TORSION SUBSPACES OF COMPLEX LIE GROUPS
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Abstract. Let G be a simply-connected semisimple complex Lie group. For a positive integer ℓ, we study the space $G[\ell]$ of ℓ-torsion elements of G. We establish a canonical bijection between the components of $G[\ell]$ and the elements of the long root lattice lying in the fundamental alcove $\ell A_\circ$. Under this bijection, the face of $\ell A_\circ$ on which a given lattice point lies determines the topology of the corresponding component. Applications include quasipolynomial growth in ℓ of components of a given topology.

1. Introduction

Given a complex reductive group G and positive integer ℓ, we can consider the space $G[\ell]$ of ℓ-torsion elements. In this paper, we attempt to address the question "what does $G[\ell]$ look like?" In the remainder of this introduction, we explain in down-to-earth terms some simple cases of this problem, which nonetheless carry many representative ideas of the more general cases considered in the remainder of the paper. In particular, the expert reader may wish to skip directly to the next section.

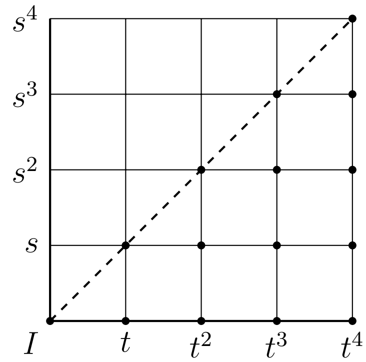
Let us begin with the simplest reductive group, the multiplicative group $\mathbb{C}^\times$. In this case, the subspace $\mathbb{C}^\times[\ell]$ of ℓ-torsion elements is just the subgroup of ℓ^{th} roots of unity μ_ℓ. This is a discrete space with ℓ elements distributed on the unit circle.

We now consider a slightly less trivial case, that of GL_2. Given an element of order ℓ, a bit of thought shows that it can be diagonalized. As such, up to a change of basis, we can write such an element in the form

$$\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$$

for some $\xi_1, \xi_2 \in \mu_\ell$. This form is canonical up to a swap of ξ_1 and ξ_2. There are thus ℓ conjugacy classes of ℓ-torsion elements where $\xi_1 = \xi_2$ and $\binom{\ell}{2}$ where $\xi_1 \neq \xi_2$. Of these, the first ℓ correspond to the discrete embedding of μ_ℓ into the center of GL_2, as before. However, the others have a new feature that did not appear in GL_1 – as they are not central, their conjugacy classes are nontrivial. In other words, we although we have identified the path components of $GL_2[\ell]$, each component may have an interesting topology. However, as each component is a single orbit of GL_2, we may recover this information from the stabilizer of a chosen point in the orbit. Turning to this, the centralizer of a matrix above with $\xi_1 \neq \xi_2$ is the subgroup T of diagonal matrices, so these conjugacy classes look like GL_2/T. The action of GL_2 on pairs of lines in $\mathbb{C}^2$ exhibits GL_2/T as the open orbit $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$, where Δ is the diagonal. In particular, it fibers over $\mathbb{P}^1$ with fibers $\mathbb{A}^1$, so it is homotopy equivalent to the sphere S^2.

Looking more carefully at the way we counted the conjugacy classes of $\mathrm{GL}_2[\ell]$, we took the space $T[\ell]$ and counted its orbits under conjugation by its normalizer in GL_2, i.e. by the Weyl group Σ_2. This gave us ℓ fixed points, with discrete components, and $\binom{\ell}{2}$ pairs, with topologically nontrivial components. For example, if $\ell = 5$, we have the following picture, with $s = \left(\begin{smallmatrix} \xi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \xi \end{smallmatrix}\right)$ for some $\xi \in \mu_5$:



The components of $\mathrm{GL}_2[\ell]$ are naturally in correspondence with the marked points above. The Weyl group reflects along the dashed line, and the points on the line correspond to discrete points of $\mathrm{GL}_2[\ell]$, while the others correspond to copies of GL_2/T.

In general, we can study the components of $G[\ell]$ using the same idea, first conjugating to a maximal torus, then looking at its torsion elements and their centralizers to obtain a complete description. In particular, we will find a similar correspondence between Weyl group orbits and the connected components of G, yielding a description of the topology of each connected component via the stabilizer of the corresponding orbit.

Given the accessibility of this question, much of this work involves piecing together already-known results or specializing them to our case, especially regarding the topology of individual components. However, much of the counting is new, and there remain some combinatorial questions to be resolved to obtain a complete description of $G[\ell]$ in all cases.

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2. Statement of results

We use standard notation regarding reductive groups; we refer the reader to §3 for definitions. We have the following result on the number of components of $G[\ell]$.

Theorem 1. *For a reductive group G, there exists a canonical bijection between W-orbits in $\mu_\ell \otimes X_*(T)$ and conjugacy classes of ℓ-torsion elements in G. A choice of isomorphism $\mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\sim} \mu_\ell$ thus yields a bijection*

$$X_*(T)/(W \ltimes \ell X_*(T)) \xrightarrow{\sim} \pi_0(G[\ell]).$$

In particular, in the complex simply-connected case, we have a canonical bijection

$$X_*(T) \cap \ell A_\circ \xrightarrow{\sim} \pi_0(G[\ell])$$

where $A_\circ$ denotes the fundamental alcove.

The correspondence between W-orbits in $\mu_\ell \otimes X_*(T)$ and conjugacy classes appears in a slightly different form in [Djo80], and the relation to the affine Weyl group and fundamental alcove in the adjoint case is due to Kac, as described in [Ree10]. The general case, while potentially well-known, and somewhat implicit in [Lus95], does not appear to have been written down explicitly, nor applied to the problem of counting components.

The faces of $\ell A_\circ$ are in correspondence with subsets of the affine simple roots, and any such subset yields a pseudo-Levi subgroup. We show, in the simply-connected case:

Theorem 2. *For a face F yielding a pseudo-Levi subgroup L under the above map, a lattice point in the interior of F corresponds to a copy of G/L in $G[\ell]$.*

While we do not know an explicit reference for this result in the literature, it is readily deduced from facts appearing in [Ree10], along with connectedness of centralizers of semisimple elements in the simply-connected form.

The cohomology of a connected component of $G[\ell]$ then readily follows from standard results on the cohomology of homogeneous spaces.

Theorem 3. *Take a connected component $X \subseteq G[\ell]$ corresponding to a point on the interior of a facet of $\ell A_\circ$ indexed by some subset J of the affine simple roots. Let L be the pseudo-Levi subgroup corresponding to J, and let W_J be the corresponding Weyl group. There are canonical isomorphisms*

$$H^*(X) \cong (\operatorname{Sym} \mathfrak{t}^\vee)^{W_J} \otimes_{(\operatorname{Sym} \mathfrak{t}^\vee)^W} \mathbb{C} \cong \mathbb{C}[W]^{W_J}$$

where $\mathfrak{t}^\vee$ *has degree* 2*, and where we use the* W*-equivariant isomorphism* $H^*(G/T) \cong \mathbb{C}[W]$*. Additionally, the Euler characteristic of* X *is* $|W/W_J|$ *and its Poincaré polynomial is*

$$P(X,t) = \frac{\prod_{i=1}^r (1-t^{2d_i})}{\prod_{i=1}^r (1-t^{2e_i})}$$

where d_i *are the exponents of* G *and* e_i *are the exponents of* W*.*

Now we turn to some asymptotic questions as ℓ grows.

Corollary 4. *There are only finitely many spaces, up to isomorphism, which appear as connected components of any of the spaces* $G[\ell]$*, for varying* ℓ*.*

Corollary 5. *Take some connected component of some* $G[\ell]$*, and write it as* G/L *for some pseudo-Levi* L *of semisimple rank* d*. Then the number of components of* $G[\ell]$ *equivariantly isomorphic to* G/L *grows as a quasipolynomial in* ℓ *of degree* $r-d$ *and period dividing* $c|Z(G)|$*, where* c *is the lacing number of* G*. For a semisimple group* G *we define the lacing number to be the product of the lacing numbers of its simple factors.*

In this preliminary draft, we give some computations of the lattice-point counts in type C; in a subsequent draft we will provide further computations in type C and other types.

3. Notation and conventions

Let G be a simply-connected semisimple complex Lie group and fix a maximal torus $T \subset G$, along with a Borel subgroup $B \supset T$. Write Φ for the root system, and let Δ be the corresponding set of simple roots. The affine Weyl group W_a acts on the real span $\mathfrak{t}^\vee$ of the weight lattice by reflections across affine hyperplanes perpendicular to the affine simple roots $\tilde{\Delta}$. This action admits a fundamental domain $A_\circ$, the fundamental alcove, defined as the subset of $\mathfrak{t}^\vee$ consisting of elements ϕ such that $\langle \alpha^\vee, \phi \rangle \geq 0$ for any simple coroot $\alpha^\vee$, and such that $\langle \theta^\vee, \phi \rangle \leq 1$ for the highest coroot $\theta^\vee$.

4. Proofs

We begin by counting the number of connected components.

Proof of Theorem 1. First, note that the conjugacy classes of G are connected, and that a conjugacy class in G is ℓ-torsion just in case any one of its elements is. Thus a connected component of $G[\ell]$ is a union of conjugacy classes of G. Conversely, semisimple conjugacy classes are closed and disjoint, so the connected components are exactly the ℓ-torsion conjugacy classes of G. Any torsion element of G is semisimple, hence conjugate to an element of T. Our problem is thus reduced to counting the number of orbits of $T[\ell]$ under the adjoint action of the normalizer $N_G(T)$, which is just the action of the Weyl group W. Choosing a generator $\xi_\ell \in \mu_\ell$, we obtain a W-equivariant map

$$\begin{array}{ccc} X_*(T) & \to & T[\ell] \\ \chi & \mapsto & \chi(\xi_\ell) \end{array}$$

The kernel of this map clearly contains $\ell X_*(T)$, and choosing coordinates $(\mathbb{G}_m)^r \cong T$ makes apparent the converse. Such a choice of coordinates also allows us to write $t \in T[\ell]$ as $(\xi_\ell^{t_1}, \dots, \xi_\ell^{t_r})$ and hence exhibit a lift

$$\chi_t : \zeta \mapsto (\zeta^{t_1}, \dots, \zeta^{t_r}).$$

Thus we have $T[\ell] \cong X_*(T)/\ell X_*(T)$ as W-modules, so $T[\ell]/W \cong X_*(T)/(W \ltimes \ell X_*(T))$. As G is simply connected, $X_*(T)$ is the coroot lattice, so $W \ltimes X_*(T) = W_a$ is the affine Weyl group. Identifying

$X_*(T)$ with the long root lattice, we have $X_*(T) \cap \ell A_\circ \cong T[\ell]/W$, and the result follows. $\square$

Let us now recall some basic facts regarding the centralizers of semisimple elements. [Hum95] First, to any root $\alpha \in \Phi$, we associate the *root subgroup* U_α, a one-dimensional subgroup of G given by a map $u_\alpha : \mathbb{G}_a \to G$, determined uniquely by the condition that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for any $t \in T, x \in \mathbb{G}_a$. We have the following description of the connected centralizer of a semisimple element.

Proposition 6 ([MS03]). *For $t \in T$, let $\Phi_t := \{\alpha \in \Phi : \alpha(t) = 1\}$. Then there exists some $J \subsetneq \tilde{\Delta}$ such that:*

(i) $\mathbb{Z}J \cap \Phi$ is W-conjugate to Φ_t

(ii) $C^\circ_G(t)$ is conjugate to the reductive subgroup $L_J \subseteq G$ generated by T and $\{U_\alpha\}_{\alpha \in \mathbb{Z}J \cap \Phi}$

Any conjugate of a subgroup as in (ii) above is referred to as a *pseudo-Levi* subgroup. It is a Levi subgroup if $J \subseteq \Delta$, or, equivalently, if it is the reductive part of the Levi decomposition of a parabolic subgroup.

This characterization of centralizers enables us to give the following description of the topology of $G[\ell]$ in terms of the polyhedral geometry of $A_\circ$. Recall that the faces of $A_\circ$ are in correspondence with proper subsets of $\tilde{\Delta}$, with a face corresponding to the set of roots orthogonal to it.

Theorem 7 (restatement of Theorem 2). *Take a face $\overline{F} \subset \ell A_\circ$, with interior F, and let J be the corresponding set of simple affine roots. Then, for any $\chi \in X_*(T) \cap F$, the corresponding component of $G[\ell]$ (via Theorem 1) is isomorphic to G/L_J, with L_J defined as above.*

Proof. Take some such χ. For $\alpha \in \tilde{\Delta}$, we have $\alpha \in J$ just in case $\langle \alpha, \chi \rangle \in \ell\mathbb{Z}$ just in case $\alpha(\chi(\xi_\ell)) = 1$. The connected component of $\chi(\xi_\ell) \in G[\ell]$ is canonically isomorphic to $G/C_G(\chi(\xi_\ell))$. As we are working in the simply-connected case, the centralizer of a semisimple element is connected, so $C_G(\chi(\xi_\ell))$ is conjugate to L_J. This yields the desired isomorphism. □

Proof of Theorem 3. Let us now fix some maximal-rank subgroup $L \subseteq G$, and let $W_L \subseteq W$ denote its Weyl group. We can express the singular cohomology of G/L in terms of G-equivariant cohomology as

$$\begin{aligned} H^\bullet(G/L) &\cong H^\bullet_G(G/L) \overset{\mathbb{L}}{\otimes}_{H^\bullet_G(*)} \mathbb{C} \\ &\cong H^\bullet_L(*) \overset{\mathbb{L}}{\otimes}_{H^\bullet_G(*)} \mathbb{C} \end{aligned}$$

Letting $\mathfrak{t} := \operatorname{Lie} T$, we have $H^\bullet_G(*) \cong L^\bullet(\mathbf{B}G)$ isomorphic to the W-invariants of the symmetric algebra $\operatorname{Sym} \mathfrak{t}^\vee$ on degree-2 generators. Describing $H^\bullet_L(*)$ similarly, we obtain

$$H^\bullet(G/L) \cong (\operatorname{Sym} \mathfrak{t}^\vee)^{W_L} \overset{\mathbb{L}}{\otimes}_{(\operatorname{Sym} \mathfrak{t}^\vee)^W} \mathbb{C}.$$

By the Chevalley–Shephard–Todd theorem, $(\operatorname{Sym} \mathfrak{t})^{W_L}$ is a free algebra over $(\operatorname{Sym} \mathfrak{t}^\vee)^W$, so the tensor product gives $H^\bullet(G/L)$ as a free algebra on generators of even degree. □

Let A denote the Cartan matrix of G, label the simple roots $\{\alpha_i\}_{i \in I}$, and let θ denote the highest root. If D is the diagonal matrix with $D_{ii} = \frac{\langle \alpha_i, \alpha_i^\vee \rangle}{\langle \theta, \theta^\vee \rangle}$ then DA^{-1} is a symmetric matrix taking the basis of fundamental weights to that of simple coroots (embedded into the long root lattice). Let $\vec{n}$ be such that $\theta = \sum_{i \in I} n_i \alpha_i$. Then points of $X_*(T) \cap \ell A_\circ$ correspond to those $\vec{v} \in \mathbb{Z}^I_{\geq 0}$ such that $DA^{-1}\vec{v} \in \mathbb{Z}^I$ and $\vec{n} \cdot \vec{v} \leq \ell$. The point lies on the closure of the facet corresponding to α_i just in case $v_i = 0$, and on that corresponding to θ just in case $\vec{n} \cdot \vec{v} = \ell$. This gives us a mechanistic way of finding the number of connected components of $G[\ell]$ of each type.

Remark 8. Let r be the rank of G and c its lacing number. The theory of Ehrhart polynomials tells us that the number of connected components of $G[\ell]$ is a quasipolynomial in ℓ of degree r and period (at most) ck, so, to count them for all ℓ, it is enough to check for $\ell \in \{1, \ldots, rc|Z(G)|\}$.

Similarly, the number of components of $G[\ell]$ corresponding to some subset of $\tilde{\Delta}$ of size d is a quasipolynomial of degree $r-d$, so it is enough to count them for $\ell \in \{1, \ldots, (r-d)c|Z(G)|\}$.

Example 9. Let $G = \mathrm{Sp}_{2n}(\mathbb{C})$, for some $n \geq 2$. Label the roots $\alpha_1, \ldots, \alpha_n$, where α_n is long. We have $(DA^{-1})_{ij} = \frac{1}{2}\min\{i,j\}$ and $\vec{n} = (2, \ldots, 2, 1)$ [OV90]. Take some $S \subseteq \{1, \ldots, n-1\}$, and write F for the (closed) face of $A_\circ$ correponding to the simple roots not in $\{\alpha_i\}_{i \in I}$. For $\vec{v} \in \mathbb{Z}^n_{\geq 0}$ to correspond to a lattice point of F, we must have:

$$v_j = 0 \text{ for } j \notin S \qquad DA^{-1}\vec{v} \in \mathbb{Z}^n \qquad \vec{n} \cdot \vec{v} \leq \ell.$$

Writing out the second description more explicitly, we require

$$s_j \coloneqq \sum_{i=1}^{n} a_i \min\{i,j\} \in 2\mathbb{Z}$$

for any $j \in \{1, \ldots, n\}$. Writing $s_0 = 0$ and $s_{n+1} = s_n$, we have $a_i = 2s_i - s_{i-1} - s_{i+1}$, so we have $a_i \in 2\mathbb{Z}$ for all i. This condition is clearly sufficient, so the lattice points correspond to $\vec{v} \in \mathbb{Z}^S$ such that $4\sum_{i \in S} v_i \leq \ell$. We thus have the generating function

$$\frac{1}{1-x} \prod_{i \in S} \frac{1}{1-x^4}$$

whose x^ℓ coefficient gives the number of lattice points of F.

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