A MULTIPLICITY-FREE MACDONALD IDENTITY
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Abstract. Given an irreducible (untwisted, reduced) affine root system, let W be its affine Weyl group. Pick a set of simple roots Σ, and let δ be the unique indecomposable positive imaginary root. We show the following equality of formal characters:

$$\sum_{w\in W}\prod_{\alpha\in\Sigma}\frac{1}{1-e^{-w\alpha}}=\frac{1}{1-e^{-\delta}}.$$

The existence of such an formula, which is a multiplicity-free version of a famous identity due to Macdonald, confirms a conjecture of Dhillon–Khare (2018).

To obtain this formula, we prove the following multivariate Macdonald identity. Let Φ_m be the multiset of all roots, real and imaginary, in which each root appears with its standard multiplicity. Let Φ_m^+ (resp. Φ_m^-) be the submultiset of every instance of every positive (resp. negative) root in Φ_m. For an arbitrary submultiset B of Φ_m^+ in which each $\alpha\in\Phi_m^+$ appears $B(\alpha)$ times, let B^- denote the submultiset of Φ_m^- in which each $-\alpha\in\Phi_m^-$ appears $B(\alpha)$ times. Also let $|B|$ be the sum of the roots in B. Finally, let $\Phi_m(w,B)=\Phi_m^+\cap w(B\cup\Phi_m^-\setminus B^-)$. We show for indeterminates u_α for each $\alpha\in\Phi_m^+$ that

$$\sum_{w\in W}\prod_{\alpha\in\Phi_m^+}\frac{1-u_\alpha e^{-w\alpha}}{1-e^{-w\alpha}}=\sum_{n=0}^{\infty}e^{-n\delta}\sum_{w\in W}\sum_{\substack{B\subseteq\Phi_m^+,\\ |B|=n\delta}}(-1)^{\#B}\prod_{\alpha\in\Phi_m(w,B)}u_\alpha.$$

A specialization of each u_α to integer values recovers the first formula.

1. Introduction

The Weyl denominator identity, which shows the equality of certain polynomials associated with finite root systems, is a classical result with numerous applications in fields such as algebraic combinatorics, representation theory, and algebraic geometry. The similarly versatile Macdonald identities are affine analogues of the Weyl denominator identity that show the equality of certain power series associated with affine root systems [3].

In a recent paper, Dhillon and Khare deduce a multiplicity-free denominator identity for finite root systems [1]. Dhillon and Khare go on to conjecture the existence of an affine analogue of their identity, which would take the form of a multiplicity-free Macdonald identity.

In this paper, we prove that such an identity does in fact hold for general (untwisted) affine root systems. We do so by examining a multivariate identity of Macdonald's on root systems of finite type [4]. We generalize this identity to root systems of affine type, then specialize the variables appropriately to resolve the conjecture of Dhillon and Khare.

1.1. **Statement of results.** To formulate the multiplicity-free Macdonald identity, we need to review some mostly standard notation.

As in the abstract, fix an irreducible (untwisted, reduced) affine root system and let W be its affine Weyl group. Pick a set of simple roots Σ, and let δ be the unique indecomposable positive imaginary root. We now state our main result, which is an equality of two power series in $e^{-\alpha}$ for $\alpha\in\Sigma$.

Theorem 1.1. *The following identity holds:*

$$\sum_{w\in W}\prod_{\alpha\in\Sigma}\frac{1}{1-e^{-w\alpha}}=\frac{1}{1-e^{-\delta}}. \tag{1.1}$$

To give the reader a feel for the content of Theorem 1.1, we sketch in Section 3 direct geometric proofs in low rank cases, with accompanying diagrams tracking the remarkable cancellations occurring in the left hand side of the identity.

To provide context, we compare Theorem 1.1 to the multiplicity-free identity for finite root systems noted by Dhillon and Khare [1, Corollary 3.6]. We state their identity after reviewing a little more largely standard notation.

Fix an irreducible (reduced) finite root system and let W_0 be its Weyl group. Pick a set of simple roots Σ_0. Then the multiplicity-free identity for finite root systems reads as

$$\sum_{w\in W_0}\prod_{\alpha\in\Sigma_0}\frac{1}{1-e^{-w\alpha}}=1. \tag{1.2}$$

This differs from the usual Weyl denominator identity in that the product in each summand is taken over just the set of simple roots, rather than the set of all positive roots.

The prior arguments for Equation (1.2), which crucially rely on the finiteness of the Weyl group, are insufficient to prove Theorem 1.1. However, there is another way to derive Equation (1.2), by using a multivariate identity of Macdonald's on finite root systems [4, Theorem (2.8)]. Since a variant of this technique will eventually yield Theorem 1.1, we first explain this simpler case.

To state Macdonald's multivariate denominator identity, we need to review a little more standard notation. Let R^+ be the set of positive roots relative to Σ_0, and similarly let R^- be the set of negative roots. For each $w\in W_0$, let $R(w)=R^+\cap wR^-$. Then for indeterminates u_α with $\alpha\in R^+$, Macdonald shows that

$$\sum_{w\in W_0}\prod_{\alpha\in R^+}\frac{1-u_\alpha e^{-w\alpha}}{1-e^{-w\alpha}}=\sum_{w\in W_0}\prod_{\alpha\in R(w)}u_\alpha. \tag{1.3}$$

We now explain how Equation (1.2) follows from Equation (1.3). Both the left hand sides of Equation (1.2) and Equation (1.3) are sums over W_0 of products taken over subsets of R^+. By specializing the variables of Equation (1.3) so that $u_\alpha=1$ for all $\alpha\notin\Sigma_0$, we restrict the product in each summand to be taken over only the set of simple roots, just like each product on the left hand side of Equation (1.2). Then if we set $u_\alpha=0$ when $\alpha\in\Sigma_0$, the multiplicands in each product of the two left hand sides become equal, so that the two left hand sides become identical.

Meanwhile, we can compute the right hand side of Equation (1.3) after specializing the variables in the same way by observing that $R(w)$ contains no simple roots if and only if w is the identity element of W_0. Hence Equation (1.2) follows directly from Equation (1.3).

To provide a similar argument for Theorem 1.1, we would like an affine analogue of Equation (1.3). While such an analogue is not available in the literature, it exists nonetheless and comprises the second main result of our paper.

To state this analogous identity, we need slightly more notation. As in the abstract, for every affine root α, let $\operatorname{mult}\alpha$ be its standard multiplicity. Thus $\operatorname{mult}\alpha=1$ if α is real and $\operatorname{mult}\alpha=\#\Sigma-1$ if α is imaginary. Then let Φ_m be the multiset of all affine roots, real and imaginary, in which each root α appears $\operatorname{mult}\alpha$ times. Let Φ_m^+ be the submultiset of every instance of every positive root in Φ_m, and similarly let Φ_m^- be the submultiset of every instance of every negative root.

Furthermore, given an arbitrary submultiset B of Φ_m^+, in which each $\alpha\in\Phi_m^+$ appears $B(\alpha)$ times, let B^- denote the submultiset of Φ_m^- in which each $-\alpha\in\Phi_m^-$ appears $B(\alpha)$ times. Let $|B|$ be the sum of the roots in B, and let $\Phi_m(w,B)=\Phi_m^+\cap w(B\cup\Phi_m^-\setminus B^-)$. We now state our other main result, which is an equality of two power series in $e^{-\alpha}$ for $\alpha \in \Sigma$, as before, as well as indeterminates u_α with $\alpha \in \Phi_m^+$.

Theorem 1.2. *The following identity holds:*

$$\sum_{w \in W} \prod_{\alpha \in \Phi_m^+} \frac{1 - u_\alpha e^{-w\alpha}}{1 - e^{-w\alpha}} = \sum_{n=0}^{\infty} e^{-n\delta} \sum_{w \in W} \sum_{\substack{B \subseteq \Phi_m^+, \\ |B| = n\delta}} (-1)^{\#B} \prod_{\alpha \in \Phi_m(w,B)} u_\alpha. \tag{1.4}$$

As previously discussed, Theorem 1.2 is an affine analogue of Equation (1.3). We should mention that Macdonald also proves a multivariate analogue of Equation (1.3) [2]. His identity, however, assumes that $u_\alpha = u_{w\alpha}$ for all $w \in W$ and all roots α, which is incompatible with the aforementioned way in which we hope to specialize each u_α. This necessitates the variant above, in which the u_α's are entirely independent.

We now briefly discuss some elements of the proof of Theorem 1.2. Since we are dealing with infinite power series rather than finite polynomials, we must take care to check that the left hand side of Equation (1.4) is a well-defined multivariate power series. For this purpose, we employ techniques much like those Macdonald uses to ensure that the left hand side of his affine multivariate identity is well-defined [2, Section 3].

The rest of the proof of Theorem 1.2 mirrors Macdonald's proof of the multivariate identity for finite root systems, Equation (1.3) [4, Theorem (2.8)]. In his proof, Macdonald invokes the Weyl denominator identity for finite root systems. Similarly, to prove Theorem 1.2, we use Macdonald's own affine analogue of the Weyl denominator identity.

To obtain Theorem 1.1 from Theorem 1.2, we would like to specialize each u_α for $\alpha \in \Phi_m^+$ exactly as we did above for the finite case. However, since we are specializing indeterminates in an infinite power series rather than a polynomial, we must ensure that this specialization yields a well-defined power series in the remaining indeterminates, $e^{-\alpha}$ for $\alpha \in \Sigma$. This is beyond what is needed in Macdonald's work and requires different techniques.

In particular, we employ a geometric argument to make an observation about the action of elements of the Weyl group on the simple roots. This observation is enough to show that we can specialize the u_α's in Equation (1.4) to deduce from Theorem 1.2 that the only nonzero terms in the expansion of the left hand side of Equation (1.1) are $e^{-n\delta}$ terms for nonnegative integers n.

Finally, a further examination of the behavior of the left hand side of Equation (1.1) allows us to deduce the coefficient of $e^{-n\delta}$ for each nonnegative integer n, giving our desired result.

1.2. **Organization of the paper.** In Section 2, we establish notation and collect some preliminary results on the topological rings in which our power series live. We then ensure that certain operations on infinite sums and products of these series can be safely performed. In Section 3, we provide intuition towards Theorem 1.1 through figures depicting the identity for types $\widetilde{A}_1$ and $\widetilde{A}_2$. In Section 4, we show that the left hand side of Equation (1.4) from the statement of Theorem 1.2 is a well-defined power series in one of our rings. In Section 5, we prove Theorem 1.2. In Section 6, we examine the action of the Weyl group on the simple roots from a geometric perspective and present an argument that will allow us to specialize the u_α's appropriately in our proof of Theorem 1.1. In Section 7, we complete the proof of Theorem 1.1. Finally, in Section 8, we explore future directions of research.

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2. Review of Notation and Power Series Operations

Here we will review some mostly standard notation, including the rings in which our power series live. We then ensure that the usual operations on infinite sums and products of our power series are valid. In particular, the reader may wish to proceed directly to Section 3 and refer back to this section as necessary.

2.1. **Affine Weyl groups.** We direct the reader to the earlier sections of [3] for an exposition of affine Weyl groups and affine roots. In particular, we note that the affine imaginary roots are those fixed by every element of the affine Weyl group. We also note that the affine Weyl group W is the semidirect product of its associated finite Weyl group W_0 and the group of translations by integer combinations of finite coroots.

2.2. **Notation for affine Weyl groups.** For each $w \in W$, let

$$\Phi(w) = \Phi_m^+ \cap w\Phi_m^-.$$

Recall that the length of w is $\ell(w) = \#\Phi(w)$ and the signature of w is $\varepsilon(w) = (-1)^{\ell(w)}$. Finally, let $s(w)$ be the sum of the roots in $\Phi(w)$.

2.3. **Indeterminates; polynomial and power series rings.** We let Φ be the set of all affine roots, real and imaginary. We also let Φ^+ be the set of all positive roots relative to the simple roots Σ and let Φ^- be the set of negative roots.

Let $\{u_\alpha \mid \alpha \in \Phi^+\}$ be a collection of independent indeterminates. We let $A = \mathbb{Z}[u_\alpha]_{\alpha\in\Phi^+}$ be the ring of polynomials over the integers in the u_α's, and we let $\widehat{A} = \mathbb{Z}[[u_\alpha]]_{\alpha\in\Phi^+}$ be the ring of formal power series over the integers in the u_α's.

Additionally, let $\{e^{-\alpha} \mid \alpha \in \Sigma\}$ be another collection of independent indeterminates. We let $C = \widehat{A}[[e^{-\alpha}]]_{\alpha\in\Sigma}$ be the ring of formal power series over $\widehat{A}$ in the $e^{-\alpha}$'s, and we let $D = \mathbb{Z}[[e^{-\alpha}]]_{\alpha\in\Sigma} \subseteq C$ be the ring of formal power series over $\mathbb{Z}$ in the $e^{-\alpha}$'s.

Then when we say that a power series is well-defined or valid, we mean that it is an element of one of whatever power series ring we are working in, be it $\widehat{A}$, C, or D. In particular, we mean that all coefficients in the power series are finite integers.

We endow each power series ring ($\widehat{A}$, C, and D) with its canonical topology. In other words, for each power series ring, if we let $\mathfrak{p}$ be the ideal generated by all indeterminates in the ring, then each ring is a topological ring for which the basis of open neighborhoods of 0 are the ideals $\mathfrak{p}^m$ for all integers $m \geq 0$. So a sequence of power series $(p_i)_{i=0}^\infty$ in one of these rings converges if and only if, for all integers $m \geq 0$, the sequence $(\overline{p_i})_{i=0}^\infty$ of the residues modulo $\mathfrak{p}^m$ of the original sequence is eventually constant.

This is equivalent to saying that for each fixed monomial $\mathbf{u} \in \mathfrak{p}$, the sequence $(c_i)_{i=0}^\infty$, in which each $c_i \in \mathbb{Z}$ is the coefficient in front of $\mathbf{u}$ in the power series p_i, converges. This is because $\mathbb{Z}$ is endowed with the discrete topology, so a sequence of integers converges if and only if it stabilizes to a single integer.

When we say that an infinite sum of power series with a given order of summation is valid or well-defined, we mean that the sequence of partial sums converges to a well-defined power series under the topology of the power series ring in which the summands lie. Similarly, we say that an infinite product of power series with a given order of multiplication is valid or well-defined, we mean that the sequence of partial products converges to a well-defined power series under the topology of the power series ring in which the multiplicands lie.

2.4. **Arbitrary negative exponents; height function.** Let L^+ be the set of nonnegative integer combinations of the affine simple roots in Σ. It is well-known that $\Phi^+ \subseteq L^+$ and that for all $f \in L^+$, we can uniquely write

$$f = \sum_{\alpha \in \Sigma} f_\alpha \alpha$$

for some nonnegative integers f_α. We can thus denote by e^{-f} the monomial

$$e^{-f} = \prod_{\alpha \in \Sigma} \left(e^{-\alpha}\right)^{f_\alpha}.$$

We call the degree of this monomial the height of f; in other words,

$$\mathrm{ht}(f) = \sum_{\alpha \in \Sigma} f_\alpha.$$

Note that for a fixed nonnegative integer h, there are only a finite number of elements $f \in L^+$ satisfying $\mathrm{ht}(f) = h$.

If we let L^- denote the set of nonpositive integer combinations of the affine simple roots in Σ, we can of course extend the definition of height to L^-. For all $g \in L^-$, we have that $-g \in L^+$, so we let

$$\mathrm{ht}(g) = -\,\mathrm{ht}(-g).$$

2.5. **Positive exponents.** Occasionally we may write

$$\frac{K_0 - K_1 e^f}{K_2 - K_3 e^f}$$

for $f \in L^+$ and polynomials K_0, K_1, K_2, K_3 that are power series in all the indeterminates excluding every $e^{-\alpha}$ for $\alpha \in \Sigma$. Any such fraction should be interpreted as

$$\frac{K_0 e^{-f} - K_1}{K_2 e^{-f} - K_3},$$

assuming that $K_2 e^{-f} - K_3$ is invertible.

2.6. **Unconditional convergence of infinite sums and products.** In the course of this paper, we will often write sums or products of power series over countably infinite sets, and we wish to show that these converge unconditionally, i.e. regardless of the way in which we order the summands or multiplicands. To that end, we show the following for a fixed power series ring over the integers (e.g. $\widehat{A}$, C, or D).

Proposition 2.1. *Let I be a countably infinite set, and let*

$$i_0, i_1, i_2, \ldots \text{ and } j_0, j_1, j_2, \ldots$$

be two enumerations of I. Also, let P be a map from I to the power series ring. Then the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} P(i_n)$$

exists if and only if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} P(j_n)$$

does, in which case the two limits are equal; i.e.,

$$\lim_{N\to\infty} \sum_{n=0}^{N} P(i_n) = \lim_{N\to\infty} \sum_{n=0}^{N} P(j_n). \tag{2.1}$$

Proof. We continue to let $\mathfrak{p}$ be the ideal generated by all indeterminates in the power series ring.

Assume that the left hand side of Equation (2.1) converges to a valid power series. Then there must exist some power series P_∞ such that for all integers $m \geq 0$, there exists large enough N'_m such that for all $N \geq N'_m$, we have that

$$\sum_{n=0}^{N} P(i_n) \equiv P_\infty \pmod{\mathfrak{p}^m}.$$

This implies that for all $N > N'_m$,

$$P_\infty + P(i_N) \equiv \sum_{n=0}^{N-1} P(i_n) + P(i_N) \equiv \sum_{n=0}^{N} P(i_n) \equiv P_\infty \pmod{\mathfrak{p}^m},$$

so $P(i_N)$ is in the ideal $\mathfrak{p}^m$ for all $N > N'_m$.

In other words, for all but finitely many elements $i \in I$, we have that $P(i) \in \mathfrak{p}^m$, and P_∞ must be equivalent modulo $\mathfrak{p}^m$ to the (finite) sum of all $P(i)$ for which $P(i) \notin \mathfrak{p}^m$.

If we instead consider the ordering $j_0, j_1, j_2, \ldots$ of the distinct elements of I, there must be some integer N''_m such that for all $N > N''_m$ we have $P(j_N) \in \mathfrak{p}^m$. Hence the residue of

$$\sum_{n=0}^{N} P(j_n)$$

modulo $\mathfrak{p}^m$ is constant for all $N \geq N''_m$ and, in particular, is equivalent modulo $\mathfrak{p}^m$ of the sum of all $P(j)$ for which $P(j) \notin \mathfrak{p}^m$. But as we saw above, P_∞ is also equivalent modulo $\mathfrak{p}^m$ to the same sum, so

$$\sum_{n=0}^{N} P(j_n) \equiv P_\infty \pmod{\mathfrak{p}^m}$$

for all $N \geq N''_m$. Therefore

$$\lim_{N\to\infty} \sum_{n=0}^{N} P(j_n) = P_\infty,$$

so Equation (2.1) follows. The converse direction follows from swapping the roles of $i_0, i_1, i_2, \ldots$ and $j_0, j_1, j_2, \ldots$ in the argument above. $\square$

Thus, we can unambiguously write down a sum of power series over a countable set. If $I = \{i_0, i_1, i_2, \ldots\}$ is a countably infinite set and P is a map from I to the power series ring, we will let

$$\sum_{i\in I} P(i) = \lim_{N\to\infty} \sum_{n=0}^{N} P(i_n)$$

if such a limit exists. By Proposition 2.1, changing the order in which we label the elements of I does not change the limit on the right hand side, so this is well-defined.

As for the unconditional convergence of infinite products, we make an additional assumption, which will hold for all infinite products that we will be taking. We continue to work in a fixed power series ring over the integers, and we continue to let $\mathfrak{p}$ be the ideal generated by all indeterminates in the power series ring.

Proposition 2.2. *Let I be a countably infinite set, and let*

$$i_0, i_1, i_2, \ldots \text{ and } j_0, j_1, j_2, \ldots$$

be two enumerations of I. Also, let Q be a map from I to the power series ring such that, for any integer $m \geq 0$, for all but finitely many elements $i \in I$, we have

$$Q(i) \equiv 1 \pmod{\mathfrak{p}^m} \tag{2.2}$$

Then

$$\lim_{N\to\infty}\prod_{n=0}^{N} Q(i_n) = \lim_{N\to\infty}\prod_{n=0}^{N} Q(j_n), \tag{2.3}$$

and in particular both sides converge.

Proof. We first show that the limit on the left hand side of Equation (2.3) converges. Fix $m \geq 0$. Since Q satisfies (2.2), there exists some N'_m for which

$$Q(i_N) \equiv 1 \pmod{\mathfrak{p}^m}$$

for all $N > N'_m$. So for all $N \geq N'_m$, we have

$$\prod_{n=0}^{N} Q(i_n) \equiv \prod_{n=0}^{N+1} Q(i_n) \pmod{\mathfrak{p}^m},$$

implying that the limit stabilizes modulo $\mathfrak{p}^m$. Thus the limit converges. In particular, the limit will converge to the power series Q_∞ that, for each integer $m \geq 0$, is equivalent modulo $\mathfrak{p}^m$ to the (finite) product of all $Q(i)$ for which $Q(i) - 1 \notin \mathfrak{p}^m$. By an identical argument, the limit on the right hand side converges as well, and to the same value, as the argument is independent of the ordering of the elements in I. ☐

Thus, as long as Condition (2.2) is satisfied for the multiplicands, we can unambiguously write down a product of power series over a countable set. If $I = \{i_0, i_1, i_2, \ldots\}$ is a countably infinite set and Q is a map from I to the power series ring satisfying (2.2), we will let

$$\prod_{i\in I} Q(i) = \lim_{N\to\infty}\prod_{n=0}^{N} Q(i_n).$$

By Proposition 2.2, changing the order in which we label the elements of I does not change the limit on the right hand side, so this is well-defined.

As a shorthand, given a map Q from a countably infinite set I to the power series ring we are working in, we will say that the infinite product of power series

$$\prod_{i\in I} Q(i)$$

satisfies Condition (2.2) if, for any integer $m \geq 0$, for all but finitely many elements $i \in I$, we have

$$Q(i) \equiv 1 \pmod{p^m}.$$

Proposition 2.2 then implies that the infinite product converges.

2.7. **Distributivity of multiplication over infinite sums.** To ensure that we can pull factors in and out of infinite sums, we show the following for a fixed power series ring over the integers (e.g. $\widehat{A}$, C, or D).

Proposition 2.3. *Let P be a map from a countable set I to the power series ring, and let K be a fixed power series in the same ring. If*

$$\sum_{i\in I} P(i)$$

is a valid power series in the ring (i.e. the infinite sum converges), then

$$K\sum_{i\in I} P(i) = \sum_{i\in I} KP(i). \tag{2.4}$$

Proof. Fix an enumeration of I so that for any map Q from I to the power series ring, we have

$$\sum_{i\in I} Q(i) = \lim_{N\to\infty} \sum_{n=0}^{N} Q(i_n), \tag{2.5}$$

in which the left hand side converges to a valid power series if and only if the right hand side does.

Setting $Q = P$ above and multiplying both sides by K, we find that the power series on the left hand side of Equation (2.4), valid because it is the product of two valid power series, is equal to

$$K \lim_{N\to\infty} \sum_{n=0}^{N} P(i_n), \tag{2.6}$$

which then must be a valid power series as well.

By the continuity of multiplication in the power series ring, the map that takes each power series in the ring to its product with K is also continuous. Hence (2.6) is equal to

$$\lim_{N\to\infty} K \sum_{n=0}^{N} P(i_n) = \lim_{N\to\infty} \sum_{n=0}^{N} KP(i_n). \tag{2.7}$$

By setting $Q(i) = KP(i)$ for all $i \in I$ in Equation (2.5), we see that the right hand side of Equation (2.7) is equal to the right hand side of Equation (2.4), as desired. $\square$

2.8. **Commutativity of infinite products.** A similar argument ensures that we can combine two convergent infinite products indexed over the same set. Again, we fix a power series ring over the integers.

Proposition 2.4. *Let P and P' be two maps from a countable set I to the power series ring. If P and P' both satisfy Condition* (2.2)*, then*

$$\left(\prod_{i\in I} P(i)\right)\left(\prod_{i\in I} P'(i)\right) = \prod_{i\in I}(P(i)P'(i)). \tag{2.8}$$

Proof. By Proposition 2.2, the fact that both P and P' satisfy (2.2) implies that

$$\prod_{i\in I} P(i) \text{ and } \prod_{i\in I} P'(i)$$

both converge to valid power series in the ring.

Since I is countable, we can write it as

$$I = \{i_0, i_1, i_2, \ldots\},$$

in which case we can write a product over I as a limit of partial products, as follows: for an arbitrary map Q from a countable set I to the power series ring, we have

$$\prod_{i\in I} Q(i) = \lim_{N\to\infty} \prod_{n=0}^{N} Q(i_n), \tag{2.9}$$

in which the left hand side converges to a valid power series if and only if the right hand side does.

Setting $Q = P$ and $Q = P'$ above, we find that the power series on the left hand side of Equation (2.8), valid because it is the product of two valid power series, is equal to

$$\left(\lim_{N\to\infty} \prod_{n=0}^{N} P(i_n)\right)\left(\lim_{N\to\infty} \prod_{n=0}^{N} P'(i_n)\right), \tag{2.10}$$

which then must be a valid power series as well.

By the continuity of multiplication in the power series ring, (2.10) is equal to

$$\lim_{N\to\infty}\left(\prod_{n=0}^{N} P(i_n)\right)\left(\prod_{n=0}^{N} P'(i_n)\right) = \lim_{N\to\infty}\prod_{n=0}^{N}(P(i_n)P'(i_n)). \tag{2.11}$$

By setting $Q(i) = P(i)P'(i)$ for all $i \in I$ in Equation (2.9), we see that the right hand side of Equation (2.11) is equal to the right hand side of Equation (2.8), as desired. □

3. A visual exposition of Theorem 1.1 in two examples

In this section, we provide a 'visual proof' of Theorem 1.1 for Weyl groups $\widetilde{A}_1$ and $\widetilde{A}_2$. We hope this gives the reader some intuition for how the summands

$$\sum_{w\in W}\prod_{\alpha\in\Sigma}\frac{1}{1-e^{-w\alpha}}.$$

on the left hand side of Theorem 1.1 look like for each w and shows the enormous cancellation that occurs when they are summed together, yielding the right hand side

$$\frac{1}{1-e^{-\delta}}.$$

3.1. **Type $\widetilde{A}_1$.** We first discuss $\widetilde{A}_1$, which is presented in Figure 1. Let E be the real span of the finite roots. Recall that the affine Weyl group acts on E by affine linear transformations, and under this action the reflection in W associated to a real root acts by an affine reflection. After removing their reflection affine hyperplanes, E decomposes as a disjoint union of connected components, called alcoves, which W acts simply transitively on. The upper line depicts the Weyl alcoves of $\widetilde{A}_1$, separated by vertical ticks representing the affine root hyperplanes. The central alcove is associated with the identity Weyl element e, and alcoves associated with longer Weyl elements are further from e. Alcoves are colored by the length of the associated Weyl element.

Informally, the central image depicts, for each colored w, the area over the lattice points hit by

$$p(w) = \prod_{\alpha\in\Sigma}\frac{1}{1-e^{-w\alpha}}.$$

More precisely, for fixed $w \in W$ we shade in, with the color associated to w in the alcove picture, the cone spanned by $-(w\alpha)^+$ and $-(w\beta)^+$, where α, β are the simple roots of $\widetilde{A}_1$ and for a root γ we denote by $(\gamma)^+$ its integral multiple which is a positive root.

Observe that longer w correspond to smaller areas closer to $-\delta$. Since $p(e)$ contains $+e^f$ for all $f = m\alpha + n\beta \in L^-$ for $m, n \in \mathbb{Z}^{\leq 0}$, all points between $-\alpha$ and $-\beta$ would be covered by this area. Our diagram shows that the areas of the other $w \in W, w \neq e$, disjointly contain this same area, except for the line through $-\delta$, which is approached as length of w increases but never reached. Further, for all $w \neq e$, the e^f terms in $p(w)$ have coefficient -1. Thus, subtracting the areas of $w \neq e$ from the area of e results in the line through $-\delta$, which confirms Theorem 1.1's statement that after summation only the $e^{n\delta}_{n\in\mathbb{Z}^{\leq 0}}$ terms remain, with coefficient 1.

The lower line depicts a slice of the central image through the points $-\alpha$ and $-\beta$, i.e. its intersection with the affine line joining $-\alpha$ and $-\beta$. Note it is an informative simplification of the central image, depicting the uncolored region condensing to the δ point at the origin as we introduce effects of longer w.

3.2. **Type $\widetilde{A}_2$.** We now discuss $\widetilde{A}_2$, which is presented in Figure 2. However, we present only the analogue of the upper and lower lines of Figure 1, due to the three dimensional nature of the analogue of the central image.

The upper left image depicts the Weyl alcoves of $\widetilde{A}_2$, separated by lines representing the affine root hyperplanes. As in $\widetilde{A}_1$, the central alcove is associated with the identity Weyl element e, and

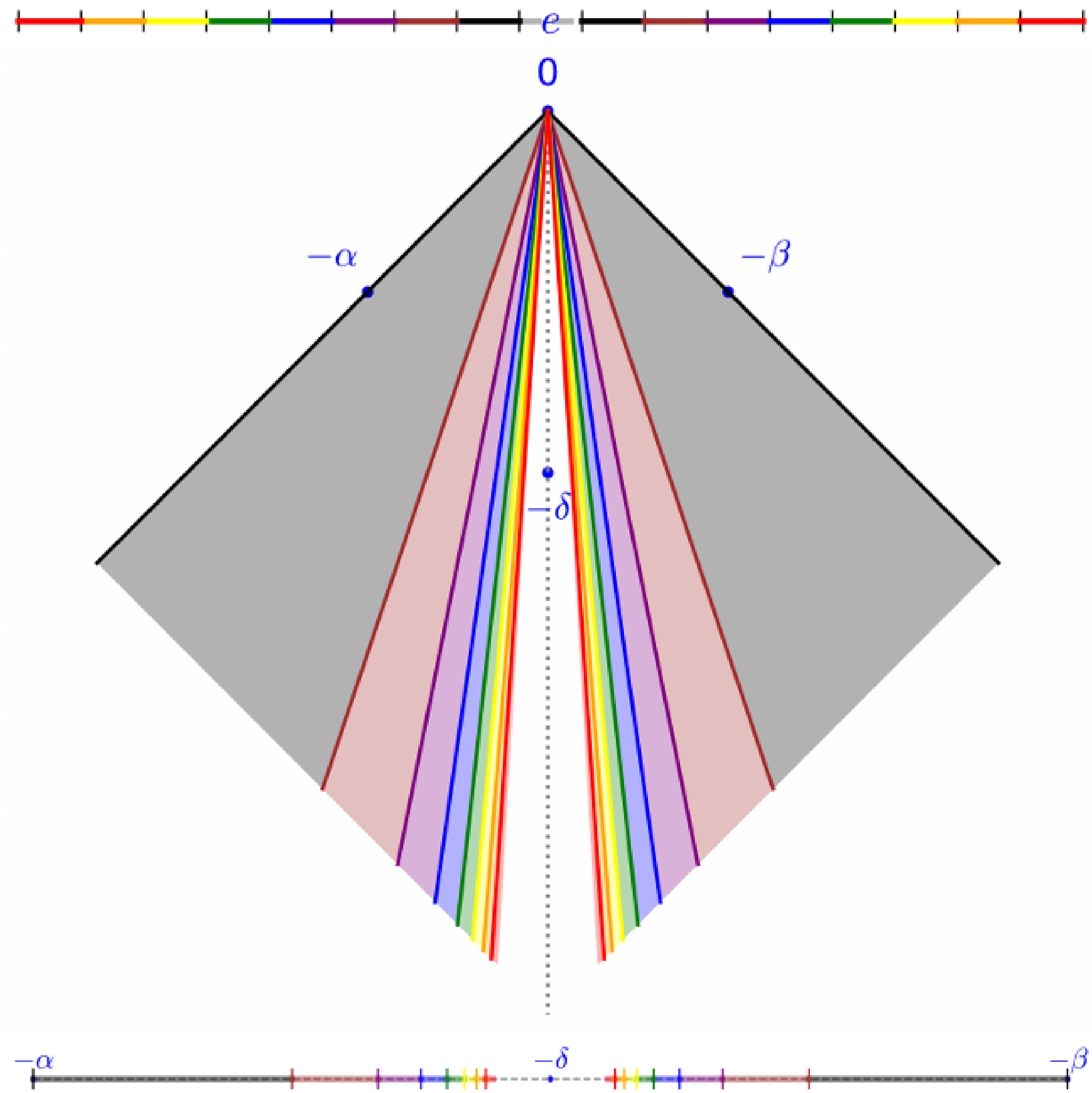

FIGURE 1. Type $\widetilde{A}_1$

alcoves associated with longer Weyl elements are further from e. Unlike previously, alcoves are colored in a more complex way. For $w \in W$, define

$$\tilde{s}(w) = \#\{\alpha \in \Sigma : w\alpha < 0\} \mod 2,$$

and note that $\tilde{s}(w)$ determines the overall sign of $p(w)$. Then, for each length $\ell \geq 0$, we distinguish by color the subset of w with $\tilde{s}(w) = 1$ and length ℓ or $\tilde{s}(w) = 0$ and length $\ell + 1$. Note that this is aligns with our coloring in Figure 1, as in $\widetilde{A}_1$ we have $\tilde{s}(w) = 0$ for all $w \neq e$.

The upper right image image is the analog of the lower line in Figure 1. I.e., for $w \in W$, one may introduce the three dimensional cone spanned by $-(w\alpha)^+$, $-(w\beta)^+$, and $-(w\gamma)^+$, where α, β, and γ are the simple roots of $\widetilde{A}_2$. One may intersect these cones with the affine plane containing $-\alpha$, $-\beta$, and $-\gamma$, to obtain convex regions in the simplex

$$-a\alpha - b\beta - c\gamma \quad \text{for} \quad a, b, c \in \mathbb{R}^{\geqslant 0},\ a + b + c = 1.$$

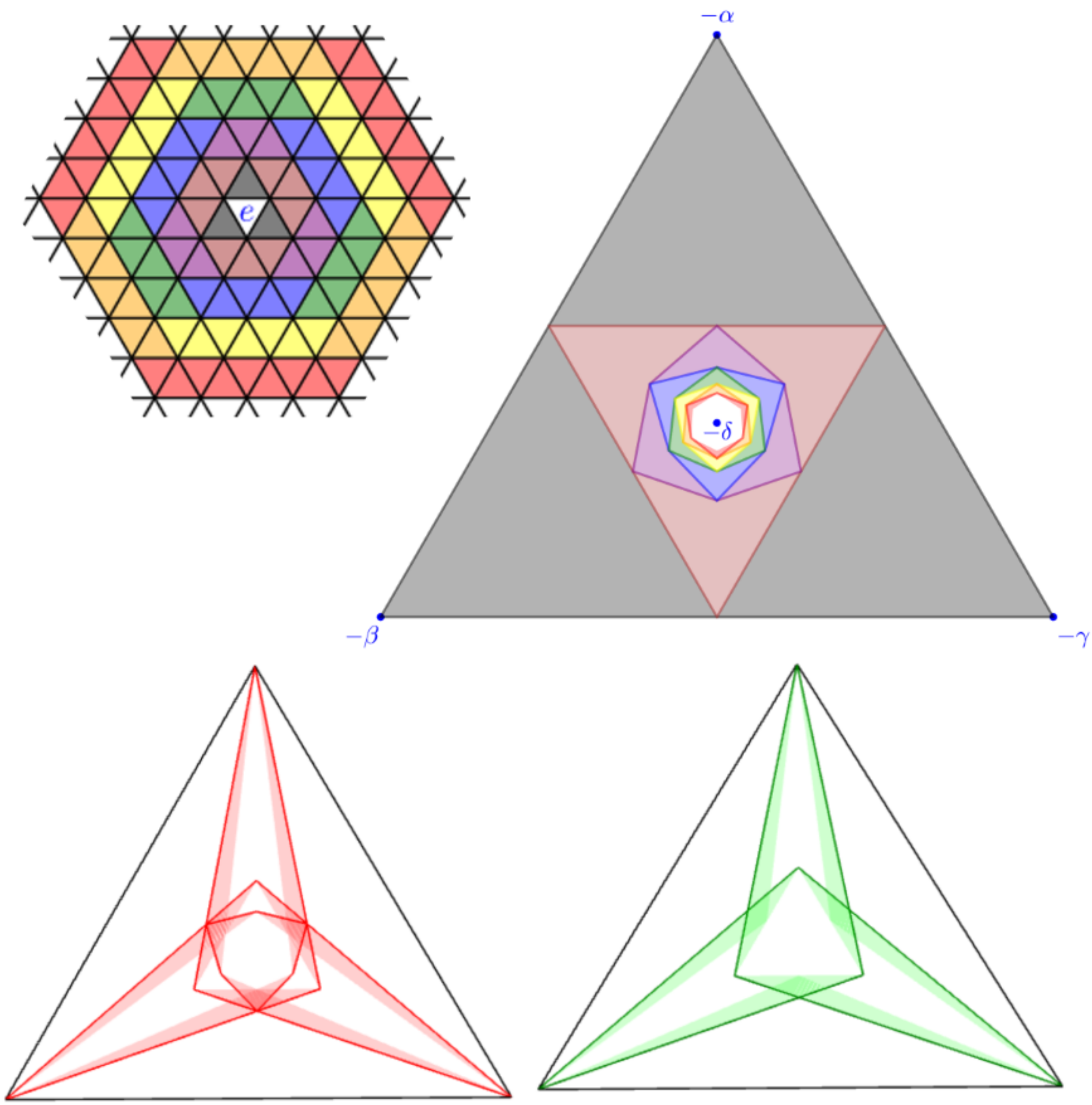

FIGURE 2. Type $\widetilde{A}_2$

As before, the sum of the $p(w)$ with w of a fixed color produces a sum over lattice points, each with coefficient -1 in the corresponding colored region of the triangle in the upper right of Figure 2. Moreover, the union of these yields the entirety of the triangle with $-\delta$ deleted. Since once again $p(e)$ assign to each lattice point of the entire triangle the coefficient 1, one combines these facts to obtain Theorem 1.1 in this case.

However, we would like to emphasize that within a region of fixed color, nontrivial cancellation occurs, as we will describe for the blue region. The lower images depicts the slice of the cones of blue w. In particular, the lower left image is of the nine elements w with $\tilde{s}(w) = 1$ and length four and the lower right is of the six elements with $\tilde{s}(w) = 0$ and length five. Observe that they differ only in the three innermost convex quadrilaterals that appear only in the left, which is their overall contribution, depicted in blue in the upper right triangle.

4. Coefficient Stabilization

In this section we will prove that the left hand side of Equation (1.4), which we denote by σ:

$$\sigma = \sum_{w \in W} \prod_{\alpha \in \Phi_m^+} \frac{1 - u_\alpha e^{-w\alpha}}{1 - e^{-w\alpha}} \tag{4.1}$$

is well-defined in the power series ring C, cf. Section 2.3. I.e., we will show that the product in each summand of Equation (4.1) is unconditionally convergent, in the sense of Proposition 2.2, and that their sum is again unconditionally convergent.

Define the following, and note that it is invertible in C:

$$\kappa = \prod_{\alpha \in \Phi_m^+} \frac{1 - e^{-\alpha}}{1 - u_\alpha e^{-\alpha}}.$$

Further, it is unconditionally convergent by the criterion of Proposition 2.2 because for any h, there are only finitely many β with height $\leq h$.

Further define the following, which if σ is well-defined in C, is equivalent to $\kappa\sigma$:

$$\Gamma = \sum_{w \in W} \left(\prod_{\alpha \in \Phi_m^+ \cap w\Phi_m^+} \frac{1 - u_{w^{-1}\alpha} e^{-\alpha}}{1 - u_\alpha e^{-\alpha}} \right) \left(\prod_{\alpha \in \Phi_m^+ \cap w\Phi_m^-} \frac{u_{-w^{-1}\alpha} - e^{-\alpha}}{1 - u_\alpha e^{-\alpha}} \right).$$

We will argue that Γ is well-defined in C, after which we show that $\sigma = \kappa^{-1}\Gamma$ so σ is well-defined in C.

For ease of notation we write

$$\Gamma = \sum_{w \in W} \prod_{\alpha \in \Phi_m^+} \Gamma_{w,\alpha}$$

where

$$\Gamma_{w,\alpha} = \begin{cases} \frac{1 - u_{w^{-1}\alpha} e^{-\alpha}}{1 - u_\alpha e^{-\alpha}}, & \text{if } \alpha \in \Phi_m^+ \cap w\Phi_m^+ \\ \frac{u_{-w^{-1}\alpha} - e^{-\alpha}}{1 - u_\alpha e^{-\alpha}}, & \text{if } \alpha \in \Phi_m^+ \cap w\Phi_m^- \end{cases}.$$

On expansion, $\Gamma_{w,\alpha}$ is a formal power series of the form

$$\sum_{r \geq 0} \gamma_{r,w,\alpha} e^{-r\alpha}$$

where $\gamma_{r,w,\alpha} \in A$ and in particular

$$\gamma_{0,w,\alpha} = \begin{cases} 1, & \text{if } \alpha \in \Phi_m^+ \cap w\Phi_m^+ \\ u_{-w^{-1}\alpha}, & \text{if } \alpha \in \Phi_m^+ \cap w\Phi_m^- \end{cases}.$$

Then $\prod_{\alpha \in \Phi_m^+} \Gamma_{w,\alpha}$ is also a formal power series, so we may write

$$\Gamma = \sum_{w \in W} \prod_{\alpha \in \Phi_m^+} \sum_{r \geq 0} \gamma_{r,w,\alpha} e^{-r\alpha} = \sum_{w \in W} \sum_{f \in L^+} \gamma_{w,f} e^{-f}$$

where

$$\gamma_{w,f} = \sum \prod_{\alpha \in \Phi_m^+} \gamma_{r_\alpha,w,\alpha} \tag{4.2}$$

is summed over all families $(r_\alpha)_{\alpha \in \Phi_m^+}$ of non-negative integers satisfying

$$\sum_{\alpha \in \Phi_m^+} r_\alpha \alpha = f. \tag{4.3}$$

We will show that $\gamma_{w,f} \in A$. Firstly, for f finite, every family $(r_\alpha)_{\alpha\in\Phi_m^+}$ satisfying Equation (4.3) contains finitely many nonzero r_α, and of the $\gamma_{0,w,\alpha}$, there are only finitely many that do not evaluate to 1, so $\prod_{\alpha\in\Phi_m^+} \gamma_{r_\alpha,w,\alpha} \in A$. Further, there are only finitely many such families $(r_\alpha)_{\alpha\in\Phi_m^+}$, so Equation (4.2) is indeed in A.

Let $\mathfrak{p}$ be the ideal in A generated by the u_α, $\alpha \in \Phi^+$. Recall that for $f \in L^+$, say $f = \sum_{\alpha\in\Sigma} n_\alpha \alpha$, the height of f is

$$\mathrm{ht}(f) = \sum_{\alpha\in\Sigma} n_\alpha.$$

From Equation (4.3) it follows that

$$\mathrm{ht}(f) = \sum_{\alpha\in\Phi_m^+} r_\alpha \mathrm{ht}(\alpha) \geq \sum_{\alpha\in\Phi_m^+} r_\alpha. \tag{4.4}$$

Let $l(w)$ be the length of w, i.e. $\mathrm{card}(\Phi^+ \cap w\Phi^-)$, and let

$$\gamma_{0,f} = \sum \gamma_{w,f} \qquad \in A$$

summed over $w \in W$ such that $l(w) \leq \mathrm{ht}(f)$; and for $m \geq 1$ let

$$\gamma_{m,f} = \sum \gamma_{w,f} \qquad \in A$$

summed over $w \in W$ such that $l(w) = \mathrm{ht}(f) + m$. From Equation (4.4) it follows that, when $l(w) = \mathrm{ht}(f) + m$, at least m of the $l(w)$ integers r_α for $\alpha \in \Phi_m^+ \cap w\Phi_m^-$ must be zero for each solution of Equation (4.3), and hence by Equation (4.2) that $\gamma_{w,f} \in \mathfrak{p}^m$. Consequently $\gamma_{m,f} \in \mathfrak{p}^m$ for all $m \geq 0$, and hence the coefficient of e^f in Γ, namely

$$\gamma_f = \sum_{m\geq 0} \gamma_{m,f},$$

is an element of $\hat{A}$. So

$$\Gamma = \sum_{f\in L^+} \gamma_f e^f$$

is a well defined element of C.

Since κ is a unit in C, it follows that $\kappa^{-1}\Gamma$ is an element of C. We will show that $\kappa^{-1}\Gamma = \sigma$.

By Proposition 2.3, we may pull κ^{-1} inside the infinite summation,

$$\kappa^{-1}\Gamma = \sum_{w\in W} \kappa^{-1} \prod_{\alpha\in\Phi_m^+} \Gamma_{w,\alpha}.$$

By Proposition 2.4, we may separately consider the product for $\alpha \in \Phi_m^+ \cap w\Phi_m^+$ and $\alpha \in \Phi_m^+ \cap w\Phi_m^-$, which evaluates to

$$\sum_{w\in W} \left(\prod_{\alpha\in\Phi_m^+\cap w\Phi_m^+} \frac{1 - u_{w^{-1}\alpha} e^{-\alpha}}{1 - e^{-\alpha}} \right) \left(\prod_{\alpha\in\Phi_m^+\cap w\Phi_m^-} \frac{u_{-w^{-1}\alpha} - e^{-\alpha}}{1 - e^{-\alpha}} \right),$$

which, by Proposition 2.2, is σ.

5. Proof of Theorem 1.2

Now that we have shown that the left hand side of Equation (1.4) from Theorem 1.2 is a valid formal power series in C, we can prove the theorem itself.

For arbitrary submultisets $B \subseteq \Phi_m^+$, recall that B^- is the image of B after each positive root is sent to its corresponding negative root, that $|B|$ is the sum of the roots in B, and that $\Phi_m(w, B) = \Phi_m^+ \cap w(B \cup \Phi_m^- \setminus B^-)$. We can think of $\Phi_m(w, -)$ as a function from the collection of submultisets of Φ_m^+ to itself.

We begin by studying the behavior of this function and its relation to another function, first defined by Macdonald [3, Section 9]. For all $w \in W$ and $f \in L^+$, we let $w \circ f = wf + s(w)$. Shortly after his introduction of the function, Macdonald proves that $\circ : W \times L^+ \to L^+$ is a group action of W on L^+.

We have the following relationship between these two functions, which we will use in our proof of Theorem 1.2.

Lemma 5.1. *For all $w \in W$ and $B \subseteq \Phi_m^+$,*

$$w \circ |B| = |\Phi_m(w, B)| \geq 0,$$

and $\#\Phi_m(w, B)$ has the same parity as $\ell(w) + \#B$.

Proof. We have that

$$\begin{aligned} w \circ |B| &= w|B| + s(w) \\ &= |\Phi_m^+ \cap wB| - |\Phi_m^+ \cap wB^-| + |\Phi_m^+ \cap w\Phi_m^-| \\ &= |\Phi_m^+ \cap w(B \cup \Phi_m^- \setminus B^-)| \\ &= |\Phi_m(w, B)| \geq 0. \end{aligned}$$

An examination of the above equalities reveals that $\#\Phi_m(w, B)$ has the same parity as $\#B + \#\Phi(w) = \#B + \ell(w)$. $\square$

Fixing $w \in W$, we make another observation regarding the function $\Phi_m(w, -)$ that we will use in our proof of Theorem 1.2.

Lemma 5.2. *The function $\Phi_m(w, -)$ is bijective. Its inverse is $\Phi_m(w^{-1}, -)$.*

Proof. Fix an arbitrary $B \subseteq \Phi_m^+$ and let

$$E = \Phi_m(w, B) = \Phi_m^+ \cap w(B \cup \Phi_m^- \setminus B^-).$$

Then

$$E^- = \Phi_m^- \cap w(B^- \cup \Phi_m^+ \setminus B),$$

so

$$\Phi_m^- \setminus E^- = \Phi_m^- \cap w(\Phi_m^- \setminus B^- \cup B),$$

which implies that

$$E \cup \Phi_m^- \setminus E^- = w(B \cup \Phi_m^- \setminus B^-).$$

Hence

$$\begin{aligned} \Phi_m(w^{-1}, E) &= \Phi_m^+ \cap w^{-1}(E \cup \Phi_m^- \setminus E^-) \\ &= \Phi_m^+ \cap (B \cup \Phi_m^- \setminus B^-) \\ &= B. \end{aligned}$$

Thus $\Phi_m(w^{-1}, -)$ is the left inverse of $\Phi_m(w, -)$. Of course, the above argument also implies that $\Phi_m((w^{-1})^{-1}, -) = \Phi_m(w, -)$ is the left inverse of $\Phi_m(w^{-1}, -)$, so we are done. $\square$

Finally, we state a result that follows directly from Macdonald's work regarding the $\circ$ action of W on L^+. We will let F denote the vector space in which the affine roots Φ in our root system live.

Lemma 5.3. *The stabilizer of any element in L^+ under the $\circ$ action is finite.*

Proof. By [3, (9.6)], there exists a map ψ from F to itself satisfying

$$\psi(w \circ f) = w(\psi(f))$$

for all $w \in W$ and $f \in L^+$.

Fix $f \in L^+$. If $w \in W$ is in the stabilizer of f under the $\circ$ action, then

$$w(\psi(f)) = \psi(w \circ f) = \psi(f),$$

which implies that w is in the stabilizer of $\psi(f)$ under the usual action. Thus the stabilizer of f under $\circ$ is contained in the stabilizer of $\psi(f)$ under the usual action. So it suffices to show that the stabilizer of an arbitrary vector in F under the usual action is finite.

For any $w \in W$, we can write

$$w = t^\lambda w_0,$$

where w_0 is in the finite Weyl group W_0 associated with our affine Weyl group W and t^λ denotes translation by a vector $\lambda \in F$.

Fix $v \in F$. Then

$$wv = (t^\lambda w_0)v = w_0 v + \lambda.$$

So if w is in the stabilizer of v under the usual action, we must have

$$\lambda = v - w_0 v.$$

Thus the stabilizer of v under the usual action is a subset of

$$\{t^{v-w_0 v} w_0 \in W \mid w_0 \in W_0\},$$

which is finite because W_0 is finite. Hence the stabilizer of any element of F under the usual action of W is finite, as desired. □

Now we are ready to prove our first main result.

Proof of Theorem 1.2. We wish to show that Equation (1.4) holds.

Since

$$\prod_{\alpha \in \Phi_m^+} \frac{1}{1 - e^{-w\alpha}} \text{ and } \prod_{\alpha \in \Phi_m^+} (1 - u_\alpha e^{-w\alpha})$$

both satisfy Condition (2.2), by Proposition 2.2 they are both valid power series (i.e. convergent infinite products), so by Proposition 2.4 their product is a valid power series equal to

$$\prod_{\alpha \in \Phi_m^+} \frac{1 - u_\alpha e^{-w\alpha}}{1 - e^{-w\alpha}}.$$

So the left hand side of Equation (1.4) can be rewritten as

$$\sum_{w \in W} \left(\prod_{\alpha \in \Phi_m^+} \frac{1}{1 - e^{-w\alpha}} \right) \left(\prod_{\alpha \in \Phi_m^+} (1 - u_\alpha e^{-w\alpha}) \right). \tag{5.1}$$

When w acts on Φ_m^+, the imaginary roots are fixed; some real roots are sent to negative roots, the roots in $\Phi(w)$ negated; and the remaining real roots are permuted. So

$$\begin{aligned} \prod_{\alpha \in \Phi_m^+} \frac{1}{1 - e^{-w\alpha}} &= \prod_{\alpha \in \Phi(w)} \frac{1}{1 - e^{\alpha}} \prod_{\alpha \in \Phi_m^+ \setminus \Phi(w)} \frac{1}{1 - e^{-\alpha}} \\ &= \varepsilon(w) e^{-s(w)} \prod_{\alpha \in \Phi_m^+} \frac{1}{1 - e^{-\alpha}}, \end{aligned}$$

which means we can rewrite (5.1) as

$$\sum_{w\in W} \Delta^{-1}\varepsilon(w)e^{-s(w)} \prod_{\alpha\in\Phi_m^+}(1-u_\alpha e^{-w\alpha}), \tag{5.2}$$

where

$$\Delta = \prod_{\alpha\in\Phi_m^+}(1-e^{-\alpha}) = \sum_{w\in W}\varepsilon(w)e^{-s(w)};$$

the latter equality is simply the Macdonald identity from [3].

By evaluating each infinite product, we can rewrite (5.2) as

$$\sum_{w\in W} \Delta^{-1}\varepsilon(w)e^{-s(w)} \sum_{E\subseteq\Phi_m^+} e^{-w|E|}\prod_{\alpha\in E}(-u_\alpha),$$

and we can apply Proposition 2.3 to bring each factor of $\varepsilon(w)e^{-s(w)}$ into the corresponding infinite sum. Recalling that $w\circ f = wf + s(w)$ for all $w\in W$ and $f\in L^+$, this yields

$$\sum_{w\in W} \Delta^{-1} \sum_{E\subseteq\Phi_m^+} \varepsilon(w)e^{-w\circ|E|}\prod_{\alpha\in E}(-u_\alpha). \tag{5.3}$$

Since Δ is invertible, we can multiply (5.3) by $\Delta^{-1}\Delta$ on the left without altering it. Proposition 2.3 allows us to bring the factor of Δ into the infinite sum, where it will cancel with the Δ^{-1} inside the sum. This means we can rewrite (5.3) as

$$\Delta^{-1}\sum_{w\in W}\sum_{E\subseteq\Phi_m^+} \varepsilon(w)e^{-w\circ|E|}\prod_{\alpha\in E}(-u_\alpha). \tag{5.4}$$

We wish to change the order of summation in (5.4), which we can safely do if each distinct monomial in the double sum appears in a summand (with a nonzero coefficient) only a finite number of times. We verify this as follows.

Each summand in (5.4) is a monomial of the form

$$\pm\, e^{-f}\prod_{\alpha\in E'} u_\alpha, \tag{5.5}$$

for some $f\in L^+$ and $E'\subseteq\Phi_m^+$, two parameters which uniquely determine such a monomial up to a sign. We claim that the number of times each distinct monomial appears (with either sign) in the double sum is finite. We prove this claim as follows.

Fix $f\in L^+$ and $E'\subseteq\Phi_m^+$. Then the associated monomial (5.5) appears as a summand corresponding to $w\in W$ and $E\subseteq\Phi_m^+$ only when $E=E'$ and $w\circ|E'|=f$. It suffices, then, to show that only finitely many $w\in W$ satisfy the latter property.

Let W_f be the set of $w\in W$ for which $w\circ|E'|=f$, so that $W_f\circ|E|=\{f\}$. If W_f is empty we are done; otherwise, fix $w_0\in W_f$ so that $w_0\circ|E|=f$ and thus $w_0^{-1}\circ f=|E|$. Then

$$W_f w_0^{-1}\circ f = W_f\circ|E| = \{f\},$$

so $W_f w_0^{-1}$ is contained in the stabilizer of $f\in L^+$ under the $\circ$ action. But by Lemma 5.3, the stabilizer of any element in L^+ under the $\circ$ action is finite, so $\#(W_f w_0^{-1}) = \#W_f$ must be finite as well, as desired. Therefore, we can safely change the order of summation of these monomials, allowing us to rewrite (5.4) as

$$\Delta^{-1}\sum_{E\subseteq\Phi_m^+}\sum_{w\in W} \varepsilon(w)e^{-w\circ|E|}\prod_{\alpha\in E}(-u_\alpha). \tag{5.6}$$

We note that both

$$\prod_{\alpha\in E}(-u_\alpha) \text{ and } \sum_{w\in W}\varepsilon(w)e^{-w\circ|E|}$$

are valid power series, as the former is a finite product and the latter is a convergent infinite sum (our earlier reasoning ensures that each summand of this sum still appears at most a finite number of times), so by Proposition 2.3 we can rewrite (5.6) as

$$\Delta^{-1} \sum_{E \subseteq \Phi_m^+} \prod_{\alpha \in E} (-u_\alpha) \sum_{w \in W} \varepsilon(w) e^{-w \circ |E|}.$$

We can bring Δ^{-1} back into the infinite sum by applying Proposition 2.3 again, which yields

$$\sum_{E \subseteq \Phi_m^+} \Delta^{-1} \prod_{\alpha \in E} (-u_\alpha) \sum_{w \in W} \varepsilon(w) e^{-w \circ |E|}. \tag{5.7}$$

By [3, Section 9],

$$\sum_{w \in W} \varepsilon(w) e^{-w \circ |E|} \neq 0$$

if and only if $|E| = x \circ n\delta$ for some $x \in W$ and $n \in \mathbb{Z}$, in which case

$$\sum_{w \in W} \varepsilon(w) e^{-w \circ |E|} = \varepsilon(x) e^{-n\delta} \Delta.$$

We note that $|E| = x \circ n\delta$ implies that

$$\begin{aligned} n\delta &= x^{-1} \circ |E| \\ &= |\Phi_m(x^{-1}, E)| \geq 0, \end{aligned}$$

where the latter equality and the inequality follow from Lemma 5.1; in particular, we must have $n \geq 0$. So we can rewrite (5.7) as

$$\sum_{n=0}^{\infty} \sum_{x \in W} \sum_{\substack{E \subseteq \Phi_m^+, \\ |E| = x \circ n\delta}} \varepsilon(x) e^{-n\delta} \prod_{\alpha \in E} (-u_\alpha). \tag{5.8}$$

After relabeling and observing that

$$e^{-n\delta}, \sum_{\substack{E \subseteq \Phi_m^+, \\ |E| = w \circ n\delta}} \varepsilon(w) \prod_{\alpha \in E} (-u_\alpha), \text{ and } \sum_{w \in W} \sum_{\substack{E \subseteq \Phi_m^+, \\ |E| = w \circ n\delta}} \varepsilon(w) \prod_{\alpha \in E} (-u_\alpha)$$

are all valid power series, we can apply Proposition 2.4 twice to rewrite (5.8) as

$$\sum_{n=0}^{\infty} e^{-n\delta} \sum_{w \in W} \sum_{\substack{E \subseteq \Phi_m^+, \\ |E| = w \circ n\delta}} \varepsilon(w) \prod_{\alpha \in E} (-u_\alpha). \tag{5.9}$$

By Lemma 5.2, $\Phi_m(w^{-1}, -)$ is a bijection with $\Phi_m(w, -)$ as its inverse, so we can make a change of variables so that $B = \Phi_m(w^{-1}, E)$. Then by Lemma 5.1,

$$w^{-1} \circ |E| = |B|,$$

so the condition that $|E| = w \circ n\delta$ or, equivalently, that $w^{-1} \circ |E| = n\delta$ can be rewritten as

$$|B| = n\delta.$$

Lemma 5.1 also tells us that $\#B = \#\Phi_m(w^{-1}, E)$ has the same parity as $\ell(w) + \#E$. Hence we can rewrite (5.9) as

$$\sum_{n=0}^{\infty} e^{-n\delta} \sum_{w \in W} \sum_{\substack{B \subseteq \Phi_m^+, \\ |B| = n\delta}} (-1)^{\#B} \prod_{\alpha \in \Phi_m(w,B)} u_\alpha,$$

which is our desired right hand side of Equation (1.4). $\square$

6. More Coefficient Stabilization

We show that taking the power series on the left hand side of Equation (1.4), which lives in the power series ring C, and specializing the u_α's appropriately yields a valid power series in $D = \mathbb{Z}[[e^{-\alpha}]]_{\alpha\in\Sigma}$. Recall that A is the ring of polynomials over the integers in the u_α's, so $A[[e^{-\alpha}]]_{\alpha\in\Sigma}$ is the power series ring of indeterminate $e^{-\alpha}$'s for over integer polynomials in the u_α's, a proper subring of C.

Proposition 6.1. *Let $\varphi : C \to C$ be the ring homomorphism that sends u_α to 0 if α is a simple root and every other integer or indeterminate in C to itself, and let*

$$\sigma = \sum_{w\in W} \prod_{\alpha\in\Phi_m^+} \frac{1 - u_\alpha e^{-w\alpha}}{1 - e^{-w\alpha}}.$$

Then $\varphi(\sigma)$ is a power series in $A[[e^{-\alpha}]]_{\alpha\in\Sigma}$.

Proof. By Theorem 1.2,

$$\sigma = \sum_{n=0}^{\infty} e^{-n\delta} \sum_{w\in W} \sum_{\substack{B\subseteq\Phi_m^+,\\ |B|=n\delta}} (-1)^{\#B} \prod_{\alpha\in\Phi_m(w,B)} u_\alpha,$$

so

$$\varphi(\sigma) = \sum_{n=0}^{\infty} e^{-n\delta} \sum_{w\in W} \sum_{\substack{B\subseteq\Phi_m^+,\\ |B|=n\delta,\\ \Sigma\cap\Phi_m(w,B)=\varnothing}} (-1)^{\#B} \prod_{\alpha\in\Phi_m(w,B)} u_\alpha$$

It suffices, then, to show that the coefficient of $e^{-n\delta}$ in the sum on the right hand side above contains only finitely many terms, which would make it a polynomial in A. There are only finitely many $B \subseteq \Phi_m^+$ for which $|B| = n\delta$, and each such B is finite, as is $\Phi_m(w, B)$, which ensures that both the inner sum and the product inside that sum within each summand associated with each $w \in W$ is finite. Thus, it is enough to show that for a fixed finite $B \subseteq \Phi_m^+$, there can only be finitely many $w \in W$ for which $\Sigma \cap \Phi_m(w, B) = \varnothing$.

Fix finite $B \subseteq \Phi_m^+$ and let

$$H = \max_{\beta\in B} \operatorname{ht}(\beta).$$

By Proposition 6.2 proved below, there are at most finitely many $w \in W$ for which

$$\max_{\alpha\in\Sigma} (-\operatorname{ht}(w\alpha)) \le H.$$

In other words, for all but finitely many $w \in W$, there exists $\alpha \in \Sigma$ for which

$$-\operatorname{ht}(w\alpha) > \operatorname{ht}(\beta)$$

for all $\beta \in B$, implying that $-w\alpha \in \Phi_m^+ \setminus B$ and thus $\alpha \in w^{-1}(\Phi_m^- \setminus B^-)$. So there are at most finitely many $w \in W$ for which

$$\Sigma \cap w^{-1}(\Phi_m^- \setminus B^-) = \varnothing;$$

equivalently, there are at most finitely many $w \in W$ for which

$$\Sigma \cap w(\Phi_m^- \setminus B^-) = \varnothing.$$

Since $\Phi_m(w, B) = \Phi_m^+ \cap w(\Phi_m^- \setminus B^-)$, this means that there are at most finitely many $w \in W$ for which

$$\Sigma \cap \Phi_m(w, B) = \varnothing,$$

as desired. ∎

Proposition 6.2. *Let K be a constant real number. There are at most finitely many $w \in W$ for which*

$$\max_{\alpha\in\Sigma}\left(-\operatorname{ht}(w\alpha)\right) \leq K.$$

We will show Proposition 6.2 by giving a w-dependent lower bound for $\max_{\alpha\in\Sigma}\left(-\operatorname{ht}(w\alpha)\right)$ and showing that for all but finitely many w, this lower bound exceeds K.

Proof. For $w \in W$, let w_0 be its finite part and t^{λ_w} be its translational part, so that

$$w = t^{\lambda_w} w_0$$

and its hyperplane passes through λ_w perpendicular to w_0.

For $\alpha \in \Sigma$, let α_0 be its finite part and k_α be its affine part, so that

$$\alpha = (\alpha_0, k_\alpha)$$

has direction α_0 in the finite root system and value k_α in the δ direction. Note that $k_\alpha = 0$ for the finite simple roots and $k_\alpha = 1$ for the affine simple root.

Then in this representation,

$$w\alpha = t^{\lambda_w} w_0(\alpha_0, k_\alpha) = t^{\lambda_w}(w_0, \alpha_0, k_\alpha) = (w_0\alpha_0, k_\alpha - \langle\lambda_w, w_0\alpha_0\rangle). \tag{6.1}$$

For $f \in L$, say $f = \sum_{\alpha\in\Sigma} n_\alpha \alpha$, where α' is the simple root with affine part, define the *delta-height* of f to be the height of f in the δ direction,

$$\operatorname{ht}_\delta(f) = \sum_{\alpha\in\Sigma} n_\alpha k_\alpha = n_{\alpha'}.$$

Note that for $f \in L^+$ we have $\operatorname{ht}_\delta(f) \leq \operatorname{ht}(f)$ since the n_α are nonnegative, while for $f \in L^-$ we have $\operatorname{ht}_\delta(f) \geq \operatorname{ht}(f)$ since the n_α are non-positive.

Then for w not the identity, there is some $w\alpha \in L^-$ so

$$\max_{\alpha\in\Sigma}\left(-\operatorname{ht}_\delta(w\alpha)\right) \leq \max_{\alpha\in\Sigma}\left(-\operatorname{ht}(w\alpha)\right). \tag{6.2}$$

Let $\angle(\cdot,\cdot)$ denote the smallest non-negative angle between two vectors, i.e. with image $[0, \pi]$. By Equation (6.1),

$$\begin{aligned}
\max_{\alpha\in\Sigma}\left(-\operatorname{ht}_\delta(w\alpha)\right) &= \max_{\alpha\in\Sigma}\left(\langle\lambda_w, w_0\alpha_0\rangle - k_\alpha\right)\\
&\geq \max_{\alpha\in\Sigma}\left(\langle\lambda_w, w_0\alpha_0\rangle\right) - 1\\
&= \max_{\alpha\in\Sigma}\left(||\lambda_w|| \cdot ||w_0\alpha_0|| \cdot \cos(\angle(\lambda_w, w_0\alpha_0))\right) - 1\\
&\geq ||\lambda_w|| \cdot \min_{\alpha\in\Sigma}(||\alpha_0||) \cdot \max_{\alpha\in\Sigma}\left(\cos(\angle(\lambda_w, w_0\alpha_0))\right) - 1
\end{aligned}$$

Using Lemma 6.3 below, we will show that for any set of simple roots there exists a constant $\theta < \pi/2$ such that for all $w \in W$,

$$\min_{\alpha\in\Sigma}\left(\angle(\lambda_w, w_0\alpha_0)\right) \leq \theta,$$

allowing us to conclude

$$\max_{\alpha\in\Sigma}\left(-\operatorname{ht}_\delta(w\alpha)\right) \geq ||\lambda_w|| \cdot \min_{\alpha\in\Sigma}(||\alpha_0||) \cdot (\cos(\theta)) - 1, \tag{6.3}$$

which is a lower bound depending only on w given a fixed Σ.

Lemma 6.3. *Let $\{\nu_j\}_{j\in J}$ be a set of vectors spanning $\mathbb{R}^n$ such that there is no hyperplane H with all $\nu_j \in V$ on the same side of H. Then there exists some $\theta < \pi/2$ such that for all unit vectors λ in this space, there is some $\nu_j \in V$ with $\angle(\lambda, \nu_j) \leq \theta$.*

Proof of Lemma 6.3. Let S^{n-1} be the set of unit vectors in $\mathbb{R}^n$. For $j \in J$, let $f_j : S^{n-1} \to [0, \pi]$ be the function such that $f_j(\lambda) = \angle(\lambda, \nu_j)$. Note that it is a composition of continuous functions:

$$f_j : S^{n-1} \xrightarrow{\langle \cdot, \nu_j \rangle} [-||\nu_j||, ||\nu_j||] \xrightarrow{\div ||\nu_j||} [-1, 1] \xrightarrow{\cos^{-1}} [0, \pi]$$

so it is again continuous.

Then let $f : S^{n-1} \to [0, \pi]$ be the function such that $f(\lambda) = \min_{j \in J} f_j(\lambda)$. Note that as the minimum of continuous functions it is also continuous.

We will show that the image of f is in fact $[0, \theta]$ for some $\theta < \pi/2$. Since S^{n-1} is compact and connected, then its image is compact and connected, hence it has form $[\alpha, \beta]$ for some $\alpha \geq 0$ and $\beta \leq \pi$. In fact, $\alpha = 0$ because the angle between a vector and itself is 0. Further, $\pi/2$ is not in the image because the assumption that there is no hyperplane H with all $\nu_j \in V$ on the same side of H implies there is no vector $v \in S^{n-1}$ such that $\angle(v, \nu_j) \geq \pi/2$ for all $\nu_j \in V$.

Thus the image of f is in fact $[0, \theta]$ for some $\theta < \pi/2$. □

We use Lemma 6.3 on the set $\{w_0 \alpha_0\}_{\alpha \in \Sigma}$ since there is no $v \in S^{n-1}$ such that $\angle(v, \alpha_0) \geq \pi/2$ for all $\alpha \in \Sigma$, so for all $w \in W$, there is no $v \in S^{n-1}$ such that $\angle(v, w_0 \alpha_0) \geq \pi/2$ for all $\alpha \in \Sigma$. (If there were such a v, then $\angle(w_0^{-1} v, \alpha_0) \geq \pi/2$ for all $\alpha \in \Sigma$, a contradiction).

Thus Equation (6.2) and Equation (6.3) give

$$\max_{\alpha \in \Sigma} (-\operatorname{ht}(w\alpha)) \geq ||\lambda_w|| \cdot \min_{\alpha \in \Sigma}(||\alpha_0||) \cdot (\cos(\theta)) - 1. \tag{6.4}$$

Thus, for w with large translational element, specifically

$$||\lambda_w|| \geq \frac{K + 1}{\min_{\alpha \in \Sigma}(||\alpha_0||) \cdot (\cos(\theta))},$$

by Equation (6.4) its action on the simple roots sends some root sufficiently negative:

$$\max_{\alpha \in \Sigma} (-\operatorname{ht}(w\alpha)) \geq ||\lambda_w|| \cdot \min_{\alpha \in \Sigma}(||\alpha_0||) \cdot (\cos(\theta)) - 1 \geq K.$$

Hence, there are only finitely many w with

$$||\lambda_w|| < \frac{K + 1}{\min_{\alpha \in \Sigma}(||\alpha_0||) \cdot (\cos(\theta))},$$

which implies there are at most finitely many $w \in W$ for which

$$\max_{\alpha \in \Sigma} (-\operatorname{ht}(w\alpha)) \leq K.$$

□

7. Proof of Theorem 1.1

We now have all the ingredients we need for our proof of Theorem 1.1 and will prove our main result.

Proof of Theorem 1.1. By Theorem 1.2,

$$\sum_{w \in W} \prod_{\alpha \in \Phi_m^+} \frac{1 - u_\alpha e^{-w\alpha}}{1 - e^{-w\alpha}} = \sum_{n=0}^{\infty} e^{-n\delta} P_n, \tag{7.1}$$

where each $P_n \in \widehat{A}$ is a power series over the integers in the indeterminates u_α for all $\alpha \in \Phi_m^+$.

By Proposition 6.1, if we set $u_\alpha = 0$ for each $\alpha \in \Sigma$ on the right hand side of Equation (7.1), it becomes

$$\sum_{n=0}^{\infty} e^{-n\delta} p_n, \tag{7.2}$$

where each $p_n \in A$ is a polynomial over the integers in the indeterminates u_α for all $\alpha \in \Phi_m^+ \setminus \Sigma$. So setting $u_\alpha = 1$ for each $\alpha \in \Phi_m^+ \setminus \Sigma$ in (7.2) gives

$$\sum_{n=0}^{\infty} e^{-n\delta} k_n,$$

where each k_n is an integer.

Thus, if we set $u_\alpha = 0$ for all $\alpha \in \Sigma$ and $u_\alpha = 1$ for all $\alpha \in \Phi_m^+ \setminus \Sigma$ on both sides Equation (7.1), the equation becomes

$$\sum_{w\in W} \prod_{\alpha\in\Sigma} \frac{1}{1 - e^{-w\alpha}} = \sum_{n=0}^{\infty} e^{-n\delta} k_n. \tag{7.3}$$

It remains to compute k_n.

Fix $w \in W$ and consider a single summand

$$\prod_{\alpha\in\Sigma} \frac{1}{1 - e^{-w\alpha}}$$

on the left hand side of Equation (7.3). We can rewrite this (finite) product as

$$\left(\prod_{\alpha\in\Sigma\cap w^{-1}\Phi^+} \frac{1}{1 - e^{-w\alpha}}\right)\left(\prod_{\alpha\in\Sigma\cap w^{-1}\Phi^-} \frac{-e^{w\alpha}}{1 - e^{w\alpha}}\right).$$

Expanding each fraction as a geometric series gives

$$\left(\prod_{\alpha\in\Sigma\cap w^{-1}\Phi^+} \sum_{r=0}^{\infty} e^{-rw\alpha}\right)\left(\prod_{\alpha\in\Sigma\cap w^{-1}\Phi^-} \sum_{s=1}^{\infty} -e^{sw\alpha}\right),$$

so for a fixed $f \in L^+$, if a e^{-f} term appears in the above product with a nonzero coefficient, then we must be able to write f in the form

$$f = \sum_{\alpha\in\Sigma\cap w^{-1}\Phi^+} r_\alpha w\alpha - \sum_{\alpha\in\Sigma\cap w^{-1}\Phi^-} s_\alpha w\alpha$$

for nonnegative integers r_α and positive integers s_α; or, equivalently, we must be able to write $w^{-1}f$ in the form

$$w^{-1}f = \sum_{\alpha\in\Sigma\cap w^{-1}\Phi^+} r_\alpha \alpha - \sum_{\alpha\in\Sigma\cap w^{-1}\Phi^-} s_\alpha \alpha \tag{7.4}$$

for nonnegative integers r_α and positive integers s_α. Note that these integers, if they exist, are unique, since Σ is a basis for L^+.

Now let $f = n\delta$, in which case $w^{-1}f = n\delta$. It is well known that we can write δ in the form

$$\delta = \sum_{\alpha\in\Sigma} c_\alpha \alpha,$$

where each $c_\alpha > 0$, so

$$n\delta = \sum_{\alpha\in\Sigma} nc_\alpha \alpha,$$

where each $nc_\alpha \geq 0$. Again, since Σ is a basis for L^+, this expression of $n\delta$ as an integer combination of roots in Σ is unique. So if $f = n\delta$ also satisfies Equation (7.4), we must have

$$-s_\alpha = nc_\alpha$$

for all $\alpha \in \Sigma \cap w^{-1}\Phi^-$. But since $s_\alpha > 0$ and $nc_\alpha \geq 0$, this can only be the case if $\Sigma \cap w^{-1}\Phi^-$ were, in fact, empty, which in turn can only be the case if w is the identity element of W. Therefore, the only summand on the left hand side of Equation (7.3) that contributes a nonzero $e^{-n\delta}$ term is the summand corresponding to w being the identity element, namely

$$\prod_{\alpha\in\Sigma}\frac{1}{1-e^{-\alpha}}=\prod_{\alpha\in\Sigma}\sum_{r=0}^{\infty}e^{-r\alpha}=\sum_{f\in L^+}e^{-f},$$

where we once again apply the fact that L^+, the set of nonnegative integer combinations of roots in Σ, has Σ as a basis. Hence the coefficient of every term in this summand is 1. Therefore, the coefficient of $e^{-n\delta}$ for each nonnegative integer n in the entire left hand side of Equation (7.3) is 1. So

$$\sum_{w\in W}\prod_{\alpha\in\Sigma}\frac{1}{1-e^{-w\alpha}}=\sum_{n=0}^{\infty}e^{-n\delta}=\frac{1}{1-e^{-\delta}}.$$

□

8. Future directions

8.1. **Generalization to all Kac-Moody algebras.** We conjecture that both Theorem 1.1 and Theorem 1.2 hold for all Kac-Moody algebra, not just root systems of finite and affine type. The proof of Theorem 1.2 for a general Kac-Moody algebra should remain largely unchanged, although one would need to verify that the $\circ$ action behaves analogously. As for the proof of Theorem 1.1, to ensure that we are able to specialize the variables, we would require a broader argument than the one presented in our proof of Proposition 6.2, which appeals directly to the geometric properties of affine root systems. On the other hand, we believe the same argument will work for twisted affine root systems.

8.2. **A combinatorial proof of Theorem 1.1.** It would be interesting to find an alternative, direct proof of Theorem 1.1 that explicitly demonstrates the cancellation of coefficients. The diagrams in Section 3 are highly indicative of deeper patterns present in the terms in the power series.

8.3. **An alternating sum over the Weyl group.** A corollary of our theorems, from the fact that each nonzero coefficient is equal to 1 in the right hand side of Equation (1.1), is that

$$\sum_{w\in W}\sum_{\substack{B\subseteq\Phi_m^+,\\ |B|=n\delta,\\ \Sigma\cap\Phi_m(w,B)=\varnothing}}(-1)^{\#B}=1.$$

It would be interesting to find a direct proof of this equality and interpret its implications.

8.4. **Relationship to Macdonald's affine multivariate identity.** As previously mentioned, Macdonald proves an identity whose left hand side is equal to the left hand side of Equation (1.4), except that $u_\alpha=u_{w\alpha}$ for all $\alpha\in\Phi_m$ and $w\in W$ and that $u_\alpha=1$ for each imaginary root α [2]. Let Φ_{re}^+ denote the set of all positive real roots. Macdonald concludes that, under such conditions,

$$\sum_{w\in W}\prod_{\alpha\in\Phi_{\mathrm{re}}^+}\frac{1-u_\alpha e^{-w\alpha}}{1-e^{-w\alpha}}=\left(\sum_{w\in W}\prod_{\alpha\in\Phi(w)}u_\alpha\right)\mathrm{ct}\left(\prod_{\alpha\in\Phi_{\mathrm{re}}^+}\frac{1-e^{-\alpha}}{1-u_\alpha e^{-\alpha}}\right)^{-1},$$

whereas from Theorem 1.2, it follows that

$$\sum_{w\in W}\prod_{\alpha\in\Phi_{\mathrm{re}}^+}\frac{1-u_\alpha e^{-w\alpha}}{1-e^{-w\alpha}}=\prod_{r\geq 0}^{\infty}(1-e^{-r\delta})^{\mathrm{mult}\,r\delta}\sum_{n=0}^{\infty}e^{-n\delta}\sum_{w\in W}\sum_{\substack{B\subseteq\Phi_{\mathrm{re}}^+,\\ |B|=n\delta}}(-1)^{\#B}\prod_{\alpha\in\Phi_m(w,B)}u_\alpha.$$

It would be interesting to find a direct proof that the right hand sides of the above equations are equal and interpret the implications of such an equality.

8.5. **Further applications of Theorem 1.2.** Finally, it would be interesting to investigate the potential for Theorem 1.2 to be used in proving other identities, as the identity is applicable whenever we seek to evaluate an infinite sum of the form

$$\sum_{w \in W} \prod_{\alpha \in S} \frac{1}{1 - e^{-w\alpha}},$$

for example, where S is some proper submultiset of Φ_m^+.

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