Diff-invariant relations on open manifolds

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1 Introduction

An h-principle is method for finding solutions to a partial differential equation given a solution to a topological problem. Let $p : X \to V$ be a smooth fiber bundle over a manifold V, and let $X^{(r)}$ be the bundle of r-jets of p. A differential relation on X is a submanifold $\mathcal{R} \subset X^{(r)}$ for some r. Any smooth section of p gives a section of $X^{(r)}$ by taking the r-jet of the section; such sections of $X^{(r)}$ are called *holonomic*. A holonomic section of $X^{(r)}$ with image in $\mathcal{R}$ is a *solution* to $\mathcal{R}$. An h-principle is a relationship between the topology of the space of solutions of $\mathcal{R}$, $Sol_{\mathcal{R}}$, and another topological space that is hopefully easier to understand.

For example, let $\mathrm{Diff}_V X$ be the group of diffeomorphisms of X that map fibers to fibers, and let $\pi : \mathrm{Diff}_V X \to \mathrm{Diff}\, V$ be the canonical projection, given by $\pi(f)(x) = p(f(p^{-1}(x)))$. We say that X is a *natural bundle* if there exists a homomorphism $j : \mathrm{Diff}\, V \to \mathrm{Diff}_X V$ that is a section of π. This gives an action of $\mathrm{Diff}\, V$ on X, and thus on $X^{(r)}$. The prototypical h-principle is,

Proposition 1.1. ***(Gromov's h-principle for open Diff-invarant differential relations on open manifolds, see [1], 7.2.3)*** *Let V be open, and let $\mathcal{R}$ be open inside $X^{(r)}$. Write $\Gamma(V, \mathcal{R})$ for the space of sections of $p|_{\mathcal{R}} : \mathcal{R} \to V$. Let X be a natural bundle. Suppose the action of Diff$V$ on $X^{(r)}$ preserves $\mathcal{R}$. Then the inclusion map*

$$Sol_{\mathcal{R}} \to \Gamma(V, \mathcal{R})$$

is a homotopy equivalence.

The above allows one to construct geometric structures on open manifolds using topological data. For example, let $\dim V = 2n$. Consider, on $(T^*V)^{(1)} \simeq T^*V \oplus T^*V \otimes T^*V \simeq T^*V \oplus \Lambda^2 T^*V \oplus \mathrm{Sym}^2 T^*V$, the relation that the determinant of the projection of a section to $\Lambda^2 T^*V$ is nonzero. This is an open Diff-invariant differential relation on T^*V, so solutions to this relation exist if sections of the relation exist. But a section of the relation exists exactly if V admits a nowhere-degenerate 2-form ω. This, in turn, is equivalent to the existence of an almost-complex structure on TV, i.e. to TV admitting the structure of a $U(n)$ bundle. Solutions of this relation are exactly one-forms α with non-degenerate $d\alpha$; since $d^2\alpha = 0$, $d\alpha$ is a symplectic form on V for any solution α.

Thus for open V^{2n}, if TV admits the structure of a $U(n)$ bundle then V admits a symplectic structure.

However, many geometric conditions on V, such as the conditions for a metric on V to be Ricci-flat or for V to admit a complex structure are *not given by open differential relations.* In this paper, we use the Gromov-Philips transversality theorem and Haefliger's construction of the classifying space of a topological groupoid to generalize Gromov's classical h-principle to the following:

Theorem 1.2. *Let M^n be an open manifold and $\mathcal{R}$ an arbitrary* Diff*-invariant differential relation on M. Let $\tau : M \to BGL(n)$ be the map classifying the tangent bundle of M. Then there exists a space $B\Gamma_{\mathcal{R}}$ and a map $\pi : B\Gamma_{\mathcal{R}} \to BGL(n)$, such that homotopy classes of solutions to $\mathcal{R}$ on M are in bijection with homotopy classes of lifts of τ from $BGL(n)$ to $B\Gamma_{\mathcal{R}}$.*

The space $B\Gamma_{\mathcal{R}}$ will be the classifying space of a topological groupoid constructed from $\mathcal{R}$, thus generalizing the remarkable observation of [2] that the problem of finding a complex structure on an open manifold is is governed by an h-principle.

The argument is directly analogous to Haefliger's argument [3] classifying foliations on open manifolds. In Section 2.1, we review definitions and standard results about topological groupoids Γ, Γ-structures, and their classifying spaces $B\Gamma$. In Section 2.2, we define the *ètale* groupoid $\Gamma_{\mathcal{R}}$ associated to a Diff-invariant differential relation $\mathcal{R}$. In Section 2.3, we discuss the notion of an *integrable* Γ-structure (for *ètale* groupoids Γ), and show that solutions to $\mathcal{R}$ correspond to an integrable $\Gamma_{\mathcal{R}}$-structures. Finally, in Section 2.4, we state the central claim (Theorem 2.27): the space $B\Gamma_{\mathcal{R}}$ admits a universal integrable $\Gamma_{\mathcal{R}}$-structure $\omega_{\mathcal{R}}$ which can be viewed as a "universal solution" to $\mathcal{R}$, in the sense that any integrable $\Gamma_{\mathcal{R}}$ on M is a pullback of $\omega_{\mathcal{R}}$ by a map $f : M \to B\Gamma_{\mathcal{R}}$ that lifts the classifying map $\tau : M \to BO(n)$ of the tangent bundle of M. Moreover, finding a lift f of τ to $B\Gamma_{\mathcal{R}}$ is *sufficient* to construct an integrable $\Gamma_{\mathcal{R}}$ on M: the existence of such a lift allows one to apply the Gromov-Philips Transversality theorem to find a map $f' : M \to B\Gamma_{\mathcal{R}}$ homotopic to f and transverse to $\omega_{\mathcal{R}}$, and the pullback of $\omega_{\mathcal{R}}$ by f' gives an integrable $\Gamma_{\mathcal{R}}$ structure on M.

At the heart of this argument are two simple observations. First, integrable $\Gamma_{\mathcal{R}}$ structures pull back through transversal maps, and in particular, solutions to Diff-invariant differential relations pull back through codimension zero immersions. Second, immersions and transversal maps from open manifolds are governed by homotopy theory. Thus we can apply the Smale-Hirsch immersion theorem to maps from M to *any* space admitting an integrable $\Gamma_{\mathcal{R}}$ structure. In Section 3, we explore this technique to find a metric on any 3-manifold that is flat outside of a neighborhood of a point (Proposition 3.5), some topological criteria for Stein surfaces to admit hyperkahler and Kahler-Einstein structures (Propositions 3.1, 3.8), and solutions to the Einstein field equations with all stress energy concentrated in a small neighborhood of a spacetime point (Proposition 3.9).

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2 *h*-principle for *Diff*-invariant relations on open manifolds

In this section, we review the language of groupoids and classifying spaces, and prove Theorem 1.2.

2.1 The Classifying Space of a Topological Groupoid

First, to fix terminology, we will review the basic notions needed to state the existence of the classifying space of a topological groupoid. This presentation follows [3].

Definition 2.1. A *groupoid* is a category whose morphisms are invertible and form a set Γ. Abusing notation, we will denote the groupoid by Γ as well. Let B be the set of objects of Γ. Let $\alpha, \beta : \Gamma \to B$ denote the source and target maps, $\gamma : \Gamma \times_B \Gamma \to \Gamma$ denote composition (where $\Gamma \times_B \Gamma$ is the fiber product of β with α), $u : B \to \Gamma$ the unit map, and $i : \Gamma \to \Gamma$ denote the map that sends a morphism to its inverse.

Definition 2.2. A *topological groupoid* is a groupoid such that Γ and B are topological spaces, and α, β, γ, u, and i are all continuous maps.

Remark 2.3. If we endow the of units U of Γ with the subspace topology, then u gives a homeomorphism $B \to U$, since α is a continuous left and right inverse.

Let X be a topological space, and let $\{U_i\}_{i\in I}$ be a covering of X. A 1-*cocycle* γ over $\{U_i\}$ with values in a topological groupoid Γ is a choice, for every pair $i, j \in I$, of a continuous map $\gamma_{ij} : U_i \cap U_j \to \Gamma$ such that $\gamma_{ij}(x)\gamma_{jk}(x) = \gamma_{ik}(x)$ for all $x \in U_i \cap U_j \cap U_k$. Note that this relation implies that γ_{ii} maps U_i to the set of units; since the set of units as a subspace of Γ is homeomorphic to B, we will at times think of γ_{ii} as maps from U_i to B, which will be denote by $\bar{\gamma}_i$. Given another 1-cocycle γ' over $\{U'_k\}_{k\in K}$, we say that γ and γ' are *equivalent* if for all $i \in I$ and $k \in K$ there exist continuous maps $\delta_{ik} : U_i \cap U_k \to \Gamma$ such that

$$\delta_{ik}(x)\gamma' kl(x) = \delta_{il}(x) \text{ for } x \in U_i \cap U'_k \cap U'_l \text{ and}$$
$$\gamma_{ji}(x)\gamma' ik(x) = \delta_{jk}(x) \text{ for } x \in U_i \cap U_j \cap U'_k$$

We will denote equivalence between two 1-cocycles γ, γ' by $\gamma = \gamma'$.

Remark 2.4. **??** By the previous equations, we have that

$$\gamma_{ij}(x)\gamma_{jj}(x)\gamma_{ij}^{-1}(x) = \gamma_{ii}(x)$$

on $x \in U_i \cap U_j$. But this means exactly that

$$\gamma_{ij} \circ \bar{\gamma}_j(x) = \bar{\gamma}_i(x).$$

Definition 2.5. A Γ-*structure* on X is an equivalence class of 1-cocycles on X with values in Γ.

If $f : X \to Y$ is a continuous map of topological spaces, then Γ-structures on Y naturally pull back to Γ-structures on X: the sets $f^{-1}(U_i)$ cover X, and the maps $\gamma_{ij} \circ f$ define the pullback Γ structure.

A map of topological groupoids $\pi : \Gamma \to \Gamma'$ defines a natural map from 1-cocycles with values in Γ to 1-cocycles with values in Γ': if γ is a 1-cocycle over the cover $\{U_i\}$ with values in Γ, then $\pi(\gamma)$ is a cocycle over $\{U_i\}$ with values in Γ' defined by $\pi(\gamma)_{ij}(x) = \pi(\gamma_{ij}(x))$. This map respects the equivalence relation on 1-cocycles, and so defines for any space X a map, also denoted π, from Γ-structures on X to Γ' structures on X.

Remark 2.6. Let $\Gamma - \text{struct} : Top \to Set$ be the functor sending a topological space X to the set of Γ-structures on X. It is easy to check that π is *natural*, in the sense that π is a natural transformation from $\Gamma - \text{struct}$ to $\Gamma' - \text{struct}$.

Definition 2.7. A *homotopy of* Γ-*structures* between Γ-structures σ_0, σ_1 on X is a Γ structure σ on $X \times [0,1]$ such that $i_0^*\sigma = \sigma_0, i_1^*\sigma = \sigma_1$ where $i_t : X \simeq X \times \{t\} \subset X \times [0,1]$ denotes the inclusion map.

To construct a classifying space for Γ-structures, one needs the following technical notion from point-set topology:

Definition 2.8. An open covering $\{U_i\}_{i \in I}$ of a topological space X is said to be *numerable* if there is a locally finite partition of unity $\{u_i\}_{i \in I}$ such that $u_i^{-1}(0,1] = U_i$. A Γ-structure on X is *numerable* if it can be defined by a 1-cocycle over a numerable covering. Two numerable Γ-structures are *numerably homotopic* if they are connected by a homotopy which is numerable.

Remark 2.9. Note that on a paracompact space, e.g. a smooth manifold, any covering is numerable.

Haefliger proves the following for all topological groupoids Γ:

Theorem 2.10. *([3], §7) There exists a space $B\Gamma$ with a numerable Γ-structure ω such that for any numerable Γ-structure σ on a space X, there exists a continuous map $f : X \to B\Gamma$ such that $f^*\omega = \sigma$. If $f_0, f_1 : X \to B\Gamma$ are two continuous maps, then $f_0^*\omega$ is numerably homotopy to $f_1^*\omega$ if and only if f_0 and f_1 are homotopic.*

Remark 2.11. If Γ_G is Lie group G viewed as a topological groupoid with one object, then a Γ-structure on X is a G-principal bundle on X.

2.2 The groupoid associated to a Diff-invariant differential relation

In this section, we define the uniquely-defined groupoid $\Gamma_{\mathcal{R}}$ associated to a Diff-invariant differential relation $\mathcal{R}$ on a manifold M.

Definition 2.12. We say that a topological groupoid Γ is *ètale* if α, the map sending morphisms to their sources, is a local homeomorphism.

Example 2.13. Let Γ_n be the groupoid of local diffeomorphisms of $\mathbb{R}^n$. This groupoid has base $B = \mathbb{R}^n$ and morphisms from $p \in \mathbb{R}^n$ to $q \in \mathbb{R}^n$ given by germs of diffeomorphisms defined on a neighborhood of p that take p to q. Each diffeomorphism between open sets $\phi : U \to V \subset \mathbb{R}^n$ defines a map $\bar{\phi} : U \to \Gamma$, sending $p \in U$ to the germ of ϕ at p. The set of morphisms is topologized in the étale topology, the weakest topology such that all the maps $\bar{\phi}$ are homeomorphisms. In this topology, Γ_n is ètale.

Remark 2.14. Given an open set $U \subset \mathbb{R}^n$, a section of $\alpha : \Gamma_n \to \mathbb{R}^n$ over U then exactly a diffeomorphism from U to some other open $V \subset \mathbb{R}^n$.

Remark 2.15. For any manifold M, there exists an analogous ètale topological groupoid Γ_M of local diffeomorphisms of M with base M. This construction makes sense even if M is non-Hausdorff.

Definition 2.16. *(The groupoid associated to a Diff-invariant differential relation)* Let X be a smooth fiber bundle over M^n, with M^n connected. Let $\mathcal{R} \subset X^{(r)}$ be a Diff-invariant differential relation on X. Take a diffeomorphism from $\mathbb{R}^n$ to a small open in M^n, and pull $\mathcal{R}$ through this diffeomorphism to a Diff-invariant differential relation $\mathcal{R}'$ on $\mathbb{R}^n$.

Let B be the sheaf of germs of holonomic sections of $\mathcal{R}'$. A point in B is thus a tuple (p, f), where p is a point of $\mathbb{R}^n$ and f is the germ of a holonomic section to $\mathcal{R}'$ defined near f (in other words, the germ of a solution to $\mathcal{R}'$ around p). The étale topology naturally endows B with the structure of a smooth non-Hausdorff n-manifold: the atlas of charts for this smooth structure is generated by the smooth holonomic sections of $\mathcal{R}'$ over all opens in $\mathbb{R}^n$.

Let $\Gamma_{\mathcal{R}}$ be the topological groupoid defined as follows. The base of $\Gamma_{\mathcal{R}}$ is B. Morphisms from (p, f) to (q, g) are germs of local diffeomorphisms h of $\mathbb{R}^n$ defined near p, such that $h(p) = q$ and $h^*g = f$. Since the set of morphisms form a sheaf over B, we topologize it with the étale topology. Then $\Gamma_{\mathcal{R}}$ is called the *groupoid associated to* $\mathcal{R}$.

It is not completely obvious that $\Gamma_{\mathcal{R}}$ is well-defined. However, given two diffeomorphisms ϕ_0, ϕ_1 from $\mathbb{R}^n$ to small opens in M, by the connectedness of M and of $\operatorname{Diff} \mathbb{R}^n$, we can find a diffeomorphism $g : M \to M$ such that $g \circ \phi_0 = \phi_1$. Since $\mathcal{R}$ is Diff-invariant, solutions to $\mathcal{R}$ over open sets in $\phi_0(\mathbb{R}^n)$ correspond to solutions to $\mathcal{R}$ over open sets in $\phi_1(\mathbb{R}^n)$ via g; moreover, given a diffeomorphism $\xi : U \to V$ betwen opens in $\phi_0(\mathbb{R}^n)$ pulling a solution over V to a solution over U, the conjugate of ξ by g pulls back the corresponding solution on $g(V)$ to the corresponding solution on $g(U)$. But solutions to $\mathcal{R}$ on opens in $\phi_i(\mathbb{R}^n)$ are exactly local solutions the pullback of $\mathcal{R}$ by ϕ_i. Following this reasoning, it is elementary to show that using g, one construct an isomorphism of ètale groupoids between the two $\Gamma_{\mathcal{R}}$ constructed from ϕ_0 and ϕ_1; so both $\mathcal{R}'$ and $\Gamma_{\mathcal{R}}$ are well-defined.

Remark 2.17. The same reasoning shows that solutions to $\mathcal{R}$ on any sufficiently small open ball $U \subset M$ are exactly pullbacks of solutions to $\mathcal{R}'$ over small balls $V \subset M$, by any diffeomorphism $U \to V$.

Remark 2.18. There is a map of topological groupoids $r : \Gamma_{\mathcal{R}} \to \Gamma_n$. Its action on the base B sends (p, f) to p, and its action on morphisms is the identity. By the definition of the smooth structure on B, r induces a local diffeomorphism $B \to \mathbb{R}^n$, which corresponds to the structure of B as the ètale space of a sheaf over $\mathbb{R}^n$.

2.3 Integrability of Γ-structures

Suppose Γ is a topological groupoid such that its base has the structure of a (possibly non-Hausdorff) manifold. This section defines *integrable* Γ*-structures*, which generalize the notion of a solution to a Diff-invariant differential relation.

Definition 2.19. A Γ*-foliation*, or an *integrable* Γ*-structure* on a manifold M is a Γ-structure admitting a representing 1-cocycle γ with $\bar{\gamma}_i$ a submersion to B for all i.

Definition 2.20. An integrable homotopy of Γ-foliations is a homotopy of Γ-structures through Γ-foliations.

Definition 2.21. Let M be a manifold with a Γ-foliation σ with a representative 1-cocycle γ and $f : N \to M$ be a smooth map. We say that f is transverse to σ if $\bar{\gamma}_i \circ f : f^{-1}(U_i) \to B$ are submersions for all i. It is straightforward to check that Γ-foliations, when viewed as Γ-structures, pull back to Γ-foliations under transverse maps, and that the property of being a Γ-foliation is independent of the choice of representative cocycle.

Example 2.22. An integrable $\Gamma_{\mathbb{R}^n}$ structure is exactly a codimension n foliation. The previous argument proves the well-known fact that foliations pull back under transverse maps.

Remark 2.23. The map from $\Gamma_{\mathcal{R}}$-structures to Γ_n structures induced by r sends integrable $\Gamma_{\mathcal{R}}$ structures to integrable Γ_n structures. This follows immediately from the fact that the action of r from the base of $\Gamma_{\mathcal{R}}$ to the base of Γ_n is a local diffeomorphism (Remark 2.18), and the fact that the composition of a submersion with a diffeomorphism is a submersion.

The same properties can be used to show that if a map is transverse to a $\Gamma_{\mathcal{R}}$-structure, then it is transverse to the induced Γ_n-structure, *and vice versa* (since a map $\xi : X \to Y$ is a submersion if and only if $\chi \circ \xi$ is a submersion, where χ is an arbitrary local diffeomorphism $Y \to Z$).

Remark 2.24. There is a map of topological groupoids $\nu : \Gamma_{\mathbb{R}^n} \to \Gamma_{GL_n(\mathbb{R})}$. The base of $\Gamma_{GL_n(\mathbb{R})}$ is a point $*$, and ν sends all elements in the base of $\Gamma_{\mathbb{R}^n}$ to $*$. Given a germ of a local diffeomorphism of $\mathbb{R}^n$ from p to q, we can take its derivative at p to get an element of $GL_n(\mathbb{R})$. The map ν sends morphisms in $\Gamma_{\mathbb{R}^n}$ to their derivatives at their source.

Example 2.25. Given an integrable $\Gamma_{\mathbb{R}^n}$-structure σ on a manifold N, the $\Gamma_{GL_n(\mathbb{R})}$-structure induced by ν defines a vector bundle ν_σ that is isomorphic to the normal bundle of the foliation corresponding to σ.

Remark 2.26. Let M^n be a manifold with Diff-invariant differential relation $\mathcal{R}$. An integrable $\Gamma_{\mathcal{R}}$-structure on M is *exactly* the data of a solution to $\mathcal{R}$. We will sketch one direction; the other direction is immediate from Remark 2.17. Suppose M admits an integrable $\Gamma_{\mathcal{R}}$-structure with representing 1-cocycle γ over an open cover $\{U_i\}$. Let B be the base of $\Gamma_{\mathcal{R}}$, and let r be the map defined in Remark 2.18. Then, $\bar{\gamma}_i$ is a submersion to B (in fact, a diffeomorphism, since B has the same dimension as M by definition). Since r is a local diffeomorphism $B \to \mathbb{R}^n$, means that after possibly refining the open cover $\{U_i\}$, $r \circ \bar{\gamma}_i : U_i \to B \to \mathbb{R}^n$ is a submersion onto an open set $V_i \subset \mathbb{R}^n$. But a point of B is a tuple (p, f), where $p \in V_i \subset \mathbb{R}^n$ and f is a germ of a solution to $\mathcal{R}'$ around p (see Definition 2.16). Thus, γ_{ii} defines a map sending $p \in V_i$ to f in the ètale space of the sheaf of solutions to $\mathcal{R}'$, i.e. a solution h_i to $\mathcal{R}'$ over V_i. Let $f_i = (r \circ \bar{\gamma}_i)^* h_i$; by Remark 2.17, for a sufficiently small U_i, f_i will be a solution to $\mathcal{R}$. But $r \circ \gamma_{ij}$ then defines a diffeomorphism from $r \circ \bar{\gamma}_j(U_i \cap U_j)$ to $r \circ \bar{\gamma}_i(U_i \cap U_j)$, with the property that $(r \circ \gamma_{ij})^* h_i = h_j$. By the functoriarity of r and the 1-cocycle equations for γ (see Remark **??**), we have that on $U_i \cap U_j$, $(r \circ \gamma_{ij}) \circ (r \circ \bar{\gamma}_j) = (r \circ (\gamma_{ij} \circ \bar{\gamma}_j) = r \circ \bar{\gamma}_i$. In other words, on $U_i \cap U_j$, $f_i = (r \circ \bar{\gamma}_i)^* h_i = ((r \circ \gamma_{ij}) \circ (r \circ \bar{\gamma}_j))^* h_i = (r \circ \bar{\gamma}_j)^* (r \circ \gamma_{ij})^* h_i = (r \circ gam\bar{m}a_j)^* h_j = f_j$; so the f_i glue together to a global solution to $\mathcal{R}$.

2.4 The main argument

Theorem 2.27. *Let M be an open manifold. Let $\mathcal{R}$ be a Diff-invariant differential relation on $\mathbb{R}^n$. Let $B\Gamma_{\mathcal{R}}$ be the space defined by Theorem 2.10, $\omega_{\mathcal{R}}$ be the $\Gamma_{\mathcal{R}}$ structure on $\Gamma_{\mathcal{R}}$, and $\nu\omega_{\mathcal{R}}$ be the vector bundle on $B\Gamma_{\mathcal{R}}$ corresponding to the $\Gamma_{GL_n(\mathbb{R})}$ structure induced from $\omega_{\mathcal{R}}$ by the compostion $\Gamma_{\mathcal{R}} \to \Gamma_{\mathbb{R}^n} \to \Gamma_{GL_n(\mathbb{R})}$.*

There is a bijection between integrable homotopy classes of $\Gamma_{\mathcal{R}}$-foliations on M and homotopy classes of epimorphisms of TM to $\nu\omega_{\mathcal{R}}$.

Proof. An integrable $\Gamma_{\mathcal{R}}$-foliation F on M is a $\Gamma_{\mathcal{R}}$ structure, and so by Theorem 2.10, $F = f^*\omega_{\mathcal{R}}$ for some $f : M \to B\Gamma_{\mathcal{R}}$. By the functoriality of ν (Remark 2.6), $f^*\nu\omega_{\mathcal{R}} \simeq \nu f^*\omega_{\mathcal{R}} \simeq \nu F$, so there is a bundle map $\phi : \nu_F \to \nu\omega_{\mathcal{R}}$ covering f that is an isomorphism on each fiber. But νF is the normal bundle to the codimension n foliation induced by F, and so admits an epimorphism $q : TM \to TM/F = \nu F$; the bundle map $\phi \circ q : TM \to \nu\omega_{\mathcal{R}}$ is the desired epimorphism. Given a homotopy of foliations foliations, this gives a homotopy of epimorphisms.

Let $\phi : TM \to \nu\omega_{\mathcal{R}}$ be an epimorphism of vector bundles whose projection is $f : M \to B\Gamma_{\mathcal{R}}$. Let $\sigma = f^*\omega_{\mathcal{R}}$. Then there exists a map $i : M \to E$ where E is a finite dimensional smooth manifold with a $\Gamma_{\mathcal{R}}$-foliation ϵ such that $i^*\epsilon = \sigma$. We will describe this construction explicitly; it is almost identical to the well-known construction of the foliated microbundle associated to a Haefliger structure. Take a defining 1-cocycle $\{\gamma_{ij}\}$ for σ, defined over a covering $\{U_i\}$ of M. Using the paracompactness of M and possibly refining the U_i, we can assume that $\bar{\gamma}_i$ maps U_i into a basis element V_i of the topology on the base B of $\Gamma_{\mathcal{R}}$; then $V_i = g_i(V_i')$, for some open subset $V_i' \subset \mathbb{R}^n$ and some solution g_i to $\mathcal{R}$ over V_i'. Let $G_i \subset U_i \times V_i'$ be the graph of $g_i^{-1} \circ \bar{\gamma}_i = r \circ \bar{\gamma}_i$, let $s_i : U_i \to U_i \times V_i'$ be the section corresponding to $g_i^{-1} \circ \bar{\gamma}_i$, and let p_k^i be the projection from G_i to the first and second factor, respectively. Let $\epsilon_i : U_i \times V_i' \to B$ be $g_i \circ p_2^i$. Let G_{ij} be the restriction of G_i to $(U_i \cap U_j) \times V_i'$. Then we have a diffeomorphism w_{ij} from some open neighborhood W_{ij} of G_{ij} inside $(U_i \cap U_j) \times V_i'$ to some open neighborhood of G_{ji} inside $(U_i \cap U_j) \times V_j'$ which can be written in the form (id, ρ_{ij}), where $\rho_{ij} = r \circ \gamma_{ij}$; then $\rho_{ij}^* g_j = g_i$ and $w_{ij} \circ s_i = s_j$.

Choose open neighborhoods W_i of G_i inside $U_i \times V_i'$ such that $W_i|_{(U_i \cap U_j) \times V_i'} \subset W_{ij}$ for all j (this is possible because of the fact that $\{U_i\}$ is a locally finite cover). The definition of a $\Gamma_{\mathcal{R}}$ structure implies that $\rho_{ij}\rho_{jk} = (r \circ \gamma_{ij})(r \circ \gamma_{jk}) = r \circ (\gamma_{ij}\gamma_{jk}) = r \circ \gamma_{ik} = \rho_{ik}$; so, $w_{ij}w_{jk} = w_{ik}$. Therefore, we can use the diffeomorphisms w_{ij} to glue the W_i into a manifold E. Because the $p_1 w_{ij} = p_1$, E inherits a projection $p : E \to M$; because the $w_{ij}s_i = s_j$, there is a section s of p, which we call i. Each of the W_i had a $\Gamma_{\mathcal{R}}$ structure given by the one-element cocycle (viewed as a map to B) $\bar{\epsilon}_i : W_i \to U_i \times V_i' \xrightarrow{p_2^i} V_i' \xrightarrow{g_i} B$, where the first arrow is the inclusion as an open submanifold. I claim that these $\Gamma_{\mathcal{R}}$ structures glue together to give a $\Gamma_{\mathcal{R}}$ structure on W_i with the desired properties. Let ϵ_{ij}' be the map from $V_i = g_i(V_i')$ to $V_j = g_j(V_j')$ acting by $v \mapsto g_j \rho_{ij} g_i^{-1} v$. But the equation $\rho_{ij}^* g_j = g_i$ implies exactly that $\epsilon_{ij}' \bar{\epsilon}_i = \bar{\epsilon}_j$ on W_{ij}. Thus, the ϵ_{ij}' together with the $\bar{\epsilon}_i$ define a $\Gamma_{\mathcal{R}}$ structure on E, which we denote by ϵ. Furthermore, $i^*\epsilon = \sigma$ since this is true over $E|_{U_j}$ for each j.

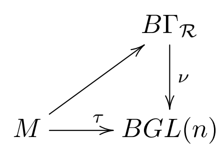
By the functoriality of ν (see Remark 2.6), $i^*\nu\epsilon \simeq \nu\sigma \simeq f^*\nu\omega_{\mathcal{R}}$. Hence, ϕ defines an epimorphism to $i^*\nu\epsilon$, and thus an epimorhism to $\nu\epsilon$ covering i. By the Gromov-Philips Transversality theorem (Proposition 4.3 and Remark 4.4), i is homotopic to a map transverse to the codimension-n foliation corresponding to ϵ. But this map is then also transverse to ϵ, and so ϵ pulls back to the desired integrable Γ-foliation on M.

Given a homotopy of epimorphisms $TM \to \nu\omega_{\mathcal{R}}$, we get an underlying homotopy of maps $M \to B\Gamma_{\mathcal{R}}$, which, by Theorem 2.10 gives a homotopy of $\Gamma_{\mathcal{R}}$ structures by pullback (since any homotopy of maps from a manifold M is numerable). A homotopy of $\Gamma_{\mathcal{R}}$-structures over M is a $\Gamma_{\mathcal{R}}$-structure over $M \times [0,1]$; applying the construction of E to the $\Gamma_{\mathcal{R}}$ structure on $M \times [0,1]$ and using the parametric version of the Gromov-Philips Transversality theorem gives a family of maps $M \to E|_{M \times t}$ transverse to the integrable integrable $\Gamma_{\mathcal{R}}$ structures on each of the $M \times t$ that is homotopic to the original inclusion $M \times [0,1] \to E$. By pullback, this gives homotopy of $\Gamma_{\mathcal{R}}$-foliations F_t on M that is homotopic by some homotopy h to the homotopy of $\Gamma_{\mathcal{R}}$ structures produced by Theorem 2.10. Thus, given a homotopy of epimorphisms $TM \to \nu\omega_{\mathcal{R}}$, we have produced a homotopy of $\Gamma_{\mathcal{R}}$-foliations on M. Applying the ν functor to h shows that that the homotopy of epimorphisms $TM \to \nu F_t \to \nu\omega_{\mathcal{R}}$ is homotopic to the original family of epimorphisms $TM \to \nu\omega_{\mathcal{R}}$, proving the hard direction in the bijection.

□

The above theorem almost immediately proves Theorem 1.2.

Proof. By Remark 2.26, an integrable $B\Gamma_{\mathcal{R}}$-structure on M^n is the same as a solution to $\mathcal{R}$. But $B\Gamma_{\mathcal{R}} = B\Gamma_{\mathcal{R}'}$ where $\mathcal{R}'$ is the pullback of $\mathcal{R}$ to $\mathbb{R}^n$ by an arbitrary diffeomorphism of M to a small ball on M. Applying Theorem **??**, this means that homotopy classes of integrable $\Gamma_{\mathcal{R}}$-structures are in bijection with homotopy classes of epimorphisms from TM to $\nu\omega_{\mathbb{R}}$. But $\nu\omega_{\mathbb{R}}$ has the same dimension as TM, so since isomorphism classes of n-dimensional vector bundles on M are in bijection with homotopy classes of maps $M \to BGL(n)$, this exactly corresponds to liftings



where τ is the map classifying TM, as desired. □

3 Examples of Geometric Structures

3.1 Hyperkähler structures on Stein manifolds

Consider a Stein manifold M of complex dimension n. Let $\tau : M \to BU(n)$ be the map classifying the tangent bundle of TM. If M admits a hyperkähler structure, then n must be even and the map τ must factor through the map $\pi : BSp(n/2) \to BU$. Suppose τ factors through π; then we say that M is *formally hyperkähler*. An easy topological computation shows the following:

Proposition 3.1. *All formally hyperkähler Stein manifolds of complex dimension* 2 *admit hyperkähler structures.*

Proof. Since $Sp(1) = SU(2) = S^3$, $\pi_i(BSp(1)) = 0$ for $i = 0 \ldots 3$. However, any open manifold of real dimension 4 is of the homotopy type of a 3-complex (Proposition 4.2), so $\tau : M \to BSp(1)$ is homotopic to a constant map; in other words, M is parallelizable. But now, by the Smale-Hirsch (Proposition 4.1) immersion theorem, this implies that M admits an immersion to $\mathbb{C}^2$. This immersion is between equidimensional spaces, so it is a local diffeomorphism; thus, the standard hyperkähler structure on $\mathbb{C}^2$ pulls back to a hyperkähler structure on M. □

This construction will generally construct metrics that are incomplete. For example in Figure 3.6 we draw an immersion $T^2 \setminus B$ (where T^2 is a torus and B is a small ball) into $\mathbb{R}^2$. The induced flat metric on $T^2 \setminus B$ is very incomplete.

Indeed, the $T^2 \setminus B$ cannot admit a complete flat metric. If it admitted a complete flat metric, then it would be the quotient of its universal cover, a simply connected complete flat manifold, by a group of isometries. But the only simply connected surface is a disk, and the only flat structure on it is the one on $\mathbb{R}^2$ by the uniformization theorem. The only isometries of $\mathbb{R}^2$ that produce orientable manifolds are translations, and any manifold quotient of $\mathbb{R}^2$ by a group generated by translations is either a cylinder or a torus.

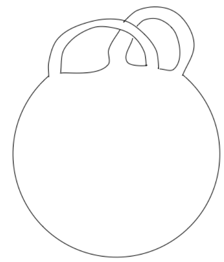


Figure 1: Immersion of $T^2 \setminus B$ into $\mathbb{R}^2$.

Remark 3.2. One could have guessed that such an immersion must exist by noting that the complement of point in a a genus g surface Σ is parallelizable. (A topological way to see this is to use that the complement of a point is homotopy equivalent to a 1-complex (Proposition 4.2), so the tangent bundle admits a nowhere zero section; but the tangent bundle to a surface admits an almost complex structure J, and so any nowhere zero section gives another linearly independent one one upon application of J.)

Definition 3.3. Given a manifold M, we say that *curvature on M can be localized* if M admits a complete Riemmanian metric that is flat outside of an arbitrarily small neighborhood of a point.

Remark 3.4. There is no obstruction to extending Riemannian metrics, so the argument in Remark 3.2 implies that curvature on any compact surface Σ can be localized. Let B be an arbitrarily small ball on Σ. We localize Σ by immersing $\Sigma \setminus B$ into $\mathbb{R}^2$, pulling back the flat Euclidean metric to $\Sigma \setminus B$, and extending the metric to Σ. This metric must be complete since the surface is compact. Notice that the resulting metric near this point must have total curvature equal to $2\pi\chi(\Sigma)$, as required by Gauss-Bonnet.

It is well known that all 3-manifolds are parallelizable; thus, applying Smale-Hirsch (Proposition 4.1) in the complement of a ball and extending the resulting flat metric, we see that curvature on any 3-manifold can be localized.

We can state the following generalizations of the argument in this section:

Proposition 3.5. *Given an open parallelizable manifold M (e.g. any M with $\dim M = 3$) and a Diff-invariant differential relation $\mathcal{R}$, if $\mathcal{R}$ admits a solution in any nonempty open set U in M then M admits a solution of $\mathcal{R}$.*

Proposition 3.6. *Given open n-manifolds M, U, a Diff-invariant differential* *relation $\mathcal{R}$ on $\mathbb{R}^n$, and a map $h : M \to U$ satisfying the commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\tau_M} & \\ \downarrow h & \searrow & \\ U & \xrightarrow{\tau_U} & BGL(n) \end{array}$$

if U admits an integrable $\Gamma_{\mathcal{R}}$ structure then so does M.

Both of these hold because by the Smale-Hirsch immersion theorem (Proposition 4.1, M can be immersed into U. Proposition 3.6 together with Theorem 1.2 immediately proves that

Proposition 3.7. *Using the notation of Proposition 3.6, if h is a homotopy equivalence, then homotopy classes of integrable $\mathcal{R}$-structures on M are in bijection with homotopy classes of integrable $\mathcal{R}$-structures on U.*

This can be rephrased loosely as "Diff-invariant differential relations on open manifolds are controlled by homotopy theory".

3.2 More complex examples

If our manifold M is not parallelizable, we can try solving Diff-invariant differential relations immersing M into a more complicated manifold than $\mathbb{R}^n$. For example,

Proposition 3.8. *Any Stein surface M with 3-divisible $c_1(M) \in H^2(M, \mathbb{Z})$ admits a Kähler-Einstein metric.*

Proof. Note that $\mathbb{P}^2$ admits a Kähler-Einstein metric given by the Fubini-Study metric, and that $c_1(\mathbb{P}^2) = 3h$, where h denotes the hyperplane class. Consider the inclusion of a complex line $i : \mathbb{P}^1 \to \mathbb{P}^2$. Let $[\mathbb{P}^1]$ denote the fundamental homology class of $\mathbb{P}^1$ and $[*]$ denote the class of a point. We know that i^*3h is some multiple n of the fundamental cohomology class $(\mathbb{P}^1)$ of $\mathbb{P}^1$. But $[\mathbb{P}^1] \cap (\mathbb{P}^1) = [*]$, so $[\mathbb{P}^1] \cap i^*3h = n[*]$. Applying i_* and using the push-pull formula for the cap product, we get that $ni_*[*] = i_*([\mathbb{P}^1] \cap i^*3h) = i_*[\mathbb{P}^1] \cap 3h = 3i_*[*]$. So $n = 3$.

Now, let $3b = c_1(M)$. There exists a 2-complex K with a homotopy equivalence $j : K \to M$. Let b be represented by a linear combination $\sum_i c_i D_i$ of 2-cells of K. Then there exists a map $K \to \mathbb{P}^1$ which pulls back the fundamental cohomology class of $\mathbb{P}^1$ to j^*b. This map can be constructed by collapsing the 1-skeleton of K to get a wedge of 2-spheres $\wedge_i S_i^2$ (with $D_i/\partial D_i$ naturally identified with S_i^2), and composing with a map $\wedge_i S_i^2 \to S^2$ that restricts to a degree c_i map on each S_i^2. This map then then pulls back three times the fundamental class of $\mathbb{P}^1$ to $3j^*b = j^*c_1(M)$. Composing with i and precomposing with the homotopy inverse of j, we get a map $l : M \to \mathbb{P}^2$ such that $l^*c_1(\mathbb{P}^2) = c_1(M)$.

This in turn implies that $l^*T\mathbb{P}^2 \simeq TM$ as complex topological bundles. First, by naturality of the Chern class, $c_1(l^*T\mathbb{P}^2) = l^*c_1(\mathbb{P}^2) = c_1(M)$. Now both of these bundles are real 4-bundles over a space homotopy equivalent to a 2 complex, so they admit a nowhere-zero section; applying the complex structure, they both admit a trivial complex subbundle. Since exact sequences of topological bundles split, we can write $l^*T\mathbb{P}^2 = L_1 \oplus \mathbb{C}$, $TM = L_2 \oplus \mathbb{C}$, where $\mathbb{C}$ denotes the trivial complex line bundle. But then by additivity of the first Chern class, $c_1(l^*T\mathbb{P}^2) = c_1(L_1), c_1(TM) = c_1(L_2)$, and since topological line bundles are determined by their first Chern class we are done.

Thus we have a continuous map $M \to \mathbb{P}^2$ and a covering bundle monomorphism $TM \to T\mathbb{P}^2$, so by the Smale-Hirsch theorem (Proposition 4.1) there exists an equidimensional immersion $M \to \mathbb{P}^2$. Pull back the Kähler-Einstein metric on $\mathbb{P}^2$ to M. □

3.3 Complete solutions to Einstein equations and localization of curvature

According to general relativity, our universe is supposed to be described by a 4-manifold M with a Lorentzian metric g and a stress-energy tensor T satisfying the Einstein field equations $\mathrm{Ric}(g) + \frac{1}{2}R(g) = (8\pi G/c^4)T$, where $\mathrm{Ric}(g)$ is the Ricci tensor and $R(g)$ is the scalar curvature. Given the importance of spinors in quantum field theory, one imagines that M admits a spin structure. A basic question is how the topology M constrains solutions to the Einstein equations.

Proposition 3.9. *Assume M^4 is compact, spin, and admits a Lorentzian metric. Then M admits a Lorentzian metric that is flat outside of an arbitrarily small ball.*

Proof. The existence of a spin structure forces $w_1(M) = w_2(M) = 0$. Pick a Riemannian metric g' on M. Let M' be the complement of a small open ball B of radius ϵ with respect to g around a point $p \in M$. Then M' is equivalent to a 3-complex (Proposition 4.2), so $H^4(M') = 0$, so $\chi(M') = p_1(M') = 0$. By the Dold-Whitney theorem, this implies that M' is parallelizable. We can thus (Proposition 4.1) immerse M' into standard Minkowski space $\mathbb{R}^{3,1}$, and pull back the standard Lorentz metric to a Lorentzian metric h on M'. I claim that for topological reasons, h restricted to the complement of a ball around p of radius $5\epsilon/4$ extends to a Lorentzian metric on M.

Indeed, $g|_{M'}$ is homotopic to h, since TM' is trivial. Let $I = [0,1]$. Pick an annulus $A = \{x \in M : d_{g'}(x,p) \in [\epsilon/4, 3\epsilon/4]\}$. Let $M'' = \{x \in M : d_{g'}(x,p) \geq \epsilon/4\}$. Since B was arbitrary, the tangent bundle of M restricted to $A \cup M'$ or to M'' is still trivial. Then $g|_{A\cup M'}$ can be viewed as a map from $A \cup M'$ to Q, the manifold of Lorentzian quadratic forms on $\mathbb{R}^4$. We have a trivial homotopy from $g|_A$ to itself; the disjoint union of this homotopy with the previous one gives a homotopy $(A \cup M') \times I \to Q$. The inclusion $A \cup M' \to M''$ is closed, and so is a cofibration, so the homotopy extension property implies that there exists an extension of this homotopy to a (continuous) homotopy $M'' \times I \to Q$. After this homotopy, the resulting map is smooth in the region $R = \{x \in M'' \times I : d_{g'}(x,p) \in (\epsilon/2, 9\epsilon/8)\}$, and so this map is homotopic to a smooth map rel

$M'' \setminus R$. The resulting map agrees with g in $\{x \in M : d(x,p) \in (\epsilon/4, \epsilon/2)\}$ and with h in $\{x \in M : d(x,p) \geq 5\epsilon/4\}$, and so we have the desired approximate extension of h to all of M. □

Remark 3.10. Notice that in this case, the metric constructed on M is complete, since M is compact.

4 Appendix

In this section, we collect a few useful theorems used in the paper.

Proposition 4.1. ***(Smale-Hirsch for equidimensional immersions from open manifolds)*** *Let V^n, W^n be manifolds with V open. If there is a map $f : V \to W$ and a bundle isomorphism $f^*TW \simeq TV$, then f is homotopic to an immersion.*

Proof. A smooth map $g : V \to W$ is an immersion if dg is nondegenerate. This condition is an open Diff-invariant differential relation on the natural bundle $W \times V \to V$, and so Proposition 1.1 applies. Thus, such a map exists exactly if this differential relation admits a section, i.e. a map $f : V \to W$ together with a bundle map $TV \to TW$ covering f that is nondegenerate. This is, in turn, equivalent to giving an isomorphism $f^*TW \simeq TV$. □

We also have the following useful observation:

Proposition 4.2. ***([1], 4.3.1)*** *Let V be an open manifold. If V is open, then there exists a polyhedron $K \subset V$, codim $K \geq 1$, such that V can be compressed by an isotopy $\phi_t : V \to V$, $t \in [0,1]$, into an arbitrarily small neighborhood U of K.*

Finally, for reference, we state a version of the Gromov-Philips Transversality theorem:

Proposition 4.3. ***(Gromov-Philips Transversality, [1], 4.6.2)*** *Let ξ be a plane field (distribution) on a q-dimensional manifold W, codim $\xi = k$. Let $n < k$. Then for any closed n-dimensional submanifold $V \cup W$ whose tangent bundle TV is homotopic insid TW to a subbundle $\tau \subset TW$ transversal to ξ, one can perturb V via an isotopy to make it transversal to ξ. The relative and parametric versions are also true.*

Remark 4.4. If ξ is the tangent bundle of a foliation, $i : V \to W$ is the inclusion, $\pi : TW \to TW/\xi$ the quotient map, and a homotopy of maps from $\pi \circ di$ to a submersion $TV \to TW/\xi$, then the theorem applies and V can be perturbed to be transverse to the foliation.

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