TRACES OF CM VALUES OF CERTAIN WEAK MAASS FORMS

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Abstract. For an automorphic form f, the *trace generating series* of f is a Fourier expansion whose coefficient of degree D is the sum of the values of f at imaginary quadratic integers of discriminant D. In [BF04], Bruinier and Funke show that when f is a modular function, the trace generating series appears in the positive exponents of the *theta lift* of f, a weight 3/2 nonholomorphic modular form for a certain congruence subgroup. Building off of their work, we give an analogous formula for the theta lift of f containing the trace generating series when f is a nonholomorphic weight 0 weak Maass form for Γ satisfying the following conditions: (1) the constant terms for the Fourier expansions of f at all cusps vanish, and (2) the cusp widths of Γ are all integer multiples of the width of the infinite cusp.

1. Introduction

Let $\tau \in \mathbb{H}$ and let $q := e^{2\pi i\tau}$. The *j-invariant* $j\colon \mathbb{H} \to \mathbb{C}$

$$j(\tau) := \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

parameterizes elliptic curves defined over the complex numbers. It is a modular of weight 0. [1] It τ is an imaginary quadratic integer, the j-invariant takes on algebraic values, in which case $j(\tau)$ then lies in the ring class field of $\mathbb{Z}[\tau]$. Such values are so special that there exists a name for them: *singular moduli*. [2]

It is natural to study the values of other modular functions at quadratic points. Let $\mathcal{Q}_D$ be the set of positive definite integral binary quadratic forms with discriminant $D > 0$. There is a right action of $\Gamma := SL_2(\mathbb{Z})$ on $\mathcal{Q}_D$ via

$$\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)(ax^2 + bxy + cy^2) := a(rx + sy)^2 + b(rx + sy)(tx + uy) + c(tx + uy)^2.$$

For $Q := Q(x, y) \in \mathcal{Q}_D$, let $\alpha_Q \in \mathbb{H}$ be its associated *CM point*, the unique root of $Q(x, 1) = 0$ in $\mathbb{H}$, and let $\bar{\Gamma}_Q$ be the image of its stabilizer in $\mathrm{PSL}_2(\mathbb{Z})$. Then, for a modular function f, we define

$$\mathrm{tr}_f(D) := \sum_{Q \in \mathcal{Q}_D/\Gamma} \frac{f(\alpha_Q)}{|\bar{\Gamma}_Q|}.$$

We can then assemble the trace generating series

$$\sum_{D<0} \mathrm{tr}_j(D) q^{-D}.$$

In 2002, Zagier proved the following landmark result:

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[1]For a comprehensive overview of modular forms, see [DS05].

[2]For more on the relationship between j-invariants and ring class fields, see Lectures 15-22 of [Sut19].

Theorem 1.1 ([Zag02]). *If $f \in \mathbb{Z}[j(z)]$ has constant term 0 in its Fourier expansion, then there is a finite sum $A_f(z) = \sum_{n\leq 0} a_f(n)q^n$ for which*

$$A_f(z) + \sum_{D>0} \mathrm{tr}_f(D)q^d$$

is a weakly holomorphic weight 3/2 modular form for $\Gamma_0(4)$.

In [BF04], Bruinier and Funke generalize Zagier's paper to modular functions and weak harmonic Maass forms. Recently, in [AS19], Alfes-Neumann and Schwagenscheidt derive a similar formula for meromorphic modular functions, and as a special case find that the trace generating series of the *reciprocal j-function*

$$\sum_{D\leq 0} \mathrm{tr}_{1/j}(D)q^{-D} = -\frac{1}{165888} + \frac{23}{331776}q^3 + \frac{1}{3456}q^4 - \frac{1}{3375}q^7 + \frac{1}{8000}q^8 + \cdots$$

is a mixed mock modular form of weight 3/2 for $\Gamma_0(4)$. The common theme in [BF04] and [AS19] is the introduction in [KM86] of the *Kudla-Millson theta lift*, the convolution

$$I_0(\tau, f) := \int_{\Gamma\backslash\mathbb{H}} f(z)\theta_0(\tau, z)$$

of f with a certain theta kernel $\theta_0(\tau, z)$ associated to a lattice L in a quadratic $\mathbb{Q}$-vector space of fixed discriminant d and signature $(1, 2)$. The most important property of the Kudla-Millson theta lift is that it is a weight 3/2 nonholomorphic modular form for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ in the variable τ. In [BF04], it is proved that subtracting from $I_0(\tau, f)$ the trace generating series for a modular function with vanishing constant coefficients at all cusps yields a finite principal part analogous to that in [Zag02]. On the other hand, in [AS19], it is shown that subtracting twice the trace generating series for $1/j$ yields a finite sum of nonholomorphic theta series.

The general idea in [BF04] and [AS19] may be summarized as

$$\underset{\text{(modular of weight 3/2)}}{\text{“Theta lift of } f\text{''}} = \text{“Trace generating series''} + \text{“Other terms''}.$$

In this paper, we continue this theme, adapting the ideas in [BF04] to the case when f is a weak (not necessarily harmonic) Maass form for Γ in which the following hold:

- The constant terms of the Fourier expansions of f at all cusps vanish.
- The cusp widths α_ℓ of Γ are all integer multiples of the infinite cusp α_∞.

Our main result, stated informally, is this:

Theorem 1.2. *Let f be a weak Maass form with eigenvalue λ for a congruence subgroup of Γ satisfying the above two conditions. Then*

$$\underset{\textit{(modular of weight 3/2)}}{I_0(\tau, f)} = \textit{“Trace generating series''} + \underset{\textit{(analogous to the other terms of [BF04, Theorem 4.5])}}{\textit{“Other terms independent of } \lambda\textit{''}}$$
$$+ \underset{\textit{(not appearing in [BF04, Theorem 4.5])}}{\textit{“Terms that depend on } \lambda\textit{''}}.$$

The formal result is stated in theorem 7.2.

The paper is split up into several parts. It follows the exposition in [BF04] closely. In Section 2, we define some notation and review complex differential forms. In Section 3, we introduce a quadratic space V of signature $(1, 2)$ and discriminant d, an even lattice $L \subseteq V$, and relate it to the classical setup of $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, on which a congruence subgroup normally acts on. In Section

4, we define weak Maass forms and derive its (nonholomorphic) Fourier expansion in terms of I- and K-Bessel functions. In Section 5, we define the theta lift of f associated to L and note some modularity and convergence properties. In Section 6, we define the trace, which is analogous to the function $\mathrm{tr}_f(D)$ defined in the introduction when $D > 0$. We extend the definition to the case of negative D; however, our definition in this case differs from that in [BF04, Definition 4.3]. Finally, in Section 7, we state and prove the main result, carrying over many ideas from [BF04, Propositions 4.10-13] (oftentimes they merely depend on f being real analytic).

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2. Conventions

2.1. **Variables.** In what follows, $\tau := u+iv$ and $z := x+iy$ are variables in $\mathbb{C}$. Let $\mathbb{H} := \{z \in \mathbb{C} : y > 0\}$ be the upper-half plane, and let $q := e^{2\pi i\tau}$.

2.2. **Differential forms.** We give a brief summary of complex differential forms. Let M be a complex manifold. Then around each point, there is a holomorphic bijection between some neighborhood of the point and and open subset of $\mathbb{C}$. The local coordinates are given by dz and $d\bar{z}$, which, in terms of real variables, are

$$dz := dx + i\,dy \qquad\qquad d\bar{z} := dx - i\,dy.$$

For a smooth function $f \in C^\infty(M)$, we define

$$\begin{aligned}\partial f &:= \frac{df}{dz}dz\\ \bar{\partial} f &:= \frac{df}{d\bar{z}}d\bar{z}\\ df &:= \partial f + \bar{\partial} f,\end{aligned}$$

where, in local coordinates, we have

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad\qquad \frac{df}{d\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Further define

$$\omega := \frac{dx \wedge dy}{y^2} = \frac{i dz \wedge d\bar{z}}{y^2} \qquad\qquad d^c := \frac{1}{4\pi i}(\partial - \bar{\partial})$$

and observe that $dd^c = -\frac{1}{2\pi i}\partial\bar{\partial}$.

When integrating over an area, the space of relevant differential forms is denoted $\Omega^{1,1}(M)$. A typical element of $\Omega^{1,1}(M)$ might, for example, look like $f(z)dz \wedge d\bar{z}$.

3. The upper half-plane as a symmetric space

Let V be an oriented quadratic space of signature $(1,2)$ viewed as an algebraic group defined over $\mathbb{Q}$.

Remark 3.1. By *oriented*, we assign one $\mathrm{GL}^+(V)$-equivalence class of the set of all ordered bases to $+1$ and assign the other to -1.

Let $(\cdot,\cdot)$ be the bilinear form, and let $q(X) := \frac{1}{2}(X,X)$ be the corresponding quadratic form. (Do not confuse $q(X)$ with q.) Let d be the discriminant of V; by definition, it is the unique square-free positive integer such that for any basis $\{v_i\}_i$ of $V(\mathbb{Q})$, the determinant of the matrix $[(v_i, v_j)]_{i,j}$ lies in $d(\mathbb{Q}^\times)^2$.

By Witt's theorem, we may identify $V \simeq \left\{\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{smallmatrix}\right) : x_i \in \mathbb{Q}\right\}$ such that $q(X) = d\det(X)$ and $(X,Y) = -d\,\mathrm{tr}(XY)$. Let $G := \mathrm{Spin}(V)$, the two-fold cover of $\mathrm{SO}(V)$. It is well-known that $G \simeq \mathrm{SL}_2$, which acts on V by conjugation. Denote this by $g.X := gXg^{-1}$. Note that this action may be identified with the adjoint action of SL_2 on its Lie algebra $\mathfrak{sl}_2$, which is precisely V, the set of trace zero matrices.

3.1. Identifying $\mathbb{H}$ in V.

Let $D := \{X\mathbb{R} : X \in V, q(X) > 0\}$ be the set of positive definite lines in V, and let $\mathrm{Iso}(V) := \{X\mathbb{R} : X \in V, q(X) = 0\}$ be the set of isotropic lines. The following statement explains why we work in V.

Proposition 3.2. *Let* $\mathrm{SL}_2(\mathbb{R})$ *act on* $\mathbb{H}$ *and* $\mathbb{P}^1(\mathbb{Q})$ *in the usual way:*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d} \qquad\qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(x:y) = (ax+by : cx+dy).$$

For $z \in \mathbb{H}$, *choose* $g_z \in \mathrm{SL}_2(\mathbb{R})$ *such that* $g_z i = z$. *Then we have bijections*

$$\begin{aligned} \mathbb{H} &\simeq D & \qquad \mathbb{P}^1(\mathbb{Q}) &\simeq \mathrm{Iso}(V) \\ g_z i &\mapsto g_z.\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\mathbb{R} & (\alpha:\beta) &\mapsto \begin{pmatrix} -\alpha\beta & \alpha^2 \\ \beta^2 & \alpha\beta \end{pmatrix}\mathbb{R} \end{aligned}$$

compatible with the SL_2*-action.*

For $X \in V$, let $D_X \in D$ be the line in $V(\mathbb{R})$ spanned by X. For $z \in \mathbb{H}$, let $X(z) := \frac{1}{\sqrt{d}y}\begin{pmatrix} -x & |z|^2 \\ -1 & x \end{pmatrix}$; it is then clear that $D_{X(z)}$ corresponds to z in the isomorphism $D \simeq \mathbb{H}$. The factor $\frac{1}{\sqrt{d}y}$ is in front to ensure that $q(X(z)) = 1$. Moreover, $X(gz) = g.X(z)$.

By explicit computation, we have

$$(X, X(z)) = -\frac{d(x_3 x - x_1)^2 + q(X)}{\sqrt{d}x_3 y} - \sqrt{d}x_3 y.$$

For convenience, further define $(X,X)_z := (X,X(z))^2 - (X,X)$.

3.2. Lattices in V.

Definition 3.3. An *even lattice* in V is a lattice L such that for all $X \in L$, $q(X) \in \mathbb{Z}$.

Let L be an even lattice; let $L^* := \{X \in V : (X,L) \in \mathbb{Z}\}$ be its dual lattice. In particular, $L \subseteq L^*$.

We define $\mathrm{Spin}(L)$ to be the elements of $\mathrm{Spin}(V)$ also act as automorphisms of L.

Let $\Gamma \subseteq \mathrm{Spin}(L)$ be a subgroup of finite index that fixes every coset $h + L \in L^*/L$, and let $\bar{\Gamma}$ be its image in $\mathrm{SO}(V)$ and let $M := \Gamma\backslash D$ be the quotient space. For $X \in L$, let G_X be the stabilizer of X in G, and let $\Gamma_X := G_X \cap \Gamma$.

Let $L_{h,m} := \{X \in L + h \colon q(X) = m\}$. Because L is discrete, the set of $m \in \mathbb{Q}$ for which $L_{h,m}$ is nonempty is discrete. This motivates the following definition:

Definition 3.4. The *level* of L is the smallest positive $k \in \mathbb{Z}$ such that $q(X) \in \frac{1}{k}\mathbb{Z}$ for all $X \in L^*$.

3.3. **Cusps.** Note that Γ acts on $\mathrm{Iso}(V)$ with finitely many orbits. Call an element $\ell \in \Gamma\backslash \mathrm{Iso}(V)$ a *cusp*. Letting $X_0 := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, observe that $\infty := (1:0) \in \mathbb{P}^1(\mathbb{Q})$ corresponds to $D_{X_0} =: \ell_0 \in \mathrm{Iso}(V)$.

Definition 3.5. For $\ell \in \mathrm{Iso}(V)$, let $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Q})$ be such that $\sigma_\ell.\ell_0 = \ell$.

We orient every $\ell \in \mathrm{Iso}(V)$, requiring that $\sigma_\ell.X_0$ be oriented positively.

The stabilizer Γ_{ℓ_0} is a discrete subgroup of $G_{\ell_0} \simeq \left\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right\}$; hence $\Gamma_{\ell_0} \simeq \left\langle \pm\left(\begin{smallmatrix} 1 & \alpha_{\ell_0} \\ 0 & 1 \end{smallmatrix}\right)\right\rangle$ for some $\alpha_{\ell_0} \in \mathbb{Q}$. Generalizing this, we find that for $\ell \in \mathrm{Iso}(V)$,

$$\sigma_\ell^{-1}\Gamma_\ell\sigma_\ell \simeq \left\langle \pm\left(\begin{smallmatrix} 1 & \alpha_\ell \\ 0 & 1 \end{smallmatrix}\right)\right\rangle$$

for some $\alpha_\ell \in \mathbb{Q}$.

Definition 3.6. For $\ell \in \Gamma\backslash \mathrm{Iso}(V)$, we call α_ℓ the *width* of the cusp ℓ. Note that α_ℓ is independent of of our choice of cusp representative.

Definition 3.7. For ℓ, let $\beta_\ell \in \mathbb{Q}_{>0}$ be such that $\ell_0 \cap \sigma_\ell^{-1}L = \left\langle \left(\begin{smallmatrix} 0 & \beta_\ell \\ 0 & 0 \end{smallmatrix}\right)\right\rangle$. The quantity β_ℓ is also independent of cusp representative.

Example 3.8. A comprehensive example relating the level 4 lattice $L := \left\{\left(\begin{smallmatrix} b & 2a \\ 2c & -b \end{smallmatrix}\right) \colon a, b, c \in \mathbb{Z}\right\}$ to Zagier's weight 3/2 Eisenstein series may be found in [Fun02, Example 3.9, p. 302].

Sometimes, when computing an integral $\int_M f$, the integrand may diverge at a cusp. Therefore, it is oftentimes necessary to introduce a *truncation*, a parameter $T >> 0$ at which point to stop integrating when in a sufficiently small neighborhood of the cusp. Formally, as in [BF04, (2.6)], we define

$$M_T := M \backslash \bigcup_{\ell \in \Gamma\backslash \mathrm{Iso}(V)} Q_\ell^{-1} D_{1/T},$$

where

$$D_{1/T} := \left\{ z \in \mathbb{C} \colon |z| < \frac{1}{2\pi T} \right\}$$
$$Q_\ell := e^{2\pi i \sigma_\ell^{-1} z/\alpha_\ell}.$$

The expression $Q_\ell^{-1}D_{1/T}$ defines a neighborhood of ℓ to delete, which can be made arbitrarily small for large T.

Example 3.9. When $\ell = \infty$, we may identify the boundary of $Q_\ell^{-1}D_{1/T}$ with the interval $[iT', \alpha_\infty + iT']$ for some large T'.

Consequently, for a differential form $f \in \Omega^{1,1}(M)$ diverging at a cusp, we define the integral

$$\int_M^{reg} f := \lim_{T\to\infty} \int_{M_T} f.$$

4. Automorphic forms

Assume the notation in the previous section, in particular identifying $\mathbb{H}$ with D.

Definition 4.1. For our purposes, a *weak Maass form* for Γ is a real-analytic function $f : \mathbb{H} \to \mathbb{C}$ such that:

(1) For all $\gamma \in \Gamma$ and $z \in \mathbb{H}$, $f(\gamma z) = f(z)$.
(2) There exists $\lambda \in \mathbb{C}$ such that $\Delta f = \lambda f$, where $\Delta := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ denotes the hyperbolic Laplacian.
(3) There exists a constant $C > 0$ such that for all $\gamma \in \Gamma$, we have $f(\gamma\tau) = O(e^{Cy})$ as $y \to \infty$.

If, in addition, $\lambda = 0$, then f is said to be a *weak harmonic Maass form*.

Remark 4.2. In [BF04, p. 28], Bruinier and Funke refer to a weak harmonic Maass form as simply a "weak Maass form" and compute the theta lift for those functions. Our work concerns Maass forms that need not be harmonic, although it is based on [BF04].

Definition 4.3. For $\nu \in \mathbb{C}$, the *Bessel functions* $I_\nu, K_\nu : \mathbb{R}_{>0} \to \mathbb{C}$ are defined as

$$I_\nu(y) := -\frac{\sin \nu\pi}{\pi}\int_0^\infty e^{-y\cosh t - \nu t}\,dt$$
$$K_\nu(y) := \frac{1}{2}\int_0^\infty t^{\nu-1}e^{(-y/2)(t+1/t)}\,dt.$$

The functions I_ν, K_ν describe solutions to the differential equation

$$y^2\frac{d^2 f}{dy^2} + y\frac{df}{dz} - (y^2+\nu^2)f = 0.$$

Proposition 4.4. *As $y \to +\infty$, we have the following asymptotics for I_ν and K_ν:*

$$I_\nu(y) \sim \frac{e^z}{\sqrt{2\pi z}} \qquad\qquad K_\nu(y) \sim \sqrt{\pi}2ze^{-z}.$$

Proof. See [AS72, p. 374]. □

Weak Maass forms admit the following Fourier expansion in terms of I_ν and K_ν:

Theorem 4.5. *Let f be a weak Maass form with eigenvalue λ, and $\nu \in \mathbb{C}$ satisfy $\lambda = 1/4 - \nu^2$. Then, at every cusp $\ell \in \Gamma\backslash \operatorname{Iso}(V)$, f has a Fourier expansion of the form*

$$f(\sigma_\ell z) = \sum_{n \in \frac{1}{\alpha_\ell}\mathbb{Z}} a_{\ell,n}(y)e^{2\pi i n x},$$

where

$$a_{\ell,n}(y) = \begin{cases} c_{\ell,n}\sqrt{y}K_\nu(2\pi|n|y) + d_{\ell,n}\sqrt{y}I_\nu(2\pi|n|y) & n \neq 0 \\ c_{\ell,0}y^{1/2-\nu} + d_{\ell,0}y^{1/2+\nu} & n = 0 \end{cases}$$

for some $c_{\ell,i}, d_{\ell,i} \in \mathbb{C}$.

Proof. We adapt [Bum97]. For convenience let $g(z) := f(\sigma_\ell z)$.

Lemma 4.6. *The function $g(z)$ is also an eigenfunction for Δ with the same eigenvalue λ.*

Proof. Let $\sigma_\ell := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then observe that $\frac{\partial \sigma_\ell(z)}{\partial z} = \frac{1}{(cz+d)^2}$ and $\Im(\sigma_\ell z) = \frac{\Im(z)}{|cz+d|^2}$. Applying the chain rule twice, we get

$$\begin{aligned}
-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)g(z) &= -4y^2\frac{\partial^2 g}{\partial \bar{z}\partial z} \\
&= -4y^2\frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z}(\sigma_\ell z)\cdot\frac{\partial \sigma_\ell}{\partial z}\right) \\
&= -4y^2\frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z}(\sigma_\ell z)\cdot\frac{1}{(cz+d)^2}\right) \\
&= -4y^2\frac{\partial^2 f}{\partial \bar{z}\partial z}(\sigma_\ell z)\cdot\frac{1}{|cz+d|^4} \\
&= -\Im(\sigma_\ell z)^2\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)(\sigma_\ell z) \\
&= (\Delta f)(\sigma_\ell z) = \lambda f(\sigma_\ell z) = \lambda g(z).
\end{aligned}$$

This completes the lemma. □

The Fourier coefficient $a_{n,\ell}(y)$ equals $\int_0^1 g(z)e^{-2\pi i n x}dx$ by definition. Then

$$\begin{aligned}
\left(\frac{1}{4}-\nu^2\right)a_{\ell,n}(y) &= \int_0^1 (\Delta g)(z)e^{-2\pi i n x}dx \\
&= -y^2\left(\int_0^1 \frac{\partial^2 g}{\partial x^2}(z)e^{-2\pi i n x}dx + \int_0^1 \frac{\partial^2 g}{\partial y^2}(z)e^{-2\pi i n x}dx\right).
\end{aligned}$$

The first term is the n-th Fourier coefficient of $\frac{\partial^2 g}{\partial x^2}$, or in other words $-4\pi n^2 a_{\ell,n}(y)$. Switching the order of integration in the second term, the expression becomes

$$\begin{aligned}
\left(\frac{1}{4}-\nu^2\right)a_{\ell,n}(y) &= 4\pi n^2 y^2 a_{\ell,n}(y) - y^2\frac{\partial^2}{\partial y^2}\int_0^1 g(z)e^{-2\pi i n x}dx \\
&= 4\pi n^2 y^2 a_{\ell,n}(y) - y^2\frac{\partial^2}{\partial y^2}a_{\ell,n}(y)
\end{aligned}$$

which yields the differential equation

$$y^2\frac{\partial^2}{\partial y^2}a_{\ell,n}(y) + \left(\frac{1}{4}-\nu^2-4\pi n^2 y^2\right)a_{\ell,n}(y) = 0.$$

Arguing as in [Bum97, p. 105], we obtain the solutions

$$a_{\ell,n}(y) = \begin{cases} c_{\ell,n}\sqrt{y}K_\nu(2\pi|n|y) + d_{\ell,n}\sqrt{y}I_\nu(2\pi|n|y) & n \neq 0 \\ c_{\ell,0}y^{1/2-\nu} + d_{\ell,0}y^{1/2+\nu} & n = 0 \end{cases}$$

for some constants $c_i, d_i \in \mathbb{C}$. Note that we do not omit the $I_\nu(2\pi|n|y)$ term because f is only required to have at most exponential growth at the cusps. This completes the proof. □

For the rest of the paper, let f be a weak Maass form with eigenvalue λ whose constant coefficients at all cusps vanish.

Let $\omega := \frac{dx\wedge dy}{y^2}$ as usual. We define

$$R(X,z) := \frac{1}{2}(X,X(z))^2 - (X,X)$$

for convenience. Kudla and Millson, in [KM86], now define the functions

$$\varphi(X,z) := \left((X,X(z))^2 - \frac{1}{2\pi}\right)e^{-\pi(X,X)_z}\omega$$
$$\varphi^0(X,z) := e^{\pi(X,X)}\varphi(X,z) = \left((X,X(z))^2 - \frac{1}{2\pi}\right)e^{-2\pi R(X,z)}\omega.$$

In [Kud97], Kudla defines a *Green function* for φ given by

$$\xi^0(X,z) := \int_1^\infty \frac{e^{-2\pi R(X,z)t}}{t}dt.$$

Proposition 5.1. *For all $\gamma \in \mathrm{SL}_2$, we have $\varphi(\gamma.X,\gamma z) = \varphi(X,z)$, $\varphi^0(\gamma.X,\gamma z) = \varphi^0(X,z)$ and $\xi^0(\gamma.X,\gamma z) = \xi^0(X,z)$.*

Proof. Compute each of the expressions and use that $(\gamma.X,\gamma.Y) = (X,Y)$ and $X(\gamma z) = \gamma.X(z)$. □

Let d,∂ and $\bar\partial$ be the complex differentials. Recall that $d^c := \frac{1}{4\pi i}(\partial - \bar\partial)$, so that $dd^c = -\frac{1}{2\pi i}\partial\bar\partial$. There following motivates Kudla's Green function:

Theorem 5.2. *On everywhere but D_X, we have $dd^c\xi^0(X,z) = \varphi^0(X,z)$.*

Proof. See [Kud97, Proposition 11.1]. □

Here is a nice corollary.

Proposition 5.3. *Fix $X \in V$ such that $q(X) > 0$. Then the differential forms $\xi^0(X,z)$, $\partial\xi^0(X,z)$, $\bar\partial\xi^0(X,z)$ and $\varphi^0(X,z)$ undergo square-exponential decay as $x \to \pm\infty$, as $y \to +\infty$ and as $y \to 0$.*

Proof. Use theorem 5.2 and stare at the formulas for $\xi^0(X,z)$, $R(X,z)$ and $(X,X(z))$. □

Now let

$$\varphi(X,\tau,z) := \varphi^0(\sqrt{v}X,z)q^m$$

where $m = q(X)$ and $q = e^{2\pi i\tau}$ as usual.

Consider the group algebra $\mathbb{C}[L^*/L]$; as a $\mathbb{C}$-vector space, it has a basis $\{\mathfrak{e}_h\}$ for $h \in L^*/L$. Define

$$\theta^0_{h,m}(v,z) := \sum_{X\in L_{h,m}} \varphi^0(\sqrt{v}X,z)$$
$$\theta_{h,m}(\tau,z) := \sum_{X\in L_{h,m}} \varphi(X,\tau,z)$$
$$\theta_h(\tau,z) := \sum_{X\in L+h} \varphi(X,\tau,z)$$
$$\Theta(\tau,z) := \sum_{h\in L^*/L} \theta_h(\tau,z)\mathfrak{e}_h.$$

The following property is important:

Theorem 5.4. *Let N be the level of L. The function $\theta_h(\tau, z)$ defines a non-holomorphic modular form of weight $3/2$ for $\Gamma(N)$ in the variable z, and if $h = 0$, we may enlarge the group to $\Gamma_0(N)$.*

Proof. See [Fun02]. □

Definition 5.5 ([KM86])**.** For an automorphic form f, the *Kudla-Millson theta lift* is defined to be

$$I(\tau, f) := \int_M f(z)\Theta(\tau, z) = \sum_{h \in L^*/L} \left(\int_M f(z)\theta_h(\tau, z) \right) \mathfrak{e}_h.$$

In addition, we define

$$I_h(\tau, f) = \int_M f(z)\theta_h(\tau, z).$$

As an integral, the Kudla-Millson theta lift is well-defined because $\theta_h(\tau, z)$ has uniform square-exponential decay at every cusp. For the proof of this, see [BF04, Proposition 4.1]. It follows from theorem 5.4 that $I_h(\tau, f)$ is also a weight $3/2$ modular form for $\Gamma(N)$.

6. Traces

6.1. **Traces of positive index.** For $m \in \mathbb{Q}_{>0}$, Γ acts on $L_{h,m}$ with finitely many orbits, and for $X \in L_{h,m}$, the stabilizer Γ_X is finite cyclic. We define

$$t_f(h, m) := \sum_{X \in \Gamma \backslash L_{h,m}} \frac{1}{|\bar{\Gamma}_X|} f(D_X).$$

6.2. **Zero index trace.** Following the convention of [BF04, Definition 4.3], we let

$$t_f(h, 0) := -\frac{\delta_{h,0}}{2\pi} \int_M^{reg} f(z) \frac{dx\,dy}{y^2}.$$

6.3. **Traces of negative index.** Consider $X \in V$ such that $q(X) =: m < 0$. If $m \notin -d(\mathbb{Q}^\times)^2$, then $X^\perp$ is non-split, $\bar{\Gamma}_X$ is infinite cyclic and we set $t_f(h, m) := 0$.

On the other hand, if $m \in -d(\mathbb{Q}^\times)^2$, then $\bar{\Gamma}_X$ is trivial and $X^\perp$ is split. Equivalently, we can find two isotropic lines $\ell_X, \tilde{\ell}_X \subset X^\perp$. Choose ℓ_X to be such that the basis $(X, \ell_X, \tilde{\ell}_X)$ is positively oriented. Then $\tilde{\ell}_X = \ell_{-X}$.

Definition 6.1. For $\ell \in \mathrm{Iso}(V)$ and $X \in V$, we denote the relation $X \sim \ell$ if $\ell = \ell_X$.

Definition 6.2. Denote, for X such that $q(X) < 0$, the following:

$$c_X := \{z \in D \colon z \perp X\}$$
$$c(X) := \Gamma_X \backslash c_X.$$

Now let $X \in L_{h,-dm^2}$. Then $\sigma_\ell^{-1} X \perp \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Observe that $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^\perp$ has generators $\left\langle \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\rangle$. Since $\det X = -m^2$, we may therefore pick an orientation of V such that for some $r \in \mathbb{Q}$,

$$\sigma_\ell^{-1} X = \left(\begin{smallmatrix} m & r \\ 0 & -m \end{smallmatrix}\right).$$

The quantity $-r/2m$ is called the *real part* of $c(X)$ and is denoted $\mathrm{Re}(c(X))$.

Let the Fourier expansion of f at ℓ be

$$\sum_{n \in \frac{1}{\alpha_\ell}\mathbb{Z}} [c_{\ell,n} \sqrt{y} K_\nu(2\pi|n|y) + d_{\ell,n} \sqrt{y} I_\nu(2\pi|n|y)] e^{2\pi i n x}.$$

We now define

$$\left\langle f, c(X)\right\rangle := -\sum_{w\in\frac{1}{\alpha_{\ell_X}}\mathbb{Z}_{<0}} \frac{d_{\ell_X}(w)}{2\pi\sqrt{|w|}} e^{-2\pi i \mathrm{Re}(c(X))w} - \sum_{w\in\frac{1}{\alpha_{\ell_{-X}}}\mathbb{Z}_{<0}} \frac{d_{\ell_{-X}}(w)}{2\pi\sqrt{|w|}} e^{-2\pi i \mathrm{Re}(c(-X))w}$$

and finally

$$t_f(h, -dm^2) := \sum_{X\in\Gamma\backslash L_{h,m}} \left\langle f, c(X)\right\rangle .$$

Remark 6.3. Note that our definition of $\left\langle f, c(X)\right\rangle$ differs slightly from the one provided in [BF04, p. 11], due to the fact f is no longer holomorphic. Moreover, only the $d_{\ell,w}$ terms appear, and there is an extra $2\pi\sqrt{|w|}$ term in the denominator. The reasons for this will appear in the proof of lemma 7.5.

7. Computing the theta lift of a weak Maass form

In this section, we find a formula for the Fourier expansion of $I(\tau, f)$ that contains the trace generating series.

Lemma 7.1. *Let f be a weak Maass form with eigenvalue λ. Then $dd^c f(z) = \frac{\lambda}{4\pi} f(z)\frac{dx\wedge dy}{y^2}$.*

Proof. We compute

$$\begin{aligned} dd^c f(z) &= -\frac{1}{2\pi i}\partial\bar\partial f = -\frac{1}{2\pi i}\frac{\partial^2 f}{\partial z\partial \bar z} d\bar z dz \\ &= -\frac{1}{8\pi i}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) d\bar z dz \\ &= -\frac{1}{4\pi} y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx dy \\ &= \frac{\lambda}{4\pi} f(z)\frac{dx\wedge dy}{y^2}. \end{aligned}$$

□

We now state and prove the main result of our paper.

Theorem 7.2. *Let f be a weak Maass form for Γ with eigenvalue λ such that at all cusps, the constant term of its Fourier expansion vanishes. In light of this, write the Fourier expansion at $\ell \in \Gamma\backslash \mathrm{Iso}(V)$ as*

$$\sum_{n\in\frac{1}{\alpha_\ell}\mathbb{Z}\backslash\{0\}} [c_{\ell,n}\sqrt{y}K_\nu(2\pi|n|y) + d_{\ell,n}\sqrt{y}I_\nu(2\pi|n|y)]e^{2\pi i n x}.$$

Further assume that the all widths α_ℓ are integer multiples of α_∞. Then, we have that

$$I_h(\tau, f) = \sum_{m\geq 0}\mathrm{tr}_f(h,m)q^m + \sum_{m>0}\mathrm{tr}_f(h,-dm^2)q^{-dm^2} + \sum_{m\neq 0}\left(\frac{\lambda}{4\pi}\sum_{X\in L_{h,m}}\int_M f(z)\xi^0(\sqrt{v}X, z)\frac{dx\wedge dy}{y^2}\right)q^m$$

where the summands are taken for $m \in \mathbb{Q}$.

Remark 7.3. It may seem strange that the exponents lie in $\mathbb{Q}$; however, note that the coefficient of q^m is nonzero if and only if $L_{h,m}$ is non-empty. The $m \in \mathbb{Q}$ represented by L^* have denominator bounded by the level of L.

Remark 7.4. The hypothesis that all widths are integer multiples of α_∞ happens, for example, when $\Gamma \in \{\Gamma(N), \Gamma_1(N), \Gamma_0(N)\}$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and then $\alpha_\infty = 1$ while $\alpha_\ell \in \mathbb{Z}$ for all other ℓ.

Proof of theorem 7.2. We follow the exposition of [BF04] closely. Observe that

$$\begin{aligned} I_h(\tau, f) &= \int_M \sum_{m \in \mathbb{Q}} f(z)\theta_{h,m}(\tau, z) \\ &= \sum_{m \in \mathbb{Q}} \left(\int_M f(z)\theta^o_{h,m}(v, z) \right) q^m \end{aligned}$$

and that

$$\begin{aligned} \int_M f(z)\theta^o_{h,m}(v, z) &= \int_M \sum_{X \in \Gamma \backslash L_{h,m}} \sum_{\gamma \in \Gamma_X \backslash \Gamma} f(z)\varphi^0(\gamma^{-1}\sqrt{v}X, z) \\ &= \sum_{X \in \Gamma \backslash L_{h,m}} \left(\int_M \sum_{\gamma \in \Gamma_X \backslash \Gamma} f(z)\varphi^0(\sqrt{v}X, \gamma z) \right). \end{aligned}$$

As in [BF04], there are four cases:

Case 1: $m > 0$. Following [BF04, Proposition 4.10], since $f(\gamma z) = f(z)$ and $\varphi^0(\gamma.X, \gamma z) = \varphi^0(X, z)$ for $\gamma \in \Gamma$, we have

$$\begin{aligned} \int_M \sum_{\gamma \in \Gamma_X \backslash \Gamma} f(z)\varphi^0(\sqrt{v}X, \gamma z) &= \int_M \sum_{\gamma \in \Gamma_X \backslash \Gamma} f(\gamma z)\varphi^0(\sqrt{v}X, \gamma z) \\ &= \frac{1}{|\bar{\Gamma}_X|} \int_D f(z)\varphi^0(\sqrt{v}X, z). \end{aligned}$$

By Stokes' theorem and lemma 7.1,

$$\begin{aligned} \frac{1}{|\bar{\Gamma}_X|} \int_D f(z)\varphi^0(\sqrt{v}X, z) &= \frac{1}{|\bar{\Gamma}_X|} \left[f(D_X) + \int_D \xi^0(\sqrt{v}X, z) dd^c f(z) \right] \\ &= \frac{1}{|\bar{\Gamma}_X|} \left[f(D_X) + \frac{\lambda}{4\pi} \int_D \xi^0(\sqrt{v}X, z) f(z) \frac{dxdy}{y^2} \right] \\ &= \frac{1}{|\bar{\Gamma}_X|} f(D_X) + \frac{\lambda}{4\pi} \int_{\Gamma_X \backslash D} \xi^0(\sqrt{v}X, z) f(z) \frac{dxdy}{y^2}. \end{aligned}$$

Hence the coefficient of q^m for $m > 0$ is

$$\sum_{X \in \Gamma \backslash L_{h,m}} \left(\int_M \sum_{\gamma \in \Gamma_X \backslash \Gamma} f(z)\varphi^0(\sqrt{v}X, \gamma z) \right) = \mathrm{tr}_f(h, m) + \left(\frac{\lambda}{4\pi} \sum_{X \in \Gamma \backslash L_{h,m}} \int_{\Gamma_X \backslash D} f(z)\xi^0(\sqrt{v}X, z) \frac{dxdy}{y^2} \right).$$

Case 2: $m < 0$ and $m \notin -d(\mathbb{Q}^\times)^2$. Everything in the proof of [BF04, Proposition 4.11] stays the same except for the last step. Namely, the proof gives

$$\int_M f(z) \sum_{\gamma \in \Gamma_X \backslash \Gamma} \varphi^0(\sqrt{v}X, \gamma z) = \int_{\Gamma_X \backslash D} \xi^0(\sqrt{v}X, z) dd^c f(z).$$

Now by lemma 7.1, this equals

$$\frac{\lambda}{4\pi}\int_{\Gamma_X\backslash D}\xi^0(\sqrt{v}X,z)f(z)\frac{dxdy}{y^2}$$

Summing over $X\in\Gamma\backslash L_{h,m}$ yields

$$\sum_{X\in\Gamma\backslash L_{h,m}}\frac{\lambda}{4\pi}\int_{\Gamma_X\backslash D}\xi^0(\sqrt{v}X,z)f(z)\frac{dxdy}{y^2}=\frac{\lambda}{4\pi}\sum_{X\in L_{h,m}}\int_M f(z)\xi^0(\sqrt{v}X,z)\frac{dx\wedge dy}{y^2}$$

for the coefficient of q^m.

Case 3: $m<0$ and $m\in -d(\mathbb{Q}^\times)^2$. Following [BF04, Proposition 4.12], write our exponent m as $-dm^2$ for some $m\in\mathbb{Q}_{>0}$. We have that Γ_X is trivial. Integrating by parts and using theorem 5.2, we obtain

$$\int_M\sum_{\gamma\in\Gamma_X\backslash\Gamma}f(z)\varphi^0(\sqrt{v}X,\gamma z)=\frac{1}{2\pi i}\int_M f(z)\bar\partial\partial\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z)$$

$$(\star)\qquad =\frac{1}{2\pi i}\int_M d\left(f(z)\partial\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z)\right)-\frac{1}{2\pi i}\int_M\bar\partial f(z)\partial\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z)$$

We turn our attention to the second term of $(\star)$. Integrating by parts, the second term equals

$$\frac{1}{2\pi i}\int_M d\left(\bar\partial f(z)\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z)\right)-\frac{1}{2\pi i}\int_M\left(\partial\bar\partial f(z)\right)\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z).$$

By Stokes' theorem, the first term vanishes, and since f is a Maass form we have for the second term

$$-\frac{1}{2\pi i}\int_M\left(\partial\bar\partial f(z)\right)\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z)=\frac{\lambda}{4\pi}\int_M f(z)\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z)\frac{dx\wedge dy}{y^2}.$$

Now we focus on the first term of $(\star)$. By Stokes' theorem again, we have

$$\frac{1}{2\pi i}\int_M d\left(f(z)\partial\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z)\right)=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\partial M_T}f(z)\sum_{\gamma\in\Gamma}\partial\xi^0(\sqrt{v}X,\gamma z).$$

By [BF04, Lemma 5.2] (which only uses that f is real-analytic), this expression equals

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\partial M_{T,\ell_X}}f(z)\sum_{\gamma\in\Gamma_{\ell_X}}\partial\xi^0(\sqrt{v}X,\gamma z)+\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\partial M_{T,\ell_{-X}}}f(z)\sum_{\gamma\in\Gamma_{\ell_{-X}}}\partial\xi^0(\sqrt{v}X,\gamma z).$$

Lemma 7.5. *Let α be the width of the cusp ℓ_X. Then we have*

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\partial M_{T,\ell_X}}f(z)\sum_{\gamma\in\Gamma_{\ell_X}}\partial\xi^0(\sqrt{v}X,\gamma z)=-\sum_{w\in\frac{1}{\alpha}\mathbb{Z}_{<0}}d_{\ell_X,w}\frac{e^{-2\pi iRe(c(X))w}}{2\pi\sqrt{|w|}}.$$

Proof. Following the conventions of [BF04, Lemma 5.3], write $g(z):=f(\sigma_{\ell_X}z)$. Translate the integral by σ_{ℓ_X}. Then, ℓ_X is sent to the cusp ∞, and hence we are integrating over the line $[iT,\alpha+iT]$.

Moreover, Γ_{ℓ_X} becomes $\left\langle \pm\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\right\rangle$. By Section 6.2, we may write $\sigma_{\ell_X}^{-1} X$ as $m\left(\begin{smallmatrix} 1 & r \\ 0 & -1 \end{smallmatrix}\right)$ for some $r \in \mathbb{Q}$. Hence $\mathrm{Re}(c(X)) = -r$. Using these as well as the Γ-invariance of ξ^0, we see that

$$(*) \quad \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\partial M_{T,\ell_X}} f(z) \sum_{\gamma\in\Gamma_{\ell_X}} \partial\xi^0(\sqrt{v}X, \gamma z) = -\frac{1}{2\pi i} \lim_{T\to\infty} \int_{iT}^{\alpha+iT} g(z) \sum_{n\in\mathbb{Z}} \partial\xi^0 \left(\sqrt{v} m \left(\begin{smallmatrix} 1 & 2(r+\alpha n) \\ 0 & -1 \end{smallmatrix}\right), z\right).$$

We now recall the following fact of Bruinier and Funke (see [BF04, p. 20]):

$$\sum_{n\in\mathbb{Z}} \partial\xi^0 \left(\sqrt{v} m \left(\begin{smallmatrix} 1 & 2(r+\alpha n) \\ 0 & -1 \end{smallmatrix}\right), z\right) = \sum_{w\in\frac{1}{\alpha}\mathbb{Z}} \frac{i}{2\alpha\sqrt{vd}m} e^{-2\pi i(x+r)w} \cdot \left(2\pi\sqrt{vd}m e^{2\pi w y} \mathrm{erfc}\left(2\sqrt{\pi v d} m + \sqrt{\pi} w y / 2\sqrt{vd} m\right) - e^{-4\pi v d m^2 - \pi w^2 y^2/4 v d m^2}\right) dz.$$

Here $\mathrm{erfc}(t) := \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx$ is the *error function* (see [AS72, p. 297] for more details).

Further recall that by theorem 4.5, we have, letting ν satisfy $\lambda = \frac{1}{4} - \nu^2$, that

$$g(z) = \sum_{n\in\frac{1}{\alpha}\mathbb{Z}\setminus\{0\}} \left(c_{\ell_X,n}\sqrt{y} K_\nu(2\pi|n|y) + d_{\ell_X,n}\sqrt{y} I_\nu(2\pi|n|y)\right) e^{2\pi i n x}.$$

Therefore, the integrand of $(*)$ is a product of two sums. Observe that:

- Since we are integrating on $[iT, \alpha + iT]$, we may replace y with T in the integrand and assume it is a constant.
- Hence, when expanding the product, each term is a constant multiple of $e^{2\pi i(n-w)x}$. Note that since both $n, w \in \frac{1}{\alpha}\mathbb{Z}$, we have $\int_0^\alpha e^{2\pi i(n-w)x} dx = \delta_{n,w}\alpha$.
- When $T \to \infty$, the term $e^{-4\pi v d m^2 - \pi w^2 T^2/4 v d m^2}$ has square-exponential decay, which dominates all other terms (namely I_ν, which is merely linear exponential in growth). Thus, we may ignore that term.

As a result, the expression $(*)$ simplifies to

$$-\frac{1}{2} \lim_{T\to\infty} \sum_{w\in\frac{1}{\alpha}\mathbb{Z}\setminus\{0\}} \left(c_{\ell_X,w}\sqrt{T} K_\nu(2\pi|w|T) + d_{\ell_X,w}\sqrt{T} I_\nu(2\pi|w|T)\right) e^{-2\pi i r w + 2\pi w T} \mathrm{erfc}\left(2\sqrt{\pi v d} m + \sqrt{\pi} w T / 2\sqrt{vd} m\right).$$

When $w > 0$, the error function has square exponential decay as $T \to \infty$ (see [AS72, p. 7.1.23]), so the limit equals 0. On the other hand, when $w < 0$, we have $\lim_{t\to-\infty} \mathrm{erfc}(t) = 2$. Combining this with proposition 4.4, we get for $w < 0$ that

$$\begin{aligned} &-\frac{1}{2} \lim_{T\to\infty} \left(c_{\ell_X,w}\sqrt{T} K_\nu(2\pi|w|T) + d_{\ell_X,w}\sqrt{T} I_\nu(2\pi|w|T)\right) e^{-2\pi i r w + 2\pi w T} \mathrm{erfc}\left(2\sqrt{\pi v d} m + \sqrt{\pi} w T / 2\sqrt{vd} m\right) \\ &= -\lim_{T\to\infty} d_{\ell_X,w}\sqrt{T} \frac{e^{2\pi|w|T}}{2\pi\sqrt{|w|T}} e^{-2\pi i r w + 2\pi w T} = -\frac{d_{\ell_X,w} e^{-2\pi i r w}}{2\pi\sqrt{|w|}}. \end{aligned}$$

Summing over all $w < 0$, we obtain our desired result. □

Therefore, for Case 3 we have

$$\int_M \sum_{\gamma\in\Gamma_X\backslash\Gamma} f(z)\varphi^0(\sqrt{v}X,\gamma z) =$$

$$-\sum_{w\in\frac{1}{\alpha}\mathbb{Z}_{<0}} d_{\ell_X,w}\frac{e^{-2\pi i\mathrm{Re}(c(X))w}}{2\pi\sqrt{|w|}} - \sum_{w\in\frac{1}{\alpha}\mathbb{Z}_{<0}} d_{\ell_{-X},w}\frac{e^{-2\pi i\mathrm{Re}(c(-X))w}}{2\pi\sqrt{|w|}} + \frac{\lambda}{4\pi}\int_M f(z)\sum_{\gamma\in\Gamma}\xi^0(\sqrt{v}X,\gamma z)\frac{dx\wedge dy}{y^2}.$$

Summing over $X \in \Gamma\backslash L_{h,-dm^2}$ gives

$$\mathrm{tr}_f(h,-dm^2) + \frac{\lambda}{4\pi}\sum_{X\in L_{h,-dm^2}}\int_M f(z)\xi^0(\sqrt{v}X,z)\frac{dx\wedge dy}{y^2}$$

for the coefficient of q^{-dm^2}.

Case 4: $m = 0$. Arguing as in [BF04, p. 15] and the proof of [BF04, Proposition 4.13], we have that the constant coefficient of $I_h(\tau, f)$, given by

$$\int_M \sum_{X\in L_{h,0}\backslash\{0\}} f(z)\varphi^0(\sqrt{v}X,z),$$

equals

$$-\frac{\delta_{h,0}}{2\pi}\int_M^{reg} f(z)\frac{dxdy}{y^2} + \int_M^{reg}\sum_{X\in L_{h,0}\backslash\{0\}} f(z)\varphi^0(\sqrt{v}X,z).$$

The first term equals $\mathrm{tr}_f(h,0)$, while by [BF04, p. 21], for each $\ell \in \Gamma\backslash\mathrm{Iso}(V)$ there exist numbers k_ℓ such that the second term equals

$$\frac{1}{2\pi}\sum_{\ell\in\Gamma\backslash\mathrm{Iso}(V),\delta_\ell(h)\neq 0}\lim_{T\to\infty}\int_{iT}^{iT+\alpha_\ell} f(z)\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{e^{-\pi vd(n\beta_\ell+k_\ell)^2/y^2}}{y}dx.$$

Here, $\delta_\ell(h) = 0$ iff ℓ does not intersect $L + h$. Using the same reasoning as in lemma 7.5, we may treat $y = T$ to be constant. Moreover, recall the Fourier expansion of f at ∞,

$$f(z) = \sum_{n\in\frac{1}{\alpha_\infty}\mathbb{Z}} a_{\infty,n}(y)e^{2\pi inx}.$$

The integrand is thus a linear combination of $\{e^{2\pi inx}\}_{n\in\frac{1}{\alpha_\infty}\mathbb{Z}}$. But by the hypothesis, α_ℓ is an integer multiple of α_∞, so $\int_0^{\alpha_\ell} e^{2\pi inx}dx = 0$ unless $n = 0$. (If we remove the hypothesis, then the limit diverges due to the exponential growth of $I_\nu(2\pi|n|y)$.) Since the constant coefficient of f vanishes at all cusps, it follows that the whole summand is 0. Therefore, the constant term of $I_h(\tau, f)$ is simply

$$\mathrm{tr}_f(h,0) = -\frac{\delta_{h,0}}{2\pi}\int_M^{reg} f(z)\frac{dxdy}{y^2}.$$

Putting all four cases together, we obtain the theorem statement. □

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