QUANTITATIVE DECAY FOR NONLINEAR WAVE EQUATIONS

SPUR FINAL PAPER, SUMMER 2018
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August 1, 2018

Abstract. In this paper, we discuss the decay rate for the solution to semi-linear energy-critical wave equation that behaves like the free wave. It is known that free waves in $\mathbb{R}^n$ have a decay rate of $(1+t)^{-\frac{n-1}{2}}$. Using the Klainerman-Sobolev inequality and Strichartz estimates, we are able to prove non-linear waves that scatter in $\mathbb{R}^3$ have a decay rate of $t^{-\frac{1}{2}}$. Moreover, we generalize the results to $\mathbb{R}^n$ to obtain a $(1+t)^{-\frac{n-2}{2}}$ decay rate, and also discuss some results that might be helpful to improve our bound.

1. Introduction

We study the decay rate for solutions to the energy-critical semi-linear wave equation:

$$\Box u := -\partial_t^2 u + \Delta u = \pm u^5$$
$$u[0] = (u(0,x), \partial_t u(0,x)) = (f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$$

in $\mathbb{R}^3$ under the scattering assumption, which means that there exists a free wave u_L that solves

$$\Box u_L = 0$$

with a possibly different initial condition than f and g, and

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as $t \to \infty$.

Remark 1.1 The equation

$$\Box u = u^5$$

is often called the defocusing equation, and the equation

$$\Box u = -u^5$$

is often called the focusing equation.

The motivation behind this problem is that any free wave u_L that has smooth and compactly supported initial data in $\mathbb{R}^3$ has a L_x^∞-norm decay rate of t^{-1}. For semi-linear waves that scatter (i.e. the difference between the semi-linear wave and the free wave tends to 0), it is tempting to think that they behave very similarly as t goes to infinity. Hence, we conjectured u should also have a decay rate of t^{-1}, and the propose of this paper is to give a bound for these semi-linear waves. The method we used is similar to the one used in the free wave equation, and we can prove a $t^{-1/2}$ decay rate for semi-linear wave equations.

Note that in this paper, we will need our solution u to exist for all time t. In the defocusing equation, this follows from the global well-posedness described in Corollary 5.2 in [2]. However, in the focusing case, not all solution u exist for all time t. (i.e. some u blows up in finite time). The assumption of small energy on the initial data is required in the focusing equation for u to exist in all time t. In this paper, we will only be discussing solutions that exist for all time t and doesn't blow up in finite time.

Remark 1.2 The intuition behind focusing equations can blow up in finite time while defocusing equations don't is due to the plus and minus sign inside the energy function. Namely, in the defocusing case, we have the energy function

$$E(u) = \int_{\mathbb{R}^3} \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{6}|u|^6 dx$$

being constant. Hence $\nabla u, u_t$, and u are all controlled by a constant that depends on the initial data. However, in the focusing case, we have the energy function

$$E(u) = \int_{\mathbb{R}^3} \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{6}|u|^6 dx$$

being constant. Since there is a minus sign in front of $|u|^6$, we can not bound any terms like we did before and therefore can not prevent u from blowing up.

Theorem 1.3 (Decay rate for $\|u(t,\cdot)\|_{L^\infty_x}$) *Let $u(t,x)$ be a solution to the following equation:*

$$\Box u = \pm u^5, u[0] = (f,g) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$$

Assume that there exists a free wave u_L with a possibly different initial condition then f and g, and

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as t goes to infinity. Then

$$\|u(t,\cdot)\|_{L^\infty_x} \le \frac{C}{\sqrt{1+t}}$$

for a constant C that only depends on u and doesn't depend on t.

We will be using the Klainerman-Sobolev inequality, Strichartz estimates, and other techniques to establish this bound. Moreover, our results can also be easily generalized into $\mathbb{R}^n$ for any $n \ge 3$. Finally, we will also provide some progress that we made towards to the t^{-1} decay rate.

2. Preliminaries, Definitions, Klainerman-Sobolev inequality, and Strichartz estimates

Throughout this paper, the default space we will be working with will always be $\mathbb{R}^3$. If we don't specify, then all the norms will also be integrating over $\mathbb{R}^3$.

To simplify things, we will be using the notation $u[t]$ as

$$u[t] = (u(t,x), \partial_t u(t,x))$$

to shorten some of our equations, and the Δ operator here will denote

$$\Delta = \sum_{i=1}^n \partial_i^2.$$

We will be using $X \lesssim Y$ to denote that $X \leq CY$ for some absolute constant C. The constant C can vary from line to line.

We will start by recalling the homogeneous and normal Sobolev spaces, which are spaces that we will be using when studying wave equations:

Definition 2.1 (Sobolev space) The Sobolev space, denoted as H^s, consists of all functions f such that

$$\|f\|_{H^s} < \infty,$$

with norm

$$\|f\|_{H^s} = \sqrt{\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi}.$$

Definition 2.2 (Homogeneous Sobolev space) The homogeneous Sobolev space, denoted as $\dot{H}^s$, is very similar to the normal Sobolev spaces. However, the $(1+|\xi|^2)$ term here will be replaced by just $|\xi|^2$ instead. Namely, $\dot{H}^s$ consists of all functions f such that

$$\|f\|_{\dot{H}^s} < \infty,$$

with norm

$$\|f\|_{\dot{H}^s} = \sqrt{\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\xi|^{2s} d\xi}.$$

Now we recall the free wave and the energy critical wave equations:

Definition 2.3 (Free wave) A free wave u_L is a function that satisfies the wave equation

$$\Box u = 0, u[0] = (f, g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$$

for some initial data (f, g).

Definition 2.4 (Energy critical wave equation) The energy critical wave equation in $\mathbb{R}^n$ has the following form:

$$\Box u = \pm u^{\frac{n+2}{n-2}} \tag{1}$$

for all $n \geq 3$.

Remark 2.5 These wave equations are called energy critical since they satisfy the scaling invariant property, which means that whenever $u(t, x)$ is a solution to (1), then

$$u_\lambda = \frac{1}{\lambda^{\frac{n-2}{2}}} u(\frac{t}{\lambda}, \frac{x}{\lambda})$$

will be a solution as well. Moreover, we also have

$$\|u_\lambda\|_{\dot{H}^1 \times L^2} = \|u\|_{\dot{H}^1 \times L^2}$$

as well, which means that the energy of u is constant no matter what λ is.

Remark 2.6 Since we will be mainly working in $\mathbb{R}^3$ here, the energy critical wave equation will then be

$$\Box u = \pm u^5.$$

Definition 2.7 (Scattering condition) We say an energy critical wave u **scatters** if there exists a free wave u_L with possibly different initial conditions that satisfy

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L_x^2} \to 0$$

as t goes to infinity.

Remark 2.8 Notice that the equation we are focusing on is

$$\Box u = \pm u^5.$$

However, the plus or minus sign doesn't matter much in our theorems and proofs since we will always be taking absolute values around u^5. Hence from now on, we will only discuss the equation

$$\Box u = u^5$$

unless otherwise stated. Throughout this paper, u will solve the above equation unless otherwise stated.

We will start off with a fundamental property for general wave equations, which is the finite speed of propagation [7]. Finite speed of propagation is an important property that allows us to control a lot of norms since we will only have to integrate over a finite volume when the initial data has compact support.

Theorem 2.9 (Finite Speed of Propagation) *Let u be a solution to the equation*

$$\Box u = F(u), u[0] = (f, g) \in C_c^\infty \times C_c^\infty,$$

with F being smooth and $F(0) = 0$. Moreover, if f, g are supported in a ball of radius R centered at the origin, then $u(x,t) = 0$ when $|x| > t + R$.

Corollary 2.10 *Let u be a solution for*

$$\Box u = u^5, u[0] = (f, g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

Suppose that f, g are supported in a ball of radius R centered at the origin, then we always have $u(x,t) = 0$ when $|x| > t + R$.

Proof. Since $F(u) = \pm u^5$ satisfies $F(0) = 0$, Theorem 2.9 proves this immediately. $\Box$

Therefore from now on, we will only need to consider x when $|x| \leq R + t$, since we will always have $u(t, x) = 0$ otherwise.

Now we introduce one of an essential theorem that will help us obtain a bound for $|\nabla u(t, x)|$, which is the Klainerman-Sobolev inequality [3]. However, we will need to define some vector fields before we can state the Klainerman-Sobolev inequality.

Definition 2.11 (Invariant vector fields) Let

$$\Gamma = \{\partial_t, \partial_i, \Omega_{ij} := x_i\partial_j - x_j\partial_i, S := t\partial_t + \sum_{i=1}^{n} x_i\partial_i, \Omega_{0i} := t\partial_i + x_i\partial_t\}$$

be a set of vector fields. These are the generators of the linear transformations that commutes with $\Box$ in the equation

$$\Box u = 0.$$

Moreover, all the vector fields in Γ actually commutes directly with $\Box$, with the exception of S. To be clear, we have

$$\Box(Su) = S(\Box u) + 2\Box u,$$

and so S commutes with $\Box$ in the free wave equation.

Remark 2.12 Notice that in $\mathbb{R}^3$, there are 5 types of different invariant vector fields up to symmetry, and a total of 14 of them.

Now we are ready to state the Klainerman-Sobolev inequality.

Theorem 2.13 (Klainerman-Sobolev inequality) *For any integer n, let $u(t,\cdot) \in C^\infty(\mathbb{R}^n)$ vanish when $|x|$ is large for all t. Then there exists a $C_n > 0$ such that*

$$(1+t+|x|)^{\frac{n-1}{2}}(1+\big|t-|x|\big|)^{\frac{1}{2}}|u(t,x)| \leq C_n \sum_{|\alpha|\leq \frac{n}{2}+1} \|\Gamma^\alpha u(t,\cdot)\|_{L^2(\mathbb{R}^n)}$$

holds for all $t \geq 0$.

Notice that this theorem gives us bounds for any smooth function u with compact support in space in terms of t and $|x|$, while sacrificing the summation of invariant vector fields applying to u. Our goal is to bound all the invariant vector fields terms to get a bound for both $|\nabla u|$ and $|u|$, and this will be done in Section 3.

Lastly, we will introduce another essential tool that will help us obtain bounds for $|\nabla u|$, which is called the Strichartz estimates [5]. The Strichartz estimates gives us an inequality that can be used in a lot of non-linear wave equations, and we will use it to bound the vector field terms coming from the Klainerman-Sobolev inequality (Theorem 2.13).

First, we will have to define what does a pair of exponents means to be wave admissible.

Definition 2.14 (Wave admissible) In the $\mathbb{R}^n$ space, a pair of number (p,q) is called wave admissible if

(1) $p, q \geq 2$
(2) $\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}$
(3) $(p,q,n) \neq (2,\infty,3)$

all holds.

Remark 2.15 The Hölder conjugate of a is denoted as a', which is defined as

$$\frac{1}{a} + \frac{1}{a'} = 1.$$

Now we can state the Strichartz estimates.

Theorem 2.16 (Strichartz estimate) *Suppose that*

(1) $n \geq 2$
(2) (p,q) *and* (a,b) *are both wave admissible*
(3) $q \neq \infty, b \neq \infty$
(4) $\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{a'} + \frac{n}{b'} - 2$

all holds. Moreover, if u solves the equation

$$\Box u = F, u[0] = (f,g),$$

then for time $0 < T < \infty$ the following inequality holds for some constant C:

$$\begin{aligned}\|u\|_{L_t^p([0,T];L_x^q(\mathbb{R}^n))} + \|u(T,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \\ \leq C(\|u[0]\|_{\dot{H}^\gamma(\mathbb{R}^n)\times \dot{H}^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L_t^{a'}([0,T];L_x^{b'}(\mathbb{R}^n))}).\end{aligned}$$

Corollary 2.17 *In $\mathbb{R}^3$, suppose that*

(1) (p,q) *and* (a,b) *are both wave admissible*
(2) $q, b \neq \infty$
(3) $\frac{1}{p} + \frac{3}{q} = \frac{1}{2} = \frac{1}{a'} + \frac{3}{b'} - 2,$

all holds. Moreover, if u solves the equation

$$\Box u = F, u[0] = (f, g),$$

then for time $0 < T < \infty$ we have:

$$\|u\|_{L_t^p([0,T];L_x^q)} + \|\nabla u(T,\cdot)\|_{L_x^2} + \|\partial_t u(T,\cdot)\|_{L_x^2}$$
$$\leq C(\|u[0]\|_{\dot{H}^1 \times L^2} + \|F\|_{L_t^{a'}([0,T];L_x^{b'})})$$

for some constant C.

Proof. Plug in $\gamma = 1$ and $n = 3$ in Theorem 2.16 and we are done. $\square$

Remark 2.18 Note that the gradient operator in this paper will always be space gradient (i.e. without the time derivative) unless otherwise stated. Hence its L^2 norm will be comparable to the sum of the L^2 norms of all derivatives in x. Therefore we can freely change between

$$\|\nabla u\|_{L_x^2}$$

and

$$\sum_{i=1}^{3} \|\partial_i u\|_{L_x^2}.$$

Now that we have all the tools, we are ready to prove our main results in the next section.

3. Quantitative decay for energy critical wave equations

Before proving the decay rate for energy critical wave equations, we will first prove the decay rate for free wave equations. Since the main ideas for proving the energy critical case follows from the free wave case, it will be helpful to understand how to prove the decay rate for free waves.

Theorem 3.1 (Decay Rate for free wave equations) *Let n be an odd integer, and u be a solution to the following equation:*

$$\Box u = 0, u[0] = (f, g) \in C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n).$$

Then the following inequality holds:

$$\|u(t,\cdot)\|_{L^\infty} \lesssim \frac{1}{t^{\frac{n-1}{2}}}.$$

Remark 3.2 The strategy here is to use the Klainerman-Sobolev inequality to first prove a bound for

$$\|\nabla u(t,\cdot)\|_{L^\infty},$$

and then use the fundamental theorem of calculus to get a bound for

$$\|u(t,\cdot)\|_{L^\infty}.$$

The reason that we don't use Klainerman-Sobolev inequality directly on u is that the term

$$\|u(t,\cdot)\|_{L^2}$$

is not controllable. We will also follow this idea when it comes to the energy critical wave equation.

Proof. To simplify notations, ∇ in this proof will denote both space and time derivative. Apply the Klainerman-Sobolev inequality to $\nabla u(t,x)$, and we obtain that

$$|\nabla u(t,x)| \leq \frac{C_n \sum_{|\alpha|\leq \frac{n+1}{2}} \|\Gamma^\alpha \nabla u(t,\cdot)\|_{L^2_x}}{(1+t+|x|)^{\frac{n-1}{2}}(1+\big|t-|x|\big|)^{\frac{1}{2}}}. \tag{2}$$

Now since Γ commutes with $\Box$, we have

$$\Box \Gamma^\alpha u = 0.$$

Combine that with commuting two Γ will only result in a linear combination of more Γ and $\partial_i \in \Gamma$, we know from (2) that

$$\begin{aligned} |\nabla u(t,x)| &\leq \frac{C_n \sum_{|\alpha|\leq \frac{n+1}{2}} \|\nabla \Gamma^\alpha u(t,\cdot)\|_{L^2_x}}{(1+t+|x|)^{\frac{n-1}{2}}(1+\big|t-|x|\big|)^{\frac{1}{2}}} \\ &= \frac{C_n \sum_{|\alpha|\leq \frac{n+1}{2}} \|\nabla \Gamma^\alpha u(0,\cdot)\|_{L^2_x}}{(1+t+|x|)^{\frac{n-1}{2}}(1+\big|t-|x|\big|)^{\frac{1}{2}}}. \end{aligned} \tag{3}$$

The last equality is the energy equality.

Notice that since we assume $f, g \in C_c^\infty$, we also have

$$\Gamma^\alpha f, \Gamma^\alpha g \in C_c^\infty$$

for any Γ. Therefore

$$\sum_{|\alpha|\leq \frac{n+1}{2}} \|\nabla \Gamma^\alpha u(0,\cdot)\|_{L^2_x} < \infty. \tag{4}$$

Hence from equation (3) and (4), we have

$$|\nabla u(t,x)| \leq \frac{C_n^{'}}{(1+t+|x|)^{\frac{n-1}{2}}(1+\big|t-|x|\big|)^{\frac{1}{2}}}$$

for some constant $C_n^{'}$.

Since n is odd, by Chapter 1, Theorem 2.2 in [5] we know that

$$u(t,x) = 0 \text{ if } \big|t-|x|\big| > R,$$

where f, g are supported in a ball centered at the origin with radius R (this is the strong Huygens principle). Therefore for any pair of (t,x), there exists a y such that

$$u(t,y) = 0 \text{ and } \big||y|-|x|\big| \leq R.$$

Now by the fundamental theorem of calculus, we now that

$$\begin{aligned} |u(t,x)| &\leq |u(t,y)| + R\frac{C_n^{'}}{(1+t)^{\frac{n-1}{2}}} \\ &\lesssim \frac{1}{(1+t)^{\frac{n-1}{2}}} \end{aligned}$$

as desired.

$\Box$

Now we will start proving the decay rate for energy critical waves. However, before proving that all the vector field bounds are finite, we need the following two lemmas:

Lemma 3.3 (Trapping Lemma) *Suppose that there exists a family of continuous functions $K_\epsilon : \mathbb{R}_0^+ \to \mathbb{R}$ that satisfy both*

(1) $K_\epsilon(0) = 0$
(2) $K_\epsilon(t) \leq C + \epsilon K_\epsilon(t)^m$ *for some fixed* $m > 1, C > 0$.

for all $\epsilon > 0$, *then*

$$\|K_\epsilon(\cdot)\|_{L^\infty(\mathbb{R}_0^+)} < \infty$$

holds for some $\epsilon > 0$.

Remark 3.4 This lemma will be used later in the scattering lemma (lemma 3.5) to show that once a function is small enough at time $t = 0$, then it will always be bounded.

Proof. If $C > 1$, choose $\epsilon = \frac{1}{(2C)^m}$. Then

$$\sup\{x - \epsilon x^m \mid x \geq 0\} \geq 2C - \frac{(2C)^m}{(2C)^m} = 2C - 1 > C.$$

If $C \leq 1$, then choose $\epsilon = \frac{1}{4^m}$ and we know that

$$\sup\{x - \epsilon x^m \mid x \geq 0\} \geq 4 - 1 = 3 > C.$$

Hence by continuity

$$K_\epsilon(t) \in [0, Q_m],$$

where Q_m is the smallest positive root for the equation $x - \epsilon x^m = C$. Therefore we can conclude that

$$\|K_\epsilon(\cdot)\|_{L^\infty(\mathbb{R}_0^+)} < \infty$$

holds for some $\epsilon > 0$ as desired. □

Notice that one of our assumption to the energy critical wave is that u scatters, which means that

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0.$$

This assumption isn't necessarily easy to use, but we can get an equivalent statement that controls some norms of u.

Lemma 3.5 (Scattering lemma)

(1) If

$$\|u\|_{L_t^a L_x^b} < \infty$$

holds for some (a, b) *such that*

$$\frac{1}{a} + \frac{3}{b} = \frac{1}{2} \text{ and } a, b \geq 1,$$

then there exists a linear wave u_L *such that*

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as t *goes to infinity.*

(2) If there exists a linear wave u_L such that

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as t goes to infinity, then

$$\|u\|_{L_t^a L_x^b} < \infty$$

holds for all (a, b) such that

$$\frac{1}{a} + \frac{3}{b} = \frac{1}{2} \text{ and } a, b \geq 1.$$

Proof. (1) follows from Theorem 6.1 in [2], so we will only be proving (2) here.

First let $p = \frac{5}{4}, q = \frac{30}{17}$, then by the Strichartz estimates and Hölder's inequality we know that

$$\begin{aligned}
\|u\|_{L_t^5 L_x^{10}} &\leq C(\|f\|_{\dot{H}^1} + \|g\|_{L^2} + \|u^5\|_{L_t^p L_x^q}) \\
&= C(K + \|(u - u_L + u_L)^5\|_{L_t^p L_x^q}) \\
&\leq C(K + \|(u - u_L)^5\|_{L_t^p L_x^q} + \|u_L^5\|_{L_t^p L_x^q}) \\
&\leq C(K + \|(u - u_L)\|_{L_t^\infty L_x^6} \|(u - u_L)\|^4_{L_t^5 L_x^{10}} + \|u_L^5\|_{L_t^p L_x^q}).
\end{aligned}$$

Now again by Strichartz estimates we know that

$$\|u_L^5\|_{L_t^p L_x^q} = \|u_L\|^5_{L_t^{5p} L_x^{5q}}$$

is bounded by initial data and therefore finite. The Sobolev inequality [3] also tells us that

$$\|u - u_L\|_{L_t^\infty L_x^6} \leq \|u - u_L\|_{L_t^\infty \dot{H}^1}.$$

Notice that for any $\epsilon > 0$, we can choose a T big enough such that

$$\|u - u_L\|_{L_t^\infty([T,\infty); \dot{H}^1)} \leq \epsilon$$

since the $\dot{H}^1$ norm goes to 0 as t goes to infinity. Combining these two inequalities while setting T as the new initial starting time and we have

$$\begin{aligned}
\|u\|_{L_t^5 L_x^{10}} &\leq C(K + \epsilon \|(u - u_L)\|^4_{L_t^5 L_x^{10}}) \\
&\leq C(K + \epsilon \|u\|^4_{L_t^5 L_x^{10}}).
\end{aligned}$$

By the trapping lemma (lemma 3.3) we know that there exists an $\epsilon > 0$ such that

$$\|u\|_{L_t^5([T,\infty); L_x^{10})}$$

is finite. (Here T depends on ϵ).

Moreover, by the finite speed of propagation, we know that

$$\|u(t, \cdot)\|_{L_x^{10}}$$

is just integrating u over a finite space, and

$$\|u\|_{L_t^5([0,T); L_x^{10})}$$

is just integrating

$$\|u(t, \cdot)\|_{L_x^{10}}$$

over a finite interval, so it must be finite. Hence

$$\|u\|_{L_t^5 L_x^{10}} = \|u\|_{L_t^5([0,T); L_x^{10})} + \|u\|_{L_t^5([T,\infty); L_x^{10})}$$

is finite.

Now that we have $\|u\|_{L_t^5 L_x^{10}}$ is finite, we can again use the Strichartz estimates to show that for all (a, b) such that

$$\frac{1}{a} + \frac{3}{b} = \frac{1}{2} \text{ and } a, b \geq 1,$$

we have

$$\|u\|_{L_t^a L_x^b} \leq C(\|f\|_{\dot{H}^1} + \|g\|_{L^2} + \|u\|^5_{L_t^5 L_x^{10}}) < \infty$$

as desired. □

With this lemma, we can use the scattering assumption to control some of the norms for u and use them in the Strichartz estimates once again. However, this time we will be putting them on the right-hand side of the inequality and try to bound the left-hand side, which consists of the invariant vector fields applied to u.

Lemma 3.6 (Bounds for $\|\nabla\Gamma^\alpha u(t, \cdot)\|_{L_x^2}$) *Let u be a solution to the equation*

$$\Box u = u^5, u[0] = (f, g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

Suppose there exists a linear wave u_L such that

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as $t \to \infty$, then

$$\|\nabla\Gamma^\alpha u(t, \cdot)\|_{L_x^2}$$

is bounded by a constant for every $|\alpha| \geq 0$ that is independent of t.

Proof. Let's first prove the case $|\alpha| = 0$, which means that we will have to bound

$$\|\nabla u(t, \cdot)\|_{L_x^2}$$

by a constant. Since

$$\Box u = u^5,$$

this is just a straightforward application of the Strichartz estimates (Corollary 2.17):

$$\begin{aligned}
\|\nabla u(T, \cdot)\|_{L_x^2} &\leq C(\|u[0]\|_{\dot{H}^1 \times L^2} + \|u^5\|_{L_t^1([0,T]; L_x^2)}) \\
&= C(\|u[0]\|_{\dot{H}^1 \times L^2} + \|u\|^5_{L_t^5([0,T]; L_x^{10})}) \\
&\leq C(\|u[0]\|_{\dot{H}^1 \times L^2} + \|u\|^5_{L_t^5([0,\infty); L_x^{10})}).
\end{aligned}$$

The last expression above is a constant independent of T since

$$\|u\|_{L_t^5([0,\infty); L_x^{10})}$$

is bounded by Lemma 3.5.

Now for $|\alpha| \geq 1$, the idea is that we apply Γ^α on both sides of the original equation $\Box u = u^5$, which will give us

$$\Gamma^\alpha \Box u = \Gamma^\alpha u^5.$$

Notice that all Γ except for S commutes with $\Box$. Moreover, even when $\Gamma = S$ we have

$$S(\Box u) = \Box(Su) - 2\Box u = \Box(Su) - 2u^5.$$

When there are two S we have

$$S^2(\Box u) = S(\Box(Su) - 2u^5) = \Box(S^2 u) - 2\Box(Su) - 2Su^5,$$

and so on. Combine that with all the Γ being linear; we know that after applying Γ^α to the original equation, it will have the form of

$$\Box(\Gamma^\alpha u) = P(\Gamma^\alpha u, \Gamma^{\alpha-1}u, \dots, \Gamma u, u). \tag{5}$$

Where P is homogeneous,

$$P \in \mathbb{Z}[\Gamma^\alpha u, \Gamma^{\alpha-1}u, \dots, \Gamma u, u], \quad \deg(P) = 5,$$

and $\Gamma^{\alpha-m}u$ means there are m less Γ applied to the equation. Moreover, the term $u^4\Gamma^\alpha u$ will always have a nonzero coefficient in P.

By the Strichartz estimates applying on (5) we know that

$$\begin{aligned}&\|\Gamma^\alpha u\|_{L^5_t([0,T];L^{10}_x)} + \|\nabla\Gamma^\alpha(T,\cdot)u\|_{L^2_x}\\ &\le C(\|\Gamma^\alpha u[0]\|_{\dot{H}^1\times L^2} + \|P\|_{L^1_t([0,T];L^2_x)}).\end{aligned}$$

Now by Lemma 3.5 we know that

$$\|u\|_{L^5_t([0,\infty);L^{10}_x)} < \infty.$$

Therefore for any $\epsilon > 0$, we can find a series of number

$$0 = T_1 < T_2 < T_3 < \cdots < T_M = \infty$$

such that

$$\|u\|_{L^5_t([T_i,T_{i+1}];L^{10}_x)} < \epsilon$$

for all $1 \le i \le M-1$. Notice that we can make any $T_i \ne T_M$ be the initial time, then from the above inequality we know that for all $T \in [T_i, T_{i+1}]$, the following inequality holds:

$$\begin{aligned}&\|\Gamma^\alpha u\|_{L^5_t([T_i,T];L^{10}_x)} + \|\nabla\Gamma^\alpha(T,\cdot)u\|_{L^2_x}\\ &\le C(\|\Gamma^\alpha u[T_i]\|_{\dot{H}^1\times L^2} + \|P\|_{L^1_t([T_i,T];L^2_x)}).\end{aligned}$$

By Hölder's inequality and there is a term in P that consists of both $\Gamma^\alpha u$ and u, we can continue the above inequality:

$$\begin{aligned}&\le C(\|\Gamma^\alpha u[T_i]\|_{\dot{H}^1\times L^2}\\ &+\|\Gamma^\alpha u\|_{L^5_t([T_i,T];L^{10}_x)}\|u\|_{L^5_t([T_i,T];L^{10}_x)}\|P_1\|_{L^{5/3}_t([T_i,T];L^{10/3}_x)} + \|P_2\|_{L^1_t([T_i,T];L^2_x)}).\end{aligned} \tag{6}$$

Here

$$P_1, P_2 \in \mathbb{Z}[\Gamma^{\alpha-1}u, \dots, \Gamma u, u], \quad \deg(P_1) = 3, \deg(P_2) = 5.$$

Choose ϵ small enough such that the constant for

$$\|\Gamma^\alpha u\|_{L^5_t([T_i,T];L^{10}_x)}$$

is less than $\frac{1}{2}$ for all i in the right hand side of (6), then we have

$$\frac{1}{2}\|\Gamma^\alpha u\|_{L^5_t([T_i,T];L^{10}_x)} + \|\nabla\Gamma^\alpha(T,\cdot)u\|_{L^2_x} \le C(\|\Gamma^\alpha u[T_i]\|_{\dot{H}^1\times L^2} + \|P_2\|_{L^1_t([T_i,T];L^2_x)}).$$

By the finite speed of propagation we know that the right hand side of the above equation is always finite. Let

$$Q = \max\{C(\|\Gamma^\alpha u[T_i]\|_{\dot{H}^1\times L^2} + \|P_2\|_{L^1_t([T_i,T];L^2_x)})\} \mid 1 \le i \le M-1\} < \infty,$$

then we have

$$\|\nabla\Gamma^\alpha(T,\cdot)u\|_{L^2_x} \le Q$$

for some constant Q independent of T as desired.

$\square$

The above lemma is very close to what we wanted, the only thing that is missing is that we successfully bounded

$$\|\nabla\Gamma^\alpha u(t,\cdot)\|_{L_x^2},$$

but what we have in the Klainerman Sobolev inequality is

$$\|\Gamma^\alpha \nabla u(t,\cdot)\|_{L_x^2}.$$

To fix this, we will need one more lemma:

Lemma 3.7 (Bounds for $\|\Gamma^\alpha \nabla u(t,\cdot)\|_{L_x^2}$) *Let u be a solution to the equation*

$$\Box u = u^5, u[0] = (f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

Suppose there exists a linear wave u_L such that

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as $t \to \infty$, then

$$\|\Gamma^\alpha \nabla u(t,\cdot)\|_{L_x^2}$$

is bounded by a constant for every $|\alpha| \geq 0$ that is independent of t.

Proof. By Lemma 3.6 we know that

$$\|\nabla\Gamma^\alpha u(t,\cdot)\|_{L_x^2}$$

is bounded by a constant for every $|\alpha| \geq 0$ that is independent of t. Now we know from remark 2.18 that we can change ∇ into the sums of ∂_i. Moreover, commuting ∂_i and Γ will result in a linear summation of multiple ∂_j terms ($1 \leq j \leq 3$). Hence every

$$\|\Gamma^\alpha \nabla u(t,\cdot)\|_{L_x^2}$$

can be written as a linear combination of

$$\|\nabla\Gamma^\alpha u(t,\cdot)\|_{L_x^2},$$

which means they are also bounded.

□

With the above lemmas, we can now easily prove the decay rate for ∇u by applying the Klainerman-Sobolev inequality to ∇u. However, the Klainerman-Sobolev inequality requires the function being applied on the be smooth in space and has compact space support for all $t > 0$, and we will need to show that ∇u does have these characteristics. The compact space support part is a direct corollary of the finite speed of propagation, and we will show that ∇u is smooth here.

Proposition 3.8 (Persistence of regularity) *Let u be a solution to the equation*

$$\Box u = u^5, u[0] = (f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

Suppose there exists a linear wave u_L such that

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as $t \to \infty$, then

$$\nabla u(t,\cdot) \in C^\infty(\mathbb{R}^3)$$

Proof. By Sobolev embedding, we only need to prove that

$$u(t,\cdot) \in H^s$$

for all s. Now to prove ∇u is in H^s, we will need to show that

$$\|\nabla\nabla^\alpha u(t,\cdot)\|_{L^2_x}$$

is finite for all $0 \le |\alpha| \le s$. (There is one extra ∇ since our original function already has 1). Moreover, notice that ∇ can be partitioned into 3 different ∂_i, and all of them are one of the invariant vector fields described in Definition 2.11. Hence by Lemma 3.6, we know that

$$\|\nabla\nabla^\alpha u(t,\cdot)\|_{L^2_x} \subset \|\nabla\Gamma^\alpha u(t,\cdot)\|_{L^2_x}$$

is finite for all $|\alpha| \ge 0$

Now we are all set to show the decay rate for ∇u: ∎

Theorem 3.9 *Let u be a solution to the equation*

$$\Box u = u^5, u[0] = (f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

Suppose there exists a linear wave u_L such that

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as $t \to \infty$, then

$$|\nabla u(t,x)| \lesssim \frac{1}{(1+t+|x|)(1+\big|t-|x|\big|)^{\frac{1}{2}}}.$$

Proof. Apply the Klainerman-Sobolev inequality to $\nabla u(t,x)$, and we obtain that

$$|\nabla u(t,x)| \le \frac{C_3 \sum_{|\alpha|\le 2} \|\Gamma^\alpha \nabla u(t,\cdot)\|_{L^2}}{(1+t+|x|)(1+\big|t-|x|\big|)^{\frac{1}{2}}}.$$

Now by Lemma 3.7 we know that

$$\sum_{|\alpha|\le 2} \|\Gamma^\alpha \nabla u(t,\cdot)\|_{L^2}$$

is bounded by a constant independent of t. Hence we have

$$|\nabla u(t,x)| \lesssim \frac{C}{(1+t+|x|)(1+\big|t-|x|\big|)^{\frac{1}{2}}},$$

where C is a constant only depending on u.

∎

Corollary 3.10 *Let u be a solution to the equation*

$$\Box u = u^5, u[0] = (f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

Suppose there exists a linear wave u_L such that

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as $t \to \infty$, then

$$\|\nabla u(t,\cdot)\|_{L^\infty} \lesssim \frac{1}{1+t}.$$

Proof. This is a straightforward application of Theorem 3.9. Since

$$|\nabla u(t,x)| \lesssim \frac{1}{(1+t+|x|)(1+\big|t-|x|\big|)^{\frac{1}{2}}},$$

we know that for any fixed t the following holds:

$$|\nabla u(t,x)| \lesssim \frac{1}{1+t}.$$

□

Now we are finally ready to prove Theorem 1.3, which is $\|u(t,\cdot)\|_{L^\infty_x}$ decays like $t^{-1/2}$:

Proof. By the fundamental theorem of calculus, we know that

$$\begin{aligned}|u(t,x)| &\le |u(t,\frac{x}{|x|}(R+t))| + \int_{|x|}^{R+t} |\nabla u(t,y)|dy \\ &\le |u(t,\frac{x}{|x|}(R+t))| + \int_{0}^{R+t} |\nabla u(t,y)|dy.\end{aligned}$$

Combine this with finite speed of propagation (Theorem 2.9) and the bounds for ∇u (Theorem 3.9), we obtain

$$\begin{aligned}|u(t,x)| &\le \int_0^{R+t} \frac{dy}{(1+t+y)\sqrt{1+|t-y|}} \\ &= \int_0^{t} \frac{dy}{(1+t+y)\sqrt{1+t-y}} + \int_t^{R+t} \frac{dy}{(1+t+y)\sqrt{1+y-t}} \\ &\le \left(\int_0^{t} \frac{dy}{(1+t+y)\sqrt{1+t-y}}\right) + \frac{R}{1+t}.\end{aligned}$$

Now

$$\int_0^t \frac{dy}{(1+t+y)\sqrt{1+t-y}} = \frac{\ln(1+\frac{2-2\sqrt{2(t+1)}}{2t+1}) - \ln(3-2\sqrt{2})}{\sqrt{2(1+t)}}.$$

Since $\ln(3-2\sqrt{2})$ is a constant, we will only need to try to bound $\ln(1+\frac{2-2\sqrt{2(t+1)}}{2t+1})$. Notice that

$$3-2\sqrt{2} \le 1+\frac{2-2\sqrt{2(t+1)}}{2t+1} \le 1$$

holds for all $t \ge 0$, which means that

$$\ln(3-2\sqrt{2}) \le \ln(1+\frac{2-2\sqrt{2(t+1)}}{2t+1}) \le 0.$$

Therefore we can conclude that

$$|u(t,x)| \le \frac{-\ln(3-2\sqrt{2})}{1\sqrt{1+t}} \lesssim \frac{1}{\sqrt{1+t}}$$

as desired.

□

4. Discussion

In this section, we will show some results we have that could lead us to prove the $\frac{1}{1+t}$ decay rate.

From Theorem 3.9 we have a bound for $|\nabla u|$ that is approximately $\frac{1}{(1+t)^{1.5}}$ when $|x|$ is away from t. By the fundamental theorem of calculus it is straightforward to guess that the bound for $|u|$ using $|\nabla u|$ will be $\frac{1}{\sqrt{1+t}}$. However, if we restrict $|x|$ into some smaller area, then we will be able to obtain a better bound than $\frac{1}{\sqrt{1+t}}$.

Proposition 4.1 *Let $u(t,x)$ be a solution to the following equation:*

$$\Box u = \pm u^5, u[0] = (f,g) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$$

Assume that there exists a free wave u_L with a possibly different initial condition then f and g, and

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as t goes to infinity. Then for any $\beta < 1$, there exists an $\epsilon = \frac{1-\beta}{2.5} > 0$ such that

$$\|u(t,\cdot)\|_{L_x^\infty(|x| \geq t - t^\beta)} \leq \frac{C}{(1+t)^{0.5+\epsilon}}$$

holds for a constant C that only depends on u.

Proof. From Theorem 3.9 we know that

$$|\nabla u(t,x)| \lesssim \frac{1}{(1+t+|x|)(1+\big|t-|x|\big|)^{\frac{1}{2}}}.$$

The finite speed of propagation tells us that we only need to consider

$$|x| \leq R + t.$$

Hence by the fundamental theorem of calculus, we know that when $t - t^\beta \leq |x| \leq R + t$, the following inequality holds

$$\begin{aligned} |u(t,x)| &\leq \Big|u(t, \frac{x}{|x|}(R+t))\Big| + \int_{|x|}^{R+t} |\nabla u(t,y)| dy \\ &\leq \int_{t-t^\beta}^{t} |\nabla u(t,y)| dy + \int_{t}^{R+t} |\nabla u(t,y)| dy. \end{aligned}$$

Now choose

$$a_1 = 0.5 - \epsilon, a_{i+1} = \frac{a_i}{2} + 0.5 - \epsilon.$$

We know that

$$\lim_{n\to\infty} a_n = 1 - 2\epsilon > \beta,$$

since $\epsilon = \frac{1-\beta}{2.5}$. Hence we have

$$|u(t,x)| \leq \int_{t-t^{a_1}}^{t} |\nabla u(t,y)| dy + \sum_{i=1}^{N} \int_{t-t^{a_{i+1}}}^{t-t^{a_i}} |\nabla u(t,y)| dy + \int_{t}^{R+t} |\nabla u(t,y)| dy$$

for some N large enough such that $a_N > \beta$. Now each term from the above inequality can be bounded as follow:

$$\int_{t-t^{a_1}}^{t} |\nabla u(t,y)| dy \lesssim \frac{t^{a_1}}{(1+t)} \leq \frac{1}{(1+t)^{0.5+\epsilon}}$$

$$\int_{t-t^{a_{i+1}}}^{t-t^{a_i}} |\nabla u(t,y)|dy \lesssim \frac{t^{a_{i+1}}}{(1+t)\sqrt{1+t^{a_i}}} \lesssim \frac{t^{a_{i+1}}}{(1+t)(1+t)^{\frac{a_i}{2}}} \leq \frac{1}{(1+t)^{0.5+\epsilon}}$$

$$\int_{t}^{R+t} |\nabla u(t,y)|dy \lesssim \frac{R}{1+t} \lesssim \frac{1}{1+t}.$$

Since N is finite, we can sum up the three inequalities above and proved that

$$|u(t,x)| \lesssim \frac{1}{(1+t)^{0.5+\epsilon}}$$

for some $\epsilon > 0$ when $|x| \geq t - t^{\beta}$. Therefore

$$\|u(t,\cdot)\|_{L_x^\infty(|x|\geq t-t^\beta)} \leq \frac{C}{(1+t)^{0.5+\epsilon}}$$

holds.

□

Remark 4.2 The above proposition tells us that the decay rate for energy critical waves is better than $\frac{1}{\sqrt{1+t}}$ when $|x|$ is near t. However, the volume where $|x| \leq t-t^\beta$ is still grows like t^3 no matter how close β is to 1.

Theorem 1.3 gives us the $\frac{1}{\sqrt{1+t}}$ bound, which is not quite the $\frac{1}{1+t}$ we expected. The above proposition gave us a slightly better $\frac{1}{(1+t)^{0.5+\epsilon}}$ bound for certain areas. Now we show that the ϵ we improved can actually be very beneficial for us to improve the bound towards $\frac{1}{1+t}$. Namely, the global $\frac{1}{(1+t)^{0.5+\epsilon}}$ decay will give us the global $\frac{1}{1+t}$ decay immediately:

Proposition 4.3 *Let $u(t,x)$ be a solution to the following equation:*

$$\Box u = u^5, u[0] = (f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

Assume that there exists a linear wave $u_L(t)$ such that

$$\|u(t) - u_L(t)\|_{\dot{H}^1\times L^2} \to 0$$

as t goes to infinity, and

$$\|u(t,\cdot)\|_{L_x^\infty} \lesssim \frac{1}{(1+t)^{\frac{1}{2}+\epsilon}}$$

for some $\epsilon > 0$, then

$$\|u(t,\cdot)\|_{L_x^\infty} \leq \frac{1}{1+t}.$$

Remark 4.4 Assume that we already have the bound improved to $\frac{1}{(1+t)^{\frac{1}{2}+\epsilon}}$, then we can use this result to improve the bound to $\frac{1}{1+t}$. This implies that if the decay rate for $\|u(t,\cdot)\|_{L_x^\infty}$ is a polynomial in t, then it can only be $\frac{1}{1+t}$ or $\frac{1}{\sqrt{1+t}}$.

Proof. Suppose that we already have

$$\|u(t,\cdot)\|_{L_x^\infty} \lesssim \frac{1}{(1+t)^{\frac{1}{2}+\epsilon}}$$

holds for a fixed $\epsilon > 0$, then by the Duhamel's formula [2] we know that

$$|u(t,x) - u_L(t,x)| \leq \int_t^\infty \left| \int_{\mathbb{R}^3} e^{i\pi x} \frac{\sin((t-s)|\xi|)}{|\xi|} \widehat{u^5}(s,\xi) d\xi \right| ds$$
$$= \int_t^\infty \frac{1}{4\pi(s-t)} \int_{|x'=x|=|s-t|} u^5(s,x') dS(x') ds.$$

The equality above comes from equation (7) from [6]. Now from our assumption for the decay rate of u, we know that

$$\int_t^\infty \frac{1}{4\pi(s-t)} \int_{|x'-x|=|s-t|} u^5(s,x') dS(x') ds.$$

$$\lesssim \int_t^\infty \frac{4\pi(s-t)^2}{4\pi(s-t)} \frac{1}{(1+s)^{2.5+5\epsilon}} ds = \int_t^\infty \frac{(s-t)}{(1+s)^{2.5+5\epsilon}} ds \lesssim \frac{1}{(1+s)^{0.5+5\epsilon}}.$$

Hence

$$|u(t,x) - u_L(t,x)| \lesssim \frac{1}{(1+t)^{0.5+5\epsilon}}.$$

We already know from Theorem 3.1 that the free wave u_L decays like $\frac{1}{1+t}$, which implies that $\|u(t,\cdot)\|_{L_x^\infty}$ has a decay rate of $\frac{1}{(1+t)^{0.5+5\epsilon}}$. Do the above progress for K times and we will be able to prove that

$$\|u(t,\cdot)\|_{L_x^\infty} \lesssim \frac{1}{(1+t)^{\min\{0.5+5^K\epsilon,1\}}} = \frac{1}{1+t}$$

if we take K large enough. □

The above argument gave us the $\frac{1}{1+t}$ bound but required some additional assumptions. On the other hand, we can also prove the desired $\frac{1}{1+t}$ decay rate without any additional conditions inside any compact space using Hardy's inequality [4].

Theorem 4.5 (Hardy's inequality) *Let f be a smooth function with compact support, then the following inequality holds:*

$$\left\| \frac{f(x)}{|x|} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

Theorem 4.6 (Decay rate within compact support) *Let $u(t,x)$ be a solution to the following equation:*

$$\Box u = u^5, u[0] = (f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

Assume that there exists a linear wave u_L with possibly different initial conditions then f and g, and

$$\|u(t) - u_L(t)\|_{\dot{H}^1 \times L^2} \to 0$$

as t goes to infinity. Let K be any compact space, then

$$||u(t,\cdot)||_{L_x^\infty(K)} \lesssim \frac{1}{1+t}.$$

Proof. Since K is a compact set, we know that there exists a real number R such that $K \subset B(0,R)$.

Let χ_R be a function that is constant in t and smooth in x. Moreover, χ_R is 1 inside $B(0,R)$, 0 outside of $B(0,R+1)$, and is decreasing in between them. Then by Theorem 4.5, we know that

$$\frac{1}{R+1}\|\chi_R u(t,\cdot)\|_{L^2_x} \le \left\|\frac{\chi_R u(t,\cdot)}{|x|}\right\|_{L^2_x} \le \frac{2}{n-2}\|\nabla(\chi_R u(t,\cdot))\|_{L^2_x}.$$

Hence we know that

$$\|\chi_R u(t,\cdot)\|_{L^2_x} \lesssim \|\nabla(\chi_R u(t,\cdot))\|_{L^2_x}.$$

Now since $\chi_R u$ is a smooth function, its derivative is bounded be a constant. Combine that with Theorem 1.3 and Theorem 3.9 we know that

$$\begin{aligned}
\|\nabla(\chi_R u(t,\cdot))\|_{L^2_x} &= \|\nabla(\chi_R u(t,\cdot))\|_{L^2_x(B(0,R+1))} \\
&\le \|\nabla u(t,\cdot)\|_{L^2_x(B(0,R+1))} + \|(\nabla \chi_R)u(t,\cdot)\|_{L^2_x(B(0,R+1))} \\
&\lesssim \frac{1}{(1+t)^{1.5}} + \|u(t,\cdot)\|_{L^2_x(B(0,R+1))} \\
&\lesssim \frac{1}{(1+t)^{1.5}} + \frac{1}{\sqrt{1+t}}.
\end{aligned}$$

Note that we have

$$|\nabla u(t,x)| \le \frac{1}{(1+t)^{1.5}}$$

since we are only considering a compact set, and therefore t is far away from $|x|$ when t is big enough.

Therefore we proved that

$$\|\chi_R u(t,\cdot)\|_{L^2_x(\mathbb{R}^n)} \lesssim \frac{1}{\sqrt{1+t}.} \tag{7}$$

For the other invariant vector field terms, we know that

$$\|\Gamma^\alpha \chi_R u(t,\cdot)\|_{L^2_x}$$

will be bounded like the one above since $|x|$ is finite, except when Γ consists of the weight t.

Notice that the weight t always pairs up with ∂_t or ∂_i in Γ.

Now when we apply $t\partial_t$ on χ_R, it will simply vanish since χ_R is constant in t. Moreover, when $t\partial_t$ is applied to u, it won't change the order of u since we will gain $\frac{1}{t}$ from ∂_t and t back right after.

For $t\partial_i$, when we apply it on χ_R, we will gain a t; when we applied to u, it won't change the order of u since we will again gain $\frac{1}{t}$ from ∂_i and t back right after.

Therefore the worst possible term is

$$\|\Gamma^\alpha \chi_R u(t,\cdot)\|_{L^2_x},$$

where $|\alpha| = 2$ and both Γ consists of $t\partial_i$. Let $t\partial_i = \Omega$, then

$$\begin{aligned}
\left\|\Omega^2 \chi_R u(t,\cdot)\right\|_{L^2_x} &\lesssim \|\Omega t u(t,\cdot)\|_{L^2_x} + \|\Omega \chi_R u(t,\cdot)\|_{L^2_x} \\
&\lesssim \left\|t^2 \nabla u(t,\cdot)\right\|_{L^2_x} + \|t u(t,\cdot)\|_{L^2_x} + \|t\nabla u(t,\cdot)\|_{L^2_x} \\
&\lesssim \sqrt{1+t} + \sqrt{1+t} + \frac{1}{1+t} \\
&\lesssim \sqrt{1+t}.
\end{aligned} \tag{8}$$

Since $\chi_R u(t,x)$ is smooth in space and has compact space support, we can use the Klainerman-Sobolev inequality on $\chi_R u(t,x)$ and obtain that

$$|\chi_R u(t,x)| \lesssim \frac{\sum_{|\alpha|\leq 2} \|\Gamma^\alpha \chi_R u(t,\cdot)\|_{L^2}}{(1+t+x)\sqrt{1+\big|t-|x|\big|}}. \tag{9}$$

Now by $|x| \leq R+1$, (7), (8), and (9) we know that

$$|\chi_R u(t,x)| \lesssim \frac{\sum_{|\alpha|\leq 2} \|\Gamma^\alpha \chi_R u(t,\cdot)\|_{L^2}}{(1+t)^{1.5}} \lesssim \frac{1}{1+t}.$$

Finally, since $\chi_R u(t,x) = u(t,x)$ when $x \in K$, we proved that

$$||u(t,\cdot)||_{L^\infty_x(K)} \lesssim \frac{1}{1+t}.$$

$\square$

For the last part of the paper, I would like to point out that almost all the theorems we proved above can be easily generalized from $\mathbb{R}^3$ into $\mathbb{R}^n$ for all $n \geq 3$. Recall the energy critical equation in $\mathbb{R}^n$ is

$$\Box u = u^{\frac{n+2}{n-2}}.$$

Using the same argument, we have the following two decay rates for u:

Theorem 4.7 *Let u be a solution to the equation*

$$\Box u = u^5, u[0] = (f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3).$$

Suppose there exists a linear wave u_L such that

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as $t \to \infty$, then

$$|\nabla u(t,x)| \lesssim \frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+\big|t-|x|\big|)^{\frac{1}{2}}}.$$

Theorem 4.8 (Decay rate for $\|u(t,\cdot)\|_{L^\infty_x}$ in $\mathbb{R}^n$) *Let $u(t,x)$ be a solution to the following equation:*

$$\Box u = u^{\frac{n+2}{n-2}}, u[0] = (f,g) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$$

Assume that there exists a free wave u_L with a possibly different initial condition then f and g, and

$$\|u[t] - u_L[t]\|_{\dot{H}^1 \times L^2} \to 0$$

as t goes to infinity. Then

$$\|u(t,\cdot)\|_{L^\infty_x} \leq \frac{C}{(1+t)^{\frac{n-2}{2}}}$$

for a constant C that only depends on u.

The proof for both theorems is entirely analogous to the $\mathbb{R}^3$ case. The only difference is when using the Klainerman Sobolev inequality, we get more powers of $(1+t+|x|)$ in the denominator and therefore have a bound that depends on n.

5. Acknowledgements

This paper is written for the MIT SPUR(Summer Program in Undergraduate Research). I would like to thank professor Andrew Lawrie for suggesting the topic and provide very useful suggestions and insights throughout this summer, and professor Casey Rodriguez for helping me start and wrap up the project, as well as pointing out some possible approaches to the problem. I will also like to thank my mentor RuoXuan Yang for explaining a lot of the new materials I studied, coming up with new ideas and methods to tackle the problem, and checking my paper and work carefully. Finally, I would like to thank professor Ankur Moitra and Davesh Maulik for giving me great suggestions during our weekly talk.

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