WILF EQUIVALENCE IN WEYL GROUPS AND SIGNED PERMUTATIONS
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Abstract. This paper consists of two separate components. In the first half, we study pattern avoidance in Weyl groups, defined by Billey and Postnikov to understand smoothness of Schubert varieties. Specifically, we find all subspace root systems in root systems of classical types, describe the restriction map explicitly from Weyl groups of classical types, and define and explore Wilf equivalence in Weyl groups.

In the second half, with a slightly different notion of pattern avoidance, we show that the number of signed permutations avoiding 1234 equals the number of signed permutations avoiding 2143 (also called vexillary signed permutations), resolving a conjecture by Anderson and Fulton.

1. Introduction

Permutation pattern avoidance has been a popular line of research for many years. Denote the symmetric group on n elements by S_n. We say that a permutation $w \in S_n$ *avoids* a pattern $\pi \in S_k$ if there does not exist indices $1 \leq a_1 < \cdots < a_k \leq n$ such that $w(a_i) < w(a_j)$ if and only if $\pi(i) < \pi(j)$. Let $S_n(\pi)$ denote the set of permutations $w \in S_n$ that avoid π. Two permutations π, π' are called *Wilf equivalent* if $|S_n(\pi)| = |S_n(\pi')|$ for all n. The study of growth rate of $|S_n(\pi)|$ and the study of nontrivial Wilf equivalence classes have been fruitful.

This paper consists of two separate parts that are mildly related.

In the first part (Section 2), we view the symmetric group as Weyl group of type A, and study the generalization of pattern avoidance and Wilf equivalence to any Weyl groups. Such generalization is first introduced to understand smoothness of Schubert varieties [2]. However, the combinatorics of pattern avoidance in Weyl groups is less explored.

In the second part (Section 3), we focus on pattern avoidance in the signed permutation group, with a different notion of pattern avoidance. We show that the number of signed permutations that avoid 1234 equals the number of signed permutations that avoid 2143 (also called vexillary signed permutations), resolving a conjecture by Anderson and Fulton [1]. An important technique that we use is the generating tree developed by West [15], which transfers the desired enumeration to that of certain lattice paths. We finish the proof algebraically.

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The two parts of the paper are written in a self-contained manner, with more background material involving motivation and preliminaries at the beginning of the respective sections. Readers are free to go through the paper in any order.

2. Subspace root systems and pattern avoidance in classical types

Billey and Postnikov [2] introduced pattern avoidance for general Weyl groups to study the smoothness of Schubert varieties. In particular, they provided a short list of patterns and showed that a Schubert variety X_w is (rationally) smooth if and only if w avoids this list of patterns. Their notion of pattern avoidance generalizes the classical pattern avoidance in the symmetric group, which is also known as the Weyl group of type A. Many geometric and combinatorial properties of permutations can be naturally generalized to any Weyl groups and some of them can be described via the generalized notion of pattern avoidance, including properties on the Bruhat order [11], on the weak order [5] and on hyperplane arrangements [12].

The usage of pattern avoidance for certain properties has proven to be fruitful but from the point of view of enumeration and the natural generalization of Wilf equivalence, little is known. The goal of the first part of the paper is to explore this area. We first introduce the basic concepts. Readers are referred to [9] for an introduction to root systems and Weyl groups. The definitions in this section regarding pattern avoidance in general Weyl groups mostly follow [2].

Let $\Phi \subset \mathbb{R}^n$ be a root system with positive roots $\Phi_+ \subset \Phi$, negative roots $\Phi_- \subset \Phi$, and simple roots $\Delta \subset \Phi_+$. We omit the axiomatic definition of root systems here. Intuitively, a root system is a finite subset of vectors in the Euclidean space that is "highly symmetric" in some precise sense. A subset of roots $S \subset \Phi_+$ is said to be *biconvex* if

- $\alpha, \beta \in S$, $\alpha + \beta \in \Phi$, then $\alpha + \beta \in S$,
- $\alpha, \beta \notin S$, $\alpha + \beta \in \Phi$, then $\alpha + \beta \notin S$.

Let $\mathcal{W}_\Phi$ be the Weyl group of Φ. The *inversion set* of a Weyl group element $w \in \mathcal{W}_\Phi$ is defined to be

$$I_\Phi(w) := \{\alpha \in \Phi_+ : w(\alpha) \in \Phi_-\}.$$

The following proposition is well-known (see for example Proposition 2.1 of [8]), which says that we can identify elements of the Weyl group with their inversion sets in a unique way.

Proposition 2.1. *The function I_Φ provides a bijection between $\mathcal{W}_\Phi$ and biconvex subsets of Φ_+.*

For a vector subspace $V \subset \mathbb{R}^n$, let $\Phi_V = \Phi \cap V$. It is easy to see from the axioms that Φ_V is a root system, and any root system obtained this way is called a *subspace root system* of V. We will endow Φ_V with the natural choice of positive roots $\Phi_{V,+} = \Phi_+ \cap V$ which corresponds to a unique choice of simple roots, denoted Δ_V. We remark that it is usually not the case that $\Delta_V = \Delta \cap V$. A subset of Φ that forms a root system will be called a *subset root system*. Any subspace root system is a subset root system, but there may be subset root systems that are not subspace root systems. Such examples include subset root systems $A_1 \times A_1$ in B_2 and A_3 in B_3.

With the tools of biconvex sets, we can define a restriction map from the Weyl group $\mathcal{W}_\Phi$ to the Weyl group $\mathcal{W}_{\Phi_V}$.

Definition 2.2. For $w \in \mathcal{W}_\Phi$, define its *restriction* w_V to be the unique $w' \in \mathcal{W}_{\Phi_V}$ such that $I_{\Phi_V}(w') = I_\Phi(w) \cap V$.

The above definition is possible since $I_\Phi(w) \cap V$ is a biconvex set in Φ_V, so it is the inversion set of a unique element $w_V \in \mathcal{W}_{\Phi_V}$ by Proposition 2.1.

We are now equipped to define pattern avoidance for root systems.

Definition 2.3. Fix a root system Θ and $\pi \in \mathcal{W}_\Theta$. For a root system Φ and an element $w \in \mathcal{W}_\Phi$, we say that w *avoids* π if there is no subspace root system Φ_V for which there is a root system isomorphism between Φ_V and Θ that preserves positive roots and sends w_V to π.

If w does not avoid π, we say it *contains* π.

Classically, two permutation patterns are said to be Wilf equivalent if for all n, the number of permutations in S_n avoiding one is the same as the number of permutations avoiding the other. In complete analogy, we make the following definition.

Definition 2.4. Two patterns $\pi_1, \pi_2 \in \mathcal{W}_\Theta$ are said to be *root system Wilf equivalent* if for all root systems Φ, the number of elements of $\mathcal{W}_\Phi$ avoiding π_1 is the same as the number of elements of $\mathcal{W}_\Phi$ avoiding π_2, denoted $\pi_1 \sim \pi_2$.

This entire story could also be repeated for subset root systems instead of subspace root systems, giving a different notion of pattern avoidance. In this paper, we only consider the case of subspace root systems, but it is evident from the analysis that the differences between these two are minor.

We adopt the following convention for root systems of classical types. Note that the ordering of the indices is somewhat atypical. The Dynkin diagrams can be seen in figure 1. In the following, e_i is the canonical unit vector at the i^{th} coordinate.

- Type A_n ($n \geq 1$): $\Phi = \{e_i - e_j : n+1 \geq i \neq j \geq 1\} \subset \mathbb{R}^{n+1}/(1, \ldots, 1)$, $\Phi_+ = \{e_i - e_j : n+1 \geq i > j \geq 1\}$, $\Delta = \{e_{i+1} - e_i : n \geq i \geq 1\}$. The Weyl group $\mathcal{W}_{A_n}$ is isomorphic to the symmetric group S_{n+1}, where a permutation acts by permuting coordinates.
- Type B_n ($n \geq 2$): $\Phi = \{\pm e_i \pm e_j, \pm e_i : n \geq i > j \geq 1\} \subset \mathbb{R}^n$, $\Phi_+ = \{e_i \pm e_j, e_i : n \geq i > j \geq 1\}$, $\Delta = \{e_{i+1} - e_i : n-1 \geq i \geq 1\} \cup \{e_1\}$. The Weyl group $\mathcal{W}_{B_n}$, also denoted B_n, is isomorphic to $\mathbb{Z}_2 \wr S_n = (\mathbb{Z}_2)^n \rtimes S_n$, the group of signed permutations, consisting of permutations σ on $\{-n, , \ldots, -1, 1, \ldots, n\}$ such that $\sigma(-i) = -\sigma(i)$.
- Type C_n ($n \geq 3$): $\Phi = \{\pm e_i \pm e_j, \pm 2e_i : n \geq i > j \geq 1\} \subset \mathbb{R}^n$, $\Phi_+ = \{e_i \pm e_j, 2e_i : n \geq i > j \geq 1\}$, $\Delta = \{e_{i+1} - e_i : n-1 \geq i \geq 1\} \cup \{2e_1\}$. The Weyl group is the same as for B_n.
- Type D_n ($n \geq 4$). $\Phi = \{\pm e_i \pm e_j : n \geq i > j \geq 1\} \subset \mathbb{R}^n$, $\Phi_+ = \{e_i \pm e_j : n \geq i > j \geq 1\}$, $\Delta = \{e_{i+1} - e_i : n-1 \geq i \geq 1\} \cup \{e_2 + e_1\}$. The Weyl group $\mathcal{W}_{D_n}$ is isomorphic to $(\mathbb{Z}_2)^{n-1} \rtimes S_n$, the group of signed permutations on n elements with an even number of sign changes.

We will often write a signed permutation in two-line notation, with the indices on the first line and their images under the signed permutation below them. An example is $\begin{matrix} -2 & -1 & 1 & 2 \\ 1 & -2 & 2 & -1 \end{matrix}$.

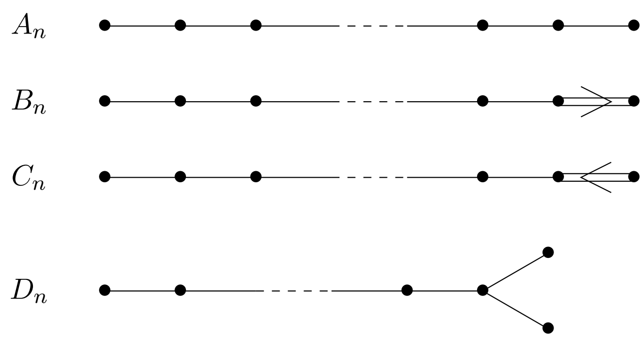

FIGURE 1. The Dynkin diagrams of root systems of classical types

In this section, we first explicitly find all irreducible subspace root systems in root systems of classical types in 2.1, including an enumeration. In 2.2, we derive explicit descriptions of the Weyl group element restriction maps, depending on the types of root systems involved. In 2.3, we reduce any instance of root system pattern avoidance (for classical types) to avoiding a set of patterns in a permutation in the usual sense or to avoiding a set of signed patterns in a signed permutations. We finish in 2.4 by discussing Wilf equivalence for root system pattern avoidance. We provide brief proofs for trivial Wilf equivalences, and mention numerical data showing that there are no nontrivial Wilf equivalences for short patterns of type A. In particular, surprisingly 1234 and 2143 are not root system Wilf equivalent.

2.1. **Subspace root systems of root systems of classical types.** In this subsection, we will characterize all irreducible subspace root systems of root systems of classical types. We will first go over some basic facts about root systems to state and prove a few general lemmas that facilitate the case analysis, and then go over the case analysis for all the main types.

Given a subset root system $\Theta \subset \Phi$, its set of simple roots is a subset of the positive roots of Φ. As the simple roots generate the entire root system via reflections, the simple roots uniquely determine the subspace root system. Given a set of positive roots, in complete analogy with the construction of the Dynkin diagram, we may form a graph in which the vertices correspond to the roots and the multiplicity of the edge between two vertices corresponding to roots α, β is $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$. It is well known that this number is always either $0, 1, 2$, or 3 and if it is not 0 or 1, the two roots have different lengths, and we will make the corresponding multiedge directed towards the vertex corresponding to the shorter root. We will call this the *characteristic graph* of this set of roots. This graph uniquely determines the spatial configuration of these roots (by determining all angles and ratios of lengths), so if some set of roots has a graph which is isomorphic to the Dynkin diagram of some root system Θ, then these vectors are isomorphic to the simple basis of this root system, so they generate by reflections a root system isomorphic to Θ. To conclude, we have proved the following lemma:

Lemma 2.5. *For root systems Φ and Θ, a subset of positive vectors of Φ is the simple basis of a subset root system isomorphic to Θ if and only if the characteristic graph of that subset is isomorphic to the Dynkin diagram of Θ, and such a simple basis determines the subset root system completely.*

For an irreducible root system Φ with given positive roots Φ_+ and simple roots Δ, there is a unique maximal root $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$. Let us denote the maximal coefficient by m_Φ, that is, $m_\Phi = \max_{\alpha\in\Delta} k_\alpha$.

Lemma 2.6. *If $m_\Phi < m_\Theta$, there are no subset root systems in Φ that are isomorphic to Θ.*

Proof. Say for a contradiction that there is a subset root system in Φ that is isomorphic to Θ. The simple roots of Θ are all positive roots of Φ, so each is a linear combination of simple roots of Φ with positive integer coefficients. Consider the maximal root of Θ. When written in terms of the basis for Θ, some root has a coefficient of m_Θ. Hence, if we expand this expression in terms of the simple roots of Φ, some simple root will have a coefficient of at least m_Θ, as all coefficients in the linear combination are positive. Hence, this root has a coefficient of at least m_Θ, so the maximal root of Φ has a coefficient of at least m_Θ, so $m_\Phi \geq m_\Theta$, a contradiction. □

Note that $m_{A_n} = 1$, $m_{B_n} = 2$, $m_{C_n} = 2$, $m_{D_n} = 2$, and if Φ is an exceptional type, $m_\Phi \geq 3$. As an immediate application of the lemma, we obtain the following corollary.

Corollary 2.7. *No exceptional type root system can be a subset root system in a classical type root system. All subsystems of A_n are of type A.*

We can further decrease the number of cases left to consider by noticing that root systems with roots of multiple lengths cannot be contained in root systems with roots of only a single length, as a root system isomorphism preserves ratios of lengths. This further rules out containing B_n or C_n in D_n. The remaining cases are considered below.

2.1.1. $A_k \subset A_n$.

Lemma 2.8. *The subset root systems $A_k \subset A_n$ are precisely those given by a simple basis of the form $e_{i_{k+1}} - e_{i_k}, \ldots, e_{i_2} - e_{i_1}$ for any decreasing sequence of indices $n+1 \geq i_{k+1} > i_k > \ldots > i_2 > i_1 \geq 1$. The number of A_k in A_n is $\binom{n+1}{k+1}$.*

Proof. By Lemma 2.5, all we need to check is that these are precisely all the sets of k roots with characteristic graph equal to the Dynkin diagram of A_k. For a set of roots of the form given above, all the inner products and lengths are the same as for the standard construction of A_k, so clearly the characteristic graph is the Dynkin diagram of A_k. Conversely, given a set of roots whose characteristic graph is the Dynkin diagram of A_k (a path with k

vertices), let $e_y - e_x$ be one of the endpoints. The only positive roots this would have a single edge to are roots of the form $e_z - e_y$ or $e_x - e_w$ (positivity requires $w < x < y < z$). If the next root is $e_x - e_w$ and this is not already the other endpoint of the path, then the next root could only be $e_q - e_x$ or $e_v - e_w$. $e_q - e_x$ is ruled out since it would have an edge to $e_y - e_x$, so this has to be $e_v - e_w$. Continuing by induction, we get that the path is of the form given in the statement of the lemma. The other case of picking $e_z - e_y$ is exactly the same, except the indices are added in the other direction, again giving the form given in the statement of the lemma. To enumerate $A_k \subset A_n$, we then just need to count the number of ways one can pick the sequence of $k+1$ indices from $[n+1]$, which is $\binom{n+1}{k+1}$. $\square$

Lemma 2.9. *Each of the subset root systems from Lemma 2.8 is also a subspace root system.*

Proof. Take a subset root system $A_k \subset A_n$, and take the span of the vectors in the subset. Let this be the vector subspace V. V has dimension k, so the subspace root system in V is A_k by Corollary 2.7. Hence, the subspace contains no vectors other than those in our initial subset root system A_k. Hence, this initial A_k is a subspace root system. $\square$

2.1.2. $A_k \subset D_n$.

Lemma 2.10. *For $k \neq 3$, the subset root systems $A_k \subset D_n$ are precisely those given by a simple basis of the form $e_{i_\ell} - e_{i_{\ell-1}}, \ldots e_{i_2} - e_{i_1}, e_{i_1} + e_{j_1}, e_{j_2} - e_{j_1}, \ldots, e_{j_m} - e_{j_{m-1}}$ with $n \geq i_\ell > i_{\ell-1} > \ldots > i_1 \geq 1$, $n \geq j_m > j_{m-1} > \ldots > j_1 > 1$, $i_1 > j_1$, $\{i_\alpha\}_{\alpha\in[\ell]} \cap \{j_\beta\}_{\beta\in[m]} = \emptyset$, and $\ell + m - 1 = k$. The number of such $A_k \subset D_n$ is $\binom{n}{k+1}2^k$. For $k = 3$, D_n contains A_3 of the above form, and the only additional $A_3 \subset D_n$ have simple roots $e_j + e_i, e_k - e_j, e_j - e_i$ for any $k > j > i$. The number of these additional A_3 is $\binom{n}{3}$.*

Proof. We again just need to check that these are precisely the sets of roots with characteristic graph equal to the Dynkin diagram of A_k. It is an explicit computation to see that the above construction gives the desired characteristic graph. It remains to show that any set of roots with the right characteristic graph is of this form. Suppose we have a set of positive roots forming a simple basis for $A_k \subset D_n$. The sum of all simple roots in A_k is a root, so the sum of the roots in our set has to be a root of D_n. Hence, our set contains at most one root with a plus sign, i.e., of the form $e_i + e_j$ (all other roots are of the form $e_i - e_j$). If the set contains no root with a plus sign, then our A_k is contained in the set of all $e_i - e_j$, which are just the positive roots of A_{n-1}. For such A_k, Lemma 2.8 gives that these are of the desired form. If the set contains a root with a plus sign, let this be $e_{i_1} + e_{j_1}$, where $i_1 > j_1$. We can cut the Dynkin diagram into two segments at this vertex (not including this vertex in either segment), and both of these segments only contain roots with a negative sign, so both segments (are either empty or) form bases for type A root systems in the aforementioned A_{n-1}, so both segments are of the form given in Lemma 2.8. The two vertices adjacent to $e_{i_1} + e_{j_1}$ in the Dynkin diagram have to have an edge to it, so they can only be of the form $e_{i_2} - e_{i_1}$, $e_{j_2} - e_{j_1}$. It is also clear that an index cannot appear in both segments, as there is no way for two roots with a minus to share an index and have inner product 0. Let us now consider if the indices i_1 and j_1 can reappear in one of the segments after the first element. If they cannot, we just have the main case given by the lemma, and the proof is done. As for the other case, the only way one of them is present in a root but there is no edge between that root and $e_{i_1}+e_{j_1}$ is when that root is $e_{i_1}-e_{j_1}$. From the previous considerations already see that if this happens, then the first vertex of the other segment is not present, so one of the segments is empty. As for the segment which is nonempty, any vertex with an edge to $e_{i_1}-e_{j_1}$ would give a nonzero inner product with $e_{i_1}+e_{j_1}$. Hence, any vertex adjacent to $e_{i_1}-e_{j_1}$ would also have to be adjacent to $e_{i_1}+e_{j_1}$, so the only option is that there is one vertex between the two, and this can only be $e_k-e_{i_1}$. This is exactly the extra case for A_3 from the statement of the lemma. Hence, any A_k always has to be of the form given in the statement. The only thing left is the enumeration. For the regular case, we can choose the subsystem by first picking the $k+1$ indices that appear, j_1 has to be the smallest, and for the k other indices we can choose which sequence they are in in any way. This gives a total of $\binom{n}{k+1}2^k$ options. As for the extra case for A_3, the number of these is the number of ways one can pick the set of 3 indices, which is $\binom{n}{3}$. □

Lemma 2.11. *Each of the subset root systems from Lemma 2.10 is also a subspace root system.*

Proof. Suppose not, then some $A_k \subset D_n$ is such that the span of the roots in this A_k contains a subspace root system which is not A_k. By Corollary 2.7 and the discussion immediately after it, this root system has to be D_k. We see from Lemma 2.10 that there are no subset root systems $A_k \subset D_k$ (note that D_n is only defined for $n \geq 4$), so this is a contradiction. □

2.1.3. *$A_k \subset B_n$.* The case of A_1 is trivial – any vector forms a simple basis for A_1. For other k, it turns out that the A_k in the standard B_n have the exact same form as those in the standard D_n.

Lemma 2.12. *For $k \neq 1, 3$, the subset root systems $A_k \subset B_n$ are precisely those given by a simple basis of the form $e_{i_\ell}-e_{i_{\ell-1}}, \dots e_{i_2}-e_{i_1}, e_{i_1}+e_{j_1}, e_{j_2}-e_{j_1}, \dots, e_{j_m}-e_{j_{m-1}}$ with $n \geq i_\ell > i_{\ell-1} > \dots > i_1 \geq 1$, $n \geq j_m > j_{m-1} > \dots > j_1 > 1$, $i_1 > j_1$, $\{i_\alpha\}_{\alpha\in[\ell]} \cap \{j_\beta\}_{\beta\in[m]} = \emptyset$, and $\ell+m-1 = k$. The number of such $A_k \subset B_n$ is $\binom{n}{k+1}2^k$. For $k = 3$, B_n contains A_3 of the above form, and the only additional $A_3 \subset B_n$ have simple roots $e_j+e_i, e_k-e_j, e_j-e_i$ for any $k > j > i$. The number of these additional A_3 is $\binom{n}{3}$.*

Proof. The positive roots of B_n can be partitioned into 2 groups: the roots with length $\sqrt{2}$ (these are $e_i \pm e_j$) and the roots with length 1 (these are e_i). As all vectors in A_k have the same length, our A_k has to be contained entirely in the set of vectors of length $\sqrt{2}$ or contained entirely in the set of vectors of length 1. For $k > 1$, the second option is impossible, as there exist nonorthogonal nonproportional pairs of roots in A_k, but all pairs of vectors of the form $\pm e_i$ are proportional or orthogonal. Hence, A_k has to be contained entirely in the set of vectors of length $\sqrt{2}$, and this set contains the same vectors as D_n for $n \geq 4$. But for $n < 4$, the set of vectors in B_n is still explicitly the same as what the construction for D_n would be for lower n. In any case, the proof of Lemma 2.10 still goes through with almost no changes, giving that the A_k in B_n are the same as in Lemma 2.10. $\square$

For $k = 1$, all subset root systems are obviously also subspace root systems. For $k > 1$, the only subset root systems which are not subspace root systems are the extra ones for A_3. This is the content of the following lemma.

Lemma 2.13. *For $k > 1$, the subspace root systems $A_k \subset B_n$ are precisely those given by a simple basis of the form $e_{i_\ell} - e_{i_{\ell-1}}, \ldots e_{i_2} - e_{i_1}, e_{i_1} + e_{j_1}, e_{j_2} - e_{j_1}, \ldots, e_{j_m} - e_{j_{m-1}}$ with $n \geq i_\ell > i_{\ell-1} > \ldots > i_1 \geq 1$, $n \geq j_m > j_{m-1} > \ldots > j_1 > 1$, $i_1 > j_1$, $\{i_\alpha\}_{\alpha\in[\ell]} \cap \{j_\beta\}_{\beta\in[m]} = \emptyset$, and $\ell + m - 1 = k$.*

Proof. We need to check which subset root systems are subspace root systems. The extra A_3 of the form $e_i + e_j, e_k - e_j, e_j - e_i$ are not subspace root systems, as the $\mathbb{R}$-span of these 3 vectors contains the root $e_j = \frac{e_i+e_j+e_j-e_i}{2}$, which is not in the A_3 (the length is not $\sqrt{2}$). As for the regular case, let us first consider $k > 3$. Suppose such a subset root system A_k is not a subspace root system. Then the root system in its span is some other root system of the same dimension. By Corollary 2.7, this has to be B_k, C_k, or D_k. In any case, for $k \geq 4$ both B_k and C_k contain D_k, and furthermore A_k has to be contained in this D_k by the length argument from the proof of Lemma 2.12. In any case, this would imply having a subset root system $A_k \subset D_k$, which is impossible by Lemma 2.10. This is a contradiction. For each $k \leq 3$, one can explicitly write out all roots in the subspace spanned by the subset given in the statement of this lemma, and check that these are precisely the roots in the subset. $\square$

2.1.4. *$A_k \subset C_n$.* Both the subset and subspace root systems correspond exactly to those in B_n. For $k = 1$, each vector forms a basis for A_1. For $k > 1$, the same length argument as in the proof of Lemma 2.12 gives that the subset root systems $A_k \subset C_n$ are the same as for B_n, and these are given in 2.12. The subspace root system argument for $A_k \subset C_n$ is also exactly the same as for $A_k \subset B_n$ in the proof of Lemma 2.13.

2.1.5. *$D_k \subset D_n$.*

Lemma 2.14. *The subset root systems $D_k \subset D_n$ are precisely those given by a simple basis of the form $e_{i_k} - e_{i_{k-1}}, \ldots, e_{i_2} - e_{i_1}, e_{i_2} + e_{i_1}$ for any indices $n \geq i_k > \ldots > i_1 \geq 1$. The number of $D_k \subset D_n$ is $\binom{n}{k}$.*

Proof. The characteristic graph of such a set of roots is clearly the Dynkin diagram of D_k. For the other direction, suppose we have a set of positive roots such that their characteristic graph is the Dynkin diagram of D_k. Then in particular, there is a vertex that is connected to 3 pairwise disconnected vertices. It is not hard to see that 2 of these 3 vertices must be $e_{i_2} + e_{i_1}$ and $e_{i_2} - e_{i_1}$. Then, the vertex connected to both of these can only be $e_{i_3} - e_{i_2}$. The third vertex connected to this can only be $e_{i_4} - e_{i_3}$. It is impossible for one of $e_{i_2} + e_{i_1}$ and $e_{i_2} - e_{i_1}$ to have an adjacent vertex without this vertex also giving a nonzero inner product with the other root in this pair. Hence, the only vertex that may have adjacent vertices is $e_{i_4} - e_{i_3}$. The sum of all roots in D_k is a root in D_k, so it is also a root in our

D_n. Hence, at most one of the simple roots for our D_k can have a plus sign. So all vertices other than $e_{i_2} + e_{i_1}$ have negative signs, so they are all elements of the set of all $e_i - e_j$, which is an A_{n-1}. From our previous considerations about the Dynkin diagram, we know that the rest of the vertices form an A_{k-1}, so we have a A_{k-1} in A_{n-1}, and the form of this was given in Lemma 2.8. This implies that the set we are considering is of the desired form. The number of suck sets is the number sets of k indices from $[n]$, which is $\binom{n}{k}$. □

Lemma 2.15. *Each of the subset root systems in Lemma 2.14 is a subspace root system.*

Proof. Consider the root system in the span of the subset root system. By Corollary 2.7, the discussion after it, and the fact that it contains D_k, the only option is for it to be D_k. This is what we wanted to show. □

2.1.6. *$D_k \subset B_n$.* The subset root systems $D_k \subset B_n$ are the same as $D_k \subset D_n$.

Lemma 2.16. *The subset root systems $D_k \subset B_n$ are precisely those given by a simple basis of the form $e_{i_k} - e_{i_{k-1}}, \ldots, e_{i_2} - e_{i_1}, e_{i_2} + e_{i_1}$ for any indices $n \geq i_k > \ldots > i_1 \geq 1$. The number of $D_k \subset B_n$ is $\binom{n}{k}$.*

Proof. The fact that $D_k \subset D_n$ and $D_k \subset B_n$ are the same follows from the same length and orthogonality considerations as those in the proof of Lemma 2.12. The present lemma then follows from Lemma 2.14. □

Lemma 2.17. *There are no subspace root systems $D_k \subset B_n$.*

Proof. By Lemma 2.1.6, any such subset root system $D_k \subset B_n$ contains $e_{i_2} + e_{i_1}$ and $e_{i_2} - e_{i_1}$, so the $\mathbb{R}$-span contains the additional root $e_{i_2} = \frac{(e_{i_2}+e_{i_1})+(e_{i_2}-e_{i_1})}{2}$ of B_n. Hence, root system in the span is not D_k. Hence, no subset root system $D_k \subset B_n$ is a subspace root system. □

2.1.7. *$D_k \subset C_n$.* The results and proofs are the same as for $D_k \subset B_n$.

2.1.8. *$B_k \subset B_n$.*

Lemma 2.18. *The subset root systems $B_k \subset B_n$ are precisely those given by a simple basis of the form $e_{i_k} - e_{i_{k-1}}, \ldots, e_{i_2} - e_{i_1}, e_{i_1}$ for any indices $n \geq i_k > \ldots > i_1 \geq 1$. The number of $B_k \subset B_n$ is $\binom{n}{k}$.*

Proof. It is an easy computation to check that a set of vectors of the given form satisfies the problem. To show that any simple basis of B_k in B_n has this form, note that the Dynkin diagram of B_k has roots of two different lengths, and so does B_n. The only short root in the Dynkin diagram of B_k must be a short root of B_n, so denote it e_{i_1}. The rest of the roots in the Dynkin diagram are long, so the rest of the roots in our set are of the form $e_i - e_j$ or $e_i + e_j$. The latter is impossible, since the sum of all long simple roots and twice the short simple root is a root of $B_k \subset B_n$, so it is also a root of B_n. Hence, all the long roots in our set are of the form $e_i - e_j$, so the long roots form an $A_{k-1} \subset A_{n-1}$. Furthermore, the root adjacent to e_{i_1} in the Dynkin diagram can only be $e_{i_2} - e_{i_1}$. Given that this is an endpoint for our A_{k-1}, by Lemma 2.8, the only options are for the other indices to form an increasing sequence from i_2 or a decreasing sequence from i_1. The presence of the short root i_1 makes it impossible for $i_1 - i_0$ to fit into the Dynkin diagram, ruling out the latter case. Hence, any set of vertices forming B_k is of the desired form. As for the enumeration, it is the number of ways to pick k indices from $[n]$, which is $\binom{n}{k}$. $\square$

Lemma 2.19. *Each of the subset root systems in Lemma 2.18 is a subspace root system.*

Proof. The root system in the $\mathbb{R}$-span of the subset root system B_k contains B_k, so it contains roots of different lengths. Together with Corollary 2.7, the only options this leaves us with is for it to be B_k or C_k. C_k has the same number of roots as B_k, so C_k does not contain B_k, so the subspace root system in the span must be B_k. $\square$

2.1.9. *$C_k \subset C_n$.* This case is essentially the same as $B_k \subset B_n$. The proof of the next lemma is omitted, as the proof of Lemma 2.18 applies with minor changes.

Lemma 2.20. *The subset root systems $C_k \subset C_n$ are precisely those given by a simple basis of the form $e_{i_k} - e_{i_{k-1}}, \ldots, e_{i_2} - e_{i_1}, 2e_{i_1}$ for any indices $n \geq i_k > \ldots > i_1 \geq 1$. The number of $B_k \subset B_n$ is $\binom{n}{k}$. All these subset root systems are subspace root systems.*

2.1.10. *$B_k \subset C_n$.* The cases $k = 2$ and $k > 2$ will be treated separately.

Lemma 2.21. *The subset root systems $B_2 \subset C_n$ are precisely those given by a simple basis of the form $e_{i_2} - e_{i_1}, 2e_{i_1}$ with $n \geq i_2 > i_1 \geq 1$. All these are also subspace root systems.*

Proof. It is easy to check that these give the correct characteristic graph. For the other direction, we can pick a more convenient construction for B_2 without changing what subset root systems of our explicit construction for C_n are isomorphic to B_2. So let our construction for B_2 have the simple roots $e_2 - e_1, 2e_1$ (one can think of this as the construction for type C with $n = 2$). For this construction, the proof of Lemma 2.18 applies with minor changes to show that the B_2 given in the lemma are indeed all the subset root systems $B_2 \subset C_n$. The proof of lemma 2.19 applies to show that these are also subspace root systems. $\square$

Lemma 2.22. *For $k > 2$, there are no subset root systems $B_k \subset C_n$.*

Proof. B_k and C_n both have roots of two lengths, so the short roots of B_k would have to be contained in the short roots of C_n and the long roots of B_k would have to be contained in the long roots of C_n. For $k \geq 2$, among the long roots of B_k, there are some nonproportional nonorthogonal pairs, but there are no such pairs among the long roots of C_n. Hence, there are no B_k in C_n. $\square$

2.1.11. *$C_k \subset B_n$.*

Lemma 2.23. *There are no subset root systems $C_k \subset B_n$.*

Proof. The proof of Lemma 2.22 applies with minor changes. $\square$

This completes the case analysis.

2.2. **Explicit restriction maps for classical types.** In this subsection, we give an explicit description of the restriction map from $\mathcal{W}_\Phi$ to $\mathcal{W}_\Theta$ for a subspace root system Θ. For a general Θ, there may be multiple ways it is isomorphic to the standard construction of Θ. Each of these ways may give a different restriction map from $\mathcal{W}_\Phi$ to $\mathcal{W}_\Theta$, as the image of the restriction set may consist of different vectors in the explicit construction. Nevertheless, any isomorphism can be written as a composition of a fixed isomorphism and an automorphism preserving the positive roots, so to characterize all restriction maps, it suffices to understand the restriction map for just one isomorphism and also the map each automorphism induces on the Weyl group (via inversion sets). The automorphisms preserving positive roots are in bijection with Dynkin diagram automorphisms, so it suffices to analyze these. This is what we will do first.

2.2.1. *Type A_k automorphisms.* The Dynkin diagram is a path, so there is exactly one automorphism. It is well known that this automorphism induces the reverse complement map on the Weyl group.

2.2.2. *Type B_k and C_k automorphisms.* There are no automorphisms of the Dynkin diagram.

2.2.3. *Type D_k automorphisms.* The Dynkin diagram of D_4 has 6 automorphisms and will not be considered (although it is possible to explicitly write out the effect of each automorphism on each of the Weyl group elements). For D_k with $k \geq 5$, the Dynkin diagram has one automorphism. Our initial Dynkin diagram has roots $e_n - e_{n-1}, \dots, e_2 - e_1, e_2 + e_1$, and the automorphism switches $e_2 - e_1$ and $e_2 + e_1$ and leaves all other simple roots unchanged. We see that the vector space isomorphism sending e_1 to $-e_1$ and leaving all other basis vectors unchanged acts in this way on the simple roots, so since the simple roots span the space, this isomorphism is the isomorphism inducing the automorphism of Dynkin diagrams.

Given an initial Weyl group element π, we are looking for the Weyl group element whose inversion set is the image of the inversion set of π. Note that $e_i + e_1$ is in the inversion set for the image if and only if $e_i - e_1$ is in the inversion set initially, and vice versa. We can talk about the inversion set for any map from a root system to itself, with the exact same definition. With this terminology, note that the map attained from the initial Weyl group element by changing the sign of the image of 1 and -1 has the desired inversion set, but it is not a Weyl group element for D_k since the number of sign changes is now odd. However, we can also change the sign of the image with the smallest absolute value (1), as this does not change whether any of the roots is inverted (only the sign of the larger index matters, and the smallest index is not the larger index in any pair). After these two changes, we have a map corresponding to a Weyl group element again, since the number of sign changes is even. This is what we were looking for. We have proved the following lemma.

Lemma 2.24. *For $k \geq 5$, the D_k automorphism induces via inversion sets the map on the Weyl group that switches the sign of* 1 *(as an image in the signed permutation) and the sign of the image of* 1*.*

Example 2.25. The automorphism sends

$$\begin{matrix} -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 \\ 1 & -2 & 4 & -3 & 3 & -4 & 2 & -1 \end{matrix}$$

to

$$\begin{matrix} -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 \\ -1 & -2 & 4 & 3 & -3 & -4 & 2 & 1. \end{matrix}$$

2.2.4. *A restriction map for $A_k \subset A_n$.* The following is well known. Lemma 2.8 gave that any $A_k \subset A_n$ is $e_{i_{k+1}} - e_{i_k}, \ldots, e_{i_2} - e_{i_1}$. We will give this the standard construction (in the corresponding order) $f_{k+1} - f_k, f_k - f_{k-1}, \ldots, f_2 - f_1$. In this case, the restriction map is just the restriction map for usual pattern avoidance. That is, the restriction map takes the substring of the permutation at the indices $i_1, i_2, \ldots i_{k+1}$ and constructs the restriction by replacing the smallest image by 1, the second smallest by 2, and so on. To check that this restriction map is really the right one, one just has to check that it gives exactly the desired inversion set. Some positive root $f_\ell - f_m$ is inverted in the proposed restriction if and only if the image of ℓ is less than the image of m. By definition of the proposed restriction map, this happens iff the image of i_ℓ is less than the image of i_m, which happens if and only if $e_{i_\ell} - e_{i_m}$ is inverted, which happens if and only if $f_\ell - f_m$ is in the inversion set. Hence, our proposed restriction has the right inversion set, so it is the actual restriction.

2.2.5. *A restriction map for $B_k \subset B_n$.* In this case, there is a unique indexing (there are no automorphisms), and by essentially the same proof as in the previous subsection, the restriction map is just picking a subset of indices in the signed permutation with the property that if i is picked, then so is $-i$, and restricting to that subset in the usual sense. This is also equivalent to the usual notion of signed patterns in signed permutations.

Example 2.26. For the signed permutation

$$\begin{matrix} -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 \\ 1 & -2 & 4 & -3 & 3 & -4 & 2 & -1, \end{matrix}$$

restricting to the indices $-3, -1, 1, 3$ gives the substring

$$\begin{matrix} -3 & -1 & 1 & 3 \\ -2 & -3 & 3 & 2 \end{matrix}$$

which reduces to the signed permutation

$$\begin{matrix} -2 & -1 & 1 & 2 \\ -1 & -2 & 2 & 1. \end{matrix}$$

2.2.6. *A restriction map for $C_k \subset C_n$.* This is the same as $B_k \subset B_n$.

2.2.7. *A restriction map for $D_k \subset D_n$.* The restriction map for $B_k \subset B_n$ would give a map with the correct inversion set in this case as well, but this map might not be a Weyl group element for D_k, depending on the number of sign switches. In 2.2.3, we argued that changing the sign of the image with the smallest absolute value does not change the inversion set. This allows for the parity of the number of sign changes to be fixed again while retaining the correct inversion set. Hence, the restriction rule is doing the restriction for $B_k \subset B_n$ and then if the number of sign changes is odd, also changing the sign of the image with the smallest absolute value.

2.2.8. *A restriction map for $A_k \subset D_n, B_n, C_n$.* The restriction map for the case of the irregular $A_3 \subset D_n$ is not described. The irregular case is not a subspace root system for B_n or C_n, so we nevertheless have a full description for the these two cases. We also have full description for all cases for $k \geq 4$. For the regular $A_k \subset D_n$ given in Lemma 2.13, it turns out that for D_n, B_n, and C_n, the explicit restriction maps are the same. The regular $A_k \subset D_k, B_k, C_k$ is given by two disjoint sequences of indices. The restriction map is given by treating the sequence $i_\ell > i_{\ell-1} > \ldots > i_1$ as giving (absolute values of) negative indices and $j_1 < j_2 < \ldots < j_m$ as giving positive indices, and restricting to these indices in the signed permutation in the usual sense. Again, this rule can be explicitly checked to give the right inversion set. Note that the only restriction on picking indices is that it is not possible to simultaneously pick i and $-i$, as the two sequences have to be disjoint.

Example 2.27. For the signed permutation

$$\begin{array}{cccccccc} -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 \\ 1 & -2 & 4 & -3 & 3 & -4 & 2 & -1, \end{array}$$

restricting to the indices $-3, -2, 1, 4$ gives

$$\begin{array}{cccc} -3 & -2 & 1 & 4 \\ -2 & 4 & 3 & -1, \end{array}$$

which reduces to the permutation

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{array}$$

2.3. **A reduction of all cases of root system pattern avoidance to usual pattern avoidance.**

2.3.1. *$A_k \subset A_n$.* The following proposition is well known and included for completeness.

Proposition 2.28. *A Weyl group element $w \in \mathcal{W}_{A_n}$ avoids $\pi \in \mathcal{W}_{A_n}$ if and only if as a permutation, w avoids the permutation π and its reverse complement.*

Hence, this case of Weyl group pattern avoidance reduces to avoiding a set of at most 2 patterns in the usual sense of pattern avoidance.

2.3.2. *$B_k \subset B_n$ and $C_k \subset C_n$.* Weyl group pattern avoidance is just the same as the usual notion of signed pattern avoidance for signed permutations, as we saw in 2.2.5. The same is true for $C_k \subset C_n$.

2.3.3. *$D_k \subset D_n$.* For an explicit signed permutation in D_n, avoiding a pattern of type D_k is equivalent to the same permutation avoiding a set of patterns of type B_k. This set of patterns is derived from the original pattern by the D_k automorphism and changing the sign of the smallest image.

2.3.4. $A_k \subset B_n$.

Lemma 2.29. *A signed permutation in B_n avoiding a pattern π of type A_k is equivalent to it avoiding the set of patterns of type B_{k+1} that contain π.*

Proof. If the signed permutation contains a pattern of type B_{k+1} containing π, then we can combine the restrictions to get that it also contains π. As for the other direction, for any A_k, there is a B_{k+1} involving the same indices, and the restriction to this B_{k+1} contains π. So containing π implies containing a pattern of type B_{k+1} containing π. This completes the proof. □

Similar lemmas also hold for the remaining cases $A_k \subset C_n, D_n$. In the case of D_n, one can first state the lemma in terms of avoiding some D_{k+1} and then translate that to usual signed pattern avoidance.

2.4. **Root system Wilf equivalence.** The following three lemmas present three trivial cases of Wilf equivalence.

Lemma 2.30. *For a root system Θ, suppose there are $\pi_1, \pi_2 \in \mathcal{W}_\Theta$ such that there is a root system automorphism φ of Θ that takes positive roots to positive roots and π_1 to π_2. Then $\pi_1 \sim \pi_2$.*

Proof. For a root system Φ, if some $w \in \mathcal{W}_\Phi$ does not avoid π_1, then there is a subspace V such that there is an isomorphism between Φ_V and Θ taking w_V to π_1. Since composition with the automorphism φ gives an isomorphism taking w_V to π_2, w also does not avoid π_2. Analogously, composition with φ^{-1} gives that any w containing π_2 also contains π_1. Hence, the $w \in \mathcal{W}_\Phi$ avoiding π_1 and π_2 are the same. So their number is also the same. □

Given $\pi \in \mathcal{W}_\Theta$, we define its complement $\overline{\pi}$ via the following construction. $I_\Theta(\pi)$ is biconvex, so by definition so is $\Theta \setminus I_\Theta(\pi)$. Hence, $\Theta \setminus I_\Theta(\pi)$ is the inversion set of a unique element of $\mathcal{W}_\Theta$, and we write $\overline{\pi}$ for this element. In fact, $\overline{\pi} = w_0\pi$, where w_0 is the longest element in the Weyl group $\mathcal{W}_\Theta$

Lemma 2.31. *For $\pi \in \mathcal{W}_\Theta$, $\pi \sim \overline{\pi}$.*

Proof. As $\overline{(\overline{w})} = w$, the elements of $\mathcal{W}_\Phi$ are partitioned into element-complement pairs. Therefore, it suffices to show that if w contains π, then $\overline{w}$ contains $\overline{\pi}$. Well, if w contains π, then there is a subspace V such that there is an isomorphism between Φ_V and Θ taking $I_\Phi(w) \cap V$ to $I_\Theta(\pi)$. Hence, the same isomorphism takes $I_\Phi(\overline{w}) \cap V = (\Phi \setminus I_\Phi(w)) \cap V = \Phi_V \setminus (I_\Phi(w) \cap V)$ to $\Theta \setminus I_\Theta(\pi) = I_\Theta(\overline{\pi})$. Hence, $\overline{w}_V$ is taken to $\overline{\pi}$, so $\overline{w}$ contains $\overline{\pi}$. □

Lemma 2.32. *For $\pi \in \mathcal{W}_\Theta$, $\pi \sim \pi^{-1}$.*

Proof. Again, it suffices to show that if w contains π, then w^{-1} contains π^{-1}. If w contains π, then there is a subspace V for which there is an isomorphism φ taking Φ_V to Θ and w_V to π. Note that $I_\Phi(w^{-1}) = -wI_\Phi(w)$. Consider the restriction of w^{-1} to the subspace wV. The inversion set of this restriction is $I_{wV}\left(\left(w^{-1}\right)_{wV}\right) = -wI_\Phi(w) \cap wV = w(-I_\Phi(w) \cap V)$. Consider the map $\pi \circ \varphi \circ w^{-1}$ taking Φ_{wV} to Θ. This map is an isomorphism, and it takes

$I_{wV}\left(\left(w^{-1}\right)_{wV}\right) = w(-I_\Phi(w)\cap V)$ to $\pi\circ\varphi(-I_\Phi(w)\cap V) = -\pi I_\Theta(\pi) = I_\Theta(\pi^{-1})$. This map also takes positive roots to positive roots, as the roots inverted by w^{-1} are $-wI_\Phi(w)\cap wV$, so the negatives of the images of the inverted roots under w^{-1} are $I_\Phi(w)\cap V$, φ takes these to $I_\Theta(\pi)$, which are inverted by π^{-1}. Hence, all positive roots inverted by w^{-1} are taken to positive roots. A similar argument shows that all positive roots that are not inverted by w^{-1} are sent to roots that are not inverted by π, so they also go to positive roots under the composite map. Hence, we have an isomorphism between Φ_{wV} and Θ sending positive roots to positive roots and $\left(w^{-1}\right)_{wV}$ to π^{-1}. Hence, w^{-1} contains π^{-1}. □

For Wilf equivalence of irreducible patterns, perhaps the most interesting unstudied case is that of type A patterns. We saw before that for patterns of other types, avoiding the pattern is equivalent to avoiding a small (constant size) set of patterns in either the usual permutation in permutation or signed permutation in signed permutation sense. For patterns from A_k for $k\le 4$ (so permutations of length up to 5), we used a program to count the number of Weyl group elements avoiding each pattern for some small root systems Φ. For each pair of patterns $\pi_1,\pi_2\in A_k$ for $k\le 4$, we found a small root system Φ for which the number of elements of $\mathcal{W}_\Phi$ avoiding π_1 and the number of elements avoiding π_2 are different. Hence, there are no Wilf equivalences for patterns in A_k with $k\le 4$. In particular, 1234 and 2143 are not root system Wilf equivalent. This data also leads us to the following conjecture.

Conjecture 2.33. There are no root system Wilf equivalences between patterns of type A.

3. 1234-avoiding and vexillary signed permutations

In this section, we will work with a different notion of pattern avoidance in the signed permutation group B_n. Recall that the signed permutation group B_n consists of permutations w on $\{-n,\dots,-1,1,\dots,n\}$ such that $w(i)=-w(-i)$ for all $i\in\{-n,\dots,-1,1,\dots,n\}$. We say that $w\in B_n$ *avoids* $\pi\in S_k$ if the natural embedding of w into S_{2n} avoids π in the sense of permutation pattern avoidance. In particular, let us define

$$\begin{aligned}B_n(1234) = \{w\in B_n \mid &\text{ there do not exist } -n\le a<b<c<d\le n\\ &\text{such that } w(a)<w(b)<w(c)<w(d)\},\\ B_n(2143) = \{w\in B_n \mid &\text{ there do not exist } -n\le a<b<c<d\le n\\ &\text{such that } w(b)<w(a)<w(d)<w(c)\},\end{aligned}$$

to be the set of signed permutations avoiding 1234 and 2143 respectively, which are the main objects of interest in this section.

The set of permutations avoiding 1234 and the set of permutations avoiding 2143 have been traditionally well-studied and enjoy nice combinatorial properties. Their permutation matrices are shown in Figure 2. Permutations avoiding 2143 are also called *vexillary* permutations. A permutation w avoids 1234 if and only if its shape under RSK has at most 3 columns [13]. And a permutation w avoids 2143 if and only if its associated Schubert polynomial is a flag Schur function [10]. Moreover, 1234 and 2143 are known to be Wilf

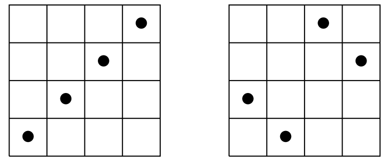

Figure 2. Permutations 1234 and 2143.

equivalent in the usual sense. Let $S_n(1234)$ and $S_n(2143)$ be the sets of permutations on n elements avoiding 1234 and avoiding 2143 respectively. West [15] showed that $|S_n(1234)| = |S_n(2143)|$ and the enumeration

$$|S_n(1234)| = \frac{1}{(n+1)^2(n+2)} \sum_{j=0}^{n} \binom{2j}{j} \binom{n+1}{j+1} \binom{n+2}{j+1}$$

appeared in many previous work [7], [6] and [3] and is now an exercise in chapter 7 of [13].

Analogously, the set of signed permutations avoiding 1234 and the set of signed permutations avoiding 2143 have similarly nice properties. In particular, the enumeration result

$$|B_n(1234)| = \sum_{j=0}^{n} \binom{n}{j}^2 C_j$$

where $C_j = \binom{2j}{j}/(j+1)$ is the j^{th} Catalan number, is given by Egge [4], using techniques involving RSK and jeu-de-taquin. Geometric and combinatorial properties of signed permutations avoiding 2143, which are also called *vexillary signed permutations*, are studied by Anderson and Fulton [1]. They also conjectured that $|B_n(1234)| = |B_n(2143)|$. The main result of this section is to answer this conjecture positively.

In fact, there are more similarities between the structures of signed permutations avoiding 1234 and signed permutations avoiding 2143. For $0 \le j \le n$, define

$$B_n^j(1234) := \{w \in B_n(1234) \mid w(i) > 0 \text{ for exactly } j \text{ indices } i \in \{1, \dots, n\}\}$$

and define $B_n^j(2143)$ in a similar way.

Theorem 3.1. *For $j \le n$, $|B_n^j(1234)| = |B_n^j(2143)|$.*

As a corollary, we obtain the following result, previously conjectured by Anderson and Fulton [1].

Corollary 3.2. *For $n \in \mathbb{Z}_{\geq 1}$, $|B_n(1234)| = |B_n(2143)|$.*

The main tool that we use is the idea of generating trees developed by West [15] to show that $|S_n(1234)| = |S_n(2143)|$. A generating tree is a rooted labeled tree for which the label at a vertex determines its descendants (their number and their labels). The generating trees in West's paper have vertices that correspond to permutations avoiding a fixed pattern, with the descendants of a vertex corresponding to all permutations with a new largest element added to some location that still avoid that pattern. The usefulness of such generating trees stems in part from the fact that it is often possible to present an isomorphic tree with vertices labeled by only a few integer statistics (instead of permutations), with a simple enough succession rule to be fit for further analysis. In the case of $S_n(1234)$ versus $S_n(2143)$, West was able to find a simple description of both trees and observed that the two are naturally isomorphic, thus proving $|S_n(1234)| = |S_n(2143)|$ bijectively. As for the case of interest in this paper, in Section 3.1, we describe the succession rules for generating trees corresponding to $B_n^j(1234)$ and $B_n^j(2143)$, using some more refined statistics that are soon made precise. However, it turns out that these two trees are far from being isomorphic. Therefore, in Section 3.2, we finish the proof by observing the succession rules more closely and comparing certain generating functions. We end in Section 3.3 with discussion on open problems.

3.1. **Generating trees for 1234 and 2143 avoiding permutations.** We will start working towards an explicit generating tree for $B_n^j(1234)$ and $B_n^j(2143)$ by first proving a structural lemma about permutations avoiding either pattern. Throughout the discussion, the reader is invited to keep the following visualization of signed permutations in mind. A signed permutation w can be represented by a point graph, where the x axis corresponds to the indices $(-n, \dots, -1, 1, \dots n)$, and the y axis corresponds to the images $(-n, \dots, -1, 1, \dots, n)$. A point appears at (x, y) if the index x is sent to the image y by the signed permutation w.

First, we have a structural lemma for 2143.

Lemma 3.3. *Suppose $w \in B_n^j(2143)$, where the j positive indices with positive images are $1 \le i_1 < i_2 < \dots < i_j \le n$. Then $w(i_1) < w(i_2) < \dots < w(i_j)$.*

In terms of the point graph of w, this lemma is saying that the j points in the top right quadrant form an increasing sequence. This implies that the j points in the bottom left quadrant form a decreasing sequence.

Proof. Suppose not. Then there exists a pair of positive indices $i < j$ with $1 \le w(j) < w(i) \le n$. Consider the pattern forming at the indices $-j, -i, i, j$. We have $w(-i) < w(-j) < 0 < w(j) < w(i)$, or in other words, the pattern is 2143. This is a contradiction with $w \in B_n^j(2143)$. □

We have a similar structural lemma for 1234.

Lemma 3.4. *Suppose $w \in B_n^j(1234)$, where the j positive indices with positive images are $1 \le i_1 < i_2 < \dots < i_j \le n$. Then $w(i_1) > w(i_2) > \dots > w(i_j)$.*

In terms of the point graph of w, this lemma is saying that the j points in the top right quadrant form a decreasing sequence. This implies that the j points in the bottom left quadrant form an increasing sequence.

Proof. Suppose not. Then there exists a pair of positive indices $i < j$ with $1 \le w(i) < w(j) \le n$. Consider the pattern forming at the indices $-j, -i, i, j$. We have $w(-j) < w(-i) < 0 < w(i) < w(j)$, or in other words, the pattern is 1234. This is a contradiction with $w \in B_n^j(1234)$. □

To motivate our method of generating signed permutations, let us first informally discuss how one can take an ordinary permutation and generate new permutations from it by simple operations. This discussion relies on [15]. We will consider inserting one element to the previous permutation. For a permutation on n indices, we will call the $n+1$ positions between indices *sites*. One can consider inserting a new element to any site. In general, this new element can also get any image between 1 and $n+1$. If we choose the image of the new element to be i, then all other images greater or equal to i will get increased by one – we think of this as making the image of the new element lie in the *gap* between the previous $i-1$ and i. In a specific scenario, one might want this generating procedure to be more constrained than just inserting any image to any site, in order to satisfy some additional properties.

With these preliminary considerations in mind, let us discuss what the generating procedure might be for our particular case. As we are dealing with signed permutations, our steps are not going to be adding just one new index and image, as this would violate the condition that the images of i and $-i$ are negatives of each other, so we would not end up with signed permutations. Instead, we will simultaneously be inserting to the site i and to the site $-i$, with images that are also negatives of each other. As we want to keep j, the number of positive indices with positive images, fixed, the only insertions we will perform are insertions of positive images to negative indices. Given these informal considerations, we will now finally define what we mean by insertions from now on.

Definition 3.5. We define the following auxiliary function (one should think of this as the function pushing images to their new locations when the gap between $\ell-1$ and ℓ gets a new image):

$$\beta_\ell(x) = \begin{cases} x & \text{if } |x| < \ell \\ x-1 & \text{if } x < -\ell \\ x+1 & \text{if } x > \ell \end{cases}.$$

For a signed permutation w on the indices $-n, \dots, -1, 1, n$, let w_ℓ^{-i} be the signed permutation with the following first half (which uniquely determines the second half):

$$\begin{array}{ccccccccc} -(n{+}1) & -n & \dots & -(i{+}1) & -i & -(i{-}1) & \dots & -1 \\ \beta_\ell(w(-n)) & \beta_\ell(w(-(n{-}1)) & \dots & \beta_\ell(w(-i)) & \ell & \beta_\ell(w(-(i{-}1))) & \dots & \beta_\ell(w(-1)). \end{array}$$

We call this inserting a new element to the site $-i$ and gap ℓ.

On any step, we will be allowed to insert a new element to any negative index site. Furthermore, we want there to be at most one way to generate each permutation. For this reason, we will only ever add images that are larger than the current largest image of among negative indices. Finally, we would like all the generated permutations to avoid 2143 (or 1234), and to generate all permutations avoiding 2143 (or 1234) To this end, we will only be adding an index if the resulting permutation avoids 2143 (or 1234). Our start point will be a permutation with exactly j positive indices with positive images, and no other positive indices. Such a permutation is unique by Lemma 3.3. In terms of our point graph representation, the points in the top right and bottom left quadrant are fixed, and all the action happens in the other two quadrants. The element we will be adding will always be the largest in the top left quadrant, and we will allow insertions to any site on the left. As for which gap the image of the new index falls in, we can think of the j initial points as cutting the top left quadrant into $j+1$ layers. When we inserted the previous point to some layer, we can attach the next one to the same layer or any layer above it. Initially, we can insert a point to any layer.

To summarize what we have arrived at, let us finally explicitly define our pattern avoidance tree.

Definition 3.6. For $\pi = 1234$ or $\pi = 2143$, define $BT^j(\pi)$, the signed permutation pattern avoidance tree for fixed number of positive indices with positive images j to be the following generating tree:

- The label of the root is the unique signed permutation on the indices $-j, \dots, -1, 1, \dots, j$ with all positive indices sent to positive images. For $\pi = 2143$, the root is

$$\begin{matrix} -j & \dots & -1 & 1 & \dots & j \\ -j & \dots & -1 & 1 & \dots & j \end{matrix}$$

  For $\pi = 1234$, the root is

$$\begin{matrix} -j & \dots & -1 & 1 & \dots & j \\ -1 & \dots & -j & j & \dots & 1 \end{matrix}$$

- The succession rule is the following. Our label is the permutation $w \in B_n$. Let m be the maximal image of a negative index in w ($m = 0$ if there are no images). Let $X(w)$ be the set of w_ℓ^{-i} for all pairs i, ℓ with $1 \le n+1$ and $m < \ell \le n+1$. The successor set of w is the subset of $X(w)$ consisting of those permutations that avoid π.

Let us confirm that this tree has the desired properties. Well, given a permutation avoiding $\pi = 1234$ with j positive indices with positive images, there is a way to reconstruct it by a sequence of such insertions. Any subpermutation of a permutation avoiding 1234 also avoids 1234. We start at the root of our tree, and then on each step we take the smallest image from our permutation that has not been inserted yet, and insert it in the tree. Each step can be made, as we are always inserting the new largest image to a negative index, and with the prescribed insertion we always end up with a signed permutation avoiding 1234. For each permutation avoiding 1234 and with j positive indices with positive images, the way it can be constructed with our procedure is unique, as the only way is to start off with the permutation containing only the positive images of positive indices, and to fill everything else in in increasing order. Additionally, as we add one index on each step, $B_n^j(\pi)$ as defined before is precisely the set of labels appearing on the $n-j$'th layer of the tree, where counting starts at 0 from the layer of the root.

Our next goal is to give an alternative description of this tree that is easier to analyze. As the first step towards this goal, we have the following pair of lemmas that allows us to ignore the bottom right quadrant.

Lemma 3.7. *For a vertex labelled w in $BT^j(2143)$, if a signed permutation x in $X(w)$ contains* 2143*, then it contains some* 2143 *at indices i_1, i_2, i_3, i_4, such that there is no i_k for which $i_k > 0$, $x(i_k) < 0$.*

Proof. Say for a contradiction that x contains 2143 but only so that it involves a positive index with a negative image. Let the indices at which one such 2143 appears be a_1, a_2, a_3, a_4. Consider the pattern appearing at the reflection of those images, that is, at $-a_4, -a_3, -a_2, -a_1$. It is easy to check explicitly that the pattern appearing at these indices is also 2143. For these indices, we know that one of them is the reflection of a positive index with a negative image, so that one is a negative index with a positive image. By our assumption, we also know that this pattern contains a positive index with a negative image. As the negative index must come before the positive one in the pattern and this pair forms a decreasing sequence, the pair must correspond to a decreasing pair in 2143. If this pair is 21, then we would need to have a 43 after the positive index with a negative image. Hence, this 43 would come at positive indices. As our 2 is a positive image, the 43 also needs to come at positive images. But this is already a contradiction, as there is no pair of positive indices with decreasing positive images, as w avoids 2143, and the positive indices with positive images are not changed by the insertion, so Lemma 3.3 applies. This completes the proof. □

Lemma 3.8. *For a vertex labelled w in $BT^j(1234)$, if a signed permutation x in $X(w)$ contains* 1234*, then it contains some* 1234 *at indices i_1, i_2, i_3, i_4, such that there is no i_k for which $i_k > 0$, $x(i_k) < 0$.*

Proof. Say for a contradiction that x contains 1234 but only so that it involves a positive index with a negative image. Let the indices at which one such 1234 appears be a_1, a_2, a_3, a_4. Consider the pattern appearing at the reflection of those images, that is, at $-a_4, -a_3, -a_2, -a_1$. It is easy to check explicitly that the pattern appearing at these indices is also 1234. For these indices, we know that one of them is the reflection of a positive index with a negative image, so that one is a negative index with a positive image. By our assumption, we also know that this pattern contains a positive index with a negative image. As the negative index must come before the positive one in the pattern and its image is larger (since it is positive and the other is negative), this pair is decreasing, but there are no decreasing pairs in 1234. This is a contradiction, completing the proof. □

We still need some more terminology before getting to presenting the generating tree. The following three permutation statistics will be important.

- **The number of sites before the first ascent (or descent).** For a signed permutation w, by a site before the first ascent (descent) we mean a site such that there is no increasing (decreasing) pair of images among the indices before this site. When w is clear from the context, we write x for the number of sites before the first ascent (descent). We say a site is after the first ascent (descent) if it is not before the first ascent (descent).
- **The number of active sites.** For a signed permutation w, we say a site with a negative index is *active* with respect to a fixed pattern π and some fixed gap ℓ if

inserting an element to the gap ℓ in that site in w results in a permutation that avoids π. When π, w, and ℓ are clear from the context, we will write y for the number of active sites.

- **The layer number.** As discussed before, the j positive indices with positive images partition the gaps between images to $j+1$ sections or layers. Visually, this corresponds to the top left quadrant being partitioned into $j+1$ horizontal strips by horizontal lines passing through the points corresponding to positive indices with positive images. For a permutation w, the layer number is the number of layers above and including the layer in which the maximal image of a negative index is in w ($j+1$ if there is no image). An equivalent way to think of the layer number is that it is the number of gaps above the maximal image of the negative indices. This is also the same as $n+1-m$ with the notation from Definition 3.6. When w is clear from the context, we will write z for the layer number.

We can now state a few lemmas that help in many of the arguments to come.

Lemma 3.9. *For $\pi = 1234$ or $\pi = 2143$ and for a vertex labelled w in $BT^j(\pi)$, suppose a site is inactive w.r.t. inserting the maximal element among the images of negative indices to a gap. Then this insertion creates a pattern π that involves the element we just inserted and no positive index with a negative image (nothing from the bottom right quadrant).*

Proof. If we ignore the two new images we created with the insertion, then the permutation still does not contain a pattern π. So the pattern π that is present after the insertion must involve at least one of the added elements. Lemmas 3.7 and 3.8 tell us that there is a π contained in the new permutation which does not involve anything from the bottom right quadrant, so in our case there must be a π in w that involves the largest element we just inserted and no positive index with a negative image. □

Lemma 3.10. *Let $\pi = 1234$ or $\pi = 2143$. Fix a layer in which we are considering active sites. If we insert the new maximal image ℓ of the negative indices to some active site, then the new active sites after this insertion are a subset of the old ones (where we think of the site where we inserted ℓ to have split into two). Furthermore, if a previously active site becomes inactive, then inserting $\ell+1$ there would create a pattern involving ℓ and $\ell+1$.*

Proof. The new active sites are a subset of the old ones, as if a site is inactive for inserting ℓ, then it is also inactive for inserting $\ell+1$, as it would create the same pattern (ℓ and $\ell+1$ relate to all other elements in the same way). If a previously active site becomes inactive, then it must be because of creating a pattern not involving anything from the bottom right quadrant by Lemmas 3.7 and 3.8. It is also impossible for this pattern just to involve elements which are not ℓ or the reflection $-\ell$, as ℓ and $\ell+1$ relate to all other elements in the same way, so if inserting ℓ at that site does not create any patterns π, then neither does inserting $\ell+1$, if we only consider patterns not involving ℓ. Hence, inserting $\ell+1$ to that site must create a pattern involving ℓ and $\ell+1$. □

The plan for the rest of the derivation of the generating trees is to first describe succession rules while staying in the same layer, then understand what happens when one passes from layer to layer, and finally combine the two into one succession rule. The next 3 lemmas describe how some statistics of a permutation avoiding π determine the same statistics for some of the successors.

Lemma 3.11. *Let the pattern we are avoiding be $\pi = 2143$. Let w be a signed permutation for which there are no images of negative indices above the ℓ'th layer. Let x be the number of elements before the first descent and y be the number of active sites for insertion as the largest element of the ℓ'th layer. If we just consider the successors of w for which we perform an insertion in the ℓ'th layer, then the pairs (x, y) for each successor form the following multiset (in terms of the values of x and y for w):*

$$\{(2, y+1), (3, y+1), \ldots, (x+1, y+1), (x, x+1), (x, x+2), \ldots, (x, y)\}$$

Proof. All sites before the first descent in w are active, as by Lemma 3.9, the largest element we add would have to be involved in 2143, and it cannot be 4 or 3 as it comes before the previous first descent, so there would be no 21, and it also cannot be a 2 or a 1 as the elements larger than it form an increasing sequence by Lemma 3.3, so there would be no 43.

If we insert to a site before the first descent, then there is a descent right after the inserted element, so the new number of sites before the first descent is 1 more than the number of sites to the left of the insertion. All previously active sites stay active, since sites to the left of the last insertion are still before the first descent, and as for sites to the right of the last insertion, for a site to become inactive inserting there would have to create a 2143 involving both elements that were last added (by Lemma 3.10). As the two elements last added are adjacent images with the smaller coming first, the only option is for them to be 2 and 3 in 2143, but then there is no 4 between them. So all active sites remain active in that case. This case gives the successor labels $(2, y+1), (3, y+1), \ldots, (x+1, y+1)$.

If we insert to a site after the first descent, then the number of sites before the first descent is unchanged. All sites after the first descent but to the left of the insertion become inactive, as inserting there would create an obvious 2143. All sites to the right of the insertion stay active, as by Lemma 3.10 for one to become inactive, inserting there would have to create a 2143 involving both last added elements, but as these are adjacent images, they would have to be the 2 and 3 in 2143. There is no 4 between them, so such a 2143 cannot be formed. This case gies the successor labels $(x, x+1), (x, x+2), \ldots, (x, y)$.

Observe that the two cases together give the desired successor set.

□

Lemma 3.12. *Let the pattern we are avoiding be $\pi = 1234$. Let us consider inserting the new largest element to the top layer. Let x be the number of elements before the first ascent and y be the number of active sites for top layer. If we just consider the successors of w for which we perform an insertion in the top layer, then the pairs (x, y) for each successor form the following multiset (in terms of the values of x and y for w):*

$$\{(2, y+1), (3, y+1), \ldots, (x+1, y+1), (x, x+1), (x, x+2), \ldots, (x, y)\}$$

Proof. All sites before the first ascent in w are active, as by Lemma 3.9, the largest element we add would have to be involved in 1234, and it is the largest element in the permutation (as we are inserting into the top layer), so it would have to be a 4. But if there is no ascent before it, then no 12 can be found before it, so it cannot be involved in a 1234.

If we insert the largest element into the first position, then the new first ascent is at $x+1$. It is obvious from Lemma 3.10 that all sites stay active. So this case gives the successor label $(x+1, y+1)$.

If we insert the largest element into some other position before the first ascent, then the new positions before the first ascent are exactly the sites preceding our inserted element. By what we argued before, all these also stay active. All active sites after the insertion also stay active, as Lemma 3.10 implies that for one to become active, there would need to be a 12 before the insertion, but this is not the case. Hence, this case gives the successors $(2, y+1), (3, y+1), \ldots, (x, y+1)$.

If we insert the largest element into some active site after the first ascent, the position of the first ascent is unchanged. It is clear that all sites after it become inactive because inserting there would create a 1234 from the ascent and the last two insertions. It is also clear from Lemma 3.10 that all active sites to the left of the insertion remain active. Hence, this case gives the successors $(x, x+1), (x, x+2), \ldots, (x, y)$.

Taking the union of the successor multisets from the three cases indeed gives the desired multiset. $\square$

Lemma 3.13. *Let the pattern we are avoiding be $\pi = 1234$. Let us consider inserting the new largest element to a layer which is not the top layer. Let x be the number of sites before the first ascent. If we just consider the successors of w for which we perform an insertion in this layer, then the positions of the first ascent form the following multiset (in terms of the value of x for w):*

$$\{2, 3, \ldots, x+1\}$$

Proof. All sites before the first ascent are active, since creating a new 1234 would have at least 2 elements to one side of the inserted element, but having these on the left is impossible since it was inserted before the first ascent, and having these on the right is impossible by Lemma 3.4. All sites after the first ascent are inactive, because inserting there would create a 1234 involving the first ascent as 12, the inserted element as 3, and the largest positive image of a positive index as 4 (this is larger than the inserted element since we are inserting to a layer below the top layer).

After inserting to the first position, the new number of sites before the first ascent is $x+1$. After inserting into some other site before the first ascent, the sites before the first ascent are the sites to the left of the insertion, giving the terms $2, 3, \ldots, x$. We see that the labels of successors are what is given in the lemma. $\square$

The following lemmas give a characterization of the number of active sites as one switches from one layer to a layer above it.

Lemma 3.14. *Let the pattern we are avoiding be* $\pi = 2143$. w *is a signed permutation avoiding* π, *for which the number of sites before the first descent is* x. *Consider a layer such that there are currently only elements inserted to layers below it. Then the number of active sites w.r.t. inserting to this layer is* x.

Proof. It is enough to show that the active sites are precisely those before the first descent. Any site after the first descent is inactive, as adding a site after a descent means we create a pattern involving the 2 elements in the first descent as 21, the new added element as 4, and the positive index positive image at which the new layer starts as 3. The sites before the first descent are all active, since the new largest element we add could only appear as 4 or 3 in a pattern 2143 (because by Lemma 3.3 everything larger than our maximal element forms an increasing sequence), and if we add a new largest element to one of the sites that are not after the first descent, there is no 21 before it to complete the 2143. Hence, the number of active sites is x. ☐

Lemma 3.15. *Let the pattern we are avoiding be* $\pi = 1234$. w *is a signed permutation avoiding* π. *If there are currently no elements in the top layer, then for inserting to the top layer, all sites are active.*

Proof. Suppose there is an inactive site in the top layer. For this to be inactive but w to avoid π, inserting to that site must create a 1234. By Lemma 3.8, we also created a 1234 even if we ignore the bottom right quadrant. The only change in the other 3 quadrants is the insertion of the new maximal element in the top layer, so there must be a 1234 involving that element. This is the largest image among all the images (as it is in the top layer), so it must be a 4 in the 1234. Hence, there is a 123 before it, but even without inserting the new largest element, the largest positive image of a positive index is greater than all previous images and comes after the 123, so it also fulfils the role of the 4, completing this 1234. Hence, w contains a 1234, which is a contradiction. So all sites in the top layer are active. ☐

We now have all the ingredients to write down the succession rule for $BT^j(2143)$. The statistics we will keep track of for 2143 are the following:

- x – the number of sites before the first descent
- y – the number of active sites in the layer given by the current layer number, or in other words the number of active sites in the lowest layer to which the maximal image of the negative indices can be inserted
- z – the layer number.

Proposition 3.16. *The generating tree given by the following:*

- *the label of the root is* $(j+1, j+1, j+1)$.
- *the succession function suc that takes a label as its input and outputs the set of successors is defined recursively as follows:*

$$suc(x,y,z) = \begin{cases} 0 & z = 0, \\ \{(2, y+1, z), (3, y+1, z), \ldots, (x+1, y+1, z) & \\ (x, x+1, z), (x, x+2, z), \ldots, (x, y, z)\} \bigcup suc(x, x, z-1) & z \geq 1. \end{cases}$$

is isomorphic as a rooted tree to $BT^j(2143)$.

Proof. The only things we have to check is that if a signed permutation w has the statistics (x, y, z), then its successors have the statistics given in the lemma, and that the statistics of the root of $BT^j(2143)$ are the same as the label of the root given in the lemma. The latter is trivial, and as for the former, all the work is already done in Lemmas 3.11 and 3.14. Lemma 3.11 gives that the successors for fixed z are those given in this lemma, and 3.14 tells us what the labels are if we consider the layer below. Inducting on z gives that this succession rule works. □

Example 3.17. Figure 3 shows the point graph of the 2143 avoiding signed permutation

$$\begin{matrix} -6 & -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 & 6 \\ -6 & 4 & -3 & 5 & 2 & 1 & -1 & -2 & -5 & 3 & -4 & 6 \end{matrix}.$$

In each layer (horizontal strip), the active sites are shown with a cross. The relevant statistics x, y, z are also displayed.

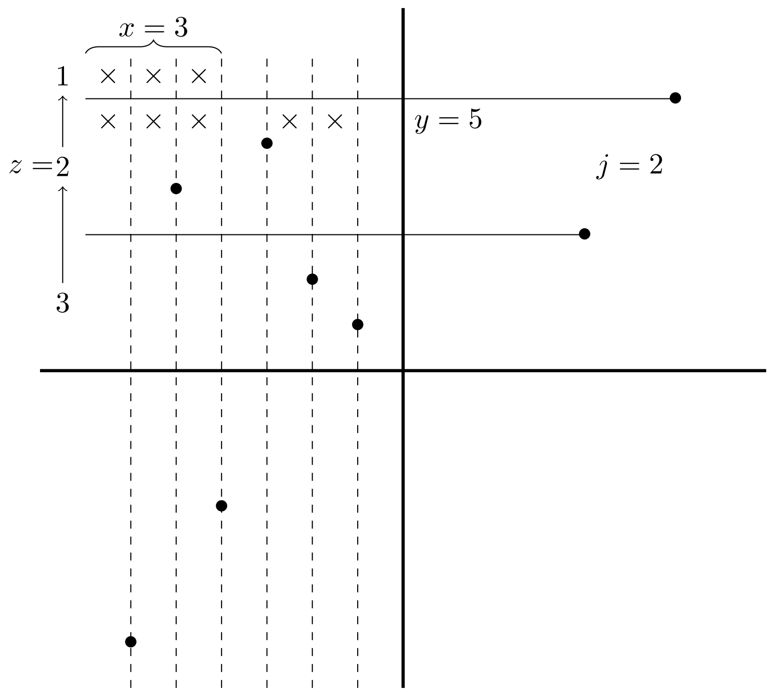

FIGURE 3. The point graph of a 2143 avoiding signed permutation with active sites and statistics

We also have the ingredients to write down the succession rule for $BT^j(1234)$. The statistics we will keep track of for 1234 are the following:

- x – the number of sites before the first ascent

- y – the number of active sites in the top layer (by Lemma 3.15, all sites are active before getting to the top layer, so this just increases by one each step as long as nothing has been inserted in the top layer yet)
- z – the layer number.

Proposition 3.18. *The generating tree given by the following:*

- *the label of the root is* $(j+1, j+1, j+1)$.
- *the succession function suc that takes a label as its input and outputs the set of successors is defined recursively as follows:*

$$suc(x,y,z) = \begin{cases} \{(2,y+1,z),(3,y+1,z),\ldots,(x+1,y+1,z) \\ (x,x+1,z),(x,x+2,z),\ldots,(x,y,z)\} & z=1, \\ \{(2,y+1,z),(3,y+1,z),\ldots,(x+1,y+1,z)\} \\ \bigcup suc(x,y,z-1) & z \geq 2. \end{cases}$$

is isomorphic as a rooted tree to $BT^j(1234)$.

Proof. The proof is analogous to the proof of Proposition 3.16. □

Example 3.19. Figure 4 shows the point graph of 1234 avoiding signed permutation

$$\begin{matrix} -6 & -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & -3 & 4 & -5 & 1 & -6 & 6 & -1 & 5 & -4 & 3 & -2 \end{matrix}.$$

In each layer, the active sites are shown with a cross. The relevant statistics x, y, z are also displayed.

These generating trees turn the problem of counting permutations avoiding certain patterns into the problem of counting lattice paths of some length in $\mathbb{Z}^3$ given by the succession rules.

3.2. **Proof of Theorem 3.1.** Proposition 3.16 and Proposition 3.18 allow us to translate the questions of enumerating $B_n^j(2143)$ and $B_n^j(1234)$ to questions of enumerating lattices paths in the integer lattice $\mathbb{Z}^3$ with specified rules. Respectively, let $\mathcal{P}^{2143}$ be the set of all lattice paths specified by the succession rule in Proposition 3.16 and let $\mathcal{P}^{1234}$ be the set of all lattice paths specified by the succession rule in Proposition 3.18. We allow arbitrary starting point (x, y, z) for those paths with $2 \leq x \leq y$ and $1 \leq z$ besides those that start at $(j+1, j+1, j+1)$. We view such a lattice path as a sequence of points connected by edges.

For a path $P \in \mathcal{P}^{2143}$ and an edge e of P that goes from (x_1, y_1, z_1) to (x_2, y_2, z_2), we say that e is *recorded*

- if $z_1 = z_2$ and $y_2 = y_1 + 1$;
- if $z_1 > z_2$ (and $y_2 = x_1 + 1$).

Notice that if $z_1 > z_2$, then we are forced to use the succession rule of (x_1, x_1, z_2) to go to (x_2, y_2, z_2) and thus $y_2 = x_1 + 1$. Analogously, for $P \in \mathcal{P}^{1234}$ and an edge e of P that goes from (x_1, y_1, z_1) to (x_2, y_2, z_2), we say that e is *recorded* if $y_2 = y_1 + 1$. In particular, if $z_2 \geq 2$, the edge is always recorded. We see from the succession rule in Section 3.1 that if

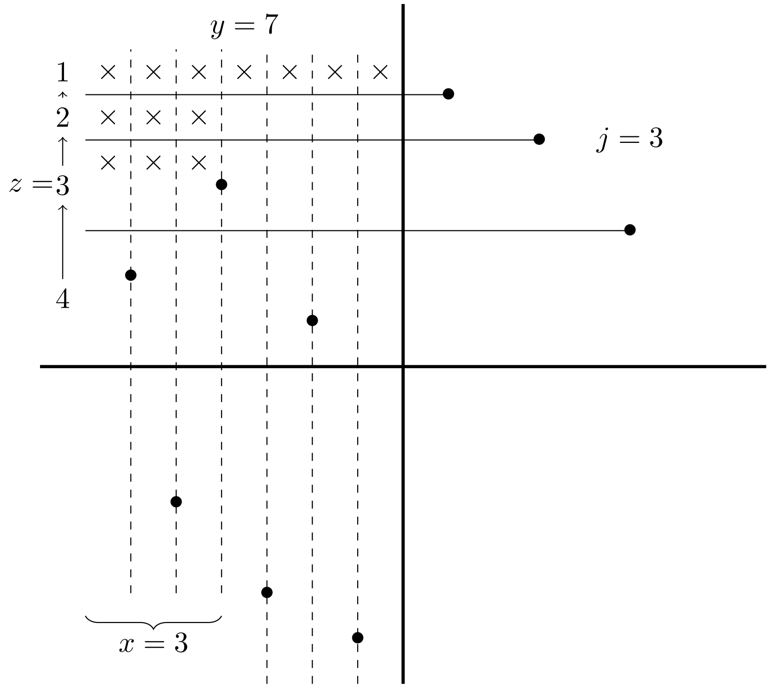

FIGURE 4. The point graph of a 1234 avoiding permutation with active sites and statistics

an edge is not recorded, then the x-coordinates are the same for the two points connected by the edge.

Definition 3.20. For a path $P \in \mathcal{P}^{\pi}$, $\pi \in \{1234, 2143\}$, define its *signature* $\operatorname{sig}(P)$ to be the tuple consists of the x-coordinate of the starting point, appended with the x-coordinates of the ending points of recorded edges in order.

Example 3.21. Consider the following paths $P \in \mathcal{P}^{2143}$ and $P' \in \mathcal{P}^{1234}$ which are

$$\begin{aligned} P =& (4,4,3) \to (3,5,3) \dashrightarrow (3,5,3) \to (4,4,2) \to (2,5,2) \dashrightarrow (2,4,2) \\ & \dashrightarrow (2,4,2) \to (2,2,1) \to (2,3,1) \dashrightarrow (2,3,1) \\ P' =& (4,4,3) \to (3,5,3) \to (4,6,3) \to (2,7,2) \to (2,8,1) \dashrightarrow (2,7,1) \\ & \dashrightarrow (2,7,1) \dashrightarrow (2,5,1) \dashrightarrow (2,4,1) \dashrightarrow (2,4,1) \to (2,5,1) \dashrightarrow (2,4,1) \end{aligned}$$

where the recorded edges are labeled with right arrows and the edges not recorded are labeled as dashed arrows. Both paths have signature $(4, 3, 4, 2, 2, 2)$.

The main goal of this section is to show that for a fixed starting point v, a fixed signature γ and $n \geq 1$, the number of paths in $\mathcal{P}^{\pi}$ that start with v, have signature γ and have length n is the same for $\pi \in \{1234, 2143\}$. To do this, let us define the corresponding generating functions. For $\pi \in \{1234, 2143\}$, $k \geq 0$, $q \geq 1$, $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}^m$, define $\mathcal{P}^{\pi}_{k,q,\gamma}$ to be the set of paths in $\mathcal{P}^\pi$ that start at $(\gamma_1, \gamma_1 + k, q)$ and have signature γ. For any path P, let its length $\ell(P)$ be the number of points that it contains. Notice that if P has signature $\gamma = (\gamma_1, \dots, \gamma_m)$, then clearly $\ell(P) \ge m$.

Definition 3.22. For $\pi \in \{1234, 2143\}$, $k \ge 0$, $q \ge 1$, $\gamma \in \mathbb{Z}^m$ with $m \ge 1$, define

$$F^\pi(k, q, \gamma) := \sum_{P \in \mathcal{P}^\pi_{k,q,\gamma}} t^{\ell(P)-m}.$$

We are going to recursively compute $F^\pi(k, q, \gamma)$ and then compare $F^{1234}(k, q, \gamma)$ with $F^{2143}(k, q, \gamma)$. As for some notations, if $\gamma \in \mathbb{Z}^m$, write $|\gamma| = m$. For convention, we say $F^\pi(k, q, \gamma) = 0$ if $q \le 0$ or $|\gamma| = 0$. Let $\gamma' = (\gamma_2, \gamma_3, \dots, \gamma_m)$, which is $\emptyset$ if $m = 1$ and let $\gamma'' = (\gamma_3, \dots, \gamma_m)$, which is $\emptyset$ if $m \le 2$. And we will restrict our attention to only those γ's that can be signatures of some valid paths in $\mathcal{P}^\pi$. Namely, we require $2 \le \gamma_{i+1} \le \gamma_i + 1$. Finally, for simplicity, let $s = 1 + t + t^2 + \cdots = 1/(1 - t)$.

Lemma 3.23. *For $k \ge 0$, $q \ge 1$, $\gamma \in \mathbb{Z}^m$ with $m \ge 1$, we have*

$$F^{2143}(k, q, \gamma) = \begin{cases} s^k & |\gamma| = 1, \\ F^{2143}(0, q{-}1, \gamma) + F^{2143}(\gamma_1{+}1{-}\gamma_2, q, \gamma') & |\gamma| \ge 2, k = 0 \\ sF^{2143}(k{-}1, q, \gamma) + sF^{2143}(\gamma_1{+}1{-}\gamma_2{+}k, q, \gamma') & \\ -sF^{2143}(\gamma_1{-}\gamma_2{+}k, q, \gamma') & |\gamma| \ge 2, k \ge 1. \end{cases}$$

Proof. We refer the readers to the succession rule in Proposition 3.16.

If $|\gamma| = 1$, then the signature has length 1 and we are summing over paths that start at $(\gamma_1, \gamma_1 + k, q)$ with no recorded edges. As soon as we decrease q, which is the z-coordinate, we need to use the succession rule for $(\gamma_1, \gamma_1, q - 1)$ and then every edge is recorded so we cannot have any edges afterwards. Therefore, the only additional points on this path come from any number of $(\gamma_1, \gamma_1{+}k, q)$ followed by any number of $(\gamma_1, \gamma_1{+}k{-}1, q)$ and so on, finally ending with any number of $(\gamma_1, \gamma_1{+}1, q)$. The resulting generating function is then $(1 + t + t^2 + \cdots)^k = s^k$.

If $k = 0$, then our paths start at (γ_1, γ_1, q). The next edge must be recorded. There are exactly two options: either decrease the z-coordinate q, or go directly to the next signature value γ_2 at the same z-coordinate. For the first option, we obtain a generating function $F^{2143}(0, q - 1, \gamma)$. For the second option, we go from (γ_1, γ_1, q) to $(\gamma_2, \gamma_1{+}1, q)$ and trim the signature so the corresponding generating function is $F^{2143}(\gamma_1{+}1{-}\gamma_2, q, \gamma')$.

The main case is $|\gamma| \ge 2$ and $k \ge 1$. Our goal is to decrease k. As $k \ge 1$, when we start at $(\gamma_1, \gamma_1{+}k, q)$, we are allowed to have an arbitrary number of $(\gamma_1, \gamma_1{+}k, q)$ first via unrecorded edges, which provide a factor of s, before we choose the next edge. Let's now compare $F^{2143}(k, q, \gamma)$ with $sF^{2143}(k{-}1, q, \gamma)$. The paths enumerated by each of them largely coincide, including those that decrease q right away. The only exception is that paths that go directly from some number of $(\gamma_1, \gamma_1{+}k, q)$ to the next recorded edge ending at $(\gamma_2, \gamma_1{+}k{+}1, q)$ are counted by $F^{2143}(k, q, \gamma)$ but not by $sF^{2143}(k{-}1, q, \gamma)$; and similarly the paths that go directly to $(\gamma_2, \gamma_1{+}k, q)$ from $(\gamma_1, \gamma_1{+}k{-}1, q)$ are counted only by $sF^{2143}(k-1, q, \gamma)$. As a result,

$$\begin{aligned}&F^{2143}(k, q, \gamma) - sF^{2143}(k-1, q, \gamma)\\ =&sF^{2143}(\gamma_1+1-\gamma_2+k, q, \gamma') - sF^{2143}(\gamma_1-\gamma_2+k, q, \gamma')\end{aligned}$$

which is equivalent to the statement that we need. □

Notice that the recursive formula provided in Lemma 3.23 can determine F^{2143} uniquely.

Lemma 3.24. *For $k \geq 0$, $q \geq 1$, $\gamma \in \mathbb{Z}^m$ with $m \geq 1$, we have*

$$F^{1234}(k, q, \gamma) = \begin{cases} s^k & |\gamma| = 1, \\ F^{2143}(k, q, \gamma) & q = 1, \\ F^{1234}(k, q-1, \gamma) + F^{1234}(\gamma_1+1-\gamma_2+k, q, \gamma') & |\gamma| \geq 2, q \geq 2. \end{cases}$$

Proof. We refer the readers to the succession rule in Proposition 3.18

If $|\gamma| = 1$, then we are considering paths that start at $(\gamma_1, \gamma_1 + k, q)$ with no recorded edges. Since every edge is recorded when $q \geq 2$, our only option is to decrease q all the way down to 1 and then use the succession rule of $(\gamma_1, \gamma_1 + k, 1)$. Now we can have an arbitrary number of $(\gamma_1, \gamma_1+k, 1)$ followed by an arbitrary number of $(\gamma_1, \gamma_1+k-1, 1)$ and so on up to an arbitrary number of $(\gamma_1, \gamma_1+1, 1)$. The generating function is thus $(1+t+t^2+\cdots)^k = s^k$.

If $q = 1$, the succession rules for $\mathcal{P}^{1234}$ and $\mathcal{P}^{2143}$ are the same so we have $\mathcal{P}^{1234}_{k,1,\gamma} = \mathcal{P}^{2143}_{k,1,\gamma}$. Therefore, $F^{1234}(k, 1, \gamma) = F^{2143}(k, 1, \gamma)$.

When $q \geq 2$ and $|\gamma| \geq 2$, for a path in $\mathcal{P}^{1234}_{k,q,\gamma}$, it starts at $(\gamma_1, \gamma_1+k, q)$. Since $q \geq 2$, all edges that keep the same z-coordinate q are recorded. So we have exactly two options: decrease q by 1, which results in the generating function $F^{1234}(k, q-1, \gamma)$, and go to $(\gamma_2, \gamma_1 + 1, q)$ indicated by the signature γ, which results in the generating function $F^{1234}(\gamma_1+1-\gamma_2+k, q, \gamma')$. Take the sum and we get the desired equation. □

With sufficient tools to determine the generating functions F^{2143} and F^{1234}, we are ready to obtain their equality.

Lemma 3.25. *For $k \geq 0$, $q \geq 1$, $\gamma \in \mathbb{Z}^m$ with $m \geq 1$,*

$$F^{1234}(k, q, \gamma) = F^{2143}(k, q, \gamma).$$

Proof. We proceed by induction on $|\gamma|$, q and k in this order. From Lemma 3.23 and Lemma 3.24, our statement is true when $|\gamma| = 1$ and is also true when $|\gamma| \geq 2$ and $q = 1$. When $|\gamma| \geq 2$, $q \geq 2$ and $k = 0$, from Lemma 3.23,

$$F^{2143}(0, q, \gamma) = F^{2143}(0, q-1, \gamma) + F^{2143}(\gamma_1 + 1 - \gamma_2, q, \gamma')$$

and from Lemma 3.24,

$$F^{1234}(0, q, \gamma) = F^{1234}(0, q-1, \gamma) + F^{1234}(\gamma_1 + 1 - \gamma_2, q, \gamma')$$

so by induction hypothesis, $F^{2143}(0, q, \gamma) = F^{1234}(0, q, \gamma)$.

Now assume that $|\gamma| \geq 2$, $q \geq 2$ and $k \geq 1$. With induction hypothesis and for the ease of notation, for the arguments that we already know the equality of F^{1234} and F^{2143}, we will just write F instead. By Lemma 3.23 and Lemma 3.24, and by induction hypothesis, we have that

$$\begin{aligned}F^{2143}(k,q,\gamma) =& sF(k-1,q,\gamma)+sF(\gamma_1+1-\gamma_2+k,q,\gamma')-sF(\gamma_1-\gamma_2+k,q,\gamma')\\ =& sF(k-1,q-1,\gamma)+sF(\gamma_1-\gamma_2+k,q,\gamma')\\ & +sF(\gamma_1+1-\gamma_2+k,q-1,\gamma')+sF(\gamma_1+2-\gamma_3+k,q,\gamma'')\\ & -sF(\gamma_1-\gamma_2+k,q-1,\gamma')-sF(\gamma_1+1-\gamma_3+k,q,\gamma'')\\ =& F(k,q-1,\gamma)+F(\gamma_1+1-\gamma_2+k,q,\gamma')\\ =& F^{1234}(k,q,\gamma)\end{aligned}$$

as desired. We also see that the above argument goes through when $|\gamma'|=1$, in which case $\gamma''=\emptyset$. Therefore, the induction step is established so we obtain the desired lemma. $\square$

With the main technical lemma (Lemma 3.25), Theorem 3.1 becomes immediate.

Proof of Theorem 3.1. For $\pi \in \{1234, 2143\}$ and $j \le n$,

$$B_n^j(\pi) = \sum_{\gamma_1=j+1} [t^{n-j-|\gamma|+1}]F^{\pi}(0,j+1,\gamma).$$

Since $F^{1234}(0,j+1,\gamma)=F^{2143}(0,j+1,\gamma)$, $B_n^j(1234)=B_n^j(2143)$. $\square$

3.3. **Open questions.** There are still many interesting questions to be asked.

First, the proof provided in this section is semi-bijective. With recursive formula provided in Lemma 3.23 and Lemma 3.24, we are able to obtain the equality of $F^{1234}(k,q,\gamma)=F^{2143}(k,q,\gamma)$. However, is there an explicit bijection between paths in $\mathcal{P}^{1234}_{k,q,\gamma}$ and $\mathcal{P}^{2143}_{k,q,\gamma}$ that is length-preserving?

Second, for a fixed $j \ge 0$, it is desirable to obtain an explicit formula for the generating function

$$\sum_{n=j}^{\infty} B_n^j(\pi)t^{n-j}$$

for either $\pi \in \{1234, 2143\}$. The case $j=0$ is the generating function for 1234 (or 2143) avoiding permutations $\sum_n S_n(1234)t^n$, which is studied in [3] and already has a complicated form.

Last but not least, can our techniques be further generalized? We make the following conjecture (see the beginning of the section for definitions).

Conjecture 3.26. For $j \le n$, $|B_n^j(12345)| = |B_n^j(21354)|$.

We have checked Conjecture 3.26 up to $n \le 7$. Notice that when $j=0$, the statement holds [14] and when $j=n$, it is not hard to see that both sides equal the Catalan number C_j.

It is known that the identity element $1,2,\ldots,k$ and $\pi = 2,1,3,\ldots,k-2,k,k-1$ are Wilf equivalent in the sense of permutations [14]. So are they Wilf equivalent in signed permutations?

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