Dynamical Stability of Translators Under Mean Curvature Flow

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Abstract

Convergence of a class of perturbations of the line in $\mathbb{R}^3$ to a line under the curve shortening flow is proven. Progress is presented towards conditions on perturbations of the Grim Reaper cylinder which guarantee convergence under mean curvature flow to the Grim Reaper cylinder.

1 Introduction

A family of smooth n-dimensional submanifolds $M_t^n \subseteq \mathbb{R}^N$ ($t \in I \subseteq \mathbb{R}$) is said to evolve by the *mean curvature flow* (MCF)–called the *curve shortening flow* (CSF) when $n = 1$–if, for each time t, any point x of the manifold M_t has time derivative the mean curvature vector of M_t at x. Systematic study of mean curvature flow was first conducted by Brakke in *The Motion of a Surface by Its Mean Curvature* [Bra78]. Notably, this work provided the first *avoidance principle*–the statement that a hypersurface contained in a convex set will remain enclosed by the set as both flow by mean curvature–for closed hypersurfaces evolving by mean curvature flow and used techniques from geometric measure theory to guarantee existence (in a certain weak sense) of a family M_t^n flowing by mean curvature–i.e. satisfying the mean curvature flow–given an initial manifold M_0^n. The application of partial differential equations to study mean curvature flow was later pioneered by Huisken [Hui84], who used PDEs to show that any closed convex hypersurface $M_0^n \subset \mathbb{R}^{n+1}$ of dimension two or greater will flow to a single point under MCF. Analogous results for convex and closed curves respectively were presented by Gage-Hamilton [GH86] and Grayson [Gra87] in the following years, and related results for noncompact hypersurfaces were provided by Ecker and Huisken [EH89, EH91].

In specific, a theorem of Ecker and Huisken [EH89, Theorem 5.1] demonstrates existence for all times $t > 0$ of a family M_t of manifolds satisfying the mean curvature flow given an initial hypersurface M_0 satisfying a linear growth condition and a bound on the growth of its curvature. If M_0 is taken to be bounded in this setting, Ecker and Huisken showed that M_0 will flow by mean curvature to a plane. We adapt the work of Ecker and Huisken [EH89, Theorem 5.1] in the case that M_0 is a bounded curve to the case that M_0 is a bounded curve in codimension two in Section 3, providing the result of Theorem 1.1:

Theorem 1.1. *Consider smooth functions $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ along with the associated curve $\gamma_0(x) = (x, u_0(x), v_0(x))$, let T_0 be the tangent vector of γ_0, and take $e_x := (1, 0, 0)$. If there exist* *positive constants C_1 and C_2 such that $\langle T_0(x), e_x\rangle \geq C_1$ and $|u_0(x)^2 + v_0(x)^2| \leq C_2$ for all $x \in \mathbb{R}$ and all derivatives of κ are bounded on γ_0, the curve shortening flow with initial data γ_0 exists for all time and there exist constants c_y, c_z such that*

$$\lim_{t\to\infty} |\gamma(x,t) - (x, c_y, c_z)| = 0$$

for all $x \in \mathbb{R}$.

Informally, this result states that a smooth, bounded perturbation γ_0 of a line in $\mathbb{R}^3$ whose tangent vector never tends towards being orthogonal to this line will flow by mean curvature for all positive time $t > 0$ and converge to a line as $t \to \infty$. Our proof of this result in codimension 2 reduces to such a proof in codiemension 1 by eliminating any mention of the binormal vector B in the computations in Section 3. We believe a result in higher codimensions to be readily attainable, requiring only our calculations to account for extra binormal vectors.

We also provide, using methods of Clutterbuck-Schnürer-Schulze [CSS07], conditions on a perturbation u_0 of the Grim Reaper cylinder and on the family u of manifolds flowing by mean curvature with initial data u_0 which ensure that u will converge under mean curvature flow–and specifically that u will, in the sense of (1.1), converge to a Grim Reaper cylinder.

Theorem 1.2. *Consider a smooth solution*

$$u : \mathbb{R}^{n-1} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times [0,\infty) \to \mathbb{R}.$$

to the graphical mean curvature flow; i.e. so that this solution can be expressed as a family of graphs of smooth functions. If the sets

$$\Omega^{\pm}_{\epsilon,t} := \left\{ (x,y) \in \mathbb{R}^{n-1} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \middle| \, u(x,y,t) - \phi(x,y,t) \geq \pm\epsilon \right\}$$

are compact and have boundary contained in $\mathbb{R}^{n-1} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ for all $\epsilon > 0$ and $t \geq 0$, we have that

$$\lim_{t\to\infty} \sup_{(x,y)\in\mathbb{R}^{n-1}\times(-\frac{\pi}{2},\frac{\pi}{2})} |u(x,y,t) - \phi(x,y,t)| = 0. \tag{1.1}$$

This result assumes strong conditions on the family u which flows by mean curvature with initial data a perturbation of the Grim Reaper cylinder. We believe a result which only makes assumptions on the initial perturbation to be achievable, but to require extra constraints on this perturbation.

Section 3 is devoted to proving Theorem 1.1, while Section 4 proves Theorem 1.2. In each case, we prove long-time existence–i.e. existence for all $t > 0$–of a family of manifolds satisfying the mean curvature flow given an initial condition in a certain set of perturbations of the line or Grim Reaper cylinder. Additionally, the condition that the mean curvature flow preserves the entire-graphicality of hypersurfaces–which was shown by Ecker-Huisken [EH89]–and preserves forwards-tangency–the codimension-2 analogue, proven in Lemma 3.2, of a graphicality condition–are discussed. These results are used throughout the proofs below to parameterize our manifolds as they evolve by mean curvature flow, and are important in presenting *avoidance principles*–statements that if our initial perturbations do not intersect with a given plane or grim reaper cylinder, then they will never cross these manifolds as they evolve by mean curvature flow–to demonstrate boundedness under the flow in the linear case and translation under the flow in the Grim Reaper cylinder case.

2 Preliminaries

Formally, a family $X : M \times [0, \infty) \to \mathbb{R}^n$ of smooth manifolds is said to flow by mean curvature if

$$\partial_t X(x,t) = \vec{H}(x,t), \tag{2.1}$$

where $\vec{H}$ is the mean curvature vector of X. We similarly say that a function $u : \Omega \times [0, \infty) \to \mathbb{R}$ is a solution to the *graphical mean curvature flow* on some open domain $\Omega \subset \mathbb{R}^n$ if we have that

$$u_t = \sqrt{1 + |\nabla u|^2}\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \tag{2.2}$$

on Ω. In this case, the graph of u will admit a paramaterization which flows by mean curvature. When u is a family of curves in $\mathbb{R}^2$, (2.2) reduces to Equation 2.3.

$$u_t = \frac{u_{xx}}{1 + u_x^2} \tag{2.3}$$

Definition 2.1. *A solution to the mean curvature flow is called a translating solution (or a translator) if every point of the solution has time derivative the same fixed vector for all time.*

The simplest example of a translator is the line or plane, which stays fixed for all time when flowed by mean curvature. In the case of curves in $\mathbb{R}^2$, the only convex translating solution other than the line is the Grim Reaper curve, which is the graph of the function

$$\phi(x,t) = \log(\sec(x)) + t \quad \textit{for} \quad (x,t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.4}$$

In higher dimensions, translators include the plane and Grim Reaper cylinder–the Cartesian product of the Grim Reaper curve with some Euclidean space $\mathbb{R}^k$. More solutions, such as the *bowl solution* to the mean curvature flow [Has15], have also been investigated.

We now present known results we use throughout the following sections. We do not reprove these statements, but proofs of the results can be found in the original papers cited. In specific, the proofs in Section 3 make use of Theorem 2.2 and Lemma 2.3 to assist in proving Theorem 1.1, while Propositions 2.4 and 2.5 are used in Section 4 to assist in proving Theorem 1.2.

Theorem 2.2, which was proved by Ecker and Huisken [EH89, Corollary 1.1], provides a maximum principle for evolving hypersurfaces on noncompact domains. We note that this result holds for not just hypersurfaces but, more generally, for any manifolds.

Theorem 2.2. *Consider a smooth submanifold $M_t \subset \mathbb{R}^n$ evolving by mean curvature flow and a function $f(x,t)$ on M_t. If f satisfies the inequality*

$$(\partial_t - \Delta) f \leq a \cdot \nabla f$$

for some vector field a–where ∇ denotes the tangential gradient on M–and

$$\sup_{M \times [0,t_1]} |a| < \infty$$

for some $t_1 > 0$, then

$$\sup_{x \in M_t} f(x,t) \leq \sup_{x \in M_0} f(x,0)$$

for all $t \in [0, t_1]$.

The second result–due to Gage and Hamilton [GH86, Lemma 3.1.3]–concerns the interchanging of partial derivatives on manifolds. Gage and Hamilton derived this formula in the codimension-one case. In the higher-codimension case, one can show that the same formula still holds by direct calculation.

Lemma 2.3. *Consider a curve c_t evolving by the curve shortening flow and a function $f(x,t)$ on c_t. We have that*

$$f_{st} = f_{ts} + \kappa^2 f_s,$$

where $\frac{\partial}{\partial s}$ is the spacial derivative with respect to the arc length parameterization of c_t and κ is the curvature of c_t.

To aid in showing longtime existence of solutions to the mean curvature flow problems we consider, we introduce estimates provided by Ecker-Huisken [EH89] of the curvature and higher derivatives of solutions to the mean curvature flow.

Proposition 2.4. *Consider a smooth function $u_0 : \Omega \to \mathbb{R}$ with bounded sup norm such that the graph M_0 of u_0 has bounded curvature. F for $x \in B_r(0) \times [0,T]$, we have that*

$$|D^m u(0,t)| \leq C, \tag{2.5}$$

where C is a constant depending on u_0, T, r, m and the dimension n.

Also a result of Ecker and Huisken [EH91, Proposition 4.1], Proposition 2.5 presents an existence condition for solutions to the mean curvature Dirichlet problem; i.e. the mean curvature flow with fixed boundary. We do not produce results concerning the Dirichlet problem, but make use of the proposition in proving Corollary 3.5 and Theorem 4.1.

Proposition 2.5. *Suppose that $F_0 : \overline{M_n} \to \mathbb{R}^{n+1}$ is a smooth isometric immersion. There exists some T_0 depending on the second fundamental form of F_0 such that there exists a unique smooth function $F : M_n \times [0,T_0] \to \mathbb{R}^{n+1}$ satisfying the mean curvature flow such that $F(p,t) = F_0(p)$ whenever $p \in \partial M_n$*

3 Curve translators

In this section, we present results generalizing those of Ecker and Huisken [EH89] regarding stability of the line in codiemnsion one to higher codimension curves. Much of the work here–in specific, the work of Subsections 3.2 and 3.3–applies directly to the higher-codimension grim reaper stability problem, but an avoidance principle is not so clear. It is believed that such a principle can likely be found by bounding the perturbed grim reaper by a pair of grim reaper cylinders and planes. It is also believed that these results and the results proved for the line in codimension 2 should be provable in codimension n using many of the same proof methods described below. Future iterations of this report may contain treatments of these problems.

Consider smooth functions $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ and the associated curve $\gamma_0(x) = (x, u_0(x), v_0(x))$ in $\mathbb{R}^3$. We say that $\gamma(x,t)$ is a solution to the curve shortening flow on an interval $I \subseteq [0,\infty)$ with the initial data γ_0 if it satisfies

$$\begin{cases} \partial_t \gamma(x,t) = -\kappa(x,t) N(x,t) & \text{for } (x,t) \in \mathbb{R} \times I \\ \gamma(x,0) = \gamma_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{3.1}$$

where $\kappa(x,t)$ is the curvature at $\gamma(x,t)$ and $N(x,t)$ is the unit principal normal to γ; i.e. N is the unit vector in the direction of $\partial_s T$. We also denote by B a unit binormal to γ–a unit vector orthogonal to T and N which varies smoothly along γ.

We first prove Lemma 3.2, showing that $\langle T_0, e_x\rangle$ having a positive lower bound allows us to pick functions $u, v : \mathbb{R}\times I$ such that $\gamma(x,t) = (x, u(x,t), v(x,t))$ on $\mathbb{R}\times I$. This is used to show Lemma 3.3–the statement that any convex shape containing γ_0 will contain $\gamma(\cdot,t)$ for all $t \in I$–and the bounds on $\langle T, e_x\rangle$ computed in Subsection 3.1 are used to show the proofs in Subsection 3.3–which show the existence of a solution γ to our curve shortening problem 3.1 and that the curvature of $\gamma(\cdot,t)$ vanishes as $t\to\infty$. In Subsection 3.4, we provide explicit evolution equations for u_t and v_t used in the proofs in Subsection 3.3, and an example of a perturbation of the line which converges by curve shortening flow to the line in Subsection 3.5.

3.1 Preservation of forwards-tangency

We first define the *forwards-tangent property.*

Definition 3.1. *A curve c in $\mathbb{R}^3$ has the forwards-tangent property if the tangent vector T_c of c satisfies*

$$\langle T_c, e_x\rangle \geq C$$

for fixed $C > 0$ at all points in the domain of c. In this case, we say that c is forwards-tangent*.*

In this section, we prove the following lemma regarding forwards-tangency:

Lemma 3.2. *Suppose $\gamma(x,t)$ ($t \in I$) is a solution to (3.1). If $\gamma_0(x)$ is forwards-tangent, then $\gamma(x,t)$ will be forwards-tangent for all times $t \in I$.*

This lemma allows us to write $\gamma(x,t) = (x, u(x,t), v(x,t))$ for some functions u, v with $u(\cdot,t) = u_0(\cdot)$ and $v(\cdot,t) = v_0(\cdot)$. Note that this paramaterization flows by

$$\gamma_t = T_s + \frac{u_x u_{xx} + v_x v_{xx}}{(1+u_x^2+v_x^2)^{\frac{3}{2}}} T = -\kappa N + \frac{u_x u_{xx} + v_x v_{xx}}{(1+u_x^2+v_x^2)^{\frac{3}{2}}} T, \tag{3.2}$$

as adding any tangential component to our flow will not impact the evolution of our curve and we can see from our evolution equation in (3.19) that adding the tangent term

$$\frac{u_x u_{xx} + v_x v_{xx}}{(1+u_x^2+v_x^2)^{\frac{3}{2}}} T$$

ensures that the x component of γ_t is 0, as desired.

Proof of Lemma 3.2. Consider the function

$$f := \langle T, e_x\rangle.$$

By metric compatibility,

$$f_t = \langle T_t, e_x\rangle + \langle T, (e_x)_t\rangle = \langle T_t, e_x\rangle. \tag{3.3}$$

Equation (3.1) and Lemma 2.3 give us that

$$T_t = \gamma_{st} = \gamma_{ts} + \kappa^2\gamma_s = (-\kappa N)_s + \kappa^2 T. \tag{3.4}$$

Similarly to in (3.3), we have

$$f_{ss} = \langle T_{ss}, e_x \rangle. \tag{3.5}$$

Now, we have from (3.1) that

$$T_{ss} = (-\kappa N)_s. \tag{3.6}$$

Combining (3.4) and (3.6), we get

$$f_t - f_{ss} = \kappa^2 f, \tag{3.7}$$

and the conclusion follows from the maximum principle presented in Theorem 2.2. □

3.2 An avoidance principle

In this section, we prove the following lemma:

Lemma 3.3. *Suppose $\gamma(x,t)$ ($t \in I$) is a solution to (3.1) and we write $\gamma(x,t) = (x, u(x,t), v(x,t))$. If $|u(\cdot,0)|$ and $|v(\cdot,0)|$ are bounded, then*

$$\sup_{x\in\mathbb{R}} |u(x,t)| \leq \sup_{x\in\mathbb{R}} |u(x,0)| \quad \text{and} \quad \sup_{x\in\mathbb{R}} |v(x,t)| \leq \sup_{x\in\mathbb{R}} |v(x,0)|$$

for all $t \in I$.

This is the statement that if our perturbation is initially contained in a rectangular prism (infinitely long in one direction) that it will remain in this prism. By rotational invariance of the flow, we can in fact contain γ in any convex set containing γ_0.

Proof of Lemma 3.3. For any $w \in \mathbb{R}^3$ and $(x,t) \in \mathbb{R} \times I$, by (3.1), we see that

$$\partial_t \langle \gamma(x,t), w \rangle = \left\langle \partial_s^2 \gamma(x,t), w \right\rangle = \partial_s^2 \langle \gamma(x,t), w \rangle .$$

Thus, $\langle \gamma(x,t), w \rangle$ is a solution to the heat equation on $\mathbb{R} \times I$. Applying Theorem 2.2 with $f = \langle \gamma(x,t), w \rangle$ gives us that

$$\sup_{x\in\mathbb{R}} \langle \gamma(x,t), w \rangle \leq \sup_{x\in\mathbb{R}} \langle \gamma(x,0), w \rangle$$

for all $t \in I$. We see that taking $w = (0,1,0)$, $\langle \gamma(x,t), w \rangle = u(x,t)$. Thus, we see taking $w = (0,1,0)$ and $w = -(0,1,0)$ respectively that

$$\sup_{x\in\mathbb{R}} u(x,t) \leq \sup_{x\in\mathbb{R}} u(x,t_0) \quad \text{and} \quad \sup_{x\in\mathbb{R}} -u(x,t) \leq \sup_{x\in\mathbb{R}} -u(x,t_0).$$

So, we have that

$$\sup_{x\in\mathbb{R}} |u(x,t)| \leq \sup_{x\in\mathbb{R}} |u(x,t_0)|.$$

Taking $w = (0,0,1)$ and $w = (0,0,-1)$ gives the analogous result that

$$\sup_{x\in\mathbb{R}} |v(x,t)| \leq \sup_{x\in\mathbb{R}} |v(x,t_0)|.$$

□

3.3 Convergence of curvature

In this subsection, we prove the following statement regarding boundedness and decay of curvature of γ as $t \to \infty$, then use this result to prove that γ converges to a line as it evolves by the curve shortening flow.

Proposition 3.4. *Suppose $\gamma(x,t)$ $(t \in I)$ is a solution to (3.1). If $\gamma_0(x)$ is forwards-tangent and all derivatives of the curvature $\kappa(\cdot,t)$ of γ_t are bounded for all $t \in I$, then for any $m \in \mathbb{N} \cup \{0\}$, we can find $C_m > 0$ only depending on γ_0 such that*

$$\sup_{\mathbb{R}} \left| \frac{\partial^m \kappa}{\partial s^m}(\cdot, t) \right|^2 \leq \frac{C_m}{t^{m+1}}$$

for all $t \in I$.

Proof of Proposition 3.4. We only present the zero-order estimate; i.e. the estimate for κ. The higher derivative estimates can be derived by a standard argument based on induction (cf. [EH89, Proposition 4.4]).

We will first compute an evolution equation for the curvature κ of γ and show boundedness of κ, then prove κ decays in time. We see that

$$\kappa = \kappa\langle N, N\rangle = -\langle T_s, N\rangle,$$

so differentiating with respect to t, applying Lemma 2.3, and noting that $\langle N, N_t\rangle = 0$ since N is a unit vector, gives us that

$$\kappa_t = -\langle T_{st}, N\rangle = -\langle T_{ts} + \kappa^2 T_s, N\rangle - \langle T_s, N_t\rangle = -\langle T_{ts}, N\rangle + \kappa^3 - 0. \tag{3.8}$$

We now compute T_{ts}. We see from (3.1) and Lemma 2.3 that

$$T_t = \gamma_{st} = \gamma_{ts} + \kappa^2\gamma_s = (-\kappa N)_s + \kappa^2 T = -\kappa_s N - \kappa N_s + \kappa^2 T,$$

so we have that

$$T_{ts} = -\kappa_{ss}N - 2\kappa_s N_s - \kappa N_{ss} + 2\kappa\kappa_s T - \kappa^3 N. \tag{3.9}$$

We also see–taking $\tau := \langle N_s, B\rangle$ and noting that $\langle N_s, T\rangle = 0 - \langle N, T_s\rangle = \kappa$ by orthogonality of N and T–that

$$N_s = \langle N_s, T\rangle T + \langle N_s, B\rangle B = \kappa T + \tau B, \tag{3.10}$$

and thus

$$N_{ss} = \kappa_s T - \kappa^2 N + \tau_s B + \tau B_s.$$

We see that $\langle B_s, B\rangle = \langle B_s, T\rangle = 0$, respectively because B is a unit vector and

$$\langle B_s, T\rangle = 0 - \langle B, T_s\rangle = -\kappa\langle B, T\rangle = 0.$$

So, $B_s = \langle B_s, N\rangle N = 0 - \tau N$, meaning that

$$N_{ss} = \kappa_s T - \kappa^2 N + \tau_s B - \tau^2 N. \tag{3.11}$$

Then, plugging in (3.10) and (3.11) to (3.9) shows us that

$$T_{ts} = (-\kappa_{ss} + \kappa^3 - \kappa^3 + \kappa\tau^2)N + (-2\kappa_s\kappa - \kappa_s\kappa + 2\kappa_s\kappa)T + (-2\kappa_s\tau - \kappa\tau_s)B.$$

This, combined with (3.8) and the orthogonality of $\{T, N, B\}$ shows that

$$\kappa_t = \kappa^3 + \kappa_{ss} - \kappa\tau^2. \tag{3.12}$$

With $v := f^{-1} = \langle T, e_x\rangle^{-1}$, take $g := \kappa^2 v^2$. We will first bound $g_t - g_{ss}$, for the purpose of applying the maximum principle.

We see that

$$g_t = 2\kappa\kappa_t v^2 + 2\kappa^2 v v_t$$

and

$$g_{ss} = (2\kappa\kappa_s v^2 + 2\kappa^2 v v_s)_s = 2\kappa_s^2 v^2 + 2\kappa\kappa_{ss} v^2 + 4\kappa\kappa_s v v_s + 4\kappa\kappa_s v v_s + 2\kappa^2 v_s^2 + 2\kappa^2 v v_{ss}.$$

So, we have that

$$g_t - g_{ss} = 2\kappa v^2(\kappa_t - \kappa_{ss}) + 2\kappa^2 v(v_t - v_{ss}) - 2(\kappa_s^2 v^2 + 4\kappa\kappa_s v v_s + \kappa^2 v_s^2).$$

Equation (3.7) gives us that

$$\begin{aligned} v_t - v_{ss} &= -f^{-2} f_t - (-f^{-2} f_s)_s \\ &= -f^{-2} f_t - 2f^{-3} f_s^2 + f^{-2} f_{ss} \\ &= -f^{-2}(f_t - f_{ss}) - 2f^{-3} f_s^2 \\ &= -\kappa^2 v - 2v^{-1} v_s^2 \end{aligned} \tag{3.13}$$

This formula and Equation (3.12) give us that

$$\begin{aligned} g_t - g_{ss} &= 2\kappa v^2(\kappa^3 - \kappa\tau^2) + 2\kappa^2 v(-\kappa^2 v - 2v^{-1} v_s^2) - 2(\kappa_s^2 v^2 + 4\kappa\kappa_s v v_s + \kappa^2 v_s^2) \\ &= 2\kappa^4 v^2 - 2\kappa^2 v^2 \tau^2 - 2\kappa^4 v^2 - 4\kappa^2 v_s^2 - 2(\kappa_s^2 v^2 + 4\kappa\kappa_s v v_s + \kappa^2 v_s^2) \\ &= -2\kappa^2 v^2 \tau^2 - 6\kappa^2 v_s^2 - 2\kappa_s^2 v^2 - 8\kappa\kappa_s v v_s. \end{aligned} \tag{3.14}$$

By the arithmetic geometric mean inequality, we see that

$$4\kappa\kappa_s v v_s = 4(\kappa^2 v_s^2 \kappa_s^2 v^2)^{\frac{1}{2}} \leq 4\frac{\kappa^2 v_s^2 + \kappa_s^2 v^2}{2},$$

so we see from (3.14) that

$$g_t - g_{ss} \leq -2\kappa^2 v^2 \tau^2 - 4\kappa^2 v_s^2 - 4\kappa\kappa_s v v_s = -2\kappa^2 v^2 \tau^2 - 2v_s v^{-1}(\kappa^2 v^2)_s = -2\kappa^2 v^2 \tau^2 - 2v_s v^{-1} g_s. \tag{3.15}$$

Noting that $g_t - g_{ss} = (\partial_t - \partial_{ss})(\kappa^2 v^2)$, we get since

$$(v^2)_t - (v^2_{ss}) = 2vv_t - 2v_s^2 - 2vv_{ss} = 2v(v_t - v_{ss}) - 2v_s^2 = -2\kappa^2 v^2 - 2v_s^2 - 2v_s^2$$

that

$$\begin{aligned} (\partial_t - \partial_{ss})(2t\kappa^2 v^2 + v^2) &= 2\kappa^2 v^2 + 2t(g_t - g_{ss}) - 2\kappa^2 v^2 - 4v_s^2 \\ &\leq 2t(-2\kappa^2 v^2 \tau^2 - 2v_s v^{-1} g_s) - 2\kappa^2 v^2 - 4v_s^2 \\ &\leq -2v_s v^{-1}(2tg)_s - 4v_s^2 \\ &\leq -2v^{-1} v_s (2tg)_s - 2v^{-1} v_s (v^2)_s \\ &= -2v^{-1} v_s (2t\kappa^2 v^2 + v^2)_s. \end{aligned} \tag{3.16}$$

Take $h := 2t\kappa^2 v^2 + v^2$ and $a := -2v^{-1}v_s$. We see that (3.16) reduces to

$$h_t - h_{ss} \leq a h_s.$$

We have that

$$a = 2v^{-1}v_s = 2f(-f^{-2}f_s) = 2\kappa \frac{\langle N, e_x \rangle}{\langle T, e_x \rangle},$$

which is bounded by the boundedness of $|\kappa|$ we assumed and the fact that $\langle T, e_x \rangle$ has a positive lower bound by Lemma 3.2. Boundedness of a means that we may apply Theorem 2.2 to show that

$$\sup_{x\in\mathbb{R}} |h(x,t)| \leq \sup_{x\in\mathbb{R}} |h(x,0)|$$

for all $t \in I$. □

Corollary 3.5. *If $\gamma_0(x)$ is forwards-tangent, u_0 and v_0 are C^∞, and all derivatives of the curvature $\kappa(\cdot,0)$ of γ_0 are bounded, then there exists a solution $\gamma(x,t)$ ($(x,t) \in \mathbb{R} \times [0,\infty)$) to (3.1) and for any $m \in \mathbb{N} \cup \{0\}$, we can find $C_m > 0$ only depending on γ_0 such that*

$$\sup_{\mathbb{R}} \left| \frac{\partial^m \kappa}{\partial s^m}(\cdot, t) \right|^2 \leq \frac{C_m}{t^{m+1}}$$

for all $t \in I = [0,\infty)$.

Proof. Proposition 3.4 gives us that if a solution to (3.1) has bounded curvature and bounded derivatives of curvature on any interval, we can find bounds on these quantities which only depend on the initial data γ_0 of (3.1). Thus, it suffices to show that a solution to (3.1) exists on some interval $[0,\varepsilon)$. Note that we know this will be the case if our solution has bounded curvature and derivatives of curvature on $[0,\varepsilon)$.

Fix $r > 0$. We consider $\gamma_{0,r} := \gamma_0|_{[-r,r]}$ alongside a solution γ_r to the Dirichlet curve shortening problem on an interval $I_r \subset [0,\infty)$ with initial data $\gamma_{0,r}$, so we have

$$\begin{cases} \partial_t \gamma_r(x,t) = -\kappa_r(x,t)N(x,t) & \text{for } (x,t) \in \mathbb{R} \times I_r \\ \gamma_r(x,0) = \gamma_{0,r}(x) & \text{for } x \in \mathbb{R} \\ \gamma_r(\pm r, t) = \gamma_{0,r}(\pm r) & \text{for } t \in I_r \end{cases}. \tag{3.17}$$

Note that κ_r is the curvature of γ_r and N is the principal normal to γ_r. We then see from Proposition 2.5–the existence theorem by Ecker and Huisken [EH91, Proposition 4.1] for the Dirichlet curve shortening problem[1]–that there exists $T_r > 0$ depending only on a bound for κ_r (which is the second fundamental form of γ_r) such that γ is a unique smooth solution to the problem (3.17) on $[0,T_r)$, so we can take $I_r = [0,T_r)$. Because each κ_r will be bounded by a bound for κ and we know by assumption that κ is bounded, we can pick the same bounds for all κ_r, giving us $T_r = T_0$ for all r. Calculations analogous to those in Subsection 3.1 show us that we can write

$$\gamma_r(x,t) = (x, u_r(x), v_r(x))$$

for functions $u_r, v_r \in C^\infty([-r,r] \times [0,T_0])$. We consider sequences $(u_r)_{r\in\mathbb{N}}, (v_r)_{r\in\mathbb{N}}$ of functions in $C^\infty(\mathbb{R} \times [0,T_0])$. $(u_r)_{r\in\mathbb{N}}, (v_r)_{r\in\mathbb{N}}$ are uniformly bounded with uniformly bounded derivatives for

[1][EH91, Proposition 4.1] is only formulated to apply to hypersurfaces, but the proof methods apply to curves in greater codimension. Thus, we can use the theorem here

fixed t by the maximum principle for parabolic partial differential equations and since our initial values u_0, v_0 are in $C^\infty(\mathbb{R})$. Thus, by Arzela-Ascoli, there are subsequences of each sequence (where we restrict each function to a given compact set $[-r', r'] \subset \mathbb{R}$) converging to continuous limits on $[-r', r']$. We then apply Arzela-Ascoli again to find a subsequence of this subsequence whose first derivatives converge to a continuous first derivative, and repeat this diagonalization argument until we have $C^\infty(\mathbb{R} \times [0, T_0])$ functions $\widetilde{u}_{r'}(x, t)$ and $\widetilde{v}_{r'}(x, t)$ which are limits of subsequences of $(u_r)_{r\in\mathbb{N}}$ and $(v_r)_{r\in\mathbb{N}}$ respectively. We now consider functions

$$u(x, t) := \widetilde{u}_{x+1}(x, t), \quad v(x, t) := \widetilde{v}_{x+1}(x, t),$$

each in $C^\infty(\mathbb{R} \times [0, T_0])$ (regularity is inherited from $\widetilde{u}_{r'}$ and $\widetilde{v}_{r'}$, and boundedness follows since $(u_r)_{r\in\mathbb{N}}, (v_r)_{r\in\mathbb{N}}$ are uniformly bounded with uniformly bounded derivatives as discussed above). We now consider

$$\gamma(x, t) := (x, u(x, t), v(x, t)).$$

It follows from the C^∞-convergence of our functions above that γ is a solution to the curve shortening flow (3.1) with initial data γ_0. We then know from (3.18) that we can express the magnitude of curvature $|\kappa| = |T_s|$ in terms of u, v, and their derivatives, so boundedness of these terms and their derivatives gives boundedness of all derivatives of κ. We may repeat this process ad infinitum to generate a sequence of intervals, all on which these bounds are satisfied and we have existence of our solution. These intervals will eventually contain the entire half-line $[0, \infty)$, as is shown by a basic openness-closedness argument. □

Now we are in a position to prove the first main result using Corollary 3.5.

Proof of Theorem 1.1. By the bounds in Corollary 3.5, the limit

$$c(x) := \left(x, \widetilde{u}(x) := \lim_{t\to\infty} u(x, t), \widetilde{v}(x) := \lim_{t\to\infty} v(x, t)\right)$$

exists. We see from Corollary 3.5 that the curvature of c vanishes for all $x \in \mathbb{R}$, so c is a line. Lemma 3.3 shows that

$$\sup_{x\in\mathbb{R}} |\widetilde{u}(x)| \le \sup_{x\in\mathbb{R}} |u(x, 0)|, \quad \sup_{x\in\mathbb{R}} |\widetilde{v}(x)| \le \sup_{x\in\mathbb{R}} |v(x, 0)|,$$

so c must be parallel to the x-axis; i.e. there exist constants c_y, c_z such that $c(x) = (x, c_y, c_z)$. □

3.4 Evolution equations

We now present explicit evolution equations for $u_t(x, t)$ and $v_t(x, t)$ as $\gamma(x, t)$ undergoes the curve shortening flow. We see that

$$\gamma_x = (1, u_x, v_x), \quad |\gamma_x| = \sqrt{1 + u_x^2 + v_x^2}.$$

Then, the tangent vector T satisfies

$$T = \frac{\gamma_x}{|\gamma_x|} = \frac{(1, u_x, v_x)}{\sqrt{1 + u_x^2 + v_x^2}}.$$

So, taking the derivative with respect to x and simplifying gives that

$$T_x = \frac{(-u_x u_{xx} - v_x v_{xx}, u_{xx} + v_x^2 u_{xx} - u_x v_x v_{xx}, v_{xx} + u_x^2 v_{xx} - u_x v_x u_{xx})}{(1 + u_x^2 + v_x^2)^{\frac{3}{2}}}.$$

Therefore, as we can see that $T_s = \frac{T_x}{|\gamma_x|}$, we have

$$T_s = \frac{(-u_x u_{xx} - v_x v_{xx}, u_{xx} + v_x^2 u_{xx} - u_x v_x v_{xx}, v_{xx} + u_x^2 v_{xx} - u_x v_x u_{xx})}{(1 + u_x^2 + v_x^2)^2}. \tag{3.18}$$

Then, by (3.1), we have that

$$\begin{aligned}
\partial_t(x, u, v) &= \partial_s T + \frac{u_x u_{xx} + v_x v_{xx}}{(1 + u_x^2 + v_x^2)^{-\frac{3}{2}}} T \\
&= \frac{(0, u_{xx} + v_x^2 u_{xx} - u_x v_x v_{xx} + u_x^2 u_{xx} + u_x v_x v_{xx}, v_{xx} + u_x^2 v_{xx} - u_x v_x u_{xx} + u_x v_x u_{xx} + v_x^2 u_{xx})}{(1 + u_x^2 + v_x^2)^2} \\
&= \frac{(0, u_{xx} + u_x^2 u_{xx} + v_x^2 u_{xx}, v_{xx} + u_x^2 v_{xx} + v_x^2 u_{xx})}{(1 + u_x^2 + v_x^2)^2} \\
&= \frac{(0, u_{xx}, v_{xx})}{1 + u_x^2 + v_x^2}.
\end{aligned} \tag{3.19}$$

3.5 An example

Most examples of curves or surfaces evolving under mean curvature flow are hard to explicitly compute. Here, we show explicitly that the helix converges to the line in codimension two. This helix in $\mathbb{R}^3$ is given by $(cx, r\cos(x), r\sin(x))$ for $x \in \mathbb{R}$ for $c, r \in \mathbb{R}$. Based on equation (3.19), we see that the radius r of the helix satisfies the differential equation

$$r_x = \frac{-r}{c + r^2}. \tag{3.20}$$

Though an explicit solution is hard to compute, we see that r converges to 0 in finite time.

4 Hypersurface translators

In this section, we consider dynamical stability under perturbations of the grim reaper cylinder as a hypersurface. Taking the grim reaper ϕ as in (2.4), we define the n-dimensional grim reaper cylinder by

$$\phi_n : \mathbb{R}^{n-1} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \ni (x, y) \to \phi(y) \in \mathbb{R}.$$

We note that ϕ_1 is the usual grim reaper curve, and that we will often write ϕ instead of ϕ_n for purposes of notation. Since the value of $\phi_n(x, y)$ is independent of x for any n, the mean curvature vector of ϕ_n–and thus how ϕ_n evolves under the mean curvature flow–depends only on $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. So, we can see that ϕ_n inherits translator properties from ϕ.

It is well known that we can express any solution of the mean curvature flow with initial condition a hypersurface as a graph on $\mathbb{R}^{n-1}$ for all time (see [EH91, Theorem 5.1]). We assume this fact throughout the proofs below.

4.1 Avoidance principle for the grim reaper cylinder

We now show that if the difference between our perturbation and the grim reaper is initially bounded, that this bound will be satisfied at all later times. We employ similar proof methods to those of Clutterbuck-Schnürer-Schulze [CSS07, Theorem 3.1].

Theorem 4.1. *Consider a smooth graph* $u_0 : \mathbb{R}^{n-1} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ *such that*

$$|u_0(x, y) - \phi(x, y, 0)| < C \tag{4.1}$$

for all $(x, y) \in \mathbb{R}^{n-1} \times (-\frac{\pi}{2}, \frac{\pi}{2})$*. There exists a solution* u *to the graphical mean curvature flow such that*

$$|u(x, y, t) - \phi(x, y, t)| \leq C \tag{4.2}$$

for all $(x, y) \in \mathbb{R}^{n-1} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $t \in [0, \infty)$*.*

This theorem gives us a solution to the mean curvature flow and a non-increasing bound on the supremum of the difference between a grim reaper cylinder and our perturbation for all time.

Proof of Theorem 4.1. We take smooth functions

$$u_0^k : \Omega_k := \overline{B}_k \times [-\frac{\pi}{2} + \frac{1}{k}, \frac{\pi}{2} - \frac{1}{k}] \to \mathbb{R}$$

such that $u_0^k(x, y) = u_0(x, y)$ when

$$(x, y) \in \overline{B}_{k/2} \times [-\frac{\pi}{2} + \frac{1}{k} + \frac{1}{k^2}, \frac{\pi}{2} - \frac{1}{k} - \frac{1}{k^2}],$$

and $u_0^k(x, y) = \phi(x, y, 0)$ when $(x, y) \in \partial\Omega_k$. Since the second fundamental form of the grim reaper cylinder is uniformly bounded, we may then apply Proposition 2.5 to find that there exists a solution u^k to the Dirichlet problem for time $[0, T_0]$ ($T_0 > 0$) equal to the grim reaper ϕ on its boundary.

Denoting $\underline{\phi} := \phi - C$ and $\overline{\phi} := \phi + C$, we have

$$\underline{\phi}(x, y, t) < u^k(x, y, t) < \overline{\phi}(x, y, t) \quad \textit{for} \quad (x, y, t) \in \partial\Omega_k. \tag{4.3}$$

Additionally, since the evolution equation of their difference $|\overline{\phi} - u^k|$ and $|\underline{\phi} - u^k|$ obey a parabolic partial differential equation, the maximum principle gives us that the maximum values of these quantities are attained on the boundary of $\Omega_k \times [0, T_0]$. Thus, we have that

$$\underline{\phi}(x, t) < u^k(x, t) < \overline{\phi}(x, t) \quad \textit{for} \quad (x, y, t) \in \Omega_k \times [0, T_0]. \tag{4.4}$$

These relations hold for any k and the time T_0 for which the flow exists is independent of k. Thus, on any compact set, we see from Proposition 2.4 that there exists a uniform bound on the time derivatives for this small time. Therefore, we can apply Arzela-Ascoli to identify a subsequence of $(u^k)_{k \in \mathbb{N}_{>0}}$ which converges to a continuous limit on this compact interval. We then apply a diagonalization result as in the proof of Corollary 3.5 to find a subsequence of $(u^k)_{k \in \mathbb{N}_{>0}}$ which converges to a C^∞ limit. As we can produce this result using any compact interval, we get a C^∞ solution u to the desired mean curvature flow problem for a small time.

Since the grim reaper cylinder is a translating solution, the second fundamental form satisfies the same bound at all times and thus we can repeat the above process starting from T_0 and taking a new $u_0' := u(x, y, T_0)$. Repeating this ad infinitum provides the desired result, as all $T_0' \in [0, \infty)$ will eventually be contained in some interval which we find our solution exists in. □

Remark 4.2. *The proof of Theorem 4.1 can be adapted to give an analogous result for the plane, as we have a similar uniform bound on the second fundamental form of the plane. In fact, this proof can be adapted to provide a similar result for any complete hypersurface with a uniform bound on its second fundamental form.*

Remark 4.3. *The result of Wang-Wo* [WW11, Theorem 1.4] *can be extended by using this approach, as it can be used to circumvent their condition of finitely many inflection points in their stability theorem at the cost of uniqueness. Considering this, we provide the addition to their result that a graph will converge to grim reaper under the curve shortening flow if the supremum of the difference between this graph and the grim reaper is bounded. We also note that a grim reaper translated such that the integral of its difference with our perturbation vanishes will satisfy that its difference with the perturbation will vanish as curve shortening is applied.*

4.2 Stability of the grim reaper cylinder

We now prove Theorem 1.2, our stability result for the Grim Reaper cylinder.

Proof of Theorem 1.2. We prove this for the supremum and note that an analogue argument works for the infimum. Assume that for all $\epsilon > 0$, there is a time t_ϵ such that for $t > t_\epsilon$, the set $\Omega^+_{\epsilon,t}$ is empty. Then our result follows. Assume for the sake of contradiction that this is not true.

Let $w(x,y,t) := u(x,y,t) - \phi(x,y,t)$ and $w_k := w(x,y,t+t_k)$ for some sequence $t_k \to \infty$. We notice that $|w_t|$ is locally uniformly bounded, so on any compact subset of our domain, there is a convergent subsequence of w_k by Arzela-Ascoli. We find this for larger and larger compact sets (applying Arzela-Ascoli on the subsequence of the last step). This gives us a solution

$$w_\infty \colon \mathbb{R}^{n-1} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \to \mathbb{R}$$

on the whole strip.

The supremum of w_∞ is independent of time, as it cannot keep decreasing. Since the set $\Omega^+_{\epsilon,t} \subset \mathbb{R}^{n-1} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ is compact and is at its lowest on its boundary, we can find a maximum in the interior and by the maximum principle, so it must be constant. Thus, it is a constant greater than ϵ–if not, $\Omega_{\epsilon,t}$ would be empty for large enough time. This contradicts that $\Omega^+_{\epsilon,t}$ is compact, completing the proof. □

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References

[Bra78] K. A. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes, vol. 20, Princeton University Press, Princeton, N.J., 1978.

[CSS07] J. Clutterbuck, O. C. Schnürer, and F. Schulze, *Stability of translating solutions to mean curvature flow*, Calc. Var. Partial Differential Equations **29** (2007), no. 3, 281–293.

[CM04] T. H. Colding and W. P. Minicozzi II, *Sharp estimates for mean curvature flow of graphs*, J. Reine Angew. Math. **574** (2004), 187–195.

[EH89] K. Ecker and G. Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math. (2) **130** (1989), no. 3, 453–471.

[EH91] ______, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569.

[GH86] M. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986), no. 1, 69–96.

[Gra87] M. A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. **26** (1987), no. 2, 285–314.

[Ham82] R. Hamilton, *Three-manifolds with positive Ricci curvature*, Journal of Differential Geometry **17** (1982), no. 2, 255 – 306, DOI 10.4310/jdg/1214436922.

[Hui84] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), no. 1, 237–266.

[NT06] M. Nara and M. Taniguchi, *Stability of a traveling wave in curvature flows for spatially non-decaying initial perturbations*, Discrete Contin. Dyn. Syst. **14** (2006), no. 1, 203–220.

[Has15] R. Haslhofer, *Uniqueness of the bowl soliton*, Geometry Topology **19** (2015), no. 4, 2393–2406, DOI 10.2140/gt.2015.19.2393.

[WW11] X. Wang and W. Wo, *On the stability of stationary line and grim reaper in planar curvature flow*, Bull. Aust. Math. Soc. **83** (2011), no. 2, 177–188.