THE SIDORENKO PROBLEM FOR DIRECTED GRAPHS IN TOURNAMENTS

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Abstract. Sidorenko's problem asks to characterize the family of undirected graphs H for which the pseudorandom graph with edge density p has asymptotically the minimum number of copies of H over all graphs on the same number of vertices and edge density. In this paper, we study the directed analogue of Sidorenko's problem, namely to determine the family of directed graphs $\vec{H}$ for which the pseudorandom tournament has asymptotically the minimum number of copies of $\vec{H}$ over all tournaments on the same number of vertices. Here we show several ways to construct directed Sidorenko graphs out of other directed Sidorenko graphs, and give all Sidorenko graphs whose underlying undirected structure is a star. It is known that transitive tournaments are the only tournaments with the Sidorenko property. We characterize when a transitive tournament minus an edge has the Sidorenko property in most cases. We also show a few other techniques we used to study the Sidorenko graphs whose underlying structure is a path.

1. Introduction

Given two undirected graphs H, G, a homomorphism $\pi \in \hom(H, G)$ is a map $V(H) \to V(G)$ such that $(\pi(i), \pi(j)) \in E(G)$ if $(i, j) \in E(H)$. In 1993 Sidorenko [1] raised the following beautiful conjecture on the homomorphism density which states as follows. We define the *homomorphism density* of H in G to be

$$t(H, G) = \frac{|\hom(H, G)|}{|V(G)|^{|V(H)|}}, \tag{1.1}$$

Conjecture 1.1 (Sidorenko [1]). *For any bipartite graph H and any graph G,*

$$t(H, G) \geq t(K_2, G)^{|E(H)|}. \tag{1.2}$$

This conjecture also has an equivalent analytic form which studies the homomorphism density of H in graphons (graphons are the limit object of graphs, which are represented by nonnegative symmetric measurable functions $[0,1]^2 \to \mathbb{R}_{\geq 0}$). We say that a graph H is *Sidorenko* if (1.2) holds for all graphs G. By considering $G = K_2$, i.e., (1.2) becomes

$$\frac{|\hom(H, K_2)|}{|V(G)|^{|V(H)|}} \geq \left(\frac{|\hom(K_2, K_2)|}{4}\right)^{|E(H)|} = 2^{-|E(H)|} > 0. \tag{1.3}$$

Thus we know that a necessary condition for H to be Sidorenko is that $\hom(H, K_2)$ is not empty, i.e., H is bipartite. Sidorenko's conjecture is that this is also a sufficient condition.

This problem has been widely studied by combinatorists. It is known that Sidorenko graphs include many families of bipartite graphs, such as hypercubes [2], paths and trees ([1], [3]), and other families [4]. Moreover, the following operations build Sidorenko graphs out of others: disjoint union [1], tensor product, Cartesian product with even cycles [5].

The equivalent analytic condition takes the following form. For any bipartite graph $H = (U, V, E)$ and any symmetric measurable function $h : [0,1]^2 \mapsto \mathbb{R}_{\geq 0}$,

$$\int_{[0,1]^{|U|+|V|}} \prod_{\substack{u \in U, v \in V,\\ (u,v) \in E}} h(x_u, y_v)\, dx^U dy^V \geq \left(\int_{[0,1]^2} h(x,y)\, dxdy \right)^{|E|}. \tag{1.4}$$

One natural relaxation of this analytic form is to consider the inequality above for nonnegative but not necessarily symmetric h. Although it is not known that these two are equivalent, it is still conjectured that the inequality (1.4) holds for any bipartite graphs and any nonnegative h. Indeed, many techniques that work for symmetric h also works for general h. In fact, when h is not symmetric, the equation (1.4) corresponds to the homomorphism density of directed graph $\vec{H}$ in h ($\vec{H}$ is obtained from H by orienting all edges from U to V), where we should interpret h as a limit object of directed graphs, as we will see once we define the homomorphism density for directed graphs in Section 2. Therefore, it becomes natural to consider a directed analogue of Sidorenko's conjecture, where we are interested in the inequality regarding the homomorphism density of any directed graph $\vec{H}$ in all directed graphs $\vec{G}$, i.e., we would like to find all directed graphs $\vec{H}$ such that for any directed graph $\vec{G}$,

$$t(\vec{H}, \vec{G}) \geq \left(t(\vec{K}_2, \vec{G}) \right)^{|E(\vec{H})|}. \tag{1.5}$$

In fact, we can show that this directed analogue is equivalent to Sidorenko's conjecture where h is not necessarily symmetric. We have already shown this equivalence when $\vec{H}$ comes from a bipartite graph $H = (U, V, E)$ and all edges are directed from U to V. We then show that if $\vec{H}$ satisfy the inequality (1.4) for general h, then $\vec{H}$ must come from a bipartite graph using the procedure above. Actually in (1.5) take $\vec{G} = \vec{K}_2$, then the RHS implies is positive, so LHS is also positive. Therefore, there exists at least one homomorphism $\pi : \vec{H} \to \vec{K}_2$. Denote the vertices of $\vec{K}_2$ by u, v and edge directed from u to v. Take $U = \pi^{-1}(u)$ and $V = \pi^{-1}(v)$, we know that U, V form a partition of vertices of H, and there are only edges from U to V (not from V to U, nor within U or V). Thus, if $\vec{H}$ satisfies (1.5) for all $\vec{G}$, it must come from a bipartite graph where all edges directed from one part to the other.

In this paper we study the another directed analogue of the Sidorenko's conjecture, which is that we consider the homomorphism density of $\vec{H}$ in all tournaments $\vec{G}$ (instead of general directed graphs).

The paper is structured as following: in Section 2 we give some preliminary definitions, and in Section 3 we will prove some basic examples and properties of directed Sidorenko graphs. In Section 4 we study Sidorenko graphs that allow us to blow-up its vertices to other Sidorenko graphs. We also show some operations that allow us to construct such Sidorenko graphs. In Section 5, we start to characterize all Sidorenko graphs with certain underlying undirected structure, e.g. stars, complete graphs. We also characterize when a transitive minus an edge has the Sidorenko property in most cases. In Section 5.2, we give a general technique for studying paths, and give a necessary condition for a directed graph to be Sidorenko. In Section 6, we present a list of open questions in this problem.

2. Definitions and Notations

Definition 2.1. A *weighted tournament* $\vec{G}$ is defined to be a directed graph such that each vertex $i \in V(\vec{G})$ has a non-negative weight $\alpha_{\vec{G}}(i)$ such that

$$\sum_{i \in V(\vec{G})} \alpha_{\vec{G}}(i) = 1. \tag{2.1}$$

Each pair (i, j) $(i, j \in V(\vec{G}))$ has a non-negative weight $\rho_{\vec{G}}(i, j)$, such that for every pair (i, j), $\rho_{\vec{G}}(i, j) + \rho_{\vec{G}}(j, i) = 1$. Each vertex has a loop of weight $1/2$ (i.e., $\rho_{\vec{G}}(i, i) = 1/2$ for all $i \in V(\vec{G})$). Let $\mathcal{G}$ be the set of all weighted tournaments.

Then we define the limit object of weighted tournaments.

Definition 2.2. A *directed graphon* $W : [0, 1]^2 \to [0, 1]$ is a measurable function satisfying that

$$W(x, y) + W(y, x) = 1 \quad \text{for a.a. } (x, y) \in [0, 1]^2 \tag{2.2}$$

Let $\mathcal{W}$ be the set of all directed graphons. Define W^T to be the directed graphon given by $W^T(x, y) = W(y, x)$.

Definition 2.3. For a weighted tournament $\vec{G}$, we define the *directed graphon associated to* $\vec{G}$ to be $W(\vec{G}) : [0, 1]^2 \to [0, 1]$ as follows: we break $[0, 1]$ into $|V(\vec{G})|$ intervals, each of length $\alpha_{\vec{G}}(i)$. Therefore $[0, 1]^2$ is divided into $|V(\vec{G})|^2$ blocks, and for $i, j \in V(\vec{G})$, let block (i, j) take value $\rho_{\vec{G}}(i, j)$ inside the block. The block boundaries are defined to be $1/2$.

Remark. Since $\rho_{\vec{G}}(i, j) + \rho_{\vec{G}}(j, i) = 1$ almost always, $W(\vec{G}) \in \mathcal{W}$ for every weighted tournament $\vec{G}$. Figure 1 provides an example.

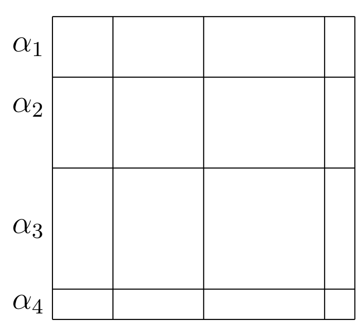


Figure 1. Example of the block graphon for a weighted tournament on 4 vertices.

Definition 2.4. The *homomorphism density* of a directed graph $\vec{H}$ in a directed graphon W (or in general, any measurable function $[0, 1]^2 \to \mathbb{R}$) is defined as

$$t(\vec{H}, W) := \int_{[0,1]^{|V(\vec{H})|}} \prod_{(i,j) \in E(\vec{H})} W(x_i, x_j)\, dx^{V(\vec{H})}. \tag{2.3}$$

The homomorphism density of a directed graph $\vec{H}$ in a weighted tournament $\vec{G} \in \mathcal{G}$ is defined as

$$t(\vec{H}, \vec{G}) := t(\vec{H}, W(\vec{G})). \tag{2.4}$$

Remark. Equivalently, we can define the homomorphism density of a directed graph $\vec{H}$ in a weighted tournament $\vec{G}$ to be the fraction of all maps from $V(\vec{H})$ to $V(\vec{G})$ that are homomorphisms: each copy is multiplied by vertex weights and edge weights. In particular,

$$t(\vec{H}, \vec{G}) = \sum_{\pi \in V(\vec{G})^{V(\vec{H})}} \prod_{i \in V(\vec{H})} \alpha_{\vec{G}}(\pi(i)) \prod_{(i,j) \in E(\vec{H})} \rho_{\vec{G}}(\pi(i), \pi(j)). \tag{2.5}$$

Definition 2.5. A directed graph $\vec{H}$ is *Sidorenko* if for every $\vec{G} \in \mathcal{G}$,

$$t(\vec{H}, \vec{G}) \geq 2^{-|E(\vec{H})|}. \tag{2.6}$$

Similarly, we define it to be *anti-Sidorenko* if for every $\vec{G} \in \mathcal{G}$,

$$t(\vec{H}, \vec{G}) \leq 2^{-|E(\vec{H})|}. \tag{2.7}$$

Remark. Although a graph is Sidorenko if the inequality holds for all weighted tournaments, a directed graph is Sidorenko if and only if the same inequality holds for all directed graphons (a larger class of objects). In other words, we have the following statement: $\vec{H}$ is *Sidorenko* if and only if for every $W \in \mathcal{W}$,

$$t(\vec{H}, W) \geq 2^{-|E(\vec{H})|}. \tag{2.8}$$

There is also a symmetric statement for anti-Sidorenko graphs which can be proved using exactly the same technique as below.

Proof. Using Definition 2.5, because $W(\vec{G}) \in \mathcal{W}$ for any $\vec{G} \in \mathcal{G}$, one direction of the statement is clear. Since we can interpret the directed graphons as the limit objects of weighted tournaments, we can approximate any given $W \in \mathcal{W}$ using a sequence of block graphons $W_i = W(\vec{G}_i)$ where $G_i \in \mathcal{G}$ for $i \in \mathbb{N}$, such that $\|W - W_i\|_{L^1} \to 0$. Then from the continuity of the integral $t(\vec{H}, W)$ in W, we conclude that

$$t(\vec{H}, W) = \lim_{i \to \infty} t(\vec{H}, \vec{G}_i) \geq 2^{-|E(\vec{H})|}, \tag{2.9}$$

hence we have both directions of the statement. $\square$

Notation. For two directed graphs $\vec{H}_1, \vec{H}_2$, let $\vec{H}_1 \cup \vec{H}_2$ be their disjoint union and $\vec{H}_1 \to \vec{H}_2$ be the directed graph defined as follows: we first take the disjoint union of them, then we add an edge from every vertex of H_1 to every vertex of H_2. We define $\vec{I}_a$ to be the directed graph of a isolated vertices, and $\vec{K}_{a,b}$ to be the complete bipartite graph where all edges are directed from the part of size a to the part of size b. In particular, $\vec{K}_{0,a} = \vec{K}_{a,0} = \vec{I}_a$.

Then we give definitions of "neighborhood" for both directed graphs and weighted tournaments.

Definition 2.6. Given a weighted tournament $\vec{G}$, for two multisets $A, B \subset V(\vec{G})$, we define the *neighborhood* of (A, B) to be a tuple $(s, \vec{N}) \in \mathbb{R}_{\geq 0} \times \mathcal{G}$ where $s = s(\vec{G}, A, B)$ is the *size* of the neighborhood, and $\vec{N}$ is the weighted tournament $\vec{N} = \vec{N}(\vec{G}, A, B)$ defined as follows: for every $i \in V(\vec{G})$, let

$$x_i = \alpha_{\vec{G}}(i) \prod_{a \in A} \rho_{\vec{G}}(a, i) \prod_{b \in B} \rho_{\vec{G}}(i, b). \tag{2.10}$$

The size is defined as $s = \sum_i x_i$, and $\vec{N}$ is defined to be the weighted tournament that has the same vertices and edge weights as $\vec{G}$, but for every vertex $i \in V(\vec{G}) = V(\vec{N})$, we assign a new vertex weight $\alpha_{\vec{N}}(i) = x_i/s$. (When $s = \sum_i x_i = 0$, we define $\vec{N}$ to be the graph with only one vertex of weight 1).

The neighborhood of a vertex in a directed graph is defined as follows. Given a directed graph $\vec{H}$ and a vertex $v \in V(\vec{H})$, the *in-neighborhood* of $\vec{H}$ is $I(v) = \{u \in V(\vec{H}) : (u, v) \in E(\vec{H})\}$, and *out-neighborhood* is $O(v) = \{u \in V(\vec{H}) : (v, u) \in E(\vec{H})\}$.

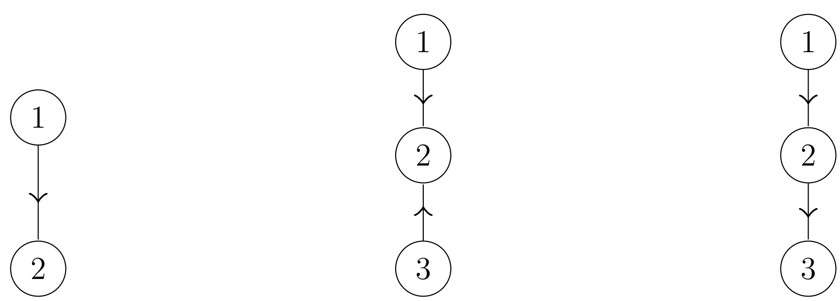
Definition 2.7. Given a directed graph $\vec{H}$, two vertices i_1, i_2 *share the same neighborhood* if they have exactly the same in- and out-neighborhood, i.e., $I(i_1) = I(i_2)$ and $O(i_1) = O(i_2)$. A set $S \subset \vec{H}$ is *neighborhood-equivalent* if elements in S pairwise share the same neighborhood.

Remark. A neighborhood-equivalent set must be an independent set.

3. Simple Examples and Basic Properties of Directed Sidorenko Graphs

Firstly, we give some examples for Sidorenko graphs and anti-Sidorenko graphs.

Example 3.1. Consider the following graphs.



(A) $\vec{H}_1$ (B) $\vec{H}_2$ (C) $\vec{H}_3$

Figure 2. Example Graphs

We prove that $\vec{H}_1$ is Sidorenko, $\vec{H}_2$ is anti-Sidorenko, and $\vec{H}_3$ is both Sidorenko and anti-Sidorenko.

Proof. The statement above is equivalent to the following three relations: for any directed graphon $W \in \mathcal{W}$,

$$t(\vec{H}_1, W) = 1/2, \quad t(\vec{H}_2, W) \geq 1/4, \quad t(\vec{H}_3, W) \leq 1/4. \tag{3.1}$$

Using the fact that $W(x,y)+W(y,x)=1$ almost everywhere, we have

$$\int_{[0,1]^2} W(x,y)\,dxdy = \frac{1}{2}\int_{[0,1]^2} 1\,dxdy = \frac{1}{2}. \tag{3.2}$$

Therefore we know that

$$t(\vec{H}_1, W) = \int_{[0,1]^2} W(x_1,x_2)\,dx_1dx_2 = \frac{1}{2}. \tag{3.3}$$

Using Cauchy-Schwarz inequality, we know that

$$\begin{aligned} t(\vec{H}_2, W) &= \int_{[0,1]^3} W(x_1,x_2)W(x_3,x_2)\,dx_1dx_2dx_3 \\ &= \int_{[0,1]} \left(\int_{[0,1]} W(x_1,x_2)\,dx_1\right)^2 dx_2 \\ &\geq \left(\int_{[0,1]^2} W(x_1,x_2)\,dx_1dx_2\right)^2 = 1/4. \end{aligned} \tag{3.4}$$

At the same time, we know that

$$\begin{aligned} t(\vec{H}_2, W) + t(\vec{H}_3, W) &= \int_{[0,1]^3} W(x_1,x_2)W(x_3,x_2)\,dx_1dx_2dx_3 \\ &\quad + \int_{[0,1]^3} W(x_1,x_2)W(x_2,x_3)\,dx_1dx_2dx_3 \\ &= \int_{[0,1]^3} W(x_1,x_2)(W(x_3,x_2)+W(x_2,x_3))\,dx_1dx_2dx_3 \\ &= \int_{[0,1]^3} W(x_1,x_2)\,dx_1dx_2dx_3 = \frac{1}{2}. \end{aligned} \tag{3.5}$$

Combining with (3.4), we conclude that

$$t(\vec{H}_3, W) \leq 1/4. \tag{3.6}$$

We therefore have all three relations in (3.1). □

Remark. Neither $\vec{H}_1$ nor $\vec{H}_2$ is both Sidorenko and anti-Sidorenko. We will see this using the graphon defined in Proposition 3.4.

Surprisingly, the $\vec{H}_3$ above is not the only non-trivial directed graph that is both Sidorenko and anti-Sidorenko. We actually have a infinite family of weakly-connected graphs that are both Sidorenko and anti-Sidorenko. Here is an example.

Example 3.2. $\vec{H}$ defined in Figure 3 is both Sidorenko and anti-Sidorenko.

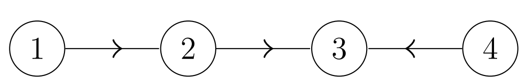


FIGURE 3. A graph $\vec{H}$ that is both Sidorenko and anti-Sidorenko

Proof. We will show that for any $W \in \mathcal{W}$, $t(\vec{H}, W) = \frac{1}{8}$, i.e.,

$$\int_{[0,1]^4} W(x_1, x_2)W(x_2, x_3)W(x_4, x_3)\, dx_1dx_2dx_3dx_4 = \frac{1}{8}. \tag{3.7}$$

Changing variable names gives

$$t(\vec{H}, W) = \int_{[0,1]^4} W(x_4, x_3)W(x_3, x_2)W(x_1, x_2)\, dx_4dx_3dx_2dx_1. \tag{3.8}$$

Therefore, using $W(x, y) + W(y, x) = 1$ almost everywhere,

$$\begin{aligned}
2t(\vec{H}, W) &= \int_{[0,1]^4} W(x_1, x_2)W(x_2, x_3)W(x_4, x_3)\, dx_1dx_2dx_3dx_4 \\
&\quad + \int_{[0,1]^4} W(x_4, x_3)W(x_3, x_2)W(x_1, x_2)\, dx_4dx_3dx_2dx_1 \\
&= \int_{[0,1]^4} W(x_1, x_2) \cdot 1 \cdot W(x_4, x_3)\, dx_1dx_2dx_3dx_4 \\
&= \left(\int_{[0,1]^2} W(x_1, x_2)\, dx_1dx_2\right)^2 = \frac{1}{4}.
\end{aligned} \tag{3.9}$$

This relation gives that $t(\vec{H}, W) = \frac{1}{8}$. Since this equality holds for all $W \in \mathcal{W}$, we conclude that $\vec{H}$ is both Sidorenko and anti-Sidorenko. $\square$

Remark. The class of directed graphs that are both Sidorenko and anti-Sidorenko has been completely classified ([6]).

Then, we prove several general properties of directed Sidorenko graphs. For the proposition below, we need the following definition.

Definition 3.3. Given a directed graph $\vec{H}$, a *topological sorting* of $\vec{H}$ is a one-to-one map $\pi : V(\vec{H}) \to \{1, \dots, |V(\vec{H})|\}$ such that if $(u, v) \in E(\vec{H})$, then $\pi(u) > \pi(v)$.

Proposition 3.4. *If $\vec{H}$ is Sidorenko, the number of topological sortings of $\vec{H}$ is at least $(|V(\vec{H})|)! \cdot 2^{-|E(\vec{H})|}$.*

Proof. Let $W_0 : [0, 1]^2 \to [0, 1]$ given by

$$W_0(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \le y \end{cases} \tag{3.10}$$

We know that $W_0(x, y) + W_0(y, x) = 1$ when $x \neq y$, so by Definition 2.2, $W_0 \in \mathcal{W}$. Then for a given Sidorenko graph $\vec{H}$,

$$t(\vec{H}, W_0) = \int_{[0,1]^{|V(\vec{H})|}} \prod_{(i,j)\in E(\vec{H})} W_0(x_i, x_j) \prod_{i\in V(\vec{H})} dx_i \ge 2^{-|E(\vec{H})|}. \tag{3.11}$$

We will show that the integrand $\prod_{(i,j)\in E(\vec{H})} W_0(x_i, x_j)$ is 1 if the ordering of $(x_i)_{i\in V(\vec{H})} \in [0, 1]^{|V(\vec{H})|}$ gives a topological sorting of $\vec{H}$, and 0 otherwise. In fact, we know that because each factor is either 0 or 1, their product is also either 0 or 1. It is 1 if and only if $W_0(x_i, x_j) =$

1 for all $(i,j) \in E(\vec{H})$, i.e., $x_i > x_j$ if $(i,j) \in E(\vec{H})$. Hence by definition of topological sorting, the integrand is 1 if and only if the ordering of $(x_i)_{i \in V(\vec{H})} \in [0,1]^{|V(\vec{H})|}$ gives a topological sorting of $\vec{H}$.

Then because we are integrating over $(x_i)_{i \in V(\vec{H})} \in [0,1]^{|V(\vec{H})|}$, notice that the probability that two indices take the same value is zero, and by symmetry each sorting appears with probability $1/n!$. Hence, if T is the number of topological sorting of $\vec{H}$, we know that

$$t(\vec{H}, W_0) = \frac{T}{n!}. \tag{3.12}$$

Together with equation (3.11), we conclude that

$$T \geq n! \cdot 2^{-|E(\vec{H})|} = (|V(\vec{H})|)! \cdot 2^{-|E(\vec{H})|}. \tag{3.13}$$

□

From the lemma above, because $(|V(\vec{H})|)! \cdot 2^{-|E(\vec{H})|} > 0$, we know $\vec{H}$ has at least one topological sorting. Thus we have the following corollary.

Corollary 3.5. *If $\vec{H}$ is Sidorenko, it does not contain a directed cycle.*

Remark. As in the undirected case, one can eliminate all graphs with an odd cycle, here we can eliminate all directed graphs with a directed cycle. However, while it has been conjectured that all graphs without an odd cycle (i.e., bipartite) is Sidorenko, not all directed graphs without a directed cycles are Sidorenko. Consider the counterexample in Figure 2c.

4. Operations for Building Sidorenko Graphs out of Others

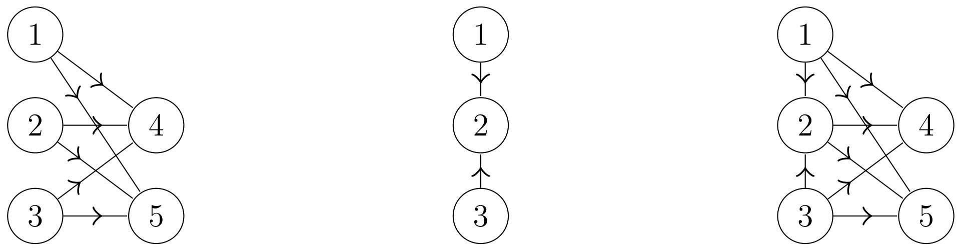
In this section, we will talk about a few operations that allows us to build Sidorenko Graphs out of other Sidorenko graphs. In particular, we study how we can blow-up a vertex of a Sidorenko graph into another Sidorenko graph so that the new graph is still Sidorenko.

Definition 4.1. A directed graph $\vec{H}$ is *blow-up Sidorenko* if for any $n = |V(\vec{H})|$ Sidorenko graphs $(\vec{H}_v)_{v \in V(\vec{H})}$, the new graph derived by blowing-up every vertex v into $\vec{H}_v$ is Sidorenko.

We start by a lemma which allows us to add edges to some Sidorenko graphs. Informally, this allows us to "insert" a Sidorenko graph into a neighborhood-equivalent set of a larger Sidorenko graph. The following lemma uses a similar idea to the argument in [7].

Lemma 4.2. *Given two Sidorenko graph $\vec{H}_1, \vec{H}_2$, let $S \subset V(\vec{H})$ be a neighborhood-equivalent set such that $|V(\vec{H}_2)| = |S|$. Given any one-to-one map $\pi : S \to V(\vec{H}_2)$, let the new graph $\vec{H}$ be the graph constructed by adding the edges between vertices in S inside a copy of $\vec{H}_1$ such that the induced subgraph on S isomorphic to $\vec{H}_2$ (see Figure 4 for an example). Then $\vec{H}$ is Sidorenko.*

Proof. Since S is neighborhood-equivalent, we know that any one-to-one map that fixes $V(\vec{H}_1) \setminus S$ and permutes S is an automorphism of $\vec{H}_1$, so it does not matter how the isomorphism $\vec{H}_1|_S \cong \vec{H}_2$ is defined. Now, let A be the in-neighborhood and B be the out-neighborhood of S. Then we can know that $\vec{H}_1$ can be written as two parts: the first one



(A) $\vec{H}_1$ (B) $\vec{H}_2$ (C) $\vec{H}$

FIGURE 4. Example construction using neighborhood-equivalent set, $S = \{1, 2, 3\}$.

$\vec{H}_1'$ is the induced subgraph on vertices other than S ($\vec{H}_1' = \vec{H}_1 - S$); the second one is a set of $|S|$ isolated vertices, and the edges between the two parts are exactly the pairs defined by $A \times S$ and $S \times B$. Moreover, all vertices of $\vec{H}_1$ can be partitioned into three disjoint parts: $S, A \cup B, C := V(\vec{H}_1) - S - A \cup B$. Then, $\pi \in V(\vec{G})^{V(\vec{H}_1)}$ can be written as $\pi = (\pi_1, \pi_2, \pi_3) \in V(\vec{G})^{A\cup B} \times V(\vec{G})^S \times V(\vec{G})^C$. Hence, $t(\vec{H}_1, \vec{G})$ for $G \in \mathcal{G}$ can be written as (using (2.5), and we fix the image of $A \cup B$ and sum over other indices and rearrange the order of summation in the following equation)

$$\begin{aligned} t(\vec{H}_1, \vec{G}) &= \sum_{\pi \in V(\vec{G})^{V(\vec{H}_1)}} \prod_{x \in V(\vec{H}_1)} \alpha_{\vec{G}}(\pi(x)) \prod_{(x,y) \in E(\vec{H}_1)} \rho_{\vec{G}}(\pi(x), \pi(y)) \\ &= \sum_{\pi_1 \in V(\vec{G})^{A \cup B}} F_1(\pi_1) G_1(\pi_1), \end{aligned} \tag{4.1}$$

where

$$F_1(\pi_1) = \sum_{\substack{\pi_2 \in V(\vec{G})^C \\ \pi = (\pi_1, \pi_2)}} \prod_{x \in V(\vec{H}_1')} \alpha_{\vec{G}}(x) \prod_{(x,y) \in E(\vec{H}_1')} \rho_{\vec{G}}(x, y), \tag{4.2}$$

and

$$\begin{aligned} G_1(\pi_1) &= \sum_{\pi_3 \in V(\vec{G})^S} \prod_{x \in S} \alpha_{\vec{G}}(\pi_3(x)) \prod_{a \in A} \rho_{\vec{G}}(\pi_1(a), \pi_3(x)) \prod_{b \in B} \rho_{\vec{G}}(\pi_3(x), \pi_1(b)) \\ &= \left(\sum_{i \in V(\vec{G})} \alpha_{\vec{G}}(i) \prod_{a \in A} \rho_{\vec{G}}(\pi_1(a), i) \prod_{b \in B} \rho_{\vec{G}}(i, \pi_1(b)) \right)^{|S|} \\ &= \left(s(\vec{G}, \pi_1(A), \pi_1(B)) \right)^{|S|}. \end{aligned} \tag{4.3}$$

The reason why we can write the summand a product of two functions on π_1 is that the other two parts C and S do not interact with each other. Similarly, we can write $t(\vec{H}, \vec{G})$

using a similar argument:

$$
\begin{aligned}
t(\vec{H},\vec{G}) &= \sum_{\pi\in V(\vec{G})^{V(\vec{H})}} \prod_{x\in V(\vec{H})} \alpha_{\vec{G}}(\pi(x)) \prod_{(x,y)\in E(\vec{H})} \rho_{\vec{G}}(\pi(x),\pi(y)) \\
&= \sum_{\pi_1\in V(\vec{G})^{A\cup B}} F(\pi_1)G(\pi_1),
\end{aligned} \tag{4.4}
$$

where

$$
F = F_1(\pi_1) = \sum_{\substack{\pi_2\in V(\vec{G})^C \\ \pi=(\pi_1,\pi_2)}} \prod_{x\in V(\vec{H}_1')} \alpha_{\vec{G}}(x) \prod_{(x,y)\in E(\vec{H}_1')} \rho_{\vec{G}}(x,y), \tag{4.5}
$$

and

$$
\begin{aligned}
G(\pi_1) = \sum_{\pi_3\in V(\vec{G})^S} &\left(\prod_{x\in S} \alpha_{\vec{G}}(\pi_3(x)) \prod_{a\in A} \rho_{\vec{G}}(\pi_1(a),\pi_3(x)) \right. \\
&\left. \cdot \prod_{b\in B} \rho_{\vec{G}}(\pi_3(x),\pi_1(b)) \right) \cdot \prod_{(x,y)\in \vec{H}_1|_S} \rho_{\vec{G}}(\pi_3(x),\pi_3(y)).
\end{aligned} \tag{4.6}
$$

Now, we can see that if we rewrite this in terms of $\vec{H}_2 \cong \vec{H}_1|_S$ and the neighborhood $(s,\vec{N})$ of $(\vec{G},\pi_1(A),\pi_1(B))$, we know that this function is actually

$$
\begin{aligned}
G(\pi_1) &= \sum_{\pi_3\in V(\vec{N})^{V(\vec{H}_2)}} \prod_{x\in V(\vec{H}_2)} s\alpha_{\vec{N}}(\pi_3(x)) \cdot \prod_{(x,y)\in E(\vec{H}_2)} \rho_{\vec{N}}(\pi_3(x),\pi_3(y)) \\
&= s^{|S|}\cdot t(\vec{H}_2,\vec{N}).
\end{aligned} \tag{4.7}
$$

Compare equations (4.3) and (4.7), because $\vec{H}_2$ is Sidorenko, and $\vec{N}$ is a weighted tournament,

$$
G(\pi_1) = s^{|S|}\cdot t(\vec{H}_2,\vec{N}) \geq s^{|S|}\cdot 2^{-|E(\vec{H}_2)|} = 2^{-|E(\vec{H}_2)|}G_1(\pi_1). \tag{4.8}
$$

Because this holds for all π_1, comparing (4.1) and (4.4) and using the fact that $\vec{H}_1$ is Sidorenko, we conclude that

$$
t(\vec{H},\vec{G}) \geq 2^{-|E(\vec{H}_2)|}t(\vec{H}_1,\vec{G}) \geq 2^{-|E(\vec{H}_2)|-|E(\vec{H}_1)|} = 2^{-|E(\vec{H})|}. \tag{4.9}
$$

In the last equation we used that $|E(\vec{H})| = |E(\vec{H}_2)| + |E(\vec{H}_1)|$, which follows directly from our construction of $\vec{H}$. □

Given the lemma above, we know that to show that a vertex in a graph can be blown-up into a Sidorenko graph, it suffices to show that it can be blown-up into a set of isolated vertices. (Then we can insert the Sidorenko graph into this set of isolated vertices.) Then we show that in particular, $\vec{K}_2$ is blow-up Sidorenko.

Lemma 4.3. *$\vec{K}_{a,b}$ is Sidorenko.*

Proof. Firstly, we show that $\vec{K}_{1,a}$ is Sidorenko. For any weighted graphon $\vec{G}$, for any given vertex $x \in V(\vec{G})$, we know that

$$\begin{aligned} t(\vec{K}_{1,a}, \vec{G}) &= \sum_{x,y_1,\dots,y_a \in V(\vec{G})} \alpha_{\vec{G}}(x) \prod_{i=1}^{a} \alpha_{\vec{G}}(y_i)\rho_{\vec{G}}(x, y_i) \\ &= \sum_{x \in V(\vec{G})} \alpha_{\vec{G}}(x) \prod_{i=1}^{a} \left(\sum_{y \in V(\vec{G})} \alpha_{\vec{G}}(y)\rho_{\vec{G}}(x, y) \right) \\ &= \sum_{x \in V(\vec{G})} \alpha_{\vec{G}}(x)(s(\vec{G}, \{x\}, \emptyset))^a. \end{aligned} \tag{4.10}$$

Now, we know that because $W(\vec{G})$ is a directed graphon, its integral on $[0, 1]^2$ is $\frac{1}{2}$. Thus,

$$\sum_{x \in V(\vec{G})} \alpha_{\vec{G}}(x)s(\vec{G}, \{x\}, \emptyset) = t(\vec{K}_{1,1}, \vec{G}) = \int W(\vec{G})(x, y)\, dxdy = \frac{1}{2}. \tag{4.11}$$

At the same time, we know that for normalized weight,

$$\sum_{x \in V(\vec{G})} \alpha_{\vec{G}}(x) = 1, \tag{4.12}$$

so we know that by Hölder's inequality,

$$\begin{aligned} &\left(\sum_{x \in V(\vec{G})} \alpha_{\vec{G}}(x)(s(\vec{G}, \{x\}, \emptyset))^a \right)^{\frac{1}{a}} \left(\sum_{x \in V(\vec{G})} \alpha_{\vec{G}}(x) \right)^{\frac{a-1}{a}} \\ &\qquad \geq \sum_{x \in V(\vec{G})} \alpha_{\vec{G}}(x)s(\vec{G}, \{x\}, \emptyset). \end{aligned} \tag{4.13}$$

Combining equations (4.11), (4.12), and inequality (4.13), we conclude that

$$t(\vec{K}_{1,a}, \vec{G}) = \sum_{x \in V(\vec{G})} \alpha_{\vec{G}}(x)(s(\vec{G}, \{x\}, \emptyset))^a \geq \left(\frac{1}{2}\right)^a. \tag{4.14}$$

This holds for all $G \in \mathcal{G}$, so $\vec{K}_{1,a}$ is indeed Sidorenko for $a \geq 1$. Then we prove that $\vec{K}_{a,b}$ is also Sidorenko. Using a similar argument as before, we know that for any $\vec{G} \in \mathcal{G}$,

$$\begin{aligned} t(\vec{K}_{a,b}, \vec{G}) &= \sum_{x_1,\dots,x_b,y_1,\dots,y_a \in V(\vec{G})} \left(\prod_{j=1}^{b} \alpha_{\vec{G}}(x_j) \right) \prod_{i=1}^{a} \left(\alpha_{\vec{G}}(y_i) \prod_{j=1}^{b} w(y_i, x_j) \right) \\ &= \sum_{x_1,\dots,x_b \in V(\vec{G})} \left(\prod_{j=1}^{b} \alpha_{\vec{G}}(x_j) \right) \left(\sum_{y \in V(\vec{G})} \alpha_{\vec{G}}(y) \prod_{j=1}^{b} w(y, x_j) \right)^a \\ &= \sum_{x_1,\dots,x_b \in V(\vec{G})} \left(\prod_{j=1}^{b} \alpha_{\vec{G}}(x_j) \right) \left(s(\vec{G}, \emptyset, \{x_1, \dots, x_n\}) \right)^a. \end{aligned} \tag{4.15}$$

Now, using the similar argument as before, we have Hölder's inequality:

$$t(\vec{K}_{a,b},\vec{G})^{\frac{1}{a}}t(\vec{K}_{0,b},\vec{G})^{\frac{a-1}{a}} \geq t(\vec{K}_{1,b},\vec{G}) \geq \left(\frac{1}{2}\right)^b, \tag{4.16}$$

where the last inequality from (4.14). Noticing that

$$t(\vec{K}_{0,b},\vec{G}) = t(\vec{I}_b,\vec{G}) = \sum_{x_1,\ldots,x_b\in V(\vec{G})}\prod_{j=1}^{b}\alpha_{\vec{G}}(x_i) = \left(\sum_{x\in V(\vec{G})}\alpha_{\vec{G}}(x)\right)^b = 1, \tag{4.17}$$

we hence conclude that

$$t(\vec{K}_{a,b},\vec{G}) \geq \left(\frac{1}{2}\right)^{ab} = 2^{-|E(\vec{K}_{a,b})|} \tag{4.18}$$

holds for any $a,b\in\mathbb{N}$ and $\vec{G}\in\mathcal{G}$, so $\vec{K}_{a,b}$ is Sidorenko. □

Then we can prove our main theorem which helps us build directed Sidorenko graphs out of others.

Theorem 4.4. *Given two directed Sidorenko graphs $\vec{H}_1,\vec{H}_2$, the two directed graphs $\vec{H}_1\cup\vec{H}_2$ and $\vec{H}_1\to\vec{H}_2$ are both Sidorenko graphs.*

Proof. For $\vec{H}_1\cup\vec{H}_2$, notice that for every $W\in\mathcal{W}$, the integral consists of two set of variables $(x_i)_{i\in V(\vec{H}_1)}$ and $(x_j)_{j\in V(\vec{H}_2)}$ and they do not interact with each other, so we have the relation using that $\vec{H}_1$ and $\vec{H}_2$ are directed Sidorenko and $|E(\vec{H}_1\cup\vec{H}_2)| = |E(\vec{H}_1)|+|E(\vec{H}_2)|$,

$$t(\vec{H}_1\cup\vec{H}_2,W) = t(\vec{H}_1,W)t(\vec{H}_2,W) \geq 2^{-|E(\vec{H}_1\cup\vec{H}_2)|}. \tag{4.19}$$

For $\vec{H}_1\to\vec{H}_2$, let $a=|V(\vec{H}_1)|$ and $b=|V(\vec{H}_2)$, then $|E(\vec{H}_1\to\vec{H}_2)| = |E(\vec{H}_1)|+|E(\vec{H}_2)|+ab$. Then we apply Lemma 4.2 with the tuple $(\vec{K}_{a,b}=\vec{I}_a\to\vec{I}_b,\vec{H}_1,V(\vec{I}_a))$. $\vec{K}_{a,b}$ is Sidorenko because of Lemma 4.3, and $\vec{H}_1$ is Sidorenko because of our assumption, and $V(\vec{I}_a)\subset V(\vec{I}_a\to\vec{I}_b)$ is indeed a neighborhood-equivalent set. Therefore, the derived graph, which is $\vec{H}_1\to\vec{I}_b$, is Sidorenko. Then we further apply Lemma 4.2 with the tuple $(\vec{H}_1\to\vec{I}_b,\vec{H}_2,V(\vec{I}_b))$, we can check that it also satisfies the conditions, so the derived graph $\vec{H}_1\to\vec{H}_2$ is also Sidorenko. □

By applying the operations $\vec{H}_1\to\vec{H}_2$ repetitively with $\vec{H}_2=\vec{I}_1$, we have the following corollary.

Corollary 4.5. *(Theorem 2.1 in* [7]*) Transitive tournaments are Sidorenko.*

Actually we have a stronger result.

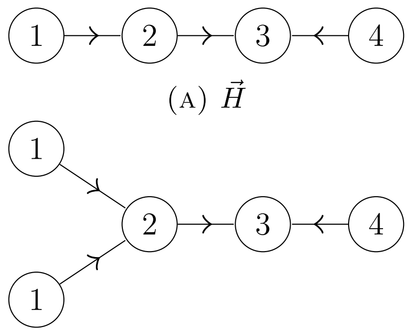
Corollary 4.6. *Let $\mathcal{B}$ be the set of directed graphs constructed using the operations from below:*

- *The directed graph of a single vertex $\vec{I}_1\in\mathcal{B}$.*
- *For two graphs $\vec{H}_1,\vec{H}_2\in\mathcal{B}$, the two other graphs $\vec{H}_1\cup\vec{H}_2$ and $\vec{H}_1\to\vec{H}_2$ are both in $\mathcal{B}$.*

Then all graphs in $\mathcal{B}$ are blow-up Sidorenko.

In particular, transitive tournaments are blow-up Sidorenko. However, we not all Sidorenko graphs are blow-up Sidorenko.

Example 4.7. In Figure 5, $\vec{H}'$ is from $\vec{H}$ by blowing up the vertex 1 into $\vec{I}_2$. We have shown in Example 3.2 that $\vec{H}$ is Sidorenko, and so is $\vec{I}_2$. However, $\vec{H}'$ is not Sidorenko, by consider the following graphon in Figure 6.



(A) $\vec{H}$

(B) $\vec{H}'$

FIGURE 5. Sidorenko graph that is not blow-up Sidorenko

1/2	1/2	3/4	3/4
1/2	1/2	3/4	3/4
1/4	1/4	1/2	3/4
1/4	1/4	1/4	1/2

FIGURE 6. Evidence for $\vec{H}'$ being not Sidorenko

5. Special Graphs

In this section, we will investigate some techniques and use them to solve determine whether specific directed graphs are Sidorenko.

5.1. **Stars.** In the case of directed graphs whose undirected structure is a star, we have the following theorem that fully characterize them.

Theorem 5.1. *Given a directed graph $\vec{H}$ which has undirected structure $H = K_{1,a}$, then $\vec{H}$ is Sidorenko if and only if $\vec{H} = \vec{K}_{1,a}$ or $\vec{H} = \vec{K}_{a,1}$.*

Proof. The one direction of the theorem follows from Lemma 4.3. But here we will present a different proof. For simplicity, let $O \in V(\vec{H})$ be the center of the star, and u, v the in-degree and out-degree of O with $u+v=a$. For any $W \in \mathcal{W}$, by definition,

$$\begin{aligned} t(\vec{H}, W) &= \int_{x, y_1, \ldots, y_u, z_1, \ldots, z_v \in [0,1]} \prod_{i=1}^{u} W(y_i, x) \prod_{j=1}^{v} W(x, z_j)\, dx dy^u dz^v \\ &= \int_{[0,1]} \left(\int_{[0,1]} W(y, x)\, dy \right)^u \left(\int_{[0,1]} W(x, z)\, dz \right)^v . \end{aligned} \tag{5.1}$$

For simplicity define

$$f(x) := \int_{[0,1]} W(y, x)\, dy. \tag{5.2}$$

Then we know that f is a function $[0,1] \mapsto [0,1]$,

$$\int_{[0,1]} f(x)\, dx = \int_{[0,1]^2} W(y, x)\, dy dx = 1/2, \tag{5.3}$$

and,

$$\int_{[0,1]} W(x, z)\, dz = \int_{[0,1]} (1 - W(z, x))\, dz = 1 - f(x). \tag{5.4}$$

Therefore, (5.1) can be rewritten as

$$t(\vec{H}, W) = \int_{[0,1]} f^u(x)(1 - f(x))^v\, dx. \tag{5.5}$$

When $v = 0$ and $a = u$, we know that

$$t(\vec{H}, W) = \int_{[0,1]} f^u(x)\, dx \geq \left(\int_{[0,1]} f(x)\, dx \right)^u = 2^{-u} = 2^{-a}; \tag{5.6}$$

when $u = 0$ and $a = v$, symmetrically

$$t(\vec{H}, W) = \int_{[0,1]} (1 - f(x))^u\, dx \geq \left(\int_{[0,1]} (1 - f(x))\, dx \right)^u = 2^{-a}. \tag{5.7}$$

Therefore $\vec{K}_{1,a}$ and $\vec{K}_{a,1}$ are Sidorenko holds. We then prove that all other orientation of stars are not Sidorenko. We give a counter example of W that make this inequality false when $u > 0$ and $v > 0$. When $u \geq v$, take W to be the following graphon W parametrized by α:

α	$1/2$	1
$1-\alpha$	0	$1/2$

FIGURE 7. Block graphon W with parameter $\alpha \in [0,1]$.

We can see that $W_\alpha + W_\alpha^T = 1$ a.e. so if $\vec{H}$ is Sidorenko,

$$t(\vec{H}, W_\alpha) \geq 2^{-a}. \tag{5.8}$$

By definition of $W = W_\alpha$, we know that

$$f(x) = \begin{cases} 1 - \alpha/2 & \text{if } 0 < x < \alpha \\ 1/2 - \alpha/2 & \text{if } \alpha < x < 1 \end{cases} \tag{5.9}$$

(We do not need the value of f at $0, 1, \alpha$.) Then by (5.5)

$$t(\vec{H}, W) = \alpha(1-\alpha/2)^u(\alpha/2)^v + (1-\alpha)(1/2-\alpha/2)^u(1/2+\alpha/2)^v. \tag{5.10}$$

Consider this as a function g of $\alpha \in [0,1]$, we know that $g(0) = \left(\frac{1}{2}\right)^{u+v}$. However, when $u \geq v > 0$,

$$g'(0) = -(u+1-v)\left(\frac{1}{2}\right)^{u+v} < 0. \tag{5.11}$$

Hence there exists $\epsilon > 0$ such that $g(\epsilon) < g(0) = \left(\frac{1}{2}\right)^{u+v}$. Therefore, the Sidorenko property does not hold for $\vec{H}$ when $u \geq v > 0$. If $v > u$, then consider W^T, by noticing that fact that if $\vec{H}$ and $\vec{H}'$ are two graphs with the same underlying undirected graph, but the directions are different on every edge, then

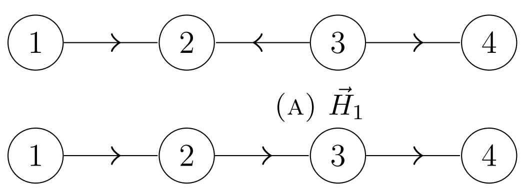
$$t(\vec{H}, W) = t(\vec{H}', W^T) \tag{5.12}$$

for all $W \in \mathcal{W}$, because the integrands are exactly the same.

In summary, $\vec{H}$ is not Sidorenko if both $u, v > 0$. Hence we derived both direction of the statement. $\square$

5.2. **Paths.** In this subsection, we will give a formula for representing the homomorphism density as a sum of subgraph homomorphism densities. We can then derive a necessary condition for a graph to be Sidorenko from the theorem. Before we get to our main result, let us start with an example.

Example 5.2. In Figure 8, $\vec{H}_1$ is Sidorenko, and $\vec{H}_2$ is anti-Sidorenko.



(A) $\vec{H}_1$

(B) $\vec{H}_2$

FIGURE 8. Two Examples of Short Paths

Proof. For any $W \in \mathcal{W}$, define $X = W - 1/2$. Then X is anti-symmetric almost everywhere because

$$X(x,y) = W(x,y) - 1/2 = (1 - W(y,x)) - 1/2 = -X(y,x). \tag{5.13}$$

Then consider

$$t(\vec{H}_1) = \int_{[0,1]^4} W(x,y)W(z,y)W(z,w)\,dxdydzdw. \tag{5.14}$$

For the integrand, we can write it as

$$\begin{aligned}
&W(x,y)W(z,y)W(z,w)\\
=&\left(\frac{1}{2}+X(x,y)\right)\left(\frac{1}{2}+X(z,y)\right)\left(\frac{1}{2}+X(z,w)\right)\\
=&1/8+1/4\big(X(x,y)+X(z,y)+X(z,w)\big)\\
&+1/2\big(X(x,y)X(z,y)+X(x,y)X(z,w)+X(z,y)X(z,w)\big)\\
&+X(x,y)X(z,y)X(z,w).
\end{aligned} \tag{5.15}$$

Then, if we put them back in the integral, we know that because X is anti-symmetric, $\int X=0$. Moreover, by renaming the variables in the integral,

$$\begin{aligned}
&\int_{[0,1]^4} X(x,y)X(z,y)X(z,w)\,dxdydzdw\\
=&\int_{[0,1]^4} X(w,z)X(y,z)X(y,x)\,dwdzdydx\\
=&(-1)^3\int_{[0,1]^4} X(x,y)X(z,y)X(z,w)\,dxdydzdw,
\end{aligned} \tag{5.16}$$

so

$$\int_{[0,1]^4} X(x,y)X(z,y)X(z,w)\,dxdydzdw=0. \tag{5.17}$$

At the same time,

$$\int_{[0,1]^4} X(x,y)X(z,w)\,dxdydzdw=\left(\int_{[0,1]^2} X(x,y)\,dxdy\right)=0. \tag{5.18}$$

Therefore most terms are zero by themselves. The terms left are

$$\begin{aligned}
&\int_{[0,1]^4} X(x,y)X(z,y)\,dxdydzdw\\
=&-\int_{[0,1]^3} X(x,y)X(y,z)\,dxdydz=-t(\vec{P}_2,X),
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
&\int_{[0,1]^4} X(z,y)X(z,w)\,dxdydzdw\\
=&-\int_{[0,1]^3} X(y,z)X(z,w)\,dydzdw=-t(\vec{P}_2,X).
\end{aligned} \tag{5.20}$$

Therefore, integrating (5.15) we derive that

$$t(\vec{H}_1)=1/8-t(\vec{P}_2,X). \tag{5.21}$$

By doing a similar argument to $\vec{H}_2$, we can derive that

$$t(\vec{H}_2)=1/8+t(\vec{P}_2,X). \tag{5.22}$$

Thus, whether or not $\vec{H}_1$ is Sidorenko depends on the sign of $t(\vec{P}_2,X)$ for any anti-symmetric $X:[0,1]\mapsto[-1/2,1/2]$. (In fact we do not need the restriction that X takes value in

$[-1/2, 1/2]$, because $t(\vec{P}_2, \epsilon X) = \epsilon^2 t(\vec{P}_2, X)$, and the coefficient ϵ^2 does not change the sign of $t(\vec{P}_2, X)$.)

Note that

$$\begin{aligned} t(\vec{P}_2, X) &= \int_{[0,1]^3} X(x,y)X(y,z)\,dxdydz \\ &= -\int_{[0,1]^3} X(x,y)X(z,y)\,dxdydz \\ &= -\int_{[0,1]} \left(\int_{[0,1]} X(x,y)\,dx\right)^2 dy \le 0. \end{aligned} \tag{5.23}$$

Therefore $t(\vec{P}_2, X)$ takes non-positive value, so we conclude that $\vec{H}_1$ is Sidorenko and $\vec{H}_2$ is anti-Sidorenko. □

Remark. In general we do not have

$$\int_{[0,1]} X(x,y)\,dx = 0 \tag{5.24}$$

for almost every value of y, $t(\vec{P}_2, X)$ can take negative values, so $\vec{H}_1$ is not anti-Sidorenko, and $\vec{H}_2$ is not Sidorenko.

Then we will start to show our main theorem. First let us give the following notations and definitions.

Notation. We define $\vec{P}_a$ to be a directed path of length a. In particular, $V(\vec{P}_a) = \{0, 1, \ldots, a\}$ and $E(\vec{P}_a) = \{(i, i+1) : 0 \le i < a\}$.

Definition 5.3. For a two directed graphs $\vec{H}_1, \vec{H}_2$ with the same undirected structure $H_1 \cong H_2$, the sign of a map that induces isomorphism on the undirected structure $\pi : V(\vec{H}_1) \to \vec{H}_2$ is defined to be

$$\operatorname{sgn}(\pi) = (-1)^{\#\{(u,v)\in E(\vec{H}_1):(\pi(v),\pi(u))\in E(\vec{H}_2)\}}. \tag{5.25}$$

The sign basically represents the parity of the number of edges that get reversed by the map π.

Definition 5.4. A *sign-reversing automorphism* of a directed graph $\vec{H}$ is a map $\pi : V(\vec{H}) \to V(\vec{H})$ that induces isomorphism on the undirected structure, i.e., $\pi : H \cong H$, satisfying that

$$\operatorname{sgn}(\pi) = -1. \tag{5.26}$$

Here we can consider π is a map between two copies of $\vec{H}$, and the sign is defined using the definition from above.

Remark. Note that whether or not π is a sign-reversing automorphism does not depend on the directed structure of $\vec{H}$. In other words, if π is a sign-reversing automorphism for some orientation of $\vec{H}$, it is sign-reversing for all orientations of $\vec{H}$.

We then prove a proposition on the condition of existence sign-reversing automorphism for $\vec{P}_a$.

Proposition 5.5. *$\vec{P}_a$ has a sign-reversing automorphism if and only if a is odd.*

Proof. Consider the possible candidate for sign-reversing automorphism $\pi : P_a \cong P_a$. Because of the structure of P_a, there are only two possible choice of π: either the identity map $v \mapsto v$, or the reflection map $v \mapsto a - v$. For the identity map, all directions of edges are not changed, so the sign is $(-1)^0 = 1$. For the reflection map, all directions get changed, so the sign is $(-1)^a$. Hence, there exists a sign-reversing automorphism if and only if a is odd. □

Then we give the following proposition on the condition of existence of sign-reversing automorphisms for disjoint union of paths. For simplicity, given a multiset A whose elements are nonnegative integers, $\vec{P}_A = \cup_{a\in A}\vec{P}_a$. Let $\mathcal{P}$ be the family of disjoint union of directed paths, and $\mathcal{P}_e$ be the family of disjoint union of directed paths of even lengths. Let $\mathcal{P}_e^k$ be the set of graphs in $\mathcal{P}_e$ that contains exactly k vertices.

Proposition 5.6. *$\vec{P}_A$ has a sign-reversing automorphism if and only if $\vec{P}_A \notin \mathcal{P}_e$.*

Proof. If $\vec{P}_A \notin \mathcal{P}_e$, we know that there is (at least) one path of odd length in the disjoint union. Therefore consider the automorphism on the underlying undirected structure that reverse this path of odd length, and keep all other vertices fixed. We know that only the edges on the path of odd length get reversed, which is an odd number of edges, so this automorphism is sign-reversing.

If $\vec{P}_A \in \mathcal{P}_e$, consider the map that induces an isomorphism on the underlying undirected structure. Because it induces and isomorphism, it must map each connected component into an isomorphic connected component. Thus each path of even length is mapped isomorphically to a path of the same length. No matter how we define this map, it always reverses even number of edges on this path. Since this holds for all paths in the disjoint union, any map reverses even number of edges, so the sign of the automorphism is always 1. Therefore there is no sign-reversing automorphism for graphs in $\mathcal{P}_e$. □

From now on, we will use $X : [0,1]^2 \to \mathbb{R}$ as a skew-symmetric function (i.e., $X(x,y) = -X(y,x)$).

Proposition 5.7. *$X : [0,1]^2 \to \mathbb{R}$ is a skew-symmetric function. For two directed graphs $\vec{H}_1, \vec{H}_2$ with an isomorphism between the undirected structures $\pi : H_1 \cong H_2$, we have the following relation:*

$$t(\vec{H}_1, X) = \operatorname{sgn}(\pi) t(\vec{H}_2, X). \tag{5.27}$$

Proof. By definition of the homomorphism density,

$$\begin{aligned}
t(\vec{H}_1, X) &= \int_{[0,1]^{|V(\vec{H}_1)|}} \prod_{(u,v)\in E(\vec{H}_1)} X(x_u, x_v)\, dx^{V\vec{H}_1} \\
&= \int_{[0,1]^{|V(\vec{H}_2)|}} \prod_{(u,v)\in E(\vec{H}_1)} X(x_{\pi(u)}, x_{\pi(v)})\, dx^{V(\vec{H}_2)} \\
&= \int_{[0,1]^{|V(\vec{H}_2)|}} \operatorname{sgn}(\pi) \prod_{(u',v')\in E(\vec{H}_2)} X(x_{u'}, x_{v'})\, dx^{V(\vec{H}_2)} \\
&= \operatorname{sgn}(\pi) t(\vec{H}_2, X).
\end{aligned} \tag{5.28}$$

In the equation above, for the second equality we just rename the variables, and the third equality is because of the definition of the sign, and every edge that gets reversed contributes to a (-1) factor to the integrand. □

Hence we have the following corollary.

Corollary 5.8. *If a directed graph $\vec{H}$ has a sign-reversing automorphism, then for any skew-symmetric X,*

$$t(\vec{H}, X) = 0. \tag{5.29}$$

Proof. Given a sign-reversing automorphism π, we know that

$$t(\vec{H}, X) = \operatorname{sgn}(\pi) t(\vec{H}, X) = -t(\vec{H}, X) \implies t(\vec{H}, X) = 0. \tag{5.30}$$

□

Then we can define the sign of the difference between two graphs.

Definition 5.9. Given two directed graphs $\vec{H}_1$ and $\vec{H}_2$ such that the undirected structures are isomorphic (i.e., there is some $\pi : H_1 \cong H_2$) and that $\vec{H}_1$ does not have a sign-reversing automorphism. Then we define

$$\operatorname{sgn}(\vec{H}_1, \vec{H}_2) = \operatorname{sgn}(\pi). \tag{5.31}$$

Remark. $\operatorname{sgn}(\vec{H}_1, \vec{H}_2)$ is well-defined. Assume that we have two different isomorphisms $\pi, \pi' : H_1 \cong H_2$, then we know that we can consider $(\pi')^{-1} \circ \pi : H_1 \cong H_1$. Moreover, we know that because sgn is basically the parity of the number of edges that gets reversed,

$$\operatorname{sgn}((\pi')^{-1} \circ \pi) = \operatorname{sgn}(\pi) \cdot \operatorname{sgn}(\pi'). \tag{5.32}$$

Since $\vec{H}_1$ does not have a sign-reversing automorphism, $\operatorname{sgn}(\pi) \cdot \operatorname{sgn}(\pi') = 1$, so $\operatorname{sgn}(\pi) = \operatorname{sgn}(\pi')$. This holds for any isomorphisms $H_1 \cong H_2$, so we conclude that this sign is well-defined.

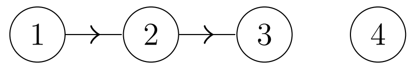
Definition 5.10. Given a directed graph $\vec{H}$, and a subset $E \subset E(\vec{H})$, then we define *edge-subgraph* $\vec{H}|_E$ to be the graph that contains all vertices of $\vec{H}$ but only the edges in E. Let $H|_E$ be the underlying undirected structure of $\vec{H}|_E$.

Definition 5.11. Given a directed graph $\vec{H}$ without sign-reversing automorphisms, and another directed graph $\vec{G}$ on the same number of vertices, let the *subgraph-count*

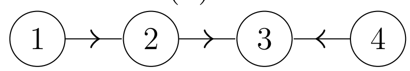
$$c(\vec{H}, \vec{G}) := \sum_{\substack{E \subset E(\vec{G}) \\ G|_E \cong H}} \operatorname{sgn}(\vec{H}, \vec{G}|_E). \tag{5.33}$$

As an example, consider the following two graphs in Figure 9. We first see that $\vec{H} = \vec{P}_{\{0,2\}} \in \mathcal{P}_e$, so $\vec{H}$ does not have a sign-reversing automorphism. Then consider the choice of $E \subset E(\vec{G})$. There are only two ways to satisfy $G|_E \cong H$: $E = \{(1, 2), (2, 3)\}$ or $E = \{(2, 3), (4, 3)\}$. Then one gives $+1$ and the other gives -1, so

$$c(\vec{H}, \vec{G}) = 0. \tag{5.34}$$



(A) $\vec{H}$



(B) $\vec{G}$

FIGURE 9. Example for Subgraph Count

For any $W \in \mathcal{W}$, $X = W - \frac{1}{2}$ is a skew-symmetric function because

$$X(x,y) = W(x,y) - \frac{1}{2} = 1 - W(y,x) - \frac{1}{2} = -X(y,x). \tag{5.35}$$

With these preparations, we can now state our main theorem.

Theorem 5.12. *For a directed graph $\vec{H}$ whose undirected structure is a path $H \cong P_k$, we have*

$$t(\vec{H}, W) = 2^{-k} \sum_{\vec{P} \in \mathcal{P}_e^{k+1}} 2^{|E(\vec{P})|} c(\vec{P}, \vec{H}) t\left(\vec{P}, W - \frac{1}{2}\right). \tag{5.36}$$

Proof. For simplicity let $X = W - \frac{1}{2}$. Then we know that

$$t(\vec{H}, W) = \int_{[0,1]^{k+1}} \prod_{(i,j) \in E(\vec{H})} \left(\frac{1}{2} + X(x_i, x_j)\right) dx^{V(\vec{H})}. \tag{5.37}$$

If we expand the integrand, we actually know that

$$\prod_{(i,j) \in E(\vec{H})} \left(\frac{1}{2} + X(x_i, x_j)\right) = \sum_{E \subset E(\vec{H})} 2^{-|E(\vec{H})| + |E|} \prod_{(i,j) \in E} X(x_i, x_j) \tag{5.38}$$

Therefore if we put this back to (5.37), we get that

$$\begin{aligned} t(\vec{H}, W) &= \sum_{E \subset E(\vec{H})} 2^{-|E(\vec{H})| + |E|} \int_{[0,1]^{k+1}} \prod_{(i,j) \in E} X(x_i, x_j)\, dx^{V(\vec{H})} \\ &= \sum_{E \subset E(\vec{H})} 2^{-|E(\vec{H})| + |E|} t(\vec{H}|_E, X). \end{aligned} \tag{5.39}$$

Then, we know that because H is a path, $H|_E$ must be a disjoint union of paths, so $H|_E \cong P_A$ for some unique multiset A. For each $E \subset E(\vec{H})$, let A_E be the corresponding multiset, then by Proposition 5.7 the summation above becomes

$$t(\vec{H}, W) = \sum_{E \subset E(\vec{H})} 2^{-|E(\vec{H})| + |E|} \operatorname{sgn}(\vec{P}_{A_E}, \vec{H}|_E) t(\vec{P}_{A_E}, X). \tag{5.40}$$

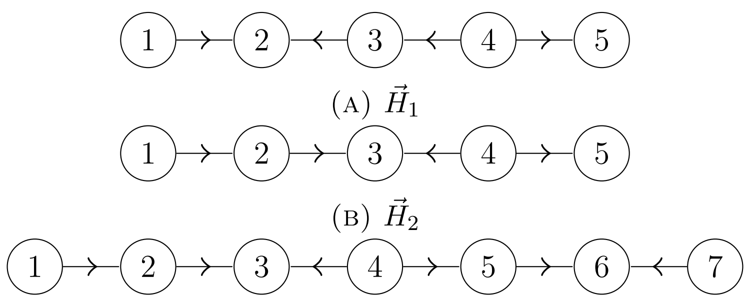
Now, when $\vec{P}_{A_E}$ has sign-reversing automorphism, $t(\vec{P}_{A_E}, X) = 0$ by Corollary 5.8, so we only need to take into account the terms $\vec{P}_{A_E} \in \mathcal{P}_e$. Instead of summing over E as above, we instead sum over $\vec{P}_A$ for $\vec{P}_A \in \mathcal{P}_e^{k+1}$ ($\vec{H}|_E$ always have $k+1$ vertices), we know that

$$\begin{aligned}
&\sum_{E \subset E(\vec{H})} 2^{-|E(\vec{H})|+|E|} \operatorname{sgn}(\vec{P}_{A_E}, \vec{H}|_E) t(\vec{P}_{A_E}, X) \\
=&\sum_{\vec{P}_A \in \mathcal{P}_e^{k+1}} \sum_{\substack{E \subset E(\vec{H}) \\ H|_E \cong P_A}} 2^{-|E(\vec{H})|+|E(\vec{P}_A)|} \operatorname{sgn}(\vec{P}_A, \vec{H}|_E) t(\vec{P}_A, X) \\
=&\, 2^{-k} \sum_{\vec{P}_A \in \mathcal{P}_e^{k+1}} 2^{|E(\vec{P}_A)|} c(\vec{P}_A, \vec{H}) t(\vec{P}_A, X).
\end{aligned} \tag{5.41}$$

This finishes our proof of the theorem. $\square$

Now, we give a few examples where we apply the theorem above to show that they are Sidorenko.

Example 5.13. The following graphs are Sidorenko.



(A) $\vec{H}_1$

(B) $\vec{H}_2$

(C) $\vec{H}_3$

FIGURE 10. Example of Sidorenko Paths

Proof. In $\mathcal{P}_e^5$, there are only 3 elements: $\vec{P}_{\{0,0,0,0,0\}} = \vec{I}_5, \vec{P}_{\{0,0,2\}}, \vec{P}_{\{4\}} = \vec{P}_4$. Then when we apply theorem to these graphs, for any $W \in \mathcal{W}$ and $X = W - \frac{1}{2}$,

$$t(\vec{H}_1, W) = \frac{1}{16}\left(1 - 4t(\vec{P}_{\{0,0,2\}}, X) + 16t(\vec{P}_4, X)\right). \tag{5.42}$$

Notice that $t(\vec{P}_{\{0,0,2\}}, X) = t(\vec{P}_0, X)^2 t(\vec{P}_2, X) = t(\vec{P}_2, X)$, to show that $t(\vec{H}_1, W) \geq 1/16$, we only need to prove that

$$-t(\vec{P}_2, X) + 4t(\vec{P}_4, X) \geq 0. \tag{5.43}$$

In fact, we will show a stronger result here where $t(\vec{P}_2, X) \leq 0$ and $t(\vec{P}_4, X) \geq 0$. Actually, for any X skew-symmetric,

$$\begin{aligned}(-1)^k t(\vec{P}_{2k}, X) &= (-1)^k \int_{[0,1]^{2k+1}} \prod_{i=0}^{2k-1} X(x_i, x_{i+1})\, dx^{2k+1} \\ &= \int_{[0,1]^{2k+1}} \prod_{i=0}^{k-1} X(x_i, x_{i+1}) \prod_{i=k}^{2k-1} X(x_{i+1}, x_i)\, dx^{2k+1} \\ &= \int_{[0,1]} \left(\int_{[0,1]^k} \prod_{i=0}^{k-1} X(x_i, x_{i+1})\, dx^k \right)^2 dx_k \geq 0.\end{aligned} \tag{5.44}$$

Therefore $\vec{H}_1$ is Sidorenko because $t(\vec{H}_1, W) \geq 1/16$ for all $W \in \mathcal{W}$.

Then we similarly expand $\vec{H}_2$ and derive that

$$\frac{1}{16}\left(1 - 4t(\vec{P}_{\{0,0,2\}}, X) - 16t(\vec{P}_4, X)\right). \tag{5.45}$$

Hence we need to show that

$$-\, t(\vec{P}_2, X) \geq 4t(\vec{P}_4, X). \tag{5.46}$$

Now we prove a lemma.

Lemma 5.14. *Given a skew-symmetric function $X : [0,1]^2 \to [-1,1]$ and a measurable function $f : [0,1] \to \mathbb{R}$, we define the operator*

$$X \cdot f(x) = \int_0^1 X(y, x) f(y)\, dy. \tag{5.47}$$

Then we have that

$$\|X \cdot f\|_{L^2} \leq \frac{1}{\sqrt{2}} \|f\|_{L^2}. \tag{5.48}$$

Proof of Lemma 5.14. For any two functions $f, g : [0,1] \to \mathbb{R}$, we prove that

$$\frac{\langle g, X \cdot f\rangle}{\|f\|_{L^2}\|g\|_{L^2}} \leq \frac{1}{\sqrt{2}}, \tag{5.49}$$

where $\langle \cdot \rangle$ is the inner product defined on $L^2([0,1])$, given by $\langle f, g\rangle = \int fg$.

In fact, because X is skew-symmetric, we know that

$$\begin{aligned}\langle g, X \cdot f\rangle &= \int_{[0,1]^2} f(x) X(x, y) g(y)\, dxdy \\ &= -\int_{[0,1]^2} f(x) X(y, x) g(y)\, dxdy.\end{aligned} \tag{5.50}$$

Thus, also noticing that the value of X lies in $[-1,1]$,

$$
\begin{aligned}
2\langle g, X\cdot f\rangle &= \int_{[0,1]^2} f(x)X(x,y)g(y)\,dxdy \\
&\quad - \int_{[0,1]^2} f(x)X(y,x)g(y)\,dxdy \\
&= \int_{[0,1]^2} (f(x)g(y)-f(y)g(x))X(x,y)\,dxdy \\
&\le \int_{[0,1]^2} |f(x)g(y)-f(y)g(x)|\,dxdy \\
&\le \left(\int_{[0,1]^2} (f(x)g(y)-f(y)g(x))^2\,dxdy\right)^{1/2}.
\end{aligned} \tag{5.51}
$$

Now, we know that

$$
\begin{aligned}
&\int_{[0,1]^2} (f(x)g(y)-f(y)g(x))^2\,dxdy \\
=&\int_{[0,1]^2} \left(f(x)^2g(y)^2+f(y)^2g(x)^2-2f(x)f(y)g(x)g(y)\right)\,dxdy \\
=&\,2\|f\|_{L^2}^2\|g\|_{L^2}^2 - 2(\langle f,g\rangle)^2 \le 2\|f\|_{L^2}^2\|g\|_{L^2}^2.
\end{aligned} \tag{5.52}
$$

Combining (5.51) and (5.52), we know that (5.49) holds. Taking $g = X\cdot f$ we have the original inequality. □

Now we can prove (5.46). Taking $(X,f) = (2X, 2X\cdot 1)$ where 1 is the constant function, we know that

$$2\|2X\cdot(2X\cdot 1)\|_{L^2}^2 \le \|2X\cdot 1\|_{L^2}^2. \tag{5.53}$$

However, in fact, we know that

$$
\begin{aligned}
\|2X\cdot 1\|_{L^2}^2 &= \int_{[0,1]}\left(\int_{[0,1]} 2X(x,y)\,dx\right)^2 dy \\
&= \int_{[0,1]^3} 2X(x,y)2X(z,y)\,dxdydz \\
&= -4\int_{[0,1]^3} X(x,y)X(y,z)\,dxdydz = -4t(\vec{P}_2, X);
\end{aligned} \tag{5.54}
$$

similarly we have another equation

$$
\begin{aligned}
\|2X\cdot(2X\cdot 1)\|_{L^2}^2 &= \int_{[0,1]}\left(\int_{[0,1]} 2X(y,z)\left(\int_{[0,1]} 2X(x,y)\,dx\right)dy\right)^2 dz \\
&= 16t(\vec{P}_4, X);
\end{aligned} \tag{5.55}
$$

Thus,

$$-4t(\vec{P}_2, X) \ge 32t(\vec{P}_4, X), \tag{5.56}$$

and thus we have (5.46), so $\vec{H}_2$ is Sidorenko.

To show that $\vec{H}_3$ is Sidorenko, with some calculation of the subgraph counts and simplifications (to get rid of isolated vertices) we would have the relation that

$$t(\vec{H}_3, W) = \frac{1}{64}\left(1 - 4t(\vec{P}_2, X) - 16t(\vec{P}_4, X) + 64t(\vec{P}_6, X) + 16t(\vec{P}_2, X)^2\right). \tag{5.57}$$

Using the relation (5.56) above, we know that

$$-4t(\vec{P}_2, X) - 16t(\vec{P}_4, X) \geq 16t(\vec{P}_4, X). \tag{5.58}$$

Then, we again apply Lemma 5.49 on $(2X, 2X \cdot (2X \cdot 1))$, and using the similar argument as above, we would see that

$$\|2X \cdot (2X \cdot (2X \cdot 1))\|_{L^2}^2 = -64t(\vec{P}^6, X), \tag{5.59}$$

so from the result of the Lemma and (5.55), we conclude that

$$16t(\vec{P}_4, X) \geq -128t(\vec{P}^6, X) \geq -64t(\vec{P}^6, X). \tag{5.60}$$

Combining (5.58), (5.60), and using the fat that $t(\vec{P}_2, X)^2 \geq 0$, we conclude that

$$t(\vec{H}_3, W) \geq 1/64. \tag{5.61}$$

This finishes our proof that $\vec{H}_3$ is Sidorenko. □

We finish this subsection with a general theorem that gives a necessary condition for graphs (not just trees) to be Sidorenko.

Theorem 5.15. *If $\vec{H}$ is Sidorenko, then*

$$c\left(\vec{P}_2 \cup \vec{I}_{|V(\vec{H}) - 3|}, \vec{H}\right) \leq 0. \tag{5.62}$$

Proof. We say an undirected graph has a sign-reversing automorphism if any orientation of the graph has a sign-reversing automorphism. This is actually equivalent to that all orientations of the graph have sign-reversing automorphisms (refer to Remark after Definition 5.4).

Let $\mathcal{A}$ be the set of undirected graphs without sign-reversing automorphisms (we regard isomorphic graphs as the same element in $\mathcal{A}$). For each element $F \in \mathcal{A}$, we take an arbitrary orientation $\vec{F}$ of the graph. In the case where $F = P_2 \cup I_a$, orient it so that $\vec{F} = \vec{P}_2 \cup \vec{I}_a$. Let $\vec{\mathcal{A}}$ be the set of these arbitrarily oriented graphs. In other words, for any undirected graph G without sign-reversing automorphisms, there is a unique element $\vec{F} \in \vec{\mathcal{A}}$ such that $G \cong F$. Denote $\vec{\mathcal{A}}^m$ be the subset of $\vec{\mathcal{A}}$ which consists of graphs on m vertices. Consequently $\vec{\mathcal{A}}^m$ is a finite set.

As in Theorem 5.12, from (5.39), we have that for any $W \in \mathcal{W}$ and $X = W - 1/2$,

$$t(\vec{H}, W) = \sum_{E \subset E(\vec{H})} 2^{-|E(\vec{H})| + |E|} t(\vec{H}|_E, X). \tag{5.63}$$

Using Corollary 5.8, only terms for which $\vec{H}|_E$ does not have sign-reversing automorphisms remain. Note that $\vec{H}|_E$ has a sign-reversing automorphism if and only if $H|_E$ has a sign-reversing automorphism, there is a unique element $\vec{F}_E \in \mathcal{A}$ such that $H|_E \cong F_E$. Then we know that

$$t(\vec{H}|_E, X) = \operatorname{sgn}(\vec{F}_E, \vec{H}|_E) t(\vec{F}_E, X). \tag{5.64}$$

Therefore, as we do in (5.41), we can rearrange the summation and get

$$\begin{aligned} t(\vec{H}, \frac{1}{2} + X) &= \sum_{E \subset E(\vec{H})} 2^{-|E(\vec{H})|+|E|} \operatorname{sgn}(\vec{F}_E, \vec{H}|_E) t(\vec{F}_E, X) \\ &= \sum_{\vec{F} \in \vec{\mathcal{A}}^{|V(\vec{H})|}} \sum_{\substack{E \subset E(\vec{H}) \\ H|_E \cong F}} 2^{-|E(\vec{H})|+|E(\vec{F})|} \operatorname{sgn}(\vec{F}, \vec{H}|_E) t(\vec{F}, X) \\ &= \sum_{\vec{F} \in \vec{\mathcal{A}}^{|V(\vec{H})|}} 2^{-|E(\vec{H})|+|E(\vec{F})|} c(\vec{F}, \vec{H}) t(\vec{F}, X). \end{aligned} \tag{5.65}$$

Now, take an arbitrary $X : [0,1]^2 \to [-1/2, 1/2]$ such that $t(\vec{P}_2, X) < 0$. (E.g. we can take $X(x,y) = 1/2$ when $x > y$ and $X(x,y) = -1/2$ when $x < y$.) For $\epsilon \in [-1,1]$ let ϵX be pointwise multiplication, then we know that

$$t(\vec{F}, \epsilon X) = \epsilon^{|E(\vec{F})|} t(\vec{F}, X). \tag{5.66}$$

This is because the integrands differ by a factor of $\epsilon^{|E(\vec{F})|}$ everywhere. Then let us consider the function $f : [-1,1] \mapsto \mathbb{R}$

$$f(\epsilon) = t(\vec{H}, 1/2 + \epsilon X) - 2^{-|E(\vec{H})|}. \tag{5.67}$$

From equation 5.65, we know that $f(\epsilon)$ is a polynomial in ϵ. Moreover, when $\epsilon = 0$, we know that the integrand of $t(\vec{H}, 1/2)$ is $2^{-|E(\vec{H})|}$ everywhere, so $f(0) = 0$.

Then let us consider the coefficient of first degree term ϵ. From (5.66), this corresponds to elements $\vec{F} \in \vec{\mathcal{A}}^{|V(\vec{H})|}$ that has exactly one edge. However, directed graphs with one edge must be in the form of $\vec{P}_1 \cup \vec{I}_a$, but $\vec{P}_1$ has a sign-reversing automorphism, so there are no element in $\vec{\mathcal{A}}^{|V(\vec{H})|}$ with only one edge. Thus, the coefficient of first degree term ϵ is zero.

Then let us consider the second degree term. Note that for a undirected graph with two edges, either the two edges share a same vertex, which takes the form $P_2 \cup I_a$, and we have specified the orientation of this to be $\vec{P}_2 \cup \vec{I}_a \in \vec{\mathcal{A}}$; or the two edges do not share any vertex, then it must be $P_1 \cup P_1 \cup I_a$, it always have a sign-reversing automorphism, so any orientation of this graph cannot be in $\mathcal{A}$. Hence we know that the coefficient of the second degree term is

$$2^{-|E(\vec{H})|+2} c(\vec{P}_2 \cup \vec{I}_{|V(\vec{H})|-3}, \vec{H}) t(\vec{F}, X) \epsilon^2. \tag{5.68}$$

Thus, we can write that as $\epsilon \to 0$,

$$f(\epsilon) = 2^{-|E(\vec{H})|+2} c(\vec{P}_2 \cup \vec{I}_{|V(\vec{H})|-3}, \vec{H}) t(\vec{F}, X) \epsilon^2 + O(\epsilon^3) \tag{5.69}$$

Now, if $\vec{H}$ is Sidorenko, we know that $f(\epsilon) \geq 0$ for all $\epsilon \in [-1,1]$. In particular, it is non-negative when ϵ close to zero. Thus a necessary condition for this to be true is that

$$2^{-|E(\vec{H})|+2} c(\vec{P}_2 \cup \vec{I}_{|V(\vec{H})|-3}, \vec{H}) t(\vec{F}, X) \geq 0. \tag{5.70}$$

Thus, if $\vec{H}$ is Sidorenko, then

$$t(\vec{P}_2 \cup \vec{I}_{|V(\vec{H})|-3}, \vec{H}) \leq 0. \tag{5.71}$$

□

5.3. **Transitive Tournaments.** In Corollary 4.5, we have already shown that all transitive tournaments are Sidorenko [7].

Proposition 5.16. *Given a directed graph $\vec{H}$ whose underlying undirected graph H is a complete graph K_n, then $\vec{H}$ is Sidorenko if and only if $\vec{H} = \mathrm{Tr}_n$.*

Proof. Firstly, we know that if $\vec{H} = \mathrm{Tr}_n$, then $\vec{H}$ is Sidorenko. On the other hand, we know that if $\vec{H}$ is Sidorenko, then $\vec{H}$ cannot contain a directed cycle by Corollary 3.5. Thus, the directed edges give a transitive relations between the vertices, so $\vec{H}$ must be the transitive tournament. □

Now, let us consider the graphs that are "close" to transitive tournaments. For simplicity, let the vertices $V(\mathrm{Tr}_n)$ be $[n] = \{1, 2, \ldots, n\}$ and all edges are directed from i to j if $i < j$. We define the length of an edge $e = (i, j) \in E(\mathrm{Tr}_n)$ to be $l(e) = j - i$. For $e \in E(\mathrm{Tr}_n)$, let $\mathrm{Tr}_n - e$ be the graph where we delete e from Tr_n and keep all other edges.

Theorem 5.17. *For $e \in E(\mathrm{Tr}_n)$, $\mathrm{Tr}_n - e$ is Sidorenko if $l(e) \neq 2$.*

Remark. $\mathrm{Tr}_n - e$ can fail to be Sidorenko when $l(e) = 2$: consider the example of $\mathrm{Tr}_3 - (1, 3)$, which corresponds to Figure 2c.

Proof. We start by proving the following lemma.

Lemma 5.18. *For any $W \in \mathcal{W}$,*

$$\int_{[0,1]^4} W(x,y)W(y,z)W(x,w)W(w,z)\, dxdydzdw \geq 1/16. \tag{5.72}$$

Proof. Let $X = W - 1/2$, then we know that $X(x, y) = -X(y, x)$. Therefore, we know that

$$\int_{[0,1]^2} X(x,y)\, dxdy = \int_{[0,1]^2} X(y,x)\, dydx = -\int_{[0,1]^2} X(x,y)\, dxdy, \tag{5.73}$$

so

$$\int_{[0,1]^2} X(x,y)\, dxdy = 0. \tag{5.74}$$

Similarly,

$$\begin{aligned} &\int_{[0,1]^4} X(x,y)X(y,z)X(z,w)\, dxdydzdw \\ = &\int_{[0,1]^4} X(w,z)X(z,y)X(y,x)\, dwdzdydx \\ = -&\int_{[0,1]^4} X(x,y)X(y,z)X(z,w)\, dxdydzdw \\ \implies &\int_{[0,1]^4} X(x,y)X(y,z)X(z,w)\, dxdydzdw = 0. \end{aligned} \tag{5.75}$$

In (5.72), we replace W with $1/2+X$, we after canceling the terms, we can derive that

$$\begin{aligned}&\int_{[0,1]^4} W(x,y)W(y,z)W(x,w)W(w,z)\,dxdydzdw\\ =&\frac{1}{16}+\int_{[0,1]^4} X(x,y)X(y,z)X(x,w)X(w,z)\,dxdydzdw\\ =&\frac{1}{16}+\int_{[0,1]^2}\left(\int_{[0,1]} X(x,y)X(y,z)\,dy\right)^2 dxdz\\ \geq&\frac{1}{16}.\end{aligned} \tag{5.76}$$

□

Given this lemma, we know that

$$\int_{[0,1]^2}\left(\int_{[0,1]} W(x,y)W(y,z)\,dy\right)^2 dxdz \geq 1/16. \tag{5.77}$$

Then using Hölder's inequality, let $f(x,z) := \int_{[0,1]} W(x,y)W(y,z)\,dy$, since f only takes nonnegative value, for $k\geq 2$

$$\left(\int_{[0,1]^2} f^k\right)^{\frac{2}{k}}\left(\int_{[0,1]^2} 1\right)^{\frac{k-2}{k}} \geq \int_{[0,1]^2} f^2 \geq \frac{1}{16}, \tag{5.78}$$

so

$$\int_{[0,1]^2} f^k \geq 2^{-2k}. \tag{5.79}$$

Actually if we write this back in terms of integral, we know that the inequality above (which holds for all $W\in\mathcal{W}$) means that the following graph is Sidorenko. Now, notice that the

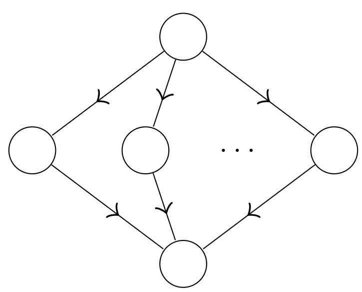


FIGURE 11. A Sidorenko graph arises from inequality, with k vertices in the middle

middle level form a neighborhood-equivalent set, we can replace them with a transitive tournament on k vertices while the new graph is still Sidorenko using Lemma 4.2. Then the new graph becomes $\mathrm{Tr}_{k+2}-(1,k+2)$, which is Sidorenko. Now, we know that $\mathrm{Tr}_{k+2}-(1,k+2)$ is Sidorenko for $k\geq 2$. At the same time, $\mathrm{Tr}_2-(1,2)=\vec{I}_2$ is also Sidorenko, so $\mathrm{Tr}_{k+2}-(1,k+2)$ is Sidorenko for all $k\neq 1, k\geq 0$. Now, we know that for $e=(i,j)\in E(\mathrm{Tr}_n)$ we can always write

$$\mathrm{Tr}_n - e = (\mathrm{Tr}_{i-1}\to(\mathrm{Tr}_{j-i+1}-(1,j-i+1)))\to\mathrm{Tr}_{n-j}\,. \tag{5.80}$$

When $j-i\neq 2$, $j-i-1\neq 1$, so letting $k=j-i-1$ we know that $\mathrm{Tr}_{j-i+1}-(1,j-i+1)$ is Sidorenko. Using Theorem 4.4 and corollary 4.5, we conclude that Tr_n-e is Sidorenko. □

6. Open Questions

In this section, we give a list of open questions that are not solved yet.

- The main problem to classify all directed Sidorenko graphs still remains open.
- In Section 4 we defined *blow-up Sidorenko* graphs. Try to classify all blow-up Sidorenko graphs. Are there blow-up Sidorenko graphs that is not in $\mathcal{B}$?
- In Section 5.1, we study which orientation of a star makes it Sidorenko. At the same time, from (5.5), we know that when $u=v$, i.e., the in-degree and out-degree of the center is the same, by AM-GM inequality,

$$f^u(1-f)^u=(f(1-f))^u\leq\left(\frac{1}{2}\right)^{2u}, \tag{6.1}$$

  so the integral $t(\vec{H},W)$ is upper bounded by $2^{-2u}=2^{-u-v}$ for all f. Therefore in this case $\vec{H}$ is anti-Sidorenko. Which orientation of a star makes it anti-Sidorenko?
- In Section 5.3 we give a proof that transitive tournament minus an edge is Sidorenko if the edge is not of length 2; when the edge is of length 2, the transitive tournament minus the edge is not Sidorenko when the transitive tournament is small. Is it always not Sidorenko when the edge deleted is of length 2?
- I conjecture that for any undirected graph H, one can always direct the edges so that the new directed graph $\vec{H}$ is Sidorenko.

  This is true for paths (make it alternating, then use Theorem 5.12 to show that every term left is non-negative), complete graphs and complete graphs minus an edge (we just need to make the deleted edge of length 1).

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