually, $\mathcal{J}$ is an *inverse category* if there is a linear extension $\mathcal{J}^{\mathrm{op}} \to \lambda$ for some ordinal λ.

Note that the dual of a direct category is an inverse category, and vice versa. In a directed category or inverse category, there is a kind of induction procedure, controlled by the latching or matching object that we now define.

Definition 3.2.2. Let $\mathcal{J}$ be a directed category, and let $\mathcal{C}$ be a category which admits small colimits, and $X : \mathcal{J} \to \mathcal{C}$ a functors. For every object $j \in \mathcal{J}$ we define the *latching space* functor $L_j(X)$ as follows. Let $\mathcal{J}_j$ be the over category of j of all non-identity maps in $\mathcal{J}$, and define L_j to be the composite

$$L_j : \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{J}_j, \mathcal{C}) \xrightarrow{\mathrm{colim}} \mathcal{C} \tag{71}$$

where the first arrow is the pullback of the canonical forgetful functor $\mathcal{J}_j \to \mathcal{J}$. Note that we have a canonical map $L_j(X) \to X_j$. Similarly, if $\mathcal{J}$ is an inverse category and $\mathcal{C}$ has all small limits, we define the *matching space* functor $M_j(X)$ to be the composite

$$M_j : \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{J}^j, \mathcal{C}) \xrightarrow{\lim} \mathcal{C} \tag{72}$$

where $\mathcal{J}^j$ is the under category of all non-identity maps in $\mathcal{J}$, and the first arrow is the pullback of the canonical forgetful functor $\mathcal{J}^j \to \mathcal{J}$. We have a canonical map $X_j \to M_j(X)$.

Example 3.2.3. Let $\widetilde{X} : \Delta^{\mathrm{op}} \to Set$ be a simplicial set, and let $\mathcal{J}$ be the subcategory of Δ with the same objects as Δ, but the morphisms are the injective morphisms in Δ. Equivalently, the morphisms in $\mathcal{J}$ are the morphisms in Δ generated by the coface maps $d^i : [n-1] \to [n]$. Let $X : \mathcal{J}^{\mathrm{op}} \to Set$ be the induced functors from $\widetilde{X}$. For every nonnegative integer n, the latching object L_nX can be identified with the collection of all degenerate simplices of $\widetilde{X}_n$.

We can use the latching object to define a model category structure on $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ for a directed category $\mathcal{J}$ and a model category $\mathcal{C}$.

Theorem 3.2.4. *Given a model category $\mathcal{C}$ and a directed category $\mathcal{J}$, there is a model structure on* $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$*, defined by*

(W) A map $f : X \to Y$ is a weak equivalence if and only if the map $X_j \to Y_j$ is for all j.

(F) A map $f : X \to Y$ is a fibration if and only if the map $X_j \to Y_j$ is for all j.

(C) A map $f : X \to Y$ is a cofibration if and only if the map the induced map $X_j \amalg_{L_j X} L_j Y \to Y_j$ is a cofibration for all j.

(WC) A map $f : X \to Y$ is a trivial cofibration if and only if the map the induced map $X_j \amalg_{L_j X} L_j Y \to Y_j$ is a trivial cofibration for all j.

Dually, if $\mathcal{J}$ is an inverse category, then we have a model structure on $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$*, defined by*

(W) A map $f : X \to Y$ is a weak equivalence if and only if the map $X_j \to Y_j$ is for all j.

(C) A map $f : X \to Y$ is a cofibration if and only if the map $X_j \to Y_j$ is for all j.

(F) A map $f : X \to Y$ is a fibration if and only if the map the induced map $X_j \to Y_j \times_{M_j Y} M_j X$ is a fibration for all j.

(WF) A map $f : X \to Y$ is a trivial fibration if and only if the map the induced map $X_j \to Y_j \times_{M_j Y} M_j X$ is a trivial fibration for all j.

To proof Theorem (3.2.4), we first prove that the lifting axiom holds. We concentrate on the direct category case, as the inverse category is dual.

Lemma 3.2.5. *Suppose that $\mathcal{J}$ is a directed category, and $\mathcal{C}$ is a model category. Suppose that we have a commutative square in* $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ *as follows,*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd} \tag{73}$$

where p is an objectwise fibration and where the map $g_j : A_j \amalg_{L_j A} L_j B \to B_j$ is a cofibration for all $j \in \mathcal{J}$. Then, if either p_j is a trivial fibration for all j or g_j is a trivial cofibration for all j, there is a lift $B \to X$.

Proof. We will only proof the case when g_i is a trivial cofibration, as the other case is similar. We will show that the required lift exists using transfinite induction. There is a linear extension $d : \mathcal{J} \to \lambda$ for some ordinal λ, and for $\beta \leq \lambda$, we define $\mathcal{J}_{<\beta}$ to be the full subcategory of $\mathcal{J}$ consisting of all j such that $d(j) < \beta$. Similarly, for $Z \in \mathrm{Fun}(\mathcal{J}, \mathcal{C})$, we let $Z_{<\beta}$ be the restriction of Z to $\mathcal{J}_{<\beta}$. We will construct a lift $h_{<\beta}$ in the diagram below,

$$\begin{tikzcd} A_{<\beta} \arrow[r] \arrow[d] & X_{<\beta} \arrow[d] \\ B_{<\beta} \arrow[r] & Y_{<\beta} \end{tikzcd} \tag{74}$$

by transfinite induction on β, such that, for all $\alpha < \beta$, the restriction of $h_{<\beta}$ to $B_{<\alpha}$ is $h_{<\alpha}$. The case $\beta = 0$ is trivial. If β is a limit ordinal and we have constructed $h_{<\alpha}$ for all $\alpha < \beta$, then we define $h_{<\beta}$

on $\mathcal{J}_{<\beta}$ as the map induced by the $h_{<\alpha}$ for $\alpha < \beta$. That is given an $j \in \mathcal{J}$ with $d(j) < \beta$, there is an $\alpha < \beta$ such that $d(j) < \alpha$, so we define $h_{<\beta}$ on X_j to be $h_{<\alpha}$ on X_j.

For the successor ordinal case, suppose we have defined $h_{<\beta}$. Then, for each element j of degree β, we have a commutative diagram as follows

$$\begin{tikzcd} A_j \amalg_{L_jA} L_jB \arrow[r] \arrow[d, "g_j"'] & X_j \arrow[d, "p_j"] \\ B_j \arrow[r] & Y_j \end{tikzcd} \tag{75}$$

where the map $L_jB \to X_j$ is defined using $h_{<\beta}$. Since g_j is a trivial cofibration, we can find a lift in this diagram. Putting these together for the different j of degree β defines an extension $h_{<\beta+1}$ of $h_{<\beta}$ as required. ◻

Lemma 3.2.6. *Let $\mathcal{J}$ be a directed category , let $\mathcal{C}$ be a model category, and let $A \to B$ be a morphisms in* $\mathrm{Fun}(\mathcal{J},\mathcal{C})$*. Let $J \subset \mathcal{J}_{/j}$ be a sieve: that is, J is a full subcategory of $\mathcal{J}_{/j}$ with the property that if $i \to i'$ is a morphism in $\mathcal{J}_{/j}$ such that $i' \in J$, then $i \in J$. Then*

(a) If the map f satisfies condition (C) of Theorem (3.2.4) for every object $i \in J$, then the induced map

$$\chi_J : \mathrm{colim}(A|J) \longrightarrow \mathrm{colim}(B|J) \tag{76}$$

is a cofibration in $\mathcal{C}$

(b) If the map f satisfies condition (WC) of Theorem (3.2.4) for every object $i \in J$, then the map χ_J is a trivial cofibration in $\mathcal{C}$.

We denote by $A|J$ image of A under the pullback functor induced by $J \to \mathcal{J}_{/j} \to \mathcal{J}$.

Proof. We will proof (a); the proof of (b) is identical. We see that the composition $J \to \mathcal{J} \to \lambda$ induces a canonical degree map on J, this is the degree map we will use through. For every triple $\delta \leq \gamma \leq \beta$, let $\chi_{\delta,\gamma,\beta}$ denote the induced map

$$\mathrm{colim}(A|J_{<\beta}) \coprod_{\mathrm{colim}(A|J_{<\delta})} \mathrm{colim}(B|J_{<\delta}) \longrightarrow \mathrm{colim}(A|J_{<\beta}) \coprod_{\mathrm{colim}(A|J_{<\gamma})} \mathrm{colim}(B|J_{<\gamma}) \tag{77}$$

We wish to prove that $\chi_{0,\lambda,\lambda}$ is a cofibration. The proof uses induction on γ. If γ is a limit ordinal, then we can write $\chi_{\delta,\gamma,\beta}$ as the transfinite composition of the maps $\{\chi_{\epsilon,\epsilon+1,\beta}\}_{\delta\leq\epsilon<\gamma}$ which are cofibrations by the inductive hypothesis. We may therefore assume that $\gamma = \gamma_0 + 1$ is a successor ordinal. If $\delta = \gamma$, then $\chi_{\delta,\gamma,\beta}$ is an isomorphism; otherwise, we have $\delta \leq \gamma_0$. In this case, we have

$$\chi_{\delta,\gamma,\beta} = \chi_{\gamma_0,\gamma,\beta} \circ \chi_{\delta,\gamma_0,\beta} \tag{78}$$

Using the inductive hypothesis, we can reduce to the case $\delta = \gamma_0$. The map $\chi_{\gamma_0,\gamma,\beta}$ is a pushout of the map $\chi_{\gamma_0,\gamma,\gamma}$. We are therefore reduced to proving that $\chi_{\gamma_0,\gamma,\gamma}$ is a cofibration. But $\chi_{\gamma_0,\gamma,\gamma}$ is a pushout of the map

$$\coprod_{d(j)=\gamma_0} \left[A_j \coprod_{L_jA} L_jB\right] \longrightarrow \coprod_{d(j)=\gamma_0} B_j \tag{79}$$

for $j \in J$. This map is a cofibration by virtue of our assumption that f satisfies (C). In particular we have for any α that the following map is a cofibration

$$\chi_{0,\alpha,\alpha} : \mathrm{colim}(A|J_{<\alpha}) \longrightarrow \mathrm{colim}(B|J_{<\alpha}) \tag{80}$$

by setting $\alpha = \lambda$ we obtained the desired result. ◻

Proof of Theorem (3.2.4). It suffices to prove the case when $\mathcal{J}$ is direct, since the isomorphism $\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{C}) \cong \mathrm{Fun}(\mathcal{J}, \mathcal{C}^{\mathrm{op}})$ converts the latching space to the matching space. The category $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ has all small colimits and limits, taken objectwise. The two-out-of-three axiom is clear.

For the moment, let us refer to a map $A \to B$ in $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ which has the property that the map $A_j \amalg_{L_j A} L_j B \to B_j$ is a trivial cofibration for all j as a good trivial cofibration. A good trivial cofibration is certainly a cofibration, and we claim that it is also a weak equivalence. Indeed, by Lemma (3.2.6), the map $L_j A \to L_j B$ is a trivial cofibration for all j. It follows that the map $A_j \to A_j \amalg_{L_j A} L_j B$ is also a trivial cofibration. Hence the map $A_j \to B_j$ is a composition of two trivial cofibrations, hence is also a trivial cofibration. Thus every good trivial cofibration is a trivial cofibration. Later in the proof, we will show that the converse is also true.

Since fibrations and weak equivalences are defined objectwise it follows that they are closer under retracts. To see that cofibrations and good trivial fibrations, note that if X is a retract of A, then $L_j X$ is a retract of $L_j A$ for al j in $\mathcal{J}$. From this, we can conclude that if $X \to Y$ is a retract of $A \to B$, then $X_j \amalg_{L_j X} L_j Y \to Y_j$ is a retract of $A_j \amalg_{L_j A} L_j B \to B_j$, it follows that cofibrations and good trivial fibrations are close under retracts. Weak equivalences are closed under retracts because they are defined objectwise and $\mathcal{C}$ is a model category.

Now we construct the functorial factorizations of maps $A \to B$. For concreteness, we will do the factorization into a good trivial cofibration followed by a fibration. The construction of the other factorization is similar. Recall that we have a degree function $d : \mathcal{J} \to \lambda$. We construct compatible functorial factorizations on $\mathrm{Fun}(\mathcal{J}_{<\beta}, \mathcal{C})$ by transfinite induction on $\beta \leq \lambda$, where $\mathcal{J}_\beta$ is the full subcategory of all j such that $d(j) < \beta$. The base case of the induction is $\beta = 1$. Here we use the functorial factorization in $\mathcal{C}$ to factor $A_j \to B_j$ for all j of degree 0. Now suppose we have constructed a functorial factorization on $\mathrm{Fun}(\mathcal{J}_{<\beta}, \mathcal{C})$. We extend this to a functorial factorization on $\mathrm{Fun}(\mathcal{J}_{<\beta+1}, \mathcal{C})$ as follows. Given a map $A \to B$ of diagrams, we have the functorial factorization $A_{<\beta} \to Z_{<\beta} \to B_{<\beta}$. Given an j of degree β, we then have a map $A_j \amalg_{L_j A} L_j Z \to B_j$. To complete the induction, we need to consider limit ordinals β. Suppose we have defined compatible functorial factorizations on $\mathrm{Fun}(\mathcal{J}_{<\gamma}, \mathcal{C})$ for all $\gamma < \beta$. Then they clearly combine to define a functorial factorization on $\mathrm{Fun}(\mathcal{J}_{<\beta}, \mathcal{C})$, as required.

To complete the proof, we must show that every trivial cofibration is a good trivial cofibration. So suppose $f : X \to Y$ is any trivial cofibration. Then we can factor it as $X \to Z \to Y$, where g is a good trivial cofibration and p is a (necessarily trivial) fibration. By lifting in the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \end{tikzcd} \tag{81}$$

we see that f is a retract of g. This implies that f is also a good trivial cofibration. $\square$

3.3 Reedy Categories

In the following section, building on the material of the previous section, we will provide a proof about the existence of a certain model structure on $\mathrm{Fun}(\Delta, \mathcal{M})$ where $\mathcal{M}$ is a model category. This model structure will be called the Reedy model structure. Its main characteristic is that the nature of Δ allows us to construct lifts of maps inductively. Moreover, as we will see, if $\mathcal{M}$ is a combinatorial model category, the Reedy model structure on $\mathrm{Fun}(\Delta, \mathcal{M})$ lies between the projective and injective model structure.

The Reedy model structure will prove to be extremely useful in this paper. In particular, it is a central tool of the proof of the main Theorems in the section of Applications. Moreover, in the next section, it will provide us with a great computational tool to compute the homotopical analogs of colimits. But more on that later.

The author learned this material from the following sources:

- Hovey - Model Categories, 5.2 [9]
- Lurie - Higher Topos Theory, A.2.9 [10]

Definition 3.3.1. A *Reedy category* is a triple $(\mathcal{J}, \mathcal{J}_+, \mathcal{J}_-)$ consisting of a small category $\mathcal{J}$ and two subcategories $\mathcal{J}_+$, and $\mathcal{J}_-$, such that there exists a functor $d : \mathcal{J} \to \lambda$, called a *degree function*, for some ordinal λ, such that every nonidentity map in $\mathcal{J}_+$ raises the degree, every nonidentity map in $\mathcal{J}_-$ lowers the degree, and every map $f \in \mathcal{J}$ can be factored uniquely as $f = g \circ h$, where $h \in \mathcal{J}_-$ and $g \in \mathcal{J}_+$. In particular, $\mathcal{J}_+$ is a direct category and $\mathcal{J}_-$ is an inverse category. By abuse of notation, we often say $\mathcal{J}$ is a Reedy category, leaving the subcategories implicit.

Example 3.3.2. The category Δ is a Reedy category with respect to the subcategories (Δ_+, Δ_-); where a morphism $f : [m] \to [n]$ belongs to Δ_+ if and only if it is injective, and it belongs to Δ_- if and only if it is surjective. Furthermore every morphisms can be factored uniquely as a morphisms in Δ_- followed by a morphisms in Δ_+. It is these properties of Δ which we abstract, to define the notion of a Reedy category.

Remark 3.3.3. We have that Δ is a Reedy category, as is Δ^{op}. Indeed, given any Reedy category $\mathcal{J}$, the category $\mathcal{J}^{\mathrm{op}}$ is also a Reedy category, where $(\mathcal{J}^{\mathrm{op}})_- = (\mathcal{J}_+)^{\mathrm{op}}$ and $(\mathcal{J}^{\mathrm{op}})_+ = (\mathcal{J}_-)^{\mathrm{op}}$.

Definition 3.3.4. Suppose $\mathcal{C}$ is a category with all small colimits and limits, and $\mathcal{J}$ is a Reedy category. For each object j of $\mathcal{J}$, we define the *latching space* functor L_j as the composite

$$\mathrm{Fun}(\mathcal{J}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{J}_+, \mathcal{C}) \xrightarrow{L_j} \mathcal{C} \tag{82}$$

where the latter functors is the latching space functor L_j defined for directed categories in Definition (3.2.2). Similarly, we define the *matching space* functor M_j as the composite

$$\mathrm{Fun}(\mathcal{J}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{J}_-, \mathcal{C}) \xrightarrow{M_j} \mathcal{C} \tag{83}$$

where the latter functor is the matching space functor defined for inverse categories in Definition (3.2.2). Note that we have natural transformations $L_jA \to A_j \to M_jA$ defined for $A \in \mathrm{Fun}(\mathcal{J}, \mathcal{C})$.

Example 3.3.5. Let $X : \Delta^{\mathrm{op}} \to \mathit{Set}$ be a simplicial set and regard Δ^{op} as a Reedy category as above. For every nonnegative integer n, the latching object L_nX can be identified with the collection of all degenerate simplices of X. In particular, the map $L_n(X) \to X_n$ is always a monomorphism.

More generally, we observe that a map of simplicial sets $f : X \to Y$ is a monomorphism if and only if, for every $n \geq 0$, the map

$$X_n \coprod_{L_n X} L_n Y \longrightarrow Y_n \tag{84}$$

is a monomorphisms of sets. The if direction is obvious. For the converse, let us suppose that f is a monomorphism; we must show that if σ is an n-simplex of X such that $f(\sigma)$ is degenerate, then σ is already degenerate. If $f(\sigma)$ is degenerate, then $f(\sigma) = \alpha^* f(\sigma) = f(\alpha^* \sigma)$, where $\alpha : [n] \to [n]$ is a map of linearly ordered sets other than the identity. Since f is a monomorphism, we deduce that $\sigma = \alpha^* \sigma$, so that σ is degenerate, as desired.

Remark 3.3.6. Let be a category with all small colimits and limits. Suppose $\mathcal{J}$ is a Reedy category, with degree function $d : \mathcal{J} \to \lambda$. Define $\mathcal{J}_{<\beta}$, for an ordinal $\beta \leq \lambda$, to be the full subcategory consisting of all j with $d(j) < \beta$. Suppose we have a functor $X : \mathcal{J}_{<\beta} \to \mathcal{C}$. For any j with $d(j) = \beta$, we then have a map $L_j X \to M_j X$. We want to show that the extension of X to a functor $X' : \mathcal{J}_{<\beta+1} \to \mathcal{C}$ is equivalent to a factorization $L_j X \to X'_j \to M_j X$ for all j such that $d(j) = \beta$. Given a nonidentity map $i \to j$, where $d(i)$ and $d(j)$ are both less or equal to β, there is a unique factorization $i \to k \to j$, where $i \to k \in \mathcal{J}_+$ and $k \to j \in \mathcal{J}$. It is then clear how to define the map $X'_i \to X'_j$, as the composite

$$X'_i \longrightarrow M_i X \longrightarrow X_k \longrightarrow L_j X \longrightarrow X'_j \tag{85}$$

Similarly, an extension of a morphisms $X \to Y$ of objects in $\mathrm{Fun}(\mathcal{J}_{<\beta}, \mathcal{C})$ is equivalent to maps $X'_j \to Y'_j$ for $d(j) = \beta$ such that the diagrams

$$\begin{tikzcd} L_j X \arrow[r] \arrow[d] & X'_j \arrow[r] \arrow[d] & M_j X \arrow[d] \\ L_j Y \arrow[r] & Y'_j \arrow[r] & M_j Y \end{tikzcd} \tag{86}$$

are commutative. The situation is even simpler with regard to limit ordinals. If β is a limit ordinal, a functor $X : \mathcal{J}_{<\beta} \to \mathcal{C}$ is equivalent to a collection of compatible functors $X_{<\gamma} : \mathcal{J}_{<\gamma} \to \mathcal{C}$ for all $\gamma < \beta$, and a natural transformation $X \to Y$ is equivalent to a collection of compatible natural transformations $X_{<\gamma} \to Y_{<\gamma}$ for all $\gamma < \beta$.

Theorem 3.3.7. *Let $\mathcal{J}$ be a Reedy category and let $\mathcal{C}$ be a model category. Then there exists a model structure on the category of functors $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ with the following properties:*

(C) A morphism $X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ is a Reedy cofibration if and only if, for every object $J \in \mathcal{J}$, the induced map $X_j \amalg_{L_j X} L_j Y \to Y_j$ is a cofibration in $\mathcal{C}$.

(F) A morphism $X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ is a Reedy fibration if and only if, for every object $j \in \mathcal{J}$, the induced map $X_j \to Y_j \times_{M_j Y} M_j X$ is a fibration in $\mathcal{C}$.

(W) A morphism $X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ is a weak equivalence if and only if, for every $j \in \mathcal{J}$, the map $X_j \to Y_j$ is a weak equivalence.

Moreover, a morphism $f : X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ is a trivial cofibration if and only if the following condition is satisfied:

(WC) For every object $j \in \mathcal{J}$, the map $X_j \amalg_{L_j X} L_j Y \to Y_j$ is a trivial cofibration in $\mathcal{C}$.

Similarly, f is a fibration if and only if it satisfies the dual condition:

(WF) For every object $j \in \mathcal{J}$, the map $X_j \to Y_j \times_{M_j Y} M_j X$ is a trivial fibration in $\mathcal{C}$.

Proof. The category $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ inherits completeness and cocompleteness from $\mathcal{C}$, by taking colimits and limits objectwise. By definition, a map is a cofibration or weak equivalence if and only if it is so in the model category $\mathrm{Fun}(\mathcal{J}_+, \mathcal{C})$ of Theorem (3.2.4). The two-out-of-three axiom for weak equivalences follows immediately, and so does the retract axiom for weak equivalences and cofibration. Similarly, a map is a fibration or weak equivalence if and only if it is so in the model category $\mathrm{Fun}(\mathcal{J}_-, \mathcal{C})$, it follows immediately that fibrations are closed under retracts.

Now suppose we have a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd} \tag{87}$$

where i is a cofibration, p is a fibration, and one of them is trivial. We must construct a lift. We proceed by transfinite induction, by extending the lift $\mathcal{J}_{<\beta}$ to $\mathcal{J}_{<\beta+1}$. The base case for $\beta = 0$ is trivial. By Remark (3.3.6) and Lemma (3.2.5), we know that extending the lift from $\mathcal{J}_{<\beta}$ to $\mathcal{J}_{<\beta+1}$ is equivalent to finding a lift of the diagram below,

$$\begin{tikzcd} A_j \amalg_{L_j A} L_j B \arrow[r] \arrow[d] & X_j \arrow[d] \\ B_j \arrow[r] & Y_j \times_{M_j Y} M_j X \end{tikzcd} \tag{88}$$

for each i of degree β. We can always find such a lift, since the left vertical map is a cofibration, the right vertical map is a fibration, and one of them is a weak equivalence.

Finally, we only need to proof the functorial factorization. Given a map $X \to Y$ we first use the functorial factorization in $\mathcal{C}$ to define $X_j \to Z_j \to Y_j$ for all j of degree 0. The limit ordinal case is clear as pointed out in Remark (3.3.6). For the successor ordinal case, suppose we have defined a partial functorial factorization $X_j \to Z_j \to Y_j$ for all j of degree $< \beta$. An extension of this is equivalent to a functorial factorization of the map

$$X_j \amalg_{L_j X} L_j Z \longrightarrow Y_j \times_{M_j Y} M_j Z \tag{89}$$

for all j of degree β, which we construct by using the functorial factorization in $\mathcal{C}$.

$\square$

Example 3.3.8. Let $\mathcal{J}$ be a Reedy category with $\mathcal{J}_- = \mathcal{J}$ and let $\mathcal{C}$ be a model category. Then we can see that $L_j X = \emptyset$ for all $j \in \mathcal{J}$ and all $X \in \mathrm{Fun}(\mathcal{J}, \mathcal{C})$. It follows that $X_j \amalg_{L_j X} L_j Y = X_j$, in particular it follows that the cofibrations coincide with the injective cofibrations in Definition (3.1.1). Since fibrations can be characterized as the morphisms with the right lifting property with respect to trivial cofibrations, it follows that the Reedy model structure on $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ coincides with the injective model structure of Theorem (3.1.2). In particular, it means we can define a injective model structure without the assumption that $\mathcal{C}$ is combinatorial, but instead we require $\mathcal{J}$ to be Reedy category. Similarly, if $\mathcal{J}_+ = \mathcal{J}$, then we can identify the Reedy model structure on $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ with the projective model structure of Theorem (3.1.2).

In the general case, we can regard the Reedy model structure on $\mathrm{Fun}(\mathcal{J},\mathcal{C})$ as a mixture of the projective and injective model structures. More precisely, we have the following

(i) A morphisms $F \to G$ in $\mathrm{Fun}(\mathcal{J},\mathcal{C})$ satisfies condition (C) of Theorem (3.3.7) if and only if the induced transformation $F|\mathcal{J}_+ \to G|\mathcal{J}_+$ is a projective cofibration in $\mathrm{Fun}(\mathcal{J}_+,\mathcal{C})$.

(ii) A morphisms $F \to G$ in $\mathrm{Fun}(\mathcal{J},\mathcal{C})$ satisfies condition (F) of Theorem (3.3.7) if and only if the induced transformation $F|\mathcal{J}_- \to G|\mathcal{J}_-$ is a injective fibration in $\mathrm{Fun}(\mathcal{J}_-,\mathcal{C})$.

(iii) A morphisms $F \to G$ in $\mathrm{Fun}(\mathcal{J},\mathcal{C})$ satisfies condition (WC) of Theorem (3.3.7) if and only if the induced transformation $F|\mathcal{J}_+ \to G|\mathcal{J}_+$ is a projective trivial cofibration in $\mathrm{Fun}(\mathcal{J}_+,\mathcal{C})$.

(iv) A morphisms $F \to G$ in $\mathrm{Fun}(\mathcal{J},\mathcal{C})$ satisfies condition (WF) of Theorem (3.3.7) if and only if the induced transformation $F|\mathcal{J}_- \to G|\mathcal{J}_-$ is a injective trivial fibration in $\mathrm{Fun}(\mathcal{J}_-,\mathcal{C})$.

Remark 3.3.9. Let $\mathcal{J}$ be a Reedy category and $\mathbf{A}$ a combinatorial model category, so that the injective and projective model structure on $\mathrm{Fun}(\mathcal{J},\mathbf{A})$ we defined independently of Theorem (3.1.2). The identity functor from $\mathrm{Fun}(\mathcal{J},\mathbf{A})$ to itself can be regarded as a left Quillen equivalence from the projective model structure to the Reedy model structure and form the Reedy model structure to the injective model structure.

Example 3.3.10. Let $\mathcal{C}$ be the category of *bisimplicial sets*, which we will identify with $\mathrm{Fun}(\Delta^{\mathrm{op}}, sSet)$ and endow it with the Reedy model structure. It follows from Example (3.3.5) that a monomorphism $f : X \to Y$ of bisimplicial sets is a Reedy cofibration if and only if it is a monomorphism. Consequently, the Reedy model structure on $\mathcal{C}$ coincided with the injective model structure on $\mathcal{C}$.

4 Homotopy Colimits

The author learned the following motivation for homotopy colimits from

- Dugger - Primer on Homotopy colimits [5]

The theory of homotopy colimits arises because of the following basic difficulty. Let I be a small category, and consider two diagrams $D, D' : I \to Top$. If one has a natural transformation $\eta : D \to D'$, then there exists an induced map $\operatorname{colim} D \to \operatorname{colim} D'$. If f is a natural weak equivalence, i.e., if $D(i) \to D'(i)$ is a weak equivalence for all $i \in I$, it unfortunately does not follow that $\operatorname{colim} D \to \operatorname{colim} D'$ is also a weak equivalence. To see this, here is an example:

Example 4.0.1. Let I be the category $\cdot \leftarrow \cdot \rightarrow \cdot$ and let D be the diagram

$$* \longleftarrow S^n \longrightarrow D^{n+1} \tag{90}$$

and let D' be the diagram

$$* \longleftarrow S^n \longrightarrow * \tag{91}$$

Let $\eta : D \to D'$ be the natural weak equivalence which is the identity on S^n and collapses D^{n+1} to a point. Then $\operatorname{colim} D \cong S^{n+1}$ and $\operatorname{colim} D' \cong *$, so the induced map $\operatorname{colim} D \to \operatorname{colim} D'$ is certainly not a weak equivalence.

So the colimit functor does not preserve weak equivalences (one sometimes says that the colimit functor is not homotopy invariant, which means the same thing). The concept of *homotopy colimit* may be thought of as a correction to the colimit, modifying it so that it is homotopy invariant.

There is one simple example of a homotopy colimit that nearly everyone has seen: the mapping cone. We generalize this slightly in the following example, which concerns homotopy pushouts.

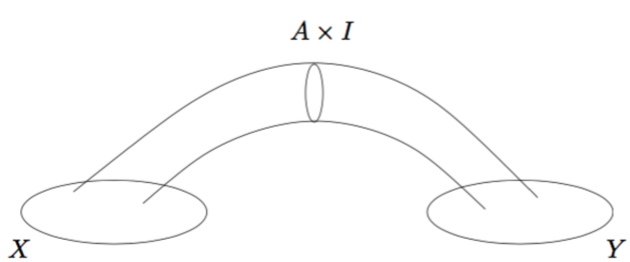
Example 4.0.2. Consider a pushout diagram of spaces $X \xleftarrow{f} A \xrightarrow{g} Y$. Call this diagram D. The pushout of D is obtained by gluing X and Y together along the images of the space A: that is, $f(a)$ is glued to $g(a)$ for every $a \in A$. The homotopy pushout, on the other hand, is constructed by gluing together X and Y up to homotopy. Specifically, we form the following quotient space:

$$\operatorname{hocolim} D = \Big[X \amalg (A \times I) \amalg Y \Big] \sim \tag{92}$$

where the equivalence relations has

$$(a, 0) \sim f(a) \quad \text{and} \quad (a, 1) \sim g(a) \qquad \text{for all } a \in A \tag{93}$$

We can depict this space by the following picture:



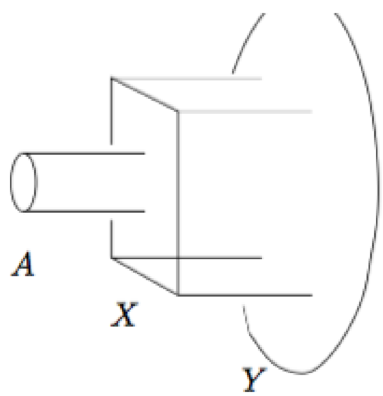
Consider the open cover $\{U, V\}$ of $\operatorname{hocolim} D$ where U is the union of X with the image of $A[0, 3/4]$, and V is the union of Y with the image of $A[1/4, 1]$. Note that U deformation retracts down to X, V deformation retracts down to Y, and that the map $A \to U \cap V$ given by $a \to (a, 1/2)$ is a homotopy equivalence. The Mayer-Vietoris sequence then gives a long exact sequence relating the homology of $\operatorname{hocolim} D$ with $H^*(X)$, $H^*(Y)$, and $H^*(A)$. Similarly, the Van Kampen theorem shows (assuming X, Y, and A are path-connected, for simplicity) that $\pi_1(\operatorname{hocolim} D)$ is the pushout of the diagram of groups $\pi_1(X) \leftarrow \pi_1(A) \to \pi_1(Y)$. The moral is that the space $\operatorname{hocolim} D$ is pretty easy to study using the standard tools of algebraic topology, in contrast to $\operatorname{colim} D$, which is much harder.

Before continuing with the next example, we should relate this past example to the mapping cones. If $f : A \to X$ is a map, then the quotient $X/f(A)$ is the pushout of $* \leftarrow A \to X$. The homotopy pushout of $* \leftarrow A \to X$, as defined above, is nothing other than the mapping cone of f.

Example 4.0.3. Consider the diagram of spaces

$$A \xrightarrow{f} X \xrightarrow{g} Y \tag{94}$$

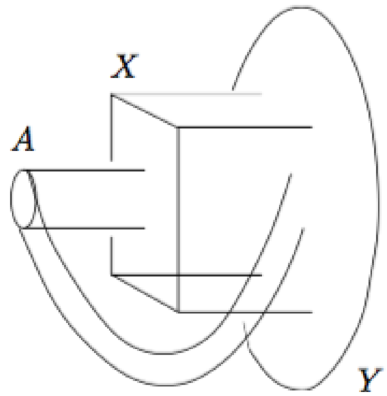
One way to construct the homotopy colimit in this case is as a double mapping cylinder as shown below



This is the space $[(A \times I) \amalg (X \times I) \amalg Y]/\sim$ in which we have identified $(a, 1) \sim (f(a), 0)$ and $(x, 1) \sim g(x)$, for all $a \in A$ and $x \in X$. Note that this space deformation retracts to Y.

Now consider the following. For the colimit of a diagram D, every map $f : D_i \to D_j$ in the diagram tells us to glue $a \in D_i$ to $f(a) \in D_j$. In the homotopy colimit we are supposed to 'glue up to homotopy', and this is what we tried to do in the double mapping cylinder above. But note that we have only done this for f and g, whereas there is a third map in our diagram – namely, the composite $g \circ f$. Maybe we should glue in a homotopy for that map, too.

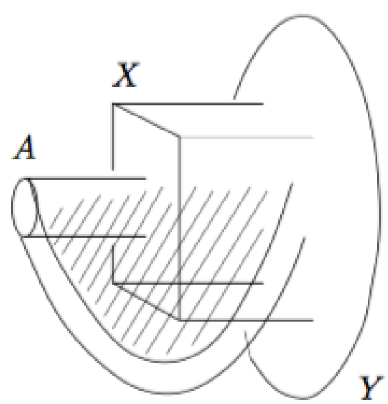
This suggests that we should do the following. Start with $A \amalg X \amalg Y$ and glue in a cylinder for f, g, and $g \circ f$. This gives us the following space, which well call W:



Unfortunately, W is clearly not homotopy equivalent to Y , and therefore not homotopy equivalent to our double mapping cylinder above. But we can fix this as follows. There is an evident map $A\times\partial\Delta^2$ into W: we have an $A\times I$ occurring in the mapping cylinders for f and $g\circ f$, forming two of the sides of $A\times\partial\Delta^2$. The third side comes the composite $A\times I \xrightarrow{g\times \mathrm{Id}} X\times I \longrightarrow W$, where the second map is the mapping cylinder for g. What we will do is take W and attach a copy of $A\times\Delta^2$ along the image of $A\times\partial\Delta^2$; that is, we form the pushout,

$$\begin{array}{ccc} A\times\partial\Delta^2 & \longrightarrow & W \\ \big\downarrow & & \big\downarrow \\ A\times\Delta^2 & \longrightarrow & W' \end{array} \tag{95}$$

It is hard to draw a picture for W', but maybe we can try something like this:



This new space W' is homotopy equivalent to the double mapping cylinder we started with: the cylinder corresponding to $g\circ f$ can be squeezed down into the double mapping cylinder, via the $A\times\Delta^2$ piece we just attached. So W' is another model for the homotopy colimit of our diagram.

The previous example suggests the following. Suppose given a small category I and a diagram $D: I\to Top$. To construct $\operatorname{hocolim} D$ we should start with $\amalg_i D(i)$, and then for every map $f: i\to j$ in I we should glue in a cylinder $D(i)\times\Delta^1$ corresponding to f. Then for every pair of composable maps

$$i \xrightarrow{f} j \xrightarrow{g} k \tag{96}$$

in I we should glue in a copy of $D(i) \times \Delta^2$. Continuing the evident pattern, for every sequence of n composable maps

$$i_0 \to i_1 \to i_2 \to \cdots \to i_n \tag{97}$$

we should glue in a copy of $D(i_0) \times \Delta^n$. Although this is not strictly true in the general case, in the end of the chapter we will proof that in certain cases this construction yields the correct homotopy colimit. The main reason why this does not work in general, is that not ever category has a notion of a Δ^n object, as the category Top has. And as we will see, this is all we need to identify the homotopy colimit with the construction defined above.

Although the discussion presented here gives a very intuitive notion of what the homotopy colimit of a certain diagram should be; in general, we can not provide such a construction for the homotopy colimit of a diagram because not every category is endowed with a operation $- \otimes \Delta^n$. To define homotopy colimits as a standard homotopical generalization of the colimit, we have presented in the first chapter a discussion on Kan extensions. We hope that this discussion motivates the definitions of the homotopy Kan extension, and makes clear how to define an homotopy colimit from here. In the last section, we present a discussion on computational tools used to compute homotopy colimits. In particular, we will present a result that states that one can indeed compute a homotopy colimit following the method described above in certain cases.

4.1 Kan Extensions

Before continuing with the definition of homotopy colimit and homotopy limits, which will be the first application of the injective and projective model structures defined above, we will take a small detour and present the theory of Kan extension. This is because we will define homotopy colimits as a special case of the homotopical analog of Kan extension, surprisingly named homotopical Kan extensions. Our main goal of this section is to present alternate definitions of limits and colimits in the language of Kan extensions, our hope is that this will provide the required motivation such that the definition of homotopy colimits feels like the natural homotopical extension. We will conclude the section with some applications of the theory of Kan extensions, since it provides clarity to many concepts in mathematics, particularly, concepts that we will be using in this paper.

The author learned this material from:

- Riehl- Categorical Homotopy Theory, Chapter 1 [15]
- McLane - Categories for the Working Mathematician, Chapter 10 [11]

Definition 4.1.1. Given functors $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$, a *left Kan extension* of F along K is a functor $\operatorname{Lan}_K F : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta : F \Rightarrow \operatorname{Lan}_K F \circ K$ such that for any other such pairs ($G : \mathcal{D} \to \mathcal{E}$ and $\gamma : F \Rightarrow GK$), we have that γ factors uniquely through η.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dashed, "\operatorname{Lan}_K F"'] & \end{tikzcd} \tag{98}$$

Dually, a *right Kan extension* of F along K is a functor $\operatorname{Ran}_K F : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\varepsilon : \operatorname{Ran}_K F \circ K \Rightarrow F$ such that for any ($G : \mathcal{D} \to \mathcal{E}$ and $\delta : G \circ K \Rightarrow F$), we have that δ factors uniquely through ε.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\varepsilon\Uparrow"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dashed, "\operatorname{Ran}_K F"'] & \end{tikzcd} \tag{99}$$

Remark 4.1.2. A left Kan extension of $F : \mathcal{C} \to \mathcal{E}$ along $K : \mathcal{C} \to \mathcal{D}$ is a representation for the functor

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(F, - \circ K) : \operatorname{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow Set \tag{100}$$

that sends a functor $\mathcal{D} \to \mathcal{E}$ to the set of natural transformations from F to its restriction along K. By the Yoneda lemma, any pair (G, γ) as in Definition (4.1.1) defines a natural transformation

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{D},\mathcal{E})}(G, -) \xRightarrow{\gamma} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(F, - \circ K) \tag{101}$$

The universal property of the pair $(\operatorname{Lan}_K F, \eta)$ is equivalent to the assertion that the corresponding map

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{D},\mathcal{E})}(\operatorname{Lan}_K F, -) \xRightarrow{\eta} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(F, - \circ K) \tag{102}$$

is a natural isomorphism, i.e., that $(\operatorname{Lan}_K F, \eta)$ represents this functor.

Extending this discussion, it follows that if for a fixed K, the left and right Kan extension of any functor $\mathcal{C} \to \mathcal{E}$ exist, then these define left and right adjoints to the pullback functor $K^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$. Particularly, we have the following isomorphisms:

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}(\mathrm{Lan}_K F, G) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(F, G \circ K) \tag{103}$$

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(G \circ K, F) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}(G, \mathrm{Ran}_K F) \tag{104}$$

or equivalently in diagram form

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C},\mathcal{E}) \arrow[r, bend left, "\mathrm{Lan}_K"] \arrow[r, bend right, "\mathrm{Ran}_K"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{Fun}(\mathcal{D},\mathcal{E}) : K^* \arrow[l] \end{tikzcd} \tag{105}$$

The universal properties of Definition (4.1.1) are precicelly thos required to define the value at a particular object $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ of a left and right adjoint to a specified functor, in this case K^*.

Example 4.1.3. We will develop the theory of colimits as a special case of Kan extensions. We will not do the same for limits since it is a completely dual discussion. We hope this serves as motivation to the definition of homotopy colimit presented in the following section. Let $[*]$ denote the final object of Cat: that is, the category with one object and only the identity morphism. For any category $\mathcal{C}$, there is a unique functor $Q : \mathcal{C} \to [*]$. If the left Kan extension of $F : \mathcal{C} \to \mathcal{E}$ along Q exists, it follows that

$$\mathrm{Hom}_{\mathrm{Fun}([*],\mathcal{E})}(\mathrm{Lan}_Q F, G) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(F, G \circ Q) \tag{106}$$

Since $\mathrm{Fun}([*], \mathcal{E}) \cong \mathcal{E}$ and $G(*) \in \mathcal{E}$, we obtain the following identity

$$\mathrm{Hom}_{\mathrm{Fun}([*],\mathcal{E})}(\mathrm{Lan}_Q F, G) \cong \mathrm{Hom}_{\mathcal{E}}(\mathrm{Lan}_Q F, G(*)) \tag{107}$$

By the universal property of $\operatorname{colim} F$ it follows that $\mathrm{Lan}_Q F \cong \operatorname{colim} F$. Or we can represent it in diagram form which makes it easier to visualize

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "Q"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\ & {[*]} \arrow[ur, dashed, "\mathrm{Lan}_Q F"'] & \end{tikzcd} \tag{108}$$

Were the pair (Lan_Q, η) satisfy the universal property of the colimit.

Finally we would like to to provide conditions under which we know the existence of Kan extensions, and moreover we will provide a formula to compute said Kan extension. As a result, we will see that the formula for the functor $Re : Pre(\mathcal{C}) \to \mathcal{D}$ given at the end of the section of presheaves is a special case of this more general framework. But first, we will need some definitions

Definition 4.1.4. The bifunctor $- \otimes - : \mathcal{C} \times Set :\to \mathcal{C}$ is called *tensor* or *copower*. If S is a set and $c \in \mathcal{C}$ then $c \otimes S$ is the S-indexed coproduct of copies of c. In particular this always exists if $\mathcal{C}$ is cocomplete. Dually, the *power* or *cotensor* c^S of $c \in \mathcal{C}$ by a set S is the S-indexed product of copies of c, this also defines a bifunctor $\mathcal{C} \times Set^{\mathrm{op}} \to \mathcal{C}$ that is contravariant in the indexing set.

Definition 4.1.5. The integral $\int^{\mathcal{C}}$ is called a *coend* is the colimit of a particular diagram constructed from a functor that is both covariant and contravariant in $\mathcal{C}$. Given $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ the coend $\int^{C} H$ is the coequalizer of the diagram

$$\int^{\mathcal{C}} H = \mathrm{coeq}\left[\coprod_{f\in \mathrm{mor}\,\mathcal{C}} H(\mathrm{cod}\, f, \mathrm{dom}\, f) \overset{f^*}{\underset{f_*}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} H(c,c)\right] \tag{109}$$

where the maps f_* and f^* are induced by a function $f : c \to c'$ and are defined by $f_* : H(c', c) \to H(c', c')$, and $f^* : H(c', c) \to H(c, c)$. Dually, an *end* is a equalizer of the following diagram

$$\int_{\mathcal{C}} H = \mathrm{eq}\left[\coprod_{f\in \mathrm{mor}\,\mathcal{C}} H(\mathrm{cod}\, f, \mathrm{dom}\, f) \overset{f^*}{\underset{f_*}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} H(c,c)\right] \tag{110}$$

Remark 4.1.6. If $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is constant in the first variable, then the coend $\int^{\mathcal{C}} H$ coincided with the colimit of a functor $H' : \mathcal{C} \to \mathcal{D}$ that coincides with H in the second variable. Dually, if H is constant in the first variable, then the end $\int_{\mathcal{C}} H$ it coincides with the functor $H' : \mathcal{C} \to \mathcal{D}$.

Example 4.1.7. Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be a cocomplete category. Recall from the section of Presheaves that for a functor $\Gamma : \mathcal{C} \to \mathcal{D}$ there exists an functor $Re : Pre(\mathcal{C}) \to \mathcal{D}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{Y}"] \arrow[dr, "\Gamma"'] & Pre(\mathcal{C}) \arrow[d, dashed, "Re"] \\ & \mathcal{D} \end{tikzcd} \tag{111}$$

where for a presheaf F, the action of Re on F can be described as $Re(F) = \int^{C} F \otimes \Gamma$, this follows directly from the definition of coend and the discussion at the end the the section of Presheaves. Where $F \otimes \Gamma : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a bifunctor with the desired properties, defined by the tensor $F(c') \otimes \Gamma(c)$, this makes sense since $F(c')$ is a set.

Theorem 4.1.8. *When $\mathcal{C}$ is small, $\mathcal{D}$ is locally small, and $\mathcal{E}$ is cocomplete, the left Kan extension of any functor $F : \mathcal{C} \to \mathcal{E}$ along any functor $K : \mathcal{C} \to \mathcal{D}$ is computed at $d \in \mathcal{D}$ by the colimit*

$$\mathrm{Lan}_K F(d) = \int^{\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(K-, d) \otimes F(-) \tag{112}$$

and in particular necessarily exists. Dually, if $\mathcal{E}$ is complete, the right Kan extension of any functor $F : \mathcal{C} \to \mathcal{E}$ along any functor $K : \mathcal{C} \to \mathcal{D}$ is computed at $d \in \mathcal{D}$ by the limit

$$\mathrm{Ran}_K F(d) = \int_{\mathcal{C}} F(-)^{\mathrm{Hom}_{\mathcal{D}}(d, K-)} \tag{113}$$

and in particular necessarily exists.

Proof. Let $L : \mathcal{D} \to \mathcal{E}$ be the functor defined pointwise by $\int^{\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(K-, d) \otimes F(-)$. It suffices to show that for any functor $G : \mathcal{D} \to \mathcal{E}$ we have the following isomorphism

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{E})}(F, G \circ K) \cong \mathrm{Hom}_{\mathsf{Fun}(\mathcal{D},\mathcal{E})}(L, G) \tag{114}$$

For this, we first note that for any two functors $G, H : \mathcal{D} \rightrightarrows \mathcal{E}$ we have the following identity

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{D},\mathcal{E})}(G, H) \cong \int_{\mathcal{D}} \mathrm{Hom}_{\mathcal{E}}(G-, H-) \tag{115}$$

this follows by construction. Now we can write the following succession of isomorphisms

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}(L,G) \cong \int_{\mathcal{D}} \mathrm{Hom}_{\mathcal{E}}(L-, G-) \tag{116}$$

$$\cong \int_{\mathcal{D}} \mathrm{Hom}_{\mathcal{E}}\left(\int^{\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(K-,-) \otimes F-, G-\right) \tag{117}$$

$$\cong \int_{\mathcal{D}} \int_{\mathcal{C}} \mathrm{Hom}_{\mathcal{E}}\left(\mathrm{Hom}_{\mathcal{D}}(K-,-) \otimes F-, G-\right) \tag{118}$$

$$\cong \int_{\mathcal{C}} \int_{\mathcal{D}} \mathrm{Hom}_{\mathcal{E}}\left(\mathrm{Hom}_{\mathcal{D}}(K-,-), \mathrm{Hom}_{\mathcal{E}}(F-, G-)\right) \tag{119}$$

$$\cong \int_{\mathcal{C}} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}\left(\mathrm{Hom}_{\mathcal{D}}(K-,-), \mathrm{Hom}_{\mathcal{E}}(F-, G-)\right) \tag{120}$$

$$\cong \int_{\mathcal{C}} \mathrm{Hom}_{\mathcal{E}}(F-, G \circ K-) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(F, G \circ K) \tag{121}$$

The first result regarding formula of left Kan extensions follows. We will not proof the dual result about right Kan extensions since it follows by exactly the same logic. □

4.2 Homotopy Kan Extensions

In this section our goal is to define the homotopical analog of Kan extensions. The theory of homotopy Kan extensions is the first application we see of the projective and injective model structures developed in the previous chapter. Although the definition provided here might feel very different from the intuition provided at the beginning of the chapter, we will see that it is the most natural definition based on the discussion on the previous section.

The author learned this material from the material from:

- Lurie - Higher Topos Theory, A.2.8 [10]

It is important to note that our definition of homotopy Kan extension is slightly weaker than the one provided in [10]. We feel that this definition is more natural and provides the flexibility needed so that in the next chapter we can identify homotopy colimits with the construction presented in the introduction of the chapter.

Let $K : \mathcal{C} \to \mathcal{C}'$ be a functor between small categories and let $\mathbf{A}$ be a combinatorial model category. Then composition with K yields a pullback functor $K^* : \mathrm{Fun}(\mathcal{C}', \mathbf{A}) \to \mathrm{Fun}(\mathcal{C}, \mathbf{A})$. Since $\mathbf{A}$ is complete and cocomplete, by Theorem (4.1.8) there exists functors $\mathrm{Lan}_K \mathrm{Fun}(\mathcal{C}, \mathbf{A}) \to \mathrm{Fun}(\mathcal{C}', \mathbf{A})$ and $\mathrm{Ran}_K \mathrm{Fun}(\mathcal{C}, \mathbf{A}) \to \mathrm{Fun}(\mathcal{C}', \mathbf{A})$. By Remark (4.1.2) it follows that Lan_K is a left adjoint to K^*, and Ran_K is a right adjoint to K^*.

Proposition 4.2.1. *Let $\mathbf{A}$ be a combinatorial model category and let $K : \mathcal{C} \to \mathcal{C}'$ be a functor between small categories. Then*

(1) The pair (Lan_K, K^*) determined a Quillen adjunction between the projective model structures of $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ and $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$

(2) The pair (K^*, Ran_K) determined a Quillen adjunction between the injective model structures of $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ and $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$

Proof. This follows from the simple observation that K^* preserves weak equivalences, projective fibrations and injective cofibrations. $\square$

Let $\mathbf{A}$ be a combinatorial model category and let $K : \mathcal{C} \to \mathcal{C}'$ be a functor between small categories. We wish to consider the left derived functor $L\,\mathrm{Lan}_K$ of the left Kan extension $K_* : \mathrm{Fun}(\mathcal{C}, \mathbf{A}) \to \mathrm{Fun}(\mathcal{C}', \mathbf{A})$. This derived functors is called the *homotopy left Kan extension* functors. The usual way of defining the left derived functor involves choosing a cofibrant replacement functor $Q : \mathrm{Fun}(\mathcal{C}, \mathbf{A}) \to \mathrm{Fun}(\mathcal{C}, \mathbf{A})$ and setting $L\,\mathrm{Lan}_K = \mathrm{Lan}_K \circ Q$. But, since intrinsically homotopy Kan extensions, as every homotopical universal construction, are supposed to be only defined up to weak equivalence, it is sometimes useful to make the extra freedom of choosing any weakly equivalent object explicit by the following definition. In particular, it the following definition takes care of the homotopy invariant problem mentioned above in a very transparent way.

Definition 4.2.2. Let $\mathcal{C}$ be a small category, and $\mathbf{A}$ be a combinatorial model category. Let $F \in \mathrm{Fun}(\mathcal{C}, \mathbf{A})$, and let $G \in \mathrm{Fun}(\mathcal{C}', \mathbf{A})$. We will say that G *is a homotopy left Kan extension of* F if for some weak equivalence $F' \to F$ where F' is projectively cofibrant in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$, the objects $\mathrm{Lan}_K F'$ and G of $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$ are isomorphic in the homotopy category of $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$. Since Lan_K preserves weak equivalences between projectively cofibrant objects, this condition is independent of the choice of F'.

Dually, let $F \in \mathrm{Fun}(\mathcal{C}, \mathbf{A})$, we will say that G *is a homotopy right Kan extension of* F is for some weak equivalence $F \to F'$ where F' is injectively fibrant in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$, the objects $\mathrm{Ran}_K F'$ and G of $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$ are isomorphic in the homotopy category of $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$. Since Ran_K preserves weak equivalences between injectively fibrant objects, this condition is independent of the choice of F'.

Remark 4.2.3. In the definition of homotopy left Kan extension, we can see it is unnecessary to pick a projective cofibrant replacement $F' \to F$, since F and F' will be in the same isomorphism class in the homotopy category. There are two main reasons why we have chosen to add detail into the definition of homotopy lefty Kan extension. The first reason, is that it becomes more transparent why our definition and the usual definition as a left derived functor of the left Kan extension coincide. The second reason is that it becomes clear why Lan_K is an homotopy invariant functor. Indeed, since Lan_K is a left Quillen functor, it preserves trivial cofibrations, therefore it preserve weak equivalences between cofibrant objects. Moreover, since the cofibrant replacement functor preserve weak equivalences, it follows that Lan_K is in fact homotopy invariant.

Definition 4.2.4. Let $[*]$ denote the final object of Cat: that is, the category with one object and only the identity morphisms. For any category $\mathcal{C}$, there is a unique functor $K : \mathcal{C} \to [*]$. If $\mathbf{A}$ is a combinatorial model category, $F : \mathcal{C} \to \mathbf{A}$ functor and $A \in \mathbf{A} \cong \mathrm{Fun}([*], \mathbf{A})$ is an object, then we will say that K^*A *is an homotopy colimit of* F if A is an homotopy left Kan extension of F. Dually, K^*A *is an homotopy limit of* F if A is an homotopy right Kan extension of F.

Remark 4.2.5. Recall that the colimit of a diagram $J \to \mathcal{C}$ is only well defined up to isomorphisms, an analogous is true for homotopy colimits. Homotopy colimits are only well defined up to isomorphism in the homotopy category.

4.3 Computational Techniques

The main goal of this section is to prove that in a particular setting the definition of homotopy colimit and the construction presented in the introduction of the chapter coincide. This is Proposition (4.3.11), and the result will be heavily used in the next chapter, in particular to show that every simplicial presheaf is an homotopy colimit of representables. The section starts by presenting some categorical definitions that the reader may not have seen before; but, we have assumed that the reader is familiar with the definition of "monoidal categories" and "enriched categories". The author decided to not include these definitions because they can be easily found elsewhere, and it will distract us from our goals. Good sources for this material are:

- Lurie - Higher Topos Theory, A.1.3 [10]
- Lurie - Higher Topos Theory, A.1.4 [10]

The author learned the material from the following resources:

- Lurie - Higher Topos Theory, A.2.9 [10]
- Lurie - Higher Topos Theory, A.3.1 [10]

We begin with some definitions that we will need.

Definition 4.3.1. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be model categories. We will say that a functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a *left Quillen bifunctor* if the following conditions are satisfied:

(a) Let $i : A \to A'$ and $j : B \to B'$ be cofibrations in $\mathcal{A}$ and $\mathcal{B}$, respectively. Then the induced map

$$i \wedge j : F(A', B) \coprod_{F(A,B)} F(A, B') \longrightarrow F(A', B') \tag{122}$$

is a cofibration in $\mathcal{C}$. Moreover, if either i or j is a trivial cofibration, then $i \wedge j$ is also a trivial cofibration.

(b) The functor F preserves small colimits separately in each variable

Remark 4.3.2. Let $\emptyset_{\mathcal{A}}$ be the initial object of $\mathcal{A}$, and denote the initial object of $\mathcal{B}$ and $\mathcal{C}$ similarly. We note that condition (b) implies that $F(\emptyset_{\mathcal{A}}, B) = F(A, \emptyset_{\mathcal{B}}) = \emptyset_{\mathcal{C}}$.

Definition 4.3.3. A *monoidal model category* is a monoidal category $\mathcal{C}$ equipped with a model structure, which satisfies the following conditions:

(i) The tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a left Quillen bifunctor

(ii) The unit object $1 \in \mathcal{C}$ is cofibrant.

(iii) The monoidal model structure on $\mathcal{C}$ is closed.

Example 4.3.4. The category of simplicial sets *sSets* is a symmetric monoidal model category with respect to the cartesian product and the Kan model structure defined in Proposition (2.3.4). A complete proof can be found in [9] as Proposition 4.2.8. A more gentle proof can also be found in [8].

Definition 4.3.5. Let $\mathcal{D}$ be a monoidal model category. A *$\mathcal{D}$-enriched model category* is an $\mathcal{D}$-enriched category $\mathcal{C}$ equipped with a model structure satisfying the following conditions:

(1) The category $\mathcal{C}$ is tensored and cotensored over $\mathcal{D}$.

(2) The tensor product $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ is a left Quillen bifunctor.

In the special case when $\mathcal{D}$ is the category of simplicial sets, we will simply refer to $\mathcal{C}$ as a simplicial model category.

Remark 4.3.6. From the fact that $\mathcal{C}$ is tensored and cotensored over $\mathcal{D}$, it follows that condition (2) is equivalent to either of the following statements:

(2′) Given any cofibration $i : C \to C'$ in $\mathcal{C}$ and any fibration $j : X \to Y$ in $\mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(C', X) \to \mathrm{Hom}_{\mathcal{C}}(C, X) \times_{\mathrm{Hom}_{\mathcal{C}}(C,Y)} \mathrm{Hom}_{\mathcal{C}}(C', Y) \tag{123}$$

is a fibration in $\mathcal{D}$, which is trivial if either i or j is a weak equivalence.

(2″) Given any cofibration $i : D \to D'$ in $\mathcal{D}$ and any fibration $j : X \to Y$ in $\mathcal{C}$, the induced map

$$X^{D'} \to X^{C} \times_{Y^C} Y^{C'} \tag{124}$$

is a fibration in $\mathcal{C}$, which is trivial if either i or j is trivial.

Now we present the connection between the definition of homotopy colimits presented in the previous section, and the discussion presented at the beginning of the chapter.

Suppose that we are given a bifunctor

$$\otimes : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C} \tag{125}$$

where $\mathcal{C}$ is a complete and cocomplete category. For any small category $\mathcal{J}$, recall we can define the *coend* functor $\int^{\mathcal{J}} \mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{A}) \times \mathrm{Fun}(\mathcal{J}, \mathcal{B}) \to \mathcal{C}$ so that the integral $\int^{\mathcal{J}} F \otimes G$ is defined to be the coequalizer of the diagram

$$\textstyle\coprod_{j' \to j} F(j) \otimes G(j') \rightrightarrows \coprod_j F(j) \otimes G(j) \tag{126}$$

We then have the following result:

Proposition 4.3.7. *Let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a left Quillen bifunctor and let $\mathcal{J}$ be a Reedy category. Then the coend functor*

$$\int^{\mathcal{J}} \mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{A}) \times \mathrm{Fun}(\mathcal{J}, \mathcal{B}) \longrightarrow \mathcal{C} \tag{127}$$

is also a left Quillen bifunctor, where we regard $\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{A})$ *and* $\mathrm{Fun}(\mathcal{J}, \mathcal{B})$ *as endowed with the Reedy model structure.*

Proof. Let $f : F \to F'$ be a Reedy cofibration in $\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{A})$ and $g : G \to G'$ a Reedy cofibration in $\mathrm{Fun}(\mathcal{J}, \mathcal{B})$. Set

$$C = \int^{\mathcal{J}} F \otimes G' \coprod_{\int^{\mathcal{J}} F \otimes G} \int^{\mathcal{J}} F' \otimes G \tag{128}$$

and

$$C' = \int^{\mathcal{J}} F' \otimes G' \tag{129}$$

We wish to show that the induced map $C \to C'$ is a cofibration, which is trivial if either f or g are trivial.

Choose a degree function $d : \mathcal{J} \to \lambda$. For $\beta \leq \lambda$ we define

$$C_\beta = \int^{\mathcal{J}_{<\beta}} F_{<\beta} \otimes G'_{<\beta} \coprod_{\int^{\mathcal{J}_{<\beta}} F_{<\beta}\otimes G_{<\beta}} \int^{\mathcal{J}_{<\beta}} F'_{<\beta} \otimes G_{<\beta} \tag{130}$$

and

$$C'_\beta = \int^{\mathcal{J}_{<\beta}} F'_{<\beta} \otimes G'_{<\beta} \tag{131}$$

we wish to show that the map

$$C \cong C_\lambda \coprod_{C_0} C'_0 \longrightarrow C_\lambda \coprod_{C_\lambda} C'_\lambda \cong C' \tag{132}$$

is a cofibration (which is trivial if either f or g is trivial). We will prove more generally that for $\delta \leq \gamma \leq \beta \leq \lambda$, the map

$$\eta_{\delta,\gamma,\beta} : C_\beta \coprod_{C_\delta} C'_\delta \longrightarrow C_\beta \coprod_{C_\gamma} C'_\gamma \tag{133}$$

is a cofibration (which is trivial if either f or g is trivial). The proof proceeds by induction on γ. If γ is a limit ordinal, then $\eta_{\delta,\gamma,\beta}$ can be obtained as a transfinite composition of maps $\{\eta_{\epsilon,\epsilon+1,\beta}\}_{\delta\leq\epsilon<\gamma}$, and the result follows form the inductive hypothesis. We may therefore assume that $\gamma = \gamma_0 + 1$ is a successor ordinal. Since $\eta_{\delta,\gamma,\beta} = \eta_{\gamma_0,\gamma,\beta} \circ \eta_{\delta,\gamma_0,\beta}$ we can use the inductive hypothesis to reduce to the case where $\delta = \gamma_0$. Since $\eta_{\delta,\gamma,\beta}$ is a pushout of $\eta_{\delta,\gamma,\gamma}$ we can also assume that $\beta = \gamma$. In other words, we are reduced to proving that the map

$$h : C_{\gamma_0+1} \coprod_{C_{\gamma_0}} C'_{\gamma_0} \longrightarrow C'_{\gamma_0} \tag{134}$$

is a cofibration, which is trivial if either f or g is trivial. Let j be the an object if $\mathcal{J}$ such that $d(j) = \gamma_0$. From a pushout diagram

$$\begin{tikzcd}
(F(j) \amalg_{L_jF} L_jF') \otimes (G(j) \amalg_{L_jG} L_jG') \arrow[r] \arrow[d] & (F(j) \amalg_{L_jF} L_jF') \otimes G'(j) \arrow[d] \\
F'(j) \otimes (G(j) \amalg_{L_jG} L_jG') \arrow[r] & X_j
\end{tikzcd} \tag{135}$$

We have an evident map $h'_j : X_j \to F'(j) \otimes G'(j)$ which is a cofibrations (trivial is either f or g is trivial) by virtue of our assumptions on f and g (together with the fact that $\otimes$ is a left Quillen bifunctor). We conclude by observing that h is a pushout of $\amalg_{d(j)=\gamma_0} h'_j$. □

Remark 4.3.8. Proposition (4.3.7) has an analog for the model structure introduced in Theorem (3.1.2). That is, suppose that $\mathcal{A}$ and $\mathcal{B}$ are combinatorial model categories and let $\mathcal{J}$ be an arbitrary small category. Then any left Quillen bifunctor $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces a left Quillen bifunctor

$$\int^{\mathcal{J}} \mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{A}) \times \mathrm{Fun}(\mathcal{J}, \mathcal{B}) \longrightarrow \mathcal{C} \tag{136}$$

where we regard $\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{A})$ as endowed with the injective model structure and $\mathrm{Fun}(\mathcal{J}, \mathcal{B})$ with the projective model structure. To see this, we must show that for any injective cofibration $f : F \to F'$

in $\mathrm{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{A})$ and any projective cofibration $g : G \to G'$ in $\mathrm{Fun}(\mathcal{J}, \mathcal{B})$, the induced map

$$h : \int^{\mathcal{J}} F \otimes G' \coprod_{\int^{\mathcal{J}} F \otimes G} \int^{\mathcal{J}} F' \otimes G \longrightarrow \int^{\mathcal{J}} F' \otimes G' \tag{137}$$

is a cofibration in $\mathcal{C}$ which is trivial if either f or g is trivial. Without loss of generality, we may suppose that g is a generating projective cofibration of the form $\mathcal{G}^j_B \to \mathcal{G}^j_{B'}$, associated to an object $j \in \mathcal{J}$ and a cofibration $B \to B'$ in $\mathcal{B}$, which is trivial if g is trivial (see proof of Theorem (3.1.2) for an explanation of this notation). Unwinding the definitions, we can see that

$$\int^{\mathcal{J}} F \otimes F' \cong F(j) \otimes B' \tag{138}$$

and more generally we can identify h with the map

$$F(j) \otimes B' \coprod_{F(j) \otimes B} F'(j) \otimes B \longrightarrow F'(j) \otimes B' \tag{139}$$

Since $B \to B'$ is a cofibration in $\mathcal{B}$ and the map $F(j) \to F'(j)$ is a cofibration in $\mathcal{A}$, we deduce that h is a cofibration in $\mathcal{C}$ (since $\otimes$ is left Quillen bifunctor) which is trivial if either i or h is trivial.

Example 4.3.9. Let $\mathcal{A}$ be a simplicial model category, so that we have the left Quillen bifunctor

$$\otimes : \mathcal{A} \times sSet \longrightarrow \mathcal{A} \tag{140}$$

The coend construction determines a left Quillen bifunctor

$$\int^{\Delta^{\mathrm{op}}} \mathrm{Fun}(\Delta, \mathcal{A}) \times \mathrm{Fun}(\Delta^{\mathrm{op}}, sSet) \longrightarrow \mathcal{A} \tag{141}$$

where $\mathrm{Fun}(\Delta, \mathcal{A})$ and $\mathrm{Fun}(\Delta^{\mathrm{op}}, sSet)$ are both endowed with the Reedy model structure. In particular, if we fix a cosimplicial object $X^\bullet \in \mathrm{Fun}(\Delta, \mathcal{A})$ which is Reedy cofibrant, then forming the coend agains $X^\bullet$ determines a left Quillen functor from the category of bisimplicial sets (with the Reedy model structure, which coincides with the injective model structure by Example (3.3.10)) to $\mathcal{A}$.

Example 4.3.10. Let $\mathcal{A}$ be a simplicial model category, so that we have a left Quillen bifunctor

$$\otimes : \mathcal{A} \times sSet \longrightarrow \mathcal{A} \tag{142}$$

and consider the coend functor

$$\int^{\Delta} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \times \mathrm{Fun}(\Delta, sSet) \longrightarrow \mathcal{A} \tag{143}$$

Let $\Delta^\bullet \in \mathrm{Fun}(\Delta, sSet)$ denote the standard simplex (that is, the functor $[n] \to \Delta^n$) and let $[*]$ denote the final object on $\mathrm{Fun}(\Delta, sSet)$ (that is, the constant functor given by $[n] \to \Delta^0$. The unique map $\Delta^\bullet \to [*]$ is a weak equivalence, and $\Delta^\bullet$ is Reedy cofibrant, we may therefore regard $\Delta^\bullet$ as a cofibrant replacement of the constant functor $[*]$.

The functor $X_\bullet \to \int^\Delta X_\bullet \otimes [*]$ can be identified with the colimit functor $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \to \mathcal{A}$. This is a left Quillen functor if $\mathrm{Fun}(\Delta, sSet)$ is endowed with the injective model structure but not the Reedy model structure. However, the *geometric realization* functor $X_\bullet \to |X_\bullet| = \int^\Delta X_\bullet \otimes \Delta^\bullet$ is a left Quillen functor with respect to the Reedy model structure.

Proposition 4.3.11. *Let $\mathcal{A}$ be a combinatorial simplicial model category and let $X_\bullet$ be a simplicial object of $\mathcal{A}$. There is a canonical map*

$$\gamma : \operatorname{hocolim} X_\bullet \longrightarrow |X_\bullet| \tag{144}$$

in the homotopy category of $\mathcal{A}$. This map is an isomorphism in the homotopy category if $X_\bullet$ is Reedy cofibrant.

Proof. Let $\Delta^\bullet$ and $[*]$ be cosimplicial objects in $sSet$ described in Example (4.3.10). Choose a weak equivalence of simplicial objects $X'_\bullet \to X_\bullet$, where $X'_\bullet$ is projectively cofibrant. We then have

$$\operatorname{hocolim} X_\bullet \cong \operatorname{colim} X'_\bullet \cong \int^\Delta X'_\bullet \otimes [*] \tag{145}$$

And also the following diagram

$$\int^\Delta X'_\bullet \otimes [*] \xleftarrow{\ \alpha\ } \int^\Delta X'_\bullet \otimes \Delta^\bullet \xrightarrow{\ \beta\ } \int^\Delta X_\bullet \otimes \Delta^\bullet \tag{146}$$

Since $X'_\bullet$ is projectively cofibrant, Remark (4.3.8) implies that the coend functor $\int^\Delta X'_\bullet \otimes -$ preserves weak equivalences between injectively cofibrant cosimplicial objects of $sSet$; in particular α is a weak equivalence in $\mathcal{A}$. This gives the desired map γ in the homotopy category. Proposition (4.3.7) implies that $\int^\Delta - \otimes \Delta^\bullet$ preserves weak equivalences between Reedy cofibrant simplicial objects of $\mathcal{A}$, which proves that γ is an isomorphisms in the homotopy category if $X_\bullet$ is Reedy cofibrant. □

Example 4.3.12. In particular, if $\mathcal{A}$ is the category of simplicial sets $sSets$, then the map γ of Proposition (4.3.11) is always a isomorphisms in the homotopy category; this follows from the fact that every bisimplicial set is Reedy cofibrant, as mentioned in Example (3.3.10).

Moreover, if $X_\bullet$ is a bisimplicial set, we claim $|X_\bullet| \cong \operatorname{diag}(X_\bullet)$; where $\operatorname{diag}(X_\bullet)$ is the diagonal simplicial set $[n] \to X_n[n]$. Too see this, we compute the geometric realization objectwise

$$X_k[k] \otimes [0 \to 1 \to \cdots \to k] = \operatorname{coeq}\left[\coprod_{m\to n} X_n[k] \otimes \Delta^m[k] \rightrightarrows \coprod_n X_n[k] \otimes \Delta^n[k]\right] \tag{147}$$

Putting this together it follows that $|X_\bullet| \cong \operatorname{diag}(X_\bullet)$. We have developed a simple computational tool to compute homotopy colimits of diagrams in $\operatorname{Fun}(\Delta^{\mathrm{op}}, sSet)$.

5 Applications

In the following chapter we will finally be able to use all of the machinery that we have developed so far to prove the following results:

(1) Every simplicial presheaf is a homotopy colimit of representables with respect to the projective model structure

(2) The category of simplicial presheaves of $\mathcal{C}$ endowed with the projective model structure is, in some sense, the universal model category related to $\mathcal{C}$.

These are, of course, analogous homotopical results to the ones mentioned in the section of Presheaves. Motivated by (2), we will denote the category $sPre(\mathcal{C})$ endowed with the projective model structure by $U\mathcal{C}$. In this chapter we will need the fact that $U\mathcal{C}$ is a simplicial model category, but we will not prove this fact.

In the following introduction, we aim to continue our discussion about why simplicial presheaves are interesting. For this, will aim to at least provide a partial answer to the question of how we can use this notion of simplicial presheaves to study homotopy category of $sShv(Sm/S)_{\mathbb{A}^1}$. Recall that there exists a model structure on $sShv(Sm/S)$, where Sm/S denotes the category of smooth schemes (of finite type), over a base scheme S, endowed with the Nisnevick topology. This model structure is due to Morel and Voevodsky [12]. The underlying idea is that it should be possible to develop a purely algebraic approach to homotopy theory by replacing the unit interval $[0,1]$, which is not an algebraic variety, with the affine line $\mathbb{A}^1$. We denote by $sShv(Sm/S)_{\mathbb{A}^1}$ the category $sShv(Sm/S)$ endowed with this model structure. We call this model structure the motivic model structure. A first answer is provided as a direct application of Theorem (5.2.9). This theorem states the following:

Theorem 5.0.1. *Any functor $\gamma : \mathcal{C} \to \mathcal{M}$ from $\mathcal{C}$ into a model category $\mathcal{M}$ may be 'factored' through $U\mathcal{C}$ in the sense that there is a Quillen adjunction $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$ and a natural weak equivalence $\eta : Re \circ \mathcal{Y} :\to \gamma$:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{Y}"] \arrow[dr, "\gamma"'] & U\mathcal{C} \arrow[d, "Re"] \arrow[dl, phantom, "\Leftarrow\eta"] \\ & \mathcal{M} \end{tikzcd} \tag{148}$$

Moreover, the category of such factorizations is contractible.

This theorem asserts that there will exist a Quillen adjunction $U(Sm_k) \rightleftarrows sShv(Sm/S)_{\mathbb{A}^1}$. Although this provides a way in which we can study $sShv(Sm/S)_{\mathbb{A}^1}$ by comparing it to $U(Sm/S)$, it may feel like an unsatisfactory answer. Fortunately, there exists a procedure known as *localization*: given a model category $\mathcal{M}$ and a set of maps S, one forms a new model structure $S^{-1}\mathcal{M}$ in which the elements of S have been added to the weak equivalences. Later in the paper, we will provide a description of the set of morphisms S in $U(Sm_k)$ that one should localize to obtain a Quillen equivalence $S^{-1}U(Sm/S) \rightleftarrows sShv(Sm/S)_{\mathbb{A}^1}$. But for now, we will just state that it is possible to obtain the desired Quillen equivalence and proceed by providing a brief discussion about the machinery of localization.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two model categories with the same underlying category. We say that $\mathcal{C}'$ is a *localization* if the following conditions are satisfies:

(C) A morphisms f of $\mathcal{C}$ is a cofibration in $\mathcal{C}$ if and only if f if is a cofibration in $\mathcal{C}'$.

(W) If a morphism f of $\mathcal{C}$ is a weak equivalence in $\mathcal{C}$, then f is a weak equivalence in $\mathcal{C}'$

It then also follows that

(F) If a morphisms f of $\mathcal{C}$ is a fibration in $\mathcal{C}'$, then f is a fibration in $\mathcal{C}$.

Let $\mathcal{C}$ be a simplicial model category. Let $h\mathcal{C}$ be the homotopy category of $\mathcal{C}$, obtained by first passing to the full subcategory $\mathcal{C}^\circ \subset \mathcal{C}$ spanned by the fibrant-cofibrant objects and he passing to the homotopy category of the simplicial category $\mathcal{C}^\circ$. We see that $h\mathcal{C}$ has a natural enrichment over the homotopy category $hsSet$: if $X, Y \in h\mathcal{C}$ are represented by fibrant-cofibrant objects $\overline{X}, \overline{Y} \in \mathcal{C}$, then we let

$$\mathrm{Hom}_{h\mathcal{C}}(X, Y) = \left[\mathrm{Hom}_{\mathcal{C}}(\overline{X}, \overline{Y})\right] \tag{149}$$

Here $[K] \in hsSet$ denotes the object of $hsSet$ represented by a Kan complex K. Note that we are using a different definition of homotopy category than the one presented before. We are denoting the homotopy category by $h\mathcal{C}$ instead of $\mathrm{Ho}\,\mathcal{C}$. It is important to know that $h\mathcal{C}$ and $\mathrm{Ho}\,\mathcal{C}$ are equivalent.

Let S be a collection of morphisms in $h\mathcal{C}$. Then

(i) We will say that an object $Z \in h\mathcal{C}$ is *S-local* if, for every morphism $f : X \to Y$ in S, the induced map

$$\mathrm{Hom}_{h\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Hom}_{h\mathcal{C}}(X, Z) \tag{150}$$

is an isomorphisms. We say that an object $\overline{Z} \in \mathcal{C}$ is S-local if the image in $h\mathcal{C}$ is S-local

(ii) We will say that a morphism $f : X \to Y$ of $h\mathcal{C}$ is an *S-equivalence* if, for every S-local object $Z \in h\mathcal{C}$, the induced map

$$\mathrm{Hom}_{h\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Hom}_{h\mathcal{C}}(X, Z) \tag{151}$$

is an isomorphism. We say that a morphism f in $\mathcal{C}$ is an S-equivalence if its image in $h\mathcal{C}$ is an S-equivalence.

If $\overline{S}$ is a collection of morphisms in $\mathcal{C}$ with image S in $h\mathcal{C}$, we will apply the same terminology: an object of $\mathcal{C}$ (or $h\mathcal{C}$) is said to be *$\overline{S}$-local* if it is S-local, and a morphism of $\mathcal{C}$ (or $h\mathcal{C}$) is said to be an S-equivalence if it is an S-equivalence.

Proposition 5.0.2. *Let $\mathcal{C}$ be a left proper combinatorial simplicial model category and let S be a (small) set of cofibrations in $\mathcal{C}$. Let $S^{-1}\mathcal{C}$ denote the same category, with the following distinguished classes of morphisms:*

(C) A morphisms g in $S^{-1}\mathcal{C}$ is a cofibration if it is a cofibration when regarded as a morphisms in $\mathcal{C}$.

(W) A morphisms g in $S^{-1}\mathcal{C}$ is a weak equivalence if it is an S-equivalence

Then

(1) The above definitions endow $S^{-1}\mathcal{C}$ with the structure of a combinatorial simplicial model category

(2) The model category $S^{-1}\mathcal{C}$ is left proper

(3) An object $X \in \mathcal{C}$ is fibrant in $S^{-1}\mathcal{C}$ if and only if X is S-local and fibrant in $\mathcal{C}$.

Now that we have defined the process of localization, let's go back to some claims we made before. We have mentioned that there exists a model structure on $sPre(Sm/S)$ that is Quillen equivalent to $sShv(Sm/S)_{\mathbb{A}^1}$, and that the model structure on $sPre(Sm/S)$ will help us understand the homotopy category of $sShv(Sm/S)_{\mathbb{A}^1}$. In particular, we said that the disadvantage of the the model structure $sShv(Sm/S)_{\mathbb{A}^1}$ is that fibrations are characterized as morphisms which satisfy the right lifting property with respect to the trivial cofibrations. Now, if we were able to find a set S of morphisms in $U(Sm/S)$ such that the localization $S^{-1}U(Sm/S)$ is Quillen equivalent to $sShv(Sm/S)_{\mathbb{A}^1}$ we will get a much better description of the fibrant objects in the homotopy category. The reason is that fibrations in $U(Sm/S)$ were described objectwise, and by Proposition (5.0.2) we know that the fibrant objects of $S^{-1}U(Sm/S)$ are just the S-local objects that are fibrant objectwise.

5.1 Homotopy Density Theorem

In this section we will prove that every simplicial presheaf is a homotopy colimit of representables. This is the homotopical analog of the density theorem proved in the section on presheaves. Unfortunately, the proof of this result will require us to be extremely careful with the notation we use, so we first introduce some notation.

Recall that the category $sPre(\mathcal{C})$ has as objects functors $\mathcal{C}^{\mathrm{op}} \to sSet$ and as morphisms natural transformation between functors. By the tensor-hom adjunction and symmetry of the closed monoidal structure on Cat we have the following equivalent formulation of $sPre(\mathcal{C})$: The objects are functors $\Delta^{\mathrm{op}} \to Pre(\mathcal{C})$ and the morphisms are the natural transformations between them. Although we can not endowed this last formulation of the definition of $sPre(\mathcal{C})$ with a model structure, at least with the tools discussed in this paper, it will be useful to have it in mind for many purposes. To be able to distinguish between this equivalent formulations of the definition of $sPre(\mathcal{C})$ we introduce the following notation: let F be a simplicial presheaf, we will denote by $F([n])$ the presheaf obtained by evaluating the functor $F : \Delta^{\mathrm{op}} \to Pre(\mathcal{C})$ at $[n]$; similarly we will denote by $F(U)$ the simplicial set we obtained by evaluating the functor $\mathcal{C}^{\mathrm{op}} \to sSet$ at some $U \in \mathcal{C}$. They should be easy to differentiate since we will put use brackets $[n]$ around the elements of Δ. To avoid ambiguity, we will say that a simplicial presheaf is constant, if it is a constant functor $\mathcal{C}^{\mathrm{op}} \to sSet$. And we will say that a simplicial presheaf is discrete if it is a constant functor $\Delta^{\mathrm{op}} \to Pre(\mathcal{C})$. The reason for this notation is that a discrete simplicial presheaf when evaluated at some $U \in \mathcal{C}$ will yield a discrete simplicial set. We can see that there is a canonical inclusion $Pre(\mathcal{C}) \hookrightarrow sPre(\mathcal{C})$, by considering presheaves as discrete simplicial presheaves.

In this section, we will use the notion of bisimplicial presheaves, as expected this is the category $biPre(\mathcal{C}) := \mathrm{Fun}(\Delta^{\mathrm{op}}, U\mathcal{C})$. We will denote a bisimplicial presheaf by $F_\bullet$, and by F_n the value that that $F_\bullet$ takes at $[n]$. And we will denote by $rX_\bullet$ the representable bisimplicial presheaves, these are the objects that are the image of the canonical inclusion $sPre(\mathcal{C}) \hookrightarrow biPre(\mathcal{C})$, of the representables objects in $sPre(\mathcal{C})$.

The category $biPre(\mathcal{C})$ can be endowed with the projective, injective, and Reedy model structure, and unless stated otherwise, we will use the Reedy model structure. The reason is the following: the main use that we will will have for this category is to proof that ever object in $sPre(\mathcal{C})$ is a homotopy colimit of representables of an object of $biPre(\mathcal{C})$, and the main computational tool to compute homotopy colimits will be Proposition (4.3.11). To be able to use Proposition (4.3.11), we will need to be able to proof that certain bisimplicial presheaves are cofibrant, for this we will use the fact that $biPre(\mathcal{C})$ is a simplicial model category. Again, we will not proof this fact, but will be used in this section.

The author learned this result from

- Dugger - Universal Homotopy Theories [4]

Although a great deal of detail has been added in this section, the bulk of the work should be credited to him.

Definition 5.1.1. A simplicial presheaf F has *free degeneracies* if there exists a sub-simplicial presheaf $N \hookrightarrow F$ such that the canonical map

$$\coprod_{\sigma} N([\sigma]) \to F([k]) \tag{152}$$

is an isomorphism: here the variable σ rages over all surjective maps in Δ of the form $[k] \to [n]$, $N([\sigma])$ denotes a copy of $N([n])$, and the map $N([\sigma]) \to F([k])$ is the one induced by $\sigma^* : F([n]) \to F([k])$. We say that N is a *splitting* of F.

Lemma 5.1.2. *If F is a simplicial presheaf that has free degeneracies then F is the colimit of the maps*

$$\operatorname{sk}_0 F \to \operatorname{sk}_1 F \to \operatorname{sk}_2 \to \cdots \tag{153}$$

where $\operatorname{sk}_0 F = N([0])$ and $\operatorname{sk}_n F$ is defined by the pushout square

$$\begin{tikzcd} N([n]) \otimes \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} F \arrow[d] \\ N([n]) \otimes \Delta^n \arrow[r] & \operatorname{sk}_n F \end{tikzcd} \tag{154}$$

where we consider $N([n])$ as a discrete simplicial presheaf.

Proof. By considering the simplicial presheaves as functors $\mathcal{C}^{\mathrm{op}} \to sSet$, we can compute the pushout square for every object $U \in \mathcal{C}$. It is clear from the fact that F has free degeneracies that $\operatorname{sk}_n F$ is indeed the pushout of the diagram. In the same way, we can see that F is the sequential colimit of the diagram $\operatorname{sk}_0 F \to \operatorname{sk}_1 F \to \cdots$ by computing the colimit objectwise. □

Corollary 5.1.3. *If F is a simplicial presheaf that has free degeneracies decomposition in which the $N([k])$ are cofibrant in $U\mathcal{C}$, as constant simplicial presheaves, then F is itself cofibrant.*

Proof. The fact that $N([k])$ is cofibrant implies that $N([k]) \otimes \partial\Delta^n \to N([k]) \otimes \Delta^n$ is a cofibration, so the map $\operatorname{sk}_{k-1} F \to \operatorname{sk}_k F$ is also a cofibration. Then F is a sequential colimit of cofibrations beginning with $\emptyset \to \operatorname{sk}_0 F$, hence F is cofibrant. □

Lemma 5.1.4. *The representable simplicial presheaves rX are cofibrant in $U\mathcal{C}$.*

Proof. Recall that in the proof of the projective model structure in Theorem (3.1.2), we proved that for any simplicial set K, and every X in $\mathcal{C}$ we have that $\mathcal{F}_X^K$ is cofibrant, this is because in the Kan model structure every simplicial set is cofibrant. In particular, we have that

$$rX \cong \mathcal{F}_X^{\Delta^0} = \coprod_{\alpha \in \operatorname{Hom}_{\mathcal{C}}(-,X)} \Delta^0 \tag{155}$$

The result follows. □

Let F be an object in $Pre(\mathcal{C})$. Define $\widetilde{Q}F$ to be the simplicial presheaf whose nth level is

$$(\widetilde{Q}F)([n]) = \coprod_{rX_n \to \cdots \to rX_0 \to F} rX_n \tag{156}$$

where the coproduct ranges over chains of composable maps in $Pre(\mathcal{C})$. The face and degeneracy maps are the obvious candidates. In particular, we can see that by the functoriality of the element category, Definition (1.1.5), we can conclude that $\widetilde{Q}$ is a functor $Pre(\mathcal{C}) \to sPre(\mathcal{C})$. Here we are using the subscript n on rX_n for indexing, not as the value of a bisimplicial presheaf.

Proposition 5.1.5. *The canonical map $\widetilde{Q}F \to F$ is a weak equivalence in $U\mathcal{C}$, when we consider F as a constant simplicial presheaf. And $\widetilde{Q}F$ is cofibrant. In other words, $\widetilde{Q}F$ is a cofibrant replacement for F.*

Proof. Recall that $\widetilde{Q}F \to F$ is a weak equivalence if and only if, for each $U \in \mathcal{C}$ the map $\widetilde{Q}F(U) \to F(U)$ is a weak equivalence of simplicial sets. By construction we can see that for every n there exists a lift

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \widetilde{Q}F(U) \arrow[d] \\ \Delta^n \arrow[r] & F(U) \end{tikzcd} \tag{157}$$

it follows that $\widetilde{Q}F(U) \to F(U)$ is a trivial fibration, it follows that $\widetilde{Q}F \to F$ is a weak equivalence.

To see that $\widetilde{Q}$ is cofibrant, first we observe that it has free degeneracies. Let N be the simplicial presheaf whose nth level is

$$N([n]) = \coprod_{rX_n \to \cdots \to rX_0 \to F} rX_n \tag{158}$$

in which no map $rX_{k+1} \to rX_k$ is an identity map. When considering each N_n as a constant simplicial presheaf, we can see that it is a coproduct of representables, therefore by Lemma(5.1.4) we have that each N_n is cofibrant. By Corollary (5.1.3) it follows that $\widetilde{Q}F$ is cofibrant. □

Definition 5.1.6. We say that a bisimplicial presheaf $F_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, sPre(\mathcal{C}))$, has *free degeneracies*, if there exists a sub-bisimplicial presheaf $N_\bullet \hookrightarrow F_\bullet$ such that the canonical map

$$N_k \hookrightarrow F_k \tag{159}$$

is a splitting of F_k, for each k. We call $N_\bullet$ a *splitting* of $F_\bullet$

Definition 5.1.7. We say that the *nth skeleton* of a bisimplicial presheaf $F_\bullet$, is the simplicial object in $sPre(\mathcal{C})$ that takes values $[k] \to \mathrm{sk}_n F_k$, and whose face and degeneracy maps are the once induced by $F_\bullet$. We denote it by $\mathrm{sk}_n F_\bullet$.

Lemma 5.1.8. *If $F_\bullet$ is a bisimplicial presheaf that has free degeneracies, then $F_\bullet$ is the colimit of maps*

$$\mathrm{sk}_0 F_\bullet \to \mathrm{sk}_1 F_\bullet \to \mathrm{sk}_2 F_\bullet \to \cdots \tag{160}$$

where $\mathrm{sk}_0 F_\bullet = N_\bullet([0])$ and $\mathrm{sk}_n F_\bullet$ is defined by the pushout square

$$\begin{tikzcd} N_\bullet([n]) \otimes \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} F_\bullet \arrow[d] \\ N_\bullet([n]) \otimes \Delta^n \arrow[r] & \mathrm{sk}_n F_\bullet \end{tikzcd} \tag{161}$$

where $N_\bullet([n])$ is a functor $\Delta^{\mathrm{op}} \to sPre(\mathcal{C})$ that takes values on discrete simplicial presheaves. Concretely it is the functor $[k] \to N_k([n])$, where we consider the presheaf $N_k[n]$ as a discrete simplicial presheaf.

Proof. By considering a bisimplicial presheaf as a functor $\Delta^{\mathrm{op}} \to sPre(\mathcal{C})$, we can compute the pushout square for every $[n] \in \Delta^{\mathrm{op}}$, we can then see that the $sk_n F_\bullet$ is indeed the pushout of the diagram by invoking Lemma (5.1.2). In the same way we can see that $F_\bullet$ is the colimit of the diagram $\mathrm{sk}_0 F_\bullet \to \mathrm{sk}_1 F_\bullet \to \cdots$ by computing the colimit objectwise and invoking Lemma (5.1.2). □

Corollary 5.1.9. *If $F_\bullet$ is a bisimplicial presheaf that has a free degeneracy decomposition, in which $N_\bullet([k])$ is cofibrant in as a bisimplicial presheaf, then $F_\bullet$ is itself cofibrant.*

Proof. The fact that $N_\bullet([k])$ is cofibrant implies that $N_\bullet([k]) \otimes \partial\Delta^n \to N_\bullet([k]) \otimes \Delta^n$ is a cofibration, so the map $\operatorname{sk}_{k-1} F_\bullet \to \operatorname{sk}_k F_\bullet$ is also a cofibration. Then $F_\bullet$ is a sequential colimit of cofibrations beginning with $\emptyset \to \operatorname{sk}_0 F_\bullet$, hence $F_\bullet$ is cofibrant. $\square$

Lemma 5.1.10. *The representable bisimplicial presheaves $rX_\bullet$ are cofibrant in $biPre(\mathcal{C})$ with the projective model structure, therefore it is also cofibrant with the Reedy model structure.*

Proof. Recall that in the proof of the projective model structure in Theorem (3.1.2), we proved that for any cofibrant simplicial presheaf G, and every $[n]$ in Δ we have that $\mathcal{F}^G_{[n]}$ is cofibrant. In particular, we have that

$$rX_\bullet = \mathcal{F}^{rX}_{[0]} = \coprod_{\alpha \in \operatorname{Hom}_\Delta(-,[0])} rX \tag{162}$$

this is true becuse $[0]$ is the final object in Δ. Moreover in Lemma (5.1.4) we showed that rX is cofibrant in $U\mathcal{C}$, the result follows. $\square$

Let F be an object of $sPre(\mathcal{C})$. Define $QF_\bullet$ to be the bisimplicial presheaf obtained in the following way: consider $F_\bullet$ as a bisimplicial presheaf that takes values on discrete simplicial presheaves; that is, $F_\bullet$ maps $[n]$ to $F([n])$ considered as a discrete simplicial presheaf. Then, apply the functor $\widetilde{Q} : Pre(\mathcal{C}) \to sPre(\mathcal{C})$ in each object $[n]$. This procedure defines a functor $Q : sPre(\mathcal{C}) \to biPre(\mathcal{C})$.

Proposition 5.1.11. *The canonical map $QF_\bullet \to F_\bullet$ is a weak equivalence in $biPre(\mathcal{C})$, and $QF_\bullet$ is cofibrant. In other words $QF_\bullet$ is a cofibrant replacement for $F_\bullet$.*

Proof. Recall that the weak equivalence in the Reedy model structure of $biPre(\mathcal{C})$ is defined objectwise, it follows form Proposition (5.1.5) that $QF_\bullet \to F_\bullet$ is a weak equivalence.

Let $N_\bullet$ be the bisimplicial presheaf, such that N_k is the splitting defined above for QF_k. It follows by definition that $N_\bullet$ is a splitting of $QF_\bullet$. By invoking Corollary (5.1.9), we see that it suffices to show that $N_\bullet([k])$ is cofibrant as a bisimplicial presheaf. Consider $N_n([k])$ as a constant bisimplicial presheaf, we can then construct $N_\bullet([k])$ inductively in the following way: Let $\operatorname{sk}_0 N_\bullet([k]) = N_0([k])$, and define $\operatorname{sk}_n N_\bullet([k])$ by the following pushout square

$$\begin{tikzcd} N_n([k]) \otimes \partial\Delta^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} N_\bullet([k]) \arrow[d] \\ N_n([k]) \otimes \Delta^n \arrow[r] & \operatorname{sk}_n N_\bullet([k]) \end{tikzcd} \tag{163}$$

If every $N_n([k])$ is cofibrant in $biPre(\mathcal{C})$, it follows that $N_n([k]) \otimes \partial\Delta^n \to N_n([k]) \otimes \Delta^n$ is a cofibration, by the simplicial model structure. Then $sk_{n-1}N_\bullet([k]) \to \operatorname{sk}_n N_\bullet([k])$ is a cofibration. Since $N_\bullet([k])$ can be considered as the sequential colimit of $\operatorname{sk}_0 N_\bullet([k]) \to \operatorname{sk}_1 N_\bullet([k]) \to \cdots$, and $\emptyset \to \operatorname{sk}_0 N_\bullet([k])$ is a cofibration, it follows that $N_\bullet([k])$ is cofibrant. Finally, we only need to proof that every $N_n([k])$ is cofibrant in $biPre(\mathcal{C})$. We note that $N_n([k])$ is just a coproduct of representables, and we invoke Lemma (5.1.10), the result follows. $\square$

Theorem 5.1.12. *Every simplicial presheaf is a homotopy colimit of representables.*

Proof. Let F be a simplicial presheaf, and let $F_\bullet$ be the bisimplicial presheaf, that maps $[k] \to F([k])$ where we consider $F([k])$ as a discrete simplicial presheaf. Recall from Proposition (4.3.11), that for a Reedy cofibrant bisimplicial presheaf $QF_\bullet$ we have that: $\operatorname{hocolim} QF_\bullet$ and $|QF_\bullet|$ are isomorphic in the homotopy category. It suffices to show that there exists a weak equivalence $|QF_\bullet| \to F$, since $QF_\bullet$ only takes values on representable simplicial presheaves. By evaluating $QF_\bullet$ at each object $U \in \mathcal{C}$, we obtain a bisimplicial set $QF_\bullet(U)$; and by Example (4.3.12), we conclude that $|QF_\bullet(U)| \cong \operatorname{diag} QF_\bullet(U)$. By construction it follows that the canonical map $\operatorname{diag} QF_\bullet(U) \to \operatorname{diag} F_\bullet(U) \cong F(U)$ is a weak equivalence. Finally, we note that since the weak equivalences are defined objectwise, it follows that $|QF_\bullet| \to F$ is a weak equivalence. $\square$

5.2 Universal Property of Simplicial Presheaves

In these section we will finally prove the result regarding the universal property of the model category $U\mathcal{C}$. This is the result that I had in mind when I wrote the short story about the physicists and the simplest physical theory. So I hope that after this chapter it becomes clear what I was trying to transmit with that short story.

The author learned this material from:

- Dugger - Universal Homotopy Theories [4]

Although we have added a great deal of detail, the bulk of the work is due to him.

Let $\mathcal{M}$ be a model category, denote by $c\mathcal{M}$ the model category $\mathrm{Fun}(\Delta, \mathcal{M})$ endowed with the Reedy model structure.

Definition 5.2.1. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, equipped with functors $r : \mathcal{C} \to \mathcal{M}$ and $\gamma : \mathcal{C} \to \mathcal{N}$, as shown in the diagram below:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[dr, "\gamma"'] & \mathcal{M} \\ & \mathcal{N} \end{tikzcd} \tag{164}$$

We define a *factorization of* γ *through* $\mathcal{M}$ to be the following data:

(i) A Quillen adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$

(ii) A natural weak equivalence $\eta : L \circ r \to \gamma$

The factorization can be denoted by the triple (L, R, η). In this paper we will usually regard a Quillen adjunction as a left Quillen functor $L : \mathcal{M} \to \mathcal{N}$. This is equivalent to the existence of a Quillen adjunction, but it makes the term 'factorization' more appropriate. Therefore, we can also denote a factorization by the pair (L, η), we will usually adopt this notation.

Definition 5.2.2. Define the *category of factorizations* $Fact_{\mathcal{M}}(\gamma)$ in the following way:

- Its objects are pairs (L, η) as above
- Its morphisms are natural transformations $L \to L'$ that make the following diagram commute

$$\begin{tikzcd} L \circ r(X) \arrow[rr] \arrow[dr, "\eta"'] & & L' \circ r(X) \arrow[dl, "\eta'"] \\ & \gamma(X) & \end{tikzcd} \tag{165}$$

Note that giving a natural transformation $L \to L'$ is equivalent, via adjointness, to giving a natural transformation $R' \to R$ or to giving two maps $L \to L'$ and $R' \to R$ which are compatible with the adjunction. So we could have adopted a more symmetric definition, but it would be equivalent to the one above.

Definition 5.2.3. Let $\mathcal{C}$ be a category with a functor $\gamma : \mathcal{C} \to \mathcal{M}$. A *cosimplicial resolution* of γ is:

(i) A functor $\Gamma : \mathcal{C} \to c\mathcal{M}$ such that $\Gamma(X)$ is Reedy cofibrant

(ii) A natural weak equivalence $\epsilon : \Gamma(X) \to X$, by regarding X as a constant functor in $c\mathcal{M}$.

The cosimplicial resolution can be denoted by the pair (Γ, ϵ).

Definition 5.2.4. Define the *category of cosimplicial resolutions* $coRes(\gamma)$ in the following way:

- Its objects are pairs (Γ, ϵ) as above
- Its morphism are natural transformations $\Gamma \to \Gamma'$ that make the following diagram commute

$$\begin{tikzcd} \Gamma(X) \arrow[rr] \arrow[dr, "\epsilon"'] & & \Gamma'(X) \arrow[dl, "\epsilon'"] \\ & X & \end{tikzcd} \tag{166}$$

  where we regard X as a constant functor in $c\mathcal{M}$.

Lemma 5.2.5. *Let $\mathcal{M}$ be a cocomplete category. There exists a bifunctor*

$$\int^{\mathcal{C}^{\mathrm{op}}} - \otimes - : \mathrm{Fun}(\mathcal{C}, c\mathcal{M}) \times \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, sSet) \longrightarrow \mathcal{M} \tag{167}$$

such that for a fixed $\Gamma \in \mathrm{Fun}(\mathcal{C}, c\mathcal{M})$, the induced functor $\int^{\mathcal{C}^{\mathrm{op}}} \Gamma \otimes - : sPre(\mathcal{C}) \to \mathcal{M}$ is a left adjoint, whose right adjoint is the functor $m \to \mathrm{Hom}_{\mathcal{M}}(\Gamma -, m)$.

Proof. As the notation suggests, the bifunctor $\int^{\mathcal{C}^{\mathrm{op}}} - \otimes -$ is the coend of a certain bifunctor $- \otimes - : c\mathcal{M} \otimes sSet \to \mathcal{M}$. We will begin by defining this bifunctor, and showing that for a fixed $X \in c\mathcal{M}$ the induced functor $X \otimes -$ is a left adjoint. Recall from Theorem (1.1.7), that for a fixed $X \in c\mathcal{M}$, we obtain the following diagram

$$\begin{tikzcd} \Delta \arrow[r, "\mathcal{Y}"] \arrow[dr, "X"'] & sSet \arrow[d, "Re_X"] \\ & \mathcal{M} \end{tikzcd} \tag{168}$$

So we define $X \otimes - : sSet \to \mathcal{M}$ to coincide the functor Re_X. Therefore $X \otimes -$ is a left adjoint functor, whose right adjoint is the functor $m \to \mathrm{Hom}_{\mathcal{M}}(X-, m)$. Moreover, by the construction of Re_X in the proof of Theorem (1.1.7), we can conclude that it is natural in X, hence $- \otimes - : c\mathcal{M} \otimes sSet \to \mathcal{M}$ is the required bifunctor.

Then we can form the coend, and obtain the following bifunctor:

$$\int^{\mathcal{C}^{\mathrm{op}}} - \otimes - : \mathrm{Fun}(\mathcal{C}, c\mathcal{M}) \times \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, sSet) \longrightarrow \mathcal{M} \tag{169}$$

We are only left to show that for a fixed $\Gamma \in \mathrm{Fun}(\mathcal{C}, c\mathcal{M})$ the functor $\int^{\mathcal{C}^{\mathrm{op}}} \Gamma \otimes - : sPre(\mathcal{C}) \to \mathcal{M}$ is a left adjoint. For this, let $F \in sPre(\mathcal{C})$, and $U \in \mathcal{C}$, and consider the following sequence of identities:

$$\mathrm{Hom}_{\mathcal{M}}\left(\int^{\mathcal{C}^{\mathrm{op}}} \Gamma \otimes F, m\right) = \int_{\mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{M}}\Big(\Gamma(U) \otimes F(U), m\Big) \tag{170}$$

$$= \int_{\mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{sSet}\Big(F(U), \mathrm{Hom}_{\mathcal{M}}(\Gamma(U)-, m)\Big) \tag{171}$$

$$= \mathrm{Hom}_{sPre(\mathcal{C})}\Big(F, \mathrm{Hom}_{\mathcal{M}}(\Gamma-, m)\Big) \tag{172}$$

This completes the proof. $\square$

Proposition 5.2.6. *Let $\mathcal{C}$ be a small category and let $\mathcal{M}$ be a model category. There exists a pair a functors*

$$\mathcal{F} : Fact_{U\mathcal{C}}(\gamma) \longrightarrow coRes(\gamma) \tag{173}$$

$$\mathcal{G} : coRes(\gamma) \longrightarrow Fact_{U\mathcal{C}}(\gamma) \tag{174}$$

Proof. First, we will define the functor $\mathcal{F} : Fact_{U\mathcal{C}}(\gamma) \to coRes(\gamma)$. Suppose we are given a factorization of $\gamma : \mathcal{C} \to \mathcal{M}$ through $U\mathcal{C}$: so we have a Quillen pair $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$ and a natural weak equivalence $Re(rX) \to \gamma(X)$. Then for each $X \in \mathcal{C}$ we get a cosimplicial resolution of γ by taking $\Gamma(X)$ to be:

$$[n] \longrightarrow Re(rX \otimes \Delta^n) \tag{175}$$

This is clearly functorial in X. We want to proof that Γ is a cosimplicial resolution of γ. It is clear that $\Gamma(X) \to Re(rX)$ is a weak equivalence that is natural in X; then all we need to do is proof that $\Gamma(X)$ is indeed cofibrant in the Reedy model structure of $c\mathcal{M}$. Recall that $L_n\Gamma(X) \cong \operatorname{colim}_{\Delta_{+,n}} Re(rX \otimes \Delta^j)$, where the category $\Delta_{+,n}$ is the subcategory of Δ generated by the injective morphisms with codomain n. Since Re is a left Quillen functor, it preserves colimits, hence $L_n\Gamma(X) \cong Re(\operatorname{colim}_{\Delta_{+,n}} rX \otimes \Delta^j)$. Moreover, we know that $-\otimes- : sPre(\mathcal{C}) \times sSet \to sPre(\mathcal{C})$ is a left Quillen bifunctor, and in particular is preserves colimits independently in each variable, it follows that $L_n\Gamma(X) \cong Re(rX \otimes \partial\Delta^n)$. Since rX is cofibrant in $U\mathcal{C}$, we conclude that $Re(rX \otimes \partial\Delta^n) \to Re(rX \otimes \Delta^n)$ is a cofibration, hence $\Gamma(X)$ is Reedy cofibrant. This defines the functor $\mathcal{F} : Fact_{U\mathcal{C}}(\gamma) \to coRes(\gamma)$.

Now, we will define the functor $\mathcal{G} : coRes(\gamma) \to Fact_{U\mathcal{C}}(\gamma)$. Suppose we start with a resolution $\Gamma : \mathcal{C} \to c\mathcal{M}$. Define the functors $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$ by the formulas

$$Re(F) = \int^{\mathcal{C}^{\mathrm{op}}} \Gamma \otimes F \qquad Sing(X) = \operatorname{Hom}_{\mathcal{M}}(\Gamma-, X) \tag{176}$$

By Lemma (5.2.5) it follows this is an adjunction. Too see that it is a Quillen pair, we will first show that the bifunctor $- \otimes - : c\mathcal{M} \times sSet \to \mathcal{M}$ satisfies the following condition: if $f : A \to B$ be a cofibration in $c\mathcal{M}$, and $g : K \to L$ is a cofibration in $sSet$, then $(A \otimes L) \amalg_{A\otimes K} (B \otimes K) \to B \otimes L$ is a cofibration in $\mathcal{M}$. And it is a trivial cofibration if f is. It suffices to show for the case when g is one of the generating cofibrations $\partial\Delta^n \to \Delta^n$. We claim that the map $A \otimes \partial\Delta^n \to A \otimes \Delta^n$ is isomorphic to $L_nA \to A_n$. This follows from the fact that $A \otimes -$ preserves colimits, and that $A \otimes \Delta_j \cong A_j$ by construction. Therefore, $(A\otimes\Delta^n) \amalg_{A\otimes\partial\Delta^n} (B\otimes\partial\Delta^n) \to B\otimes\Delta^n$ is isomorphic to $A_n \amalg_{L_nA} L_nB \to B_n$, which is a (trivial) cofibration, since f is a (trivial) cofibration. By adjunction given a cofibration $f : A \to B$ in $c\mathcal{M}$, and any (trivial) fibration $i : X \to Y$ in $c\mathcal{M}$, we conclude that the map

$$\operatorname{Hom}_{\mathcal{M}}(B, X) \longrightarrow \operatorname{Hom}_{\mathcal{M}}(A, X) \times_{\operatorname{Hom}_{\mathcal{M}}(A,Y)} \operatorname{Hom}_{\mathcal{M}}(B, Y) \tag{177}$$

is a (trivial) fibration of simplicial sets if i is. In particular, since $\Gamma : \mathcal{C}^{\mathrm{op}} \to c\mathcal{M}$ takes values in cofibrant objects, it follows that for a fixed $U \in \mathcal{C}$ the functor $c\mathcal{M} \to sSet$ defined by the rule $X \to \operatorname{Hom}_{\mathcal{M}}(\Gamma(U), X)$ preserves (trivial) fibrations. Therefore, since $U\mathcal{C}$ has the projective model structure, its (trivial) fibrations are defined objectwise, hence $Sing : \mathcal{M} \to U\mathcal{C}$ is preserves (trivial) fibrations. It follows that $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$ is a Quillen adjunction. Finally, we need to check that there is a natural weak equivalence $Re(rX) \to \gamma(X)$. But recall from the definition that $Re(rX) \cong \Gamma(X)_0$, and our cosimplicial resolution came with a weak natural weak equivalence $\Gamma(X) \to \gamma(X)$, when we consider $\gamma(X)$ as a constant functor $c\mathcal{M}$. We have defined a functor $\mathcal{G} : coRes(\gamma) \to Fact_{U\mathcal{C}}(\gamma)$. $\square$

Lemma 5.2.7. *For each n, there exists an adjunction*

$$-\otimes\Delta^n : Pre(\mathcal{C}) \rightleftarrows sPre(\mathcal{C}) : Ev_n \tag{178}$$

where $Ev_n(X) = \mathrm{Hom}_{sPre(\mathcal{C})}(-\otimes\Delta^n, X)$ and $Y\otimes\Delta^n$ is the expected simplicial presheaf.

Proof. Let Γ_n be the functor $\mathcal{C} \to sPre(\mathcal{C})$, that maps $X \to rX\otimes\Delta^n$. It follows by Theorem (1.1.7) that there exists a left adjoint functor $-\otimes\Delta^n : Pre(\mathcal{C}) \to sPre(\mathcal{C})$. Moreover, we know that its right adjoint is the functor $X \to \mathrm{Hom}_{sPre(\mathcal{C})}(-\otimes\Delta^n, X)$. □

Proposition 5.2.8. *The functors $\mathcal{F}$ and $\mathcal{G}$ defined in the proof of Proposition (5.2.6), determines an equivalence of categories*

$$\mathcal{F} : Fact_{U\mathcal{C}}(\gamma) \simeq coRes(\gamma) : \mathcal{G} \tag{179}$$

Proof. First, we will show that given a object $Re \in Fact_{U\mathcal{C}}$, there exists a natural isomorphism $Re \cong \mathcal{G}\circ\mathcal{F}Re$. First, recall that $\mathcal{F}Re$ is the cosimplicial resolution defined by the rule

$$\mathcal{F}Re(-) : [n] \longrightarrow Re(r-\otimes\Delta^n) \tag{180}$$

and $\mathcal{G}\circ\mathcal{F}Re$ is the left Quillen functor determined by

$$\mathcal{G}\circ\mathcal{F}Re(-) = \int^{\mathcal{C}^{\mathrm{op}}} \mathcal{F}Re\otimes - \tag{181}$$

We claim that for any left Quillen functor $Re : U\mathcal{C} \to \mathcal{M}$, its right adjoint is the functor $m \to \mathrm{Hom}_{\mathcal{M}}(\mathcal{F}Re-, m)$. Consider the following identity:

$$\mathrm{Hom}_{U\mathcal{C}}(F, \mathrm{Hom}_{\mathcal{M}}(\mathcal{F}Re-, m)) \cong \int_{\Delta} \mathrm{Hom}_{Pre(\mathcal{C})}(F_n, \mathrm{Hom}_{\mathcal{M}}(Re(-\otimes\Delta^n), m)) \tag{182}$$

By Lemma (5.2.7) we know that $Re(-\otimes\Delta^n) : Pre(\mathcal{C}) \to \mathcal{M}$ is a left adjoint, and by Theorem (1.1.7), we conclude that its right adjoint is $m \to \mathrm{Hom}_{\mathcal{M}}(Re(-\otimes\Delta^n), m)$. It follows that

$$\int_{\Delta} \mathrm{Hom}_{Pre(\mathcal{C})}\Big(F_n, \mathrm{Hom}_{\mathcal{M}}(Re(-\otimes\Delta^n), m)\Big) \cong \int_{\Delta} \mathrm{Hom}_{\mathcal{M}}\Big(Re(F_n\otimes\Delta^n), m\Big) \tag{183}$$

$$\cong \mathrm{Hom}_{\mathcal{M}}\Big(Re(F), m\Big) \tag{184}$$

Then, by the definition of $\mathcal{G} : coRes(\gamma) \to Fact_{U\mathcal{C}}(\gamma)$, and from the fact that adjoint functors are unique up unique isomorphism it follows that $Re \cong \mathcal{G}\circ\mathcal{F}Re$.

Next, we need to show that for a cosimplicial resolution Γ, there exists a natural isomorphism $\Gamma \cong \mathcal{F}\circ\mathcal{G}\Gamma$. For this, we will only compute the values of $\mathcal{G}\Gamma$ at the functors $rX\otimes\Delta^n$, the reason is that this are the only values that matter to determine $\mathcal{F}\circ\mathcal{G}\Gamma$. For this, consider the following identities

$$\mathcal{G}\Gamma = \int^{\mathcal{C}^{\mathrm{op}}} \Gamma\otimes(rX\otimes\Delta^n) \cong \int^{\mathcal{C}^{\mathrm{op}}} \coprod_{rX(U)} \Gamma\otimes\Delta^n \cong \int^{\mathcal{C}^{\mathrm{op}}} \Gamma_n\otimes rX \tag{185}$$

in the last term, we are considering Γ_n as a functor $\mathcal{C} \to \mathcal{M}$ and rX as an object of $Pre(\mathcal{C})$. Then, by Theorem (1.1.7) we conclude that

$$\mathcal{G}\Gamma \cong \int^{\mathcal{C}^{\mathrm{op}}} \Gamma_n\otimes rX \cong \Gamma_n(X) \tag{186}$$

Then, the functor $\mathcal{F}\circ\mathcal{G}F$ is determined by the following rule

$$\mathcal{F}\circ\mathcal{G}\Gamma : [n] \longrightarrow \Gamma_n(X) \tag{187}$$

Hence $\Gamma \cong \mathcal{F}\circ\mathcal{G}\Gamma$. We have shown that there exists an equivalence of categories. □

Theorem 5.2.9. *Any functor $\gamma : \mathcal{C} \to \mathcal{M}$ from $\mathcal{C}$ into a model category $\mathcal{M}$ may be 'factored' through $U\mathcal{C}$ in the sense that there is a Quillen adjunction $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$ and a natural weak equivalence $\eta : Re \circ \mathcal{Y} :\to \gamma$:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{Y}"] \arrow[dr, "\gamma"'] & U\mathcal{C} \arrow[d, "Re"] \arrow[dl, phantom, "\Leftarrow\eta"] \\ & \mathcal{M} \end{tikzcd} \tag{188}$$

Moreover, the category of such factorizations is contractible. In other words: the category $Fact_{U\mathcal{C}}(\gamma)$ is non empty and contractible.

Proof. To show that $Fact_{U\mathcal{C}}(\gamma)$ is non empty, by Proposition (5.2.8) it suffices to show that $coRes(\gamma)$ is non empty. From the functorial factorization axiom of $c\mathcal{M}$ it follows that $coRes(\gamma)$ is non empty. Similarly, to show that the nerve of the category $Fact_{U\mathcal{C}}(\gamma)$ denoted by $N(Fact_{U\mathcal{C}}(\gamma))$ is contractible, it suffices to show that $N(coRes(\gamma))$ is contractible. For this, denote by Γ_F cosimplicial resolution of γ obtained by using the functorial factorization axiom of $c\mathcal{M}$. Therefore we have that $\Gamma_F(X) \to \gamma(X)$ is a trivial fibration for every $X \in \mathcal{C}$. Let Γ_1 be another cosimplicial resolution of γ, then for each morphism $\Gamma_1(X) \to \gamma(X)$ there exists a lift

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \Gamma_F(X) \arrow[d] \\ \Gamma_1(X) \arrow[r] \arrow[ur, dashed] & \gamma(X) \end{tikzcd} \tag{189}$$

this induces a lift in the category $\mathrm{Fun}(\mathcal{C}, c\mathcal{M})$ as described in the following diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \Gamma_F \arrow[d] \\ \Gamma_1 \arrow[r] \arrow[ur, dashed] & \gamma \end{tikzcd} \tag{190}$$

In particular, this induced a map in $\Gamma_1 \to \Gamma_F$ in $coRes(\gamma)$. Next, for each map $\Gamma_1 \to \Gamma_2$ in $coRes(\gamma)$, we have the following commutative diagram in $\mathrm{Fun}(\mathcal{C}, c\mathcal{M})$:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \Gamma_F \arrow[d] & \emptyset \arrow[l] \arrow[d] \\ \Gamma_1 \arrow[r] \arrow[ur, dashed] \arrow[rr, bend right] & \gamma & \Gamma_2 \arrow[l] \arrow[ul, dashed] \end{tikzcd} \tag{191}$$

It follows that the morphism $\Gamma_1 \to \Gamma_2$ extends to the following commutative diagram in $coRes(\gamma)$

$$\begin{tikzcd} & \Gamma_1 \arrow[dl] \arrow[dr] & \\ \Gamma_2 \arrow[rr] & & \Gamma_F \end{tikzcd} \tag{192}$$

Since $coRes(\gamma)$ is a category, its nerve is completely determined by its 2-skeleton. Hence, $N(coRes(\gamma))$ is contractible. $\square$

6 Looking Forward

Recall that the category of simplicial sheaves on a Grothendieck site $\mathcal{C}$ is the full subcategory of $sPre(\mathcal{C})$, such that it satisfies the sheave condition when evaluated at each $[n]$. We denote this category by $sShv(\mathcal{C})$. Recall that there exists a model structure on $sShv(Sm/S)$, where Sm/S denotes the category of smooth schemes (of finite type), over a base scheme S, endowed with the Nisnevick topology. This model structure is due to Morel and Voevodsky [12]. The underlying idea is that it should be possible to develop a purely algebraic approach to homotopy theory by replacing the unit interval $[0, 1]$, which is not an algebraic variety, with the affine line $\mathbb{A}^1$. We denote by $sShv(Sm/S)_{\mathbb{A}^1}$ the category $sShv(Sm/S)$ endowed with this model structure. We will call this model structure the motivic model structure. Our goal in this section is to provide a more careful discussion regarding how one can use the category $sPre(Sm/S)$ to study $sShv(Sm/S)_{\mathbb{A}^1}$. For this, we first provide careful definitions of the required model structures.

Simplicial sets have homotopy groups. If X is a simplicial set, and x is a vertex of X, then define the homotopy groups of X at x by the following rule

$$\pi_n(X, x) \cong \pi_n(|X|, x) \tag{193}$$

A map of simplicial sets $f : X \to Y$ is a weak equivalence if and only if

(i) the function $\pi_0 X \to \pi_0 Y$ is a bijection

(ii) the maps $\pi_n(X, x) \to \pi_n(Y, f(x))$ are isomorphisms of groups for $n \geq 1$ and all $x \in X_0$.

Let $\mathcal{C}$ be a Grothendieck site, and X a simplicial presheaf on $\mathcal{C}$. We can define a presheaf $\pi_0 X$ and a group valued presheaf $\pi_n(X, x)$ for all $n \geq 1$ and all $x \in X_0$. These constructions are clearly functorial. Every $U \in \mathcal{C}$ determines a site $\mathcal{C}_{/U}$ whose objects are all the morphisms $V \to U$. And we say that a family

$$V_i \longrightarrow V \longrightarrow U \tag{194}$$

is a covering if and only if the family $V_i \to V$ covers V. Precomposition with the canonical functor $\mathcal{C}_{/U} \to \mathcal{C}$ determines a restricted simplicial presheaf $X|_U$ on $\mathcal{C}_{/U}$ for every simplicial presheaf X on $\mathcal{C}$. In the same way as before define a presheaf $\pi_0(X|_U)$ and a group valued presheaf $\pi_n(X|_U, x)$ for all n and all $x \in X_0$. These constructions are again functorial. Recall that the inclusion functor $Shv(\mathcal{C}) \hookrightarrow Pre(\mathcal{C})$ has a left adjoint $L : Pre(\mathcal{C}) \to Shv(\mathcal{C})$ which is known in the literature as the sheafification functor.

Definition 6.0.1. A morphism $f : X \to Y$ of simplicial presheaves is a *local weak equivalence* if and only if

(i) the function $L(\pi_0 X) \to L(\pi_0 Y)$ is an isomorphisms of set valued sheaves.

(ii) the maps $L(\pi_n(X, x)) \to L(\pi_n(Y, f(x)))$ are isomorphisms of groups valued sheafs for $n \geq 1$ and all $x \in X_0$.

It is important to note that every objectwise weak equivalence in $sPre(\mathcal{C})$ is also a local weak equivalence. Thus, in some sense, a map is a local weak equivalence if and only if it induces an isomorphisms in all possible sheaves of homotopy groups at all base points.

We can now define the following model structures; although, we will not prove that the description satisfies the model structure axioms.

Definition 6.0.2. Define the *local model structure* on $sPre(\mathcal{C})$, by the following maps

(W) A morphisms $f : X \to Y$ of simplicial presheaves is a weak equivalence if f is a local weak equivalence

(C) A morphisms $f : X \to Y$ of simplicial presheaves is a cofibration if it is an objectwise monomorphism

(F) A morphisms $f : X \to Y$ of simplicial presheaves is a fibration if it has the right lifting property with respect to the trivial cofibrations

Denote by $sPre(\mathcal{C})_{local}$ the category $sPre(\mathcal{C})$ endowed with the local model structure.

Remark 6.0.3. The map induced by the unit $X \to L(X)$ is a local weak equivalence.

Remark 6.0.4. The canonical inclusion $sShv(\mathcal{C}) \hookrightarrow sPre(\mathcal{C})$ defines a mode structure on $sShv(\mathcal{C})$, when we consider $sPre(\mathcal{C})$ to be equipped with the local model structure. We denote by $sShv(\mathcal{C})_{local}$, the category $sShv(\mathcal{C})$ with the model structure defined above.

It is important to note that $sShv(Sm/S)_{local}$ is not Quillen equivalent to $sShv(Sm/S)_{\mathbb{A}^1}$, but they are closely related. In particular, Morel and Voevodsky [12] define the motivic model structure to be the localization of $sShv(Sm_k)_{local}$ with respect to the projection maps $X \times \mathbb{A}^1 \to X$.

Our next goal is to find a set's worth of morphisms H in $U\mathcal{C}$, such that $H^{-1}U\mathcal{C} \rightleftarrows sPre(\mathcal{C})_{local}$ is a Quillen equivalence. If we are able to find such set of maps, then we will only have to localize the projection maps $X \times \mathbb{A}^1 \to X$ on $H^{-1}U(Sm/S)$ to obtain a Quillen equivalence $U(Sm/S)_{\mathbb{A}^1} \rightleftarrows sShv(Sm/S)_{\mathbb{A}^1}$. Where $U(Sm/S)_{\mathbb{A}^1}$ denotes the localization of $U(Sm/S)$, with respect to the hypothetical set H described above, and the projection maps $X \times \mathbb{A} \to X$. Since fibrations in $U(Sm/S)$ are described objectwise, by Proposition (5.0.2) we will obtain a complete description of the fibrant objects in $U(Sm/S)_{\mathbb{A}^1}$. This will provide us better understanding of the homotopy category of $sShv(Sm/S)_{\mathbb{A}^1}$. It is worth mentioning that all the Quillen equivalences here are induced by the Quillen adjunction of Theorem (5.2.9).

Now we will provide a description of which set of morphisms H one should localize to obtain a Quillen equivalence $H^{-1}U(\mathcal{C}) \rightleftarrows sPre(\mathcal{C})_{local}$. Unfortunately, we will not provide a proof for any of the claims we will make. We refer the reader to [6] for a more detailed exposition.

Definition 6.0.5. Let X belong to $\mathcal{C}$ and suppose that U is a simplicial presheaf, with and map $U \to X$, where we consider X as a discrete simplicial presheaf. This map is called a *hypercover* of X if each U_n is a coproduct of representables, and $U \to X$ is a trivial fibration in $sPre(\mathcal{C})_{local}$.

Morally speaking, we want to localize the hypercovers in $U\mathcal{C}$. But there is a small technical issue, the class of all hypercovers need not be a set, so we cannot apply Proposition (5.0.2). There exists a way around it, under a slight hypothesis, it is possible to prove that there exists a set H of hypercovers, such that once we localize, every hypercover becomes a weak equivalence. We will not prove this fact. We call a set of hypercovers H *dense* if it satisfies the property that: if we localize with respect to H we localize with respect to all hypercovers.

Theorem 6.0.6. *Let H be a dense set of hypercovers. Then the localization $H^{-1}U(\mathcal{C})$ exists and coincides with $sPre(\mathcal{C})_{local}$.*

It is proven in [6] that the Grothendieck site Sm/S has a dense set of hypercovers. We have completed our promise to construct a localization of $U(Sm/S)$ to obtain a category $U(Sm/S)_{\mathbb{A}^1}$ which is Quillen equivalent to $sShv(Sm/S)_{\mathbb{A}^1}$. We said that this construction was useful because we can get a complete description of the fibrant objects in $U(Sm/S)_{\mathbb{A}^1}$. By Proposition (5.0.2), we know that the fibrant objects in $U(Sm/S)_{\mathbb{A}^1}$ are the presheaves that are objectwise fibrant, and which are local with respect to the hypercovers and to the projection maps $X \times \mathbb{A}^1 \to X$. But we can do better.

Definition 6.0.7. An objectwise fibrant simplicial presheaf F *satisfies descent* for a hypercover $U \to X$ if the natural map from $F(X)$ to the homotopy limit of the diagram

$$\prod_a F(U_0^a) \rightrightarrows \prod_a F(U_1^a) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \tag{195}$$

is a weal equivalence. Here the products range over the representable summands of each U_n. If F is not objectwise fibrant, we say that it satisfies descent is some objectwise fibrant replacement for F does.

Corollary 6.0.8. *Let H be a dense set of hypercovers. A simplicial presheaf F is fibrant in $H^{-1}U(\mathcal{C})$ if and only if F is objectwise fibrant and satisfies descent for all hypercovers.*

We have obtained an even better description of the fibrant objects in $U(Sm/S)_{\mathbb{A}^1}$. The fibrant objects in $U(Sm/S)_{\mathbb{A}^1}$ are the objectwise fibrant simplicial presheaves, which satisfy descent with respect to all hypercovers, and which are local with respect to the projection maps $X \times \mathbb{A}^1 \to X$.

Finally, I would like to conclude this paper by presenting a more direct application of Theorem (5.2.9). We consider model categories, which can be obtained – up to Quillen equivalence – by starting with a universal model category $U\mathcal{C}$ and then localizing at some set of maps S. We say that these model categories have *small presentations*, since the category $\mathcal{C}$ can be thought of as a category of 'generators' and the set S as a collection of 'relations'.

Definition 6.0.9. Let $\mathcal{M}$ be a model category. A *small presentation* of $\mathcal{M}$ consists of the following data

(i) a small category $\mathcal{C}$

(ii) a choice of Quillen pair $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$

(iii) a set of maps S in $U\mathcal{C}$

and we require the properties that

(1) The left derived functor of Re takes maps in S to weak equivalences

(2) The induced Quillen pair $S^{-1}U(\mathcal{C}) \rightleftarrows \mathcal{M}$ is a Quillen equivalence.

Theorem 6.0.10. *Any combinatorial model category has small presentations.*

Corollary 6.0.11. *Any combinatorial model category is Quillen equivalent to one which is both simplicial and left proper.*

Unfortunately, we will not provide a proof for these results. We refer the reader to [3], for a more detailed exposition.

Example 6.0.12. We usually think of simplicial sets as objects formally built from Δ. The obvious map $\Delta \to Top$ gives rise to a Quillen pair $Re : U(\Delta) \rightleftarrows Top : Sing$, but this is not a Quillen equivalence. The first problem one encounters is that there is nothing in $U(\Delta)$ saying that the objects Δ^n are contractible. In fact this turns out to be the only problem. If we localize $U(\Delta)$ at the set of maps $S = \{\Delta^n \to *\}$, then our Quillen functors descend to a pair

$$Re : S^{-1}U(\Delta) \rightleftarrows Top : Sing \tag{196}$$

It can be proven that this is a Quillen equivalence. So the homotopy theory of simplicial sets is the universal homotopy theory built from Δ in which the Δ^n are contractible.

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