Uncompleting E-theory and Johnson-Wilson theory at height 2

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Abstract

We prove that there is an $\mathbf{E}_\infty$-ring structure on a particular form of Johnson-Wilson theory $E(2)$ at height 2 at any prime. This provides a realization in spectral algebraic geometry of a flat cover of the moduli stack of formal groups of height at most 2. We show this by using techniques developed by Lawson and Naumann and with Zhu's computations of power operations for Morava E-theory E_2 at height 2 to construct an $\mathbf{E}_\infty$-ring spectrum R whose $K(2)$-localization is E_2. This $\mathbf{E}_\infty$-ring has the property that taking homotopy fixed points with respect to the action of a certain finite group is an $\mathbf{E}_\infty$-form of $E(2)$.

Contents

1 Introduction

One of the slogans behind the modern algebro-geometric viewpoint of chromatic homotopy theory can be phrased roughly as: "it's the theory of formal (p-divisible) groups, only in the spectral setting". In this paper, we take a step in the direction of making this maxim precise by proving that there is an $\mathbf{E}_\infty$-ring $E(2)$ such that the affine derived scheme $\operatorname{Spec} E(2)$ realizes the flat affine cover of the moduli stack $\mathscr{M}_{\mathbf{fg}}^{\leq 2}$ parametrizing formal groups of height at most 2. (See Section 2 for a terse review of some of these terms.)

To explain this result, we must delve into more detail. The process of taking MU-homology defines a functor sending spectra to quasicoherent sheaves on the moduli stack, $\mathscr{M}_{\mathbf{fg}}$, of formal groups (the reader is referred to [Goe08] for a study of the latter category). This functor ties the category of spectra closely to the geometry of $\mathscr{M}_{\mathbf{fg}}$. Indeed, after p-localizing, the E_2-page of the BP-based Adams-Novikov spectral sequence, which converges to the graded ring of (p-local) stable homotopy groups of spheres, is precisely the bigraded ring $\mathrm{H}^*(\mathscr{M}_{\mathbf{fg}}; \omega^{\otimes *})$, where ω is the line bundle of invariant differentials on $\mathscr{M}_{\mathbf{fg}}$. This looks suspiciously like the descent spectral sequence associated to a sheaf of $\mathbf{E}_\infty$-rings on $\mathscr{M}_{\mathbf{fg}}$ whose global sections are the sphere spectrum. However, constructing such a sheaf of $\mathbf{E}_\infty$-rings on $\mathscr{M}_{\mathbf{fg}}$ has proved to be incredibly hard. The only reasonable Grothendieck topology one can impose on this stack is the *fpqc* topology, but it is known that there is no such sheaf of $\mathbf{E}_\infty$-rings on $(\mathscr{M}_{\mathbf{fg}})_{\mathrm{fpqc}}$ (see, e.g., [SVW99, Dev17]). Further obstructing this program is the realization that work of Quillen, Landweber, and Novikov, among others, suggests that $\mathscr{M}_{\mathbf{fg}}$ is presented by the "spectral Hopf algebroid" $(BP, BP \wedge BP)$. However, it is known, by [Law17, Sen17], that the homotopy commutative ring structure on BP does *not* rectify to an $\mathbf{E}_\infty$-ring structure.

Instead, one can try to approximate this candidate derived stack by constructing other derived stacks which are "as close as possible" to each of the strata in the (infinitely long) height filtration of $\mathscr{M}_{\mathbf{fg}}$. Recall that $BP_* \cong \mathbf{Z}_{(p)}[v_1, v_2, \cdots]$ for some choice of generators v_n, and let $G = \operatorname{Spec} BP_*BP$ denote the group scheme over $\operatorname{Spec} BP_*$ parametrizing isomorphisms of the universal p-typical formal group; then, these strata are precisely given by $\mathscr{M}_{\mathbf{fg}}^{\leq n} = \operatorname{Spec}(v_n^{-1} BP_*/(v_{n+1}, v_{n+2}, \cdots))//G$. It is then natural to ask: is there a derived stack $\mathscr{Y}^{\leq n}$ whose underlying stack (in *some* Grothendieck) topology is $\mathscr{M}_{\mathbf{fg}}^{\leq n}$? The study of the layers $\mathscr{M}_{\mathbf{fg}}^{\leq (n+1)} \setminus \mathscr{M}_{\mathbf{fg}}^{\leq n}$ is spectrally realized as the study of $K(n)$-local stable homotopy theory; see (this is obviously an incomplete list) [GH94, HMS94, HL13, HS99, Rez06]. Here, we will attack the following simpler question: is there a (necessarily affine) derived stack $\mathscr{X}^{\leq n}$ whose underlying stack (in the étale topology) is $\operatorname{Spec}(v_n^{-1} BP_*/(v_{n+1}, v_{n+2}, \cdots))$? We answer this question affirmatively in the case $n = 2$, and propose a method of attack for $n \geq 3$.

Before stating our main result, we will recall some of the history behind this problem. Most approaches have involved asking the analogous question for $\operatorname{Spec}(BP_*/(v_{n+1}, v_{n+2}, \cdots))$. One can always form (for nice choices of generators v_n) a *homotopy commutative* ring spectrum $BP\langle n \rangle = BP/(v_{n+1}, v_{n+2}, \cdots)$, but this structure is not even close to the amount of data encapsulated in the phrase "$\mathbf{E}_\infty$-ring". We also get a homotopy commutative ring spectrum $E(n) = v_n^{-1} BP\langle n \rangle$, so the question asked above is whether or not $E(n)$ admits an $\mathbf{E}_\infty$-structure. If there is a $BP\langle n \rangle$ which admits an $\mathbf{E}_\infty$-ring structure, then there is also an $E(n)$ which admits an $\mathbf{E}_\infty$-ring structure. The cases $n = -1, 0$ are easy: $BP\langle -1 \rangle = \mathbf{F}_p$ and $BP\langle 0 \rangle = \mathbf{Z}_{(p)}$. The more interesting case of $n = 1$ is classical; one can realize $BP\langle 1 \rangle_p^\wedge$ as the Adams summand in p-adic complex K-theory, and this can be used to reconstruct the p-local $\mathbf{E}_\infty$-ring $BP\langle 1 \rangle$. This is not indicative of the situation for larger n, though: in [Law17, Sen17], it was shown that the $BP\langle n \rangle$ for $n \geq 4$ do *not* admit $\mathbf{E}_\infty$-structures. (This does *not* imply that the $E(n)$ cannot admit the structure of an $\mathbf{E}_\infty$-ring.) The remaining cases are $n = 2, 3$. We will focus our attention on the case $n = 2$. In [HL10] (resp. [LN12]), it was proved that there is an $\mathbf{E}_\infty$-ring structure on $BP\langle 2\rangle$ at the prime 3 (resp. the prime 2). The existence of an $\mathbf{E}_\infty$-ring structure on a $BP\langle 2\rangle$ immediately implies the existence of an $\mathbf{E}_\infty$-ring structure on an $E(2)$, as mentioned above. The obvious question that remains is: what about $p \geq 5$?

A standard heuristic in homotopy theory ("large primes are easy") suggests that this problem should be easier to solve. Indeed, the work in [LN12] proved a much more general result, using Goerss-Hopkins obstruction theory: at any fixed prime p, the homotopy commutative ring spectrum $BP\langle 2\rangle$ admits an $\mathbf{E}_\infty$-ring structure if and only if a certain subring of $\pi_0 L_{K(1)} L_{K(2)} BP\langle 2\rangle$ is closed under a certain power operation. The proof at $p = 2$ utilized explicit computations of this power operation on Morava E-theory at height 2 by Rezk in [Rez08] at the prime 2 allowed Lawson and Naumann to conclude. (Interestingly, the proof of existence of an $\mathbf{E}_\infty$-ring structure at $p = 3$ did not rely on such computations.) It was not until recently that Zhu computed power operations on Morava E-theory at height 2 at any prime in [Zhu12, Zhu15a, Zhu15b]. The hope, then, is that one could use Zhu's computations to prove the existence of an $\mathbf{E}_\infty$-ring structure on $BP\langle n\rangle$ for $p \geq 5$.

In this paper, we will prove a weaker result: using the techniques of [LN12] and Zhu's computations, we prove the following.

Theorem 1.0.1. *For a particular choice of coordinate on the universal deformation* $\mathbf{G}$*, there is a* p*-complete complex-oriented* $\mathbf{E}_\infty$*-ring* R *with an action of* $G = \mathbf{F}_{p^2}^\times \rtimes \mathrm{Gal}(\mathbf{F}_{p^2}/\mathbf{F}_p)$ *by* $\mathbf{E}_\infty$*-ring maps such that*

1. *there is a* G*-equivariant equivalence of* $\mathbf{E}_\infty$*-rings between* $L_{K(2)}R$ *and the Morava* E*-theory* E_2*; and*
2. *the complex-oriented* $\mathbf{E}_\infty$*-ring* R^{hG} *is (the* p*-completion of) a solution to an* $E(2)$*-realization problem (see Definition 4.1.2).*

As a corollary, we obtain the following result.

Corollary 1.0.2. *There is a* p*-local solution to an* $E(2)$*-realization problem.*

In other words, at any prime, there is an $\mathbf{E}_\infty$-ring structure on some $E(2)$. This allows us to realize an affine derived scheme $\mathscr{X}^{\leq 2} = \operatorname{Spec} E(2)$ whose underlying stack (in the étale topology) is $\mathrm{Spec}(v_2^{-1}BP_*/(v_3, v_4, \cdots))$. We do not know if there is an $\mathbf{E}_\infty$-ring structure on some $BP\langle 2\rangle$ of which this $\mathbf{E}_\infty$-ring is a localization. (In fact, there are numerous things we do not know about the $\mathbf{E}_\infty$-ring R of Theorem 1.0.1; see Section 4.5.)

In Section 2, we recall a few basic facts about $\mathbf{E}_\infty$-rings, complex oriented cohomology theories, and Morava E-theory. Essentially no proofs are provided, since there are numerous resources with far more detailed explanations of these topics. The reader familiar with these topics should skip these sections. In Section 3, we recall the theory of power operations on $\mathbf{E}_\infty$-rings, as introduced by Ando-Hopkins-Strickland and Rezk via isogenies of formal groups (see [AHS04]). We also recall the basics of the theory of p-divisible groups, and then prove a result (Theorem 3.3.2) which is of independent interest: the power operations on $L_{K(m)}E_n$ (where E_n denotes a Morava E-theory of height n) for $m \leq n$ is determined *purely algebraically* by the subgroup structure of the associated universal deformation, regarded as a connected *p-divisible* group. Finally, in Section 4, we recall Zhu's derivation of the power operations on a particular Morava E-theory at height 2, and use this to prove Theorem 1.0.1 and Corollary 1.0.2. We conclude that section by providing a(n inexhaustive) list of open questions; in particular, in Remark 4.5.3 we provide an outline for generalizing Theorem 1.0.1 to all heights.

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1.2 Conventions

We will freely use the language of algebraic geometry (particularly, the theory of formal groups and stacks), but we will provide a terse review of a few homotopy-theoretic concepts in the next section. We shall, however, assume familiarity with spectra and the basics of the theory of homotopy commutative structures. All spaces will be CW-complexes. If A is an abelian group, we will denote by A the corresponding Eilenberg-Maclane spectrum. If A is a ring, then the same notation will be used to regard A as a discrete $\mathbf{E}_\infty$-ring. If X is a *space*, we will denote the A-cohomology of X by $\mathrm{H}^*(X; A)$ — in particular, even if $A = \mathbf{F}_p$, we will explicitly specify the coefficient. However, if X is a spectrum, then $\mathrm{H}^*(X)$ will always mean $\mathrm{H}^*(X; \mathbf{F}_p)$; the choice of prime will be clear from context.

Whenever we discuss Morava E-theory, we will be implicitly choosing a prime p, and working in the stable ∞-category of p-complete spectra. When we work p-locally, we will say so explicitly. We have attempted to be consistent in using $\mathbf{G}$ to denote a formal group, and $\mathbf{G}_f$ to denote the formal group *law* associated to $\mathbf{G}$ (obtained after picking a coordinate on $\mathbf{G}$; the choice of coordinate will be clear from context).

We will use $\mathbf{Z}_p$ to denote the p-adics (and $\mathrm{W}(k)$ to denote the ring of Witt vectors of a perfect field), $\mathbf{Z}_{(p)}$ to denote the localization of $\mathbf{Z}$ at (p), and we will distinguish between C_p, $\mathbf{Z}/p$ and $\mathbf{F}_p$ by using the C_p to denote the integers modulo p viewed as a group, $\mathbf{Z}/p$ to denote the integers modulo p viewed as a group *with a chosen generator*, and $\mathbf{F}_p$ to denote the integers modulo p viewed as a ring. The only exception will be in Section 4 (particularly, during the proof of Theorem 1.0.1), where we write $(\mathbf{Z}/p)^\times$ to denote the group of units in the field $\mathbf{F}_p$. The reason for this exception is that there will be an action of $\mathbf{F}_{p^2}^\times$ in sight, but the source of this action will be different from the source of the action of $(\mathbf{Z}/p)^\times$.

Although this notation will appear infrequently in the paper, we will use X to denote a classical stack, and $\mathscr{X}$ to denote a derived stack with underlying stack X. The exceptions will be moduli problems such as $\mathscr{M}_{\mathbf{fg}}$ and $\mathscr{M}_{\mathrm{BT}}$.

All limits and colimits should automatically assumed to be *homotopy* limits and colimits, taken in the appropriate ∞-category/model category. We will often use the term "category" to refer to an ∞-category.

2 Homotopy-theoretic prerequisites

2.1 E_∞-rings

Two classical examples of cohomology theories are given by ordinary cohomology $H^*(-; A)$ and (complex) K-theory $K(-)$. Let X be a compact Hausdorff space, and suppose A is a ring. Then the graded abelian group $H^*(X; A)$ canonically endowed with the structure of a graded ring via the cup product. Similarly, $K(X)$ is a commutative ring, since one can consider the Whitney sum and tensor product of vector bundles on X. Both of these functors are also representable by spectra, denoted (abusively, in the first case) by A, the Eilenberg-Maclane spectrum, (see Section 1.2) and K, respectively. It is natural to ask that the cup product on $H^*(-; A)$ and and tensor product on $K(-)$ are representable by ring objects in the homotopy category of spectra. Let us consider, for simplicity, the zeroth space $\Omega^\infty K = \mathbf{Z} \times BU$ of complex K-theory. The existence of a *canonical* isomorphism of vector bundles $\mathscr{F} \otimes (\mathscr{G} \oplus \mathscr{G}') \cong (\mathscr{F} \otimes \mathscr{G}) \oplus (\mathscr{F} \otimes \mathscr{G}')$ posits the existence of more structure on $\mathbf{Z} \times BU$: it is equipped with an addition (Whitney sum) and multiplication (tensor product) for which the usual axioms lead to diagrams which commute not just up to homotopy, but rather up to *coherent* homotopy.

Let us be a little more explicit in the case of $\mathbf{Z} \times BU$, where we will only consider the associativity axiom (describing the associativity isomorphism for the tensor product of vector bundles). For simplicity of notation, let us write $X = \mathbf{Z} \times BU$. The tensor product of vector bundles gives a multiplication $\mu : X \times X \to X$, along with a homotopy $\mu \circ (1 \times \mu) \simeq \mu \circ (\mu \times 1)$ — this is the assocativity. This homotopy is a map $\mu_3 : X^3 \times [0, 1] \to X$. In order to proceed, we need to consider four-fold multiplications. There are five ways to take the product of four elements (in general, if C_n denotes the nth Catalan number, then there are C_n ways to take the product of n elements), so one can think of these different products as sitting on the vertices of a pentagon P; the homotopy μ_3 connects consecutive edges of this pentagon. This begets a map $\partial P \times X^4 \to X$, and the condition that all of these different methods of taking the product of four vector bundles should be canonically isomorphic is precisely the condition that this map extends to a map $\mu_4 : P \times X^4 \to X$ (which is appropriately coherent with respect to μ and μ_3). One can proceed in this manner, and find that one obtains a coherent family of maps $\mu_n : K_n \times X^n \to X$, where the K_n are the (contractible) *Stasheff associahedra*.

One can run a similar argument when studying the commutativity axiom (for tensor products of vector bundles). In this case, one obtains a coherent family of maps $\mu_n : E\Sigma_n \times_{\Sigma_n} X^n \to X$. Of course, there was no need to work with spaces in this definition: we could equally as well have worked with the K-theory spectrum (and the Eilenberg-Maclane spectrum A). Let us make these ideas precise.

Let $\mathrm{FinSet}^{\simeq}$ denote the category of (nonempty) finite sets and bijections.

Definition 2.1.1. A symmetric sequence is a functor $\mathrm{FinSet}^{\simeq} \to \mathrm{Top}$. The category of symmetric sequences is the functor category $\mathrm{Fun}(\mathrm{FinSet}^{\simeq}, \mathrm{Top})$.

The category of symmetric sequences has a composition product, which may be described as follows. Let X and Y be two symmetric sequences. Then $X \circ Y$ is defined as

$$(X \circ Y)(S) = \coprod_{S = \coprod_{i\in I} S_i} \left(X(I) \times \prod_{i \in I} Y(S_i) \right),$$

where the coproduct is taken over all unordered partitions of S into nonempty subsets S_i. Note that this defines a monoidal structure on $\mathrm{Fun}(\mathrm{FinSet}^{\simeq}, \mathrm{Top})$ which is not symmetric. An operad $\mathscr{O}$ is defined to be a monoid in the category of symmetric sequences equipped with the composition product. Such an object defines a monad on the category Sp of spectra via $\mathbf{T}_{\mathscr{O}}(X) = \bigvee_{n\geq 0} \mathscr{O}(n) \times_{\Sigma_n} X^{\wedge n}$. There is a model structure on the category of operads (see [GH04] for more details) such that if $\mathscr{O}$ is cofibrant as an operad, then $\mathrm{Alg}_{\mathbf{T}_{\mathscr{O}}}$ admits a model structure for which there is a Quillen adjunction $\mathrm{Sp} \leftrightarrows \mathrm{Alg}_{\mathbf{T}_{\mathscr{O}}}$.

An $\mathbf{E}_\infty$-operad is an operad $\mathscr{O}$ such that $\mathscr{O}(S)$ is a contractible space with a free Σ_S-action, where Σ_S is the symmetric group on the set S.

Definition 2.1.2. An $\mathbf{E}_\infty$-ring spectrum is a $\mathbf{T}_{\mathscr{O}}$-algebra in Sp, where $\mathscr{O}$ is an $\mathbf{E}_\infty$-operad.

In this paper, we will choose a particular $\mathbf{E}_\infty$-operad, called the Barratt-Eccles operad; for this operad, the associated symmetric sequence sends S to

$E\Sigma_S$. We will often abuse terminology by referring to an $\mathbf{E}_\infty$-ring spectrum as an $\mathbf{E}_\infty$-ring.

Remark 2.1.3. In classical algebra, the only datum required to specify a ring structure on an abelian group A is a multiplication map $\mu : A \otimes A \to A$ which makes certain diagram commute; however, in the setting of $\mathbf{E}_\infty$-rings, the (homotopy) commutativity of the diagrams is not a property, but rather extra *structure*: this extra data is exactly what is specified in the definition above.

At this point, the reader might — rightly — ask *why* one needs to work with $\mathbf{E}_\infty$-rings; why do homotopy commutative rings not suffice? For an answer, we refer the reader to the incomplete list [May77, BMMS86, EKMM97, May09, Lur16] of books and papers on this topic. However, we will mention two important reasons: first, it is possible to develop a good theory of modules and algebras over $\mathbf{E}_\infty$-rings (see [EKMM97, Lur16], for instance), which allows one to construct a derived affine scheme associated to an $\mathbf{E}_\infty$-ring such that quasicoherent sheaves over such an object recovers the associated category of modules; and second, the ∞-category of $\mathbf{E}_\infty$-rings is closed under homotopy limits (which is essential in defining sheaves of $\mathbf{E}_\infty$-rings (as in Theorem 3.1.7, for instance) — it is for this reason that "sheaves" of homotopy commutative ring spectra are insufficient; one can, however, work in some appropriate model category, as described in [Goe10, Remark 2.7]).

Before concluding this subsection, we will just recall the following result, a modern proof of which can be found as [AB14, Corollary 3.2].

Theorem 2.1.4. *Let X be an infinite loop space, and let $X \to B\mathrm{GL}_1(S)$ be an infinite loop map classifying a spherical bundle on X. Then the associated Thom spectrum admits the structure of an $\mathbf{E}_\infty$-ring.*

2.2 Bousfield localization

Let E be a spectrum. Say that a map $f : X \to Y$ is an E-equivalence if it induces an isomorphism in E-homology. We would like to localize the category Sp with respect to the collection of E-equivalence. However, some care is required to make this a well-defined procedure. To do so, we need some definitions. A spectrum X is said to be E-acyclic if $X \wedge E$ is contractible.

Let $\mathrm{Sp}_E^{\mathrm{acyc}}$ denote the category of E-acyclic spectra. It is clear that $\mathrm{Sp}_E^{\mathrm{acyc}}$ is closed under shifts and colimits. Moreover, there is a small subcategory of $\mathrm{Sp}_E^{\mathrm{acyc}}$ which generates it under colimits. By general nonsense, the inclusion $\mathrm{Sp}_E^{\mathrm{acyc}} \subseteq \mathrm{Sp}$ preserves all colimits, so the adjoint functor theorem begets a functor $C_E : \mathrm{Sp} \to \mathrm{Sp}_E^{\mathrm{acyc}}$ which is right adjoint to this inclusion.

Let L_E denote the cofiber of the natural transformation $C_E \to \mathrm{id}$; this called *Bousfield localization.* Say that X is E-local if every map $Y \to X$ is null for any E-acyclic Y. Then $L_E X$ is E-local, and the map $X \to L_E X$ is an isomorphism on E-homology. In fact, these two properties characterize the functor L_E. The spectrum $L_E X$ is called the E-localization of X. In many cases, there is an equivalence $L_E X \to \lim(X \wedge E^{\wedge n+1})$.

2.3 Complex oriented cohomology theories

Recall the Thom isomorphism theorem, which states that if $\xi : E \to B$ is a rank n complex vector bundle over a base space B, then there is an isomorphism $\mathrm{H}^*(B; \mathbf{F}_2) \xrightarrow{x \mapsto x \cup u} \widetilde{\mathrm{H}}^{*+2n}(\mathrm{Th}(\xi); \mathbf{F}_2)$, where $\mathrm{Th}(\xi)$ is the Thom space of ξ, and $u \in \mathrm{H}^{2n}(\mathrm{Th}(\xi); \mathbf{F}_2)$ is the Thom class of ξ. In order to get a general result along these lines, there is no need to restrict to mod 2 cohomology. Indeed, we can attempt to study those cohomology theories E for which there exists a good notion of "Thom classes". Such a theory is said to be *complex-oriented.* One way to make this notion precise is by asking that there be an isomorphism $E^*(\mathbf{C}P^\infty) \simeq E^*[[t]]$, where t is a generator in degree 2. It is not immediately clear why this implies a theory of Thom classes, so we will briefly explain the relationship. A complex vector bundle $\xi : E \to B$ is the same as a map $B \xrightarrow{f_\xi} BU$. The classifying space BU has a filtration by the classifying spaces $BU(n)$ as n varies. If we have compatible choices of elements $c_n \in E^{2n}(BU(n))$, then we can just pull back along f_ξ^* to get Thom classes for every vector bundle. One can prove using the splitting principle for vector bundles to prove that $E^*(BU(n)) \simeq E^*(\mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty)^{\Sigma_n} \simeq E^*[[t_1, \cdots, t_n]]^{\Sigma_n} \simeq E^*[[c_1, \cdots, c_n]]$, where the $c_i \in E^{2i}(BU(n))$ denotes the ith elementary symmetric polynomial in the t_k. The element that is the "universal orientation" in $E^{2n}(BU(n))$ is therefore completely determined by our choice of $t \in E^2(\mathbf{C}P^\infty)$.

It turns out that there is an analogous Thom isomorphism for such complex-oriented cohomology theories. Let $\gamma \to BU$ be the universal complex vector bundle over BU, and let $BU \to B\mathrm{GL}_1(S)$ classify its one-point compactification. This is a map of infinite loop spaces, so by Theorem 2.1.4 begets an $\mathbf{E}_\infty$-ring MU, called the complex cobordism spectrum. It is often common to replace MU with its 2-periodic version, $MUP = \bigvee_{k \in \mathbf{Z}} \Sigma^{2k} MU$. Then, we have isomorphisms $E^*(BU) \cong E^*(MU)$. In fact, a complex orientation of E is the same data as a homotopy commutative ring map $MU \to E$.

The fact that E is a cohomology theory gives us a little more: the space $\mathbf{C}P^\infty$ has a multiplication $\mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ (corresponding to the tensor product of line bundles), which induces a map $E^*[[t]] \simeq E^*(\mathbf{C}P^\infty) \to E^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) \simeq E^*[[x, y]]$ in the opposite direction in cohomology. The image of t under this comuliplication gives us a formal group law $f(x, y) \in E^*[[x, y]]$. Forgetting about the particular choice of complex orientation t, we learn that $\mathrm{Spf}\, E^0(\mathbf{C}P^\infty)$ is itself the formal group associated to a complex-orientable homotopy commutative ring spectrum E.

A theorem of Lazard's states that a formal group law over a ring R is the same as a map $L \to R$, where L is the Lazard ring. The Lazard ring is isomorphic (noncanonically) to a polynomial ring over $\mathbf{Z}$ in infinitely many generators, and the identity map $L \to L$ gives a universal formal group law over L. In particular, any complex oriented cohomology theory E determines a map $L \to E_*$. In particular, the "universal complex-orientation" $MU \to MU$ gives a formal group law over MU_*. We then have the following miracle:

Theorem 2.3.1 (Quillen)**.** *The map $L \to MU_*$ classifying the formal group law over MU_* is an isomorphism.*

If E is a cohomology theory, the algebra of homology operations E_*E acts on the E-homology of anything; in particular, it acts on $E_* = E_*(*)$. We therefore have an action of MU_*MU on MU_*. It is useful to determine MU_*MU:

Theorem 2.3.2 (Landweber, Novikov)**.** *We can identify MU_*MU with the ring $MU_*[b_1, b_2, \cdots]$ (with $|b_i| = 2i$) parametrizing automorphisms of the universal formal group law over MU_*.*

This is precisely the group scheme of automorphisms of the universal formal group law over MU_*. This allows us to identify the stack associated with the (graded[1]) Hopf algebroid (MU_*, MU_*MU) is precisely the moduli stack $\mathscr{M}_{\mathbf{fg}}$ of formal groups. In particular, we learn that a complex *oriented* homotopy commutative ring spectrum E has a canonically associated formal group $\mathbf{G}_E$ over E_*, and that the choice of a complex *orientation* (i.e., a map $MU \to E$) is the same data as a choice of coordinate on $\mathbf{G}_E$.

Many cohomology theories are naturally complex oriented. A complex orientation of a homotopy commutative ring spectrum E can equivalently be thought of as an extension of the canonical generator of $\widetilde{E}^2(S^2)$, thought of as a map $S^2 \to \Omega^\infty E$, to a map $\mathbf{C}P^\infty \to \Omega^\infty E$. Suppose that $\pi_{\mathrm{odd}} E = 0$. The obstruction to extending the map $\mathbf{C}P^n \to \Omega^\infty E$ through $\mathbf{C}P^{n+1}$ lives in $\pi_{2n+1}E$, which vanishes by our condition. It follows that any cohomology theory concentrated in even degrees is complex-orientable. We will consider a subclass of such spectra.

Definition 2.3.3. An even periodic ring spectrum is a homotopy commutative ring spectrum E such that $\pi_{\mathrm{odd}} E = 0$ and such that $\pi_2 E$ is an invertible module over $\pi_0 E$ with inverse $\pi_{-2}E$ such that $\pi_{2k}E \cong (\pi_2 E)^{\otimes k}$ for $k \in \mathbf{Z}$ under multiplication.

As proved above, all such spectra are canonically complex-oriented.

Let us now localise everything at a fixed prime p. In that case, Quillen and Brown–Peterson showed that MU splits as $MU \simeq \bigvee \Sigma^? BP$, where BP is the Brown–Peterson spectrum. The homotopy groups of BP are also polynomial: $BP_* \simeq \mathbf{Z}_{(p)}[v_1, v_2, \cdots]$, where $|v_n| = 2(p^n - 1)$. They're much sparser than that of MU, which is useful in analyzing the BP-based Adams spectral sequence (i.e., the Adams-Novikov spectral sequence). The ring BP_* admits a moduli-theoretic interpretation very similar to that of MU_*: every formal group law over a p-local ring M is isomorphic to a p-typical formal group law (Cartier's theorem), and just like MU_* classifies formal group laws, BP_* classifies p-typical formal group laws.

We warn the reader that there are multiple *different* choices of the v_ns; see [Rav86, Theorem A2.2.3]. Nonetheless, the spectrum BP has a unique additive structure, and a unique homotopy commutative multiplication. It turns out that, as with the situation for MU, the ring of co-operations BP_*BP is polynomial over BP_* (so it is flat over BP_*).

[1]See [Mil] for a more careful study of this theory, which works out the gradings involved in detail.

Warning 2.3.4. Unlike MU, the spectrum BP is *not* an $\mathbf{E}_\infty$-ring. This was proved recently in [Law17, Sen17].

Let $I_n = (p, v_1, \cdots, v_{n-1}) \subseteq BP_*$ denote the ideal generated by the first few v_i's. These ideals are invariant under the action of BP_*BP, and in fact comprise all of the invariant prime ideals (see [Lan73]).

Bousfield proved that $L_{BP}X \simeq X_{(p)}$ for X a finite spectrum. We would like to use the connection between formal groups and homotopy theory described by Theorem 2.3.1 and Theorem 2.3.2 to find a filtration of every p-local finite spectrum. To do so, it will be useful to have an interpretation of the elements v_i. Let us briefly recall some facts about heights of formal groups (which we are assuming the reader is already familiar with); see [Goe08, Goe10].

Definition 2.3.5. Let $\mathbf{G}$ be a formal group over a (classical) $\mathbf{F}_p$-scheme S, and let $\psi^{(p)} : \mathbf{G} \to \mathbf{G}^{(p)}$ denote the relative Frobenius. Then $\mathbf{G}$ has height $\geq n$ if there is a factorization[2]

$$\begin{array}{ccccccc} \mathbf{G} & \xrightarrow{\varphi} & \mathbf{G}^{(p)} & \xrightarrow{\varphi^{(p)}} & \cdots & \xrightarrow{\varphi^{(p^{n-1})}} & \mathbf{G}^{(p^n)} \\ & & & \searrow^{[p]} & & & \downarrow T \\ & & & & & & \mathbf{G} \end{array}$$

Construction 2.3.6. The map T induces a map $T^* : \omega_{\mathbf{G}} \to \omega_{\mathbf{G}^{(p^n)}} \simeq \omega_{\mathbf{G}}^{\otimes p^n}$. As $\omega_{\mathbf{G}}$ is a line bundle, this is the same as a map $\mathscr{O}_S \to \omega_{\mathbf{G}}^{\otimes(p^n-1)}$. This defines a global section $v_n \in \omega_{\mathbf{G}}^{\otimes(p^n-1)}$, called the nth Hasse invariant.

Let $\mathbf{G}$ be a formal group over a ring R. After picking a coordinate on $\mathbf{G}$, we obtain a map $BP_* \to R$; then, one might (reasonably) expect that the v_i from Definition 2.3.5 are exactly the images of the elements $v_i \in BP_*$ under this map. This is not exactly true: the two elements coincide modulo $(p, v_1, \cdots, v_{i-1})$.

[2]It is possible to make this definition more concrete after choosing a coordinate. Let $f(x,y)$ be a formal group law over a ring R. We define $[1]_f(x) = x$, and set $[n]_f(x) = f([n-1]_f(x), x)$. The formal power series $[n]_f(x)$ is called the n-series. One can prove that if R is a commutative ring in which $p = 0$, and $f(x,y)$ a formal group law over R, then the p-series $[p]_f(x)$ of f has leading term ax^{p^n}. Let v_i denote the coefficient of x^{p^i} in $[p]_f(x,y)$. The formal group law has height $\geq n$ if $v_i = 0$ for $i < n$, and $f(x,y)$ has height exactly n if it has height $\geq n$ and v_n is invertible.

The moduli stack $\mathscr{M}_{\mathbf{fg}}$ of formal groups is stratified by height, such that the stratum $\mathscr{M}_{\mathbf{fg}}^{\leq n}$ of formal groups of height $\leq n$ can be described as follows. Let $G = \operatorname{Spec} BP_*BP$ denote the group scheme over $\operatorname{Spec} BP_*$ parametrizing isomorphisms of the universal p-typical formal group; then, $\mathscr{M}_{\mathbf{fg}}^{\leq n} = \operatorname{Spec}(v_n^{-1}BP_*/(v_{n+1}, v_{n+2}, \cdots))//G$. Similarly, $\mathscr{M}_{\mathbf{fg}}^{\geq n} = (\operatorname{Spec} BP_*/(p, v_1, \cdots, v_{n-1}))//G$. The natural suggestion for spectrally realizing this filtration $\mathscr{M}_{\mathbf{fg}}^{\leq 1} \subseteq \mathscr{M}_{\mathbf{fg}}^{\leq 2} \subseteq \cdots$, therefore, is to "kill off $v_{n+1}, v_{n+2}, \cdots$, and then invert v_n". In order make this precise, we need to describe a geometric way to cone off elements in the homotopy of a ring spectrum. If R is a ring spectrum and $x \in \pi_* R$, we can take the cofiber of the map $\Sigma^{|x|}R = S^{|x|} \wedge R \xrightarrow{x \wedge 1} R \wedge R \xrightarrow{\mu} R$, and this is denoted R/x. Likewise, we can iterate the map to get a system $R \to \Sigma^{-|x|}R \to \Sigma^{-2|x|}R \to \cdots$, and the (homotopy) colimit is denoted $x^{-1}R$, for obvious reasons.

Warning 2.3.7. In general, if R is an $\mathbf{E}_\infty$-ring spectrum, there is no reason for the cone R/x to be an $\mathbf{E}_\infty$-ring spectrum, in contrast to classical algebra. However, $x^{-1}R$ is always an $\mathbf{E}_\infty$-ring spectrum. This problem is the *raison d'être* of this paper.

One way to obtain $\mathbf{Q}$ from BP is by coning off the elements $v_1, v_2, \cdots$, and then inverting p. In other words, $\mathbf{Q} \simeq p^{-1}BP/(v_1, v_2, \cdots)$. Motivated by this, we define $E(0) = \mathbf{Q}$, and let L_0 denote Bousfield localization with respect to $E(0)$. We then define $E(n) = v_n^{-1}BP/(v_{n+1}, v_{n+2}, \cdots)$, and let L_n denote localization with respect to $E(n)$. The spectrum $E(n)$ is called Johnson-Wilson theory. We can also construct the spectrum $BP\langle n\rangle = BP/(v_{n+1}, v_{n+2}, \cdots)$, so that $v_n^{-1}BP\langle n\rangle = E(n)$. Note, however, that these spectra depend on the choices of the v_i's; see Proposition 2.4.9.

We can also define spectra which detect "exactly" v_n. The nth Morava K-theory is defined to be $K(n) = v_n^{-1}BP/(p, v_1, \cdots, v_{n-1}, v_{n+1}, \cdots)$. Let $L_{K(n)}$ denote Bousfield localization with respect to $K(n)$. For instance, $E(0) = K(0) = \mathbf{Q}$, and $E(1)$ is one of the $(p-1)$ summands of p-local complex K-theory. The spectrum $K(1)$ is a retract of mod p complex K-theory. Ravenel proved in [Rav84] that $E(n)$ is a ring spectrum. However, it is not known in general whether $E(n)$ admits the structure of an $\mathbf{E}_\infty$-ring spectrum for $n \geq 2$ — Theorem 1.0.1 provides an affirmative answer for $n = 2$ at any prime.

The following few (standard) results will be crucial in the sequel; we include the proofs for the sake of completeness. Note, however, that the proofs use more background than we can provide in this section.

Proposition 2.3.8. *Let R be a complex oriented ring spectrum (not necessarily an $\mathbf{E}_\infty$-ring). Then there is an equivalence*

$$\begin{array}{ccc} R & \longrightarrow & L_{K(n)}R \\ & \searrow & \downarrow \sim \\ & & \operatorname{holim}_{J\in\mathbf{N}^n} v_n^{-1}R/I_n^J =: R_{v_n}, \end{array}$$

where $I_n = (p, v_1, \cdots, v_{n-1}) \subseteq BP_*$ and $I_n^J = (p^{J_0}, v_1^{J_1}, \cdots, v_{n-1}^{J_{n-1}})$.

Proof. We must first show that the map $R \to R_{v_n}$ factors through $L_{K(n)}R$. It suffices to show that each $v_n^{-1}R/I_n^J$ is $K(n)$-local. The spectrum $v_n^{-1}R/I_n^J$ is built from $v_n^{-1}R/I_n$ by a finite number of cofiber sequence, so it suffices to prove that the spectrum $v_n^{-1}R/I_n$ is $K(n)$-local. This spectrum is a $v_n^{-1}BP/I_n$-module, hence $v_n^{-1}BP/I_n$-local. As $\langle v_n^{-1}BP/I_n\rangle = \langle K(n)\rangle$, it is also $K(n)$-local. To prove that the map $L_{K(n)}R \to R_{v_n}$ is an equivalence, we must show that $K(n)_*R \xrightarrow{\sim} K(n)_*R_{v_n}$. It suffices to prove this after smashing the map $R \to R_{v_n}$ with a finite complex of type n. Consider the type n complex $X = S/(p^{I_0}, v_1^{I_1}, \cdots, v_{n-1}^{I_{n-1}})$ for some cofinal $(I_0, I_1, \cdots, I_{n-1})$ coming from the Devinatz-Hopkins-Smith periodicity theorem; then $R_{v_n} \wedge X \simeq \operatorname{holim}_{J\in\mathbf{N}^n}(v_n^{-1}R/I_n^J \wedge X) \simeq v_n^{-1}R/I_n^I$. Therefore, as $K(n)_*(R \wedge X) \simeq K(n)_*(R/I_n^I)$, we learn that $K(n)_*(R\wedge X) \simeq K(n)_*(v_n^{-1}R/I_n^I) \simeq K(n)_*(R_{v_n}\wedge X)$, as desired. $\square$

Lemma 2.3.9. *Let E and F be spectra such that every E-local spectrum is F-acyclic, i.e., that $F_*(L_EX) \simeq 0$. Then there is a pullback square:*

$$\begin{array}{ccc} L_{E\vee F}X & \longrightarrow & L_EX \\ \downarrow & & \downarrow \\ L_FX & \longrightarrow & L_EL_FX. \end{array}$$

Proof. The map $L_{E\vee F}X \to L_FX$ is the unique factorization of the map $X \to L_FX$ through $L_{E\vee F}X$ since the map $X \to L_{E\vee F}X$ is an E-equivalence. The map $L_{E\vee F}X \to L_EX$ admits a similar description. To establish the existence of the pullback square, let Y denote the pullback of the diagram.

We need to show that the map $X \to Y$ is an E- and F-equivalence, and that Y is $(E \vee F)$-local. Let us first show that $X \to Y$ is an E- and F-equivalence. Since both $X \to L_E X$ and $Y \to L_E X$ are E-equivalences, we find that $X \to Y$ is an E-equivalence. Moreover, the map $X \to Y$ is an F-equivalence since, again, $X \to L_F X$ is an equivalence and $Y \to L_F X$ is an F-equivalence (since both spectra on the right hand vertical map are F-acyclic). It remains to show that Y is $(E \vee F)$-local. For this, it suffices to show that if Z is any $(E \vee F)$-acyclic spectrum, then $[Z, Y]$ is zero. This follows from the long exact sequence $\cdots \to [Z, Y] \to [Z, L_E X] \oplus [Z, L_F X] \to [Z, L_F L_E X] \to \cdots$ □

Corollary 2.3.10. *For any $m < n$, there is a pullback square*

$$\begin{array}{ccc} L_{K(n)\vee K(m)}X & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{K(m)}X & \longrightarrow & L_{K(m)}L_{K(n)}X. \end{array}$$

Proof. By Lemma 2.3.9, it suffices to prove that $K(n)_*(L_{K(m)}X)$ is zero for any spectrum X. Let Y be a $K(m)$-local spectrum. By the periodicity theorem, we can inductively construct spectra $S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$ for sufficiently large $(i_0, i_1, \cdots, i_m)$ which are type m finite spectra. In particular, $K(n)_*(S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m}))$ is not zero (since $n > m$); it therefore suffices to show that $Y \wedge S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$ is $K(n)$-acyclic. The periodicity theorem also begets a self-map $v_m^N : \Sigma^{2(p^m-1)N} S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m}) \to S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$ which is an isomorphism on $K(m)$-homology, but is zero on $K(n)$-homology. Therefore, the map $v_m^N : Y \wedge \Sigma^{2(p^m-1)N} S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m}) \to Y \wedge S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$ is zero on $K(n)$-homology, but is an isomorphism on $K(m)$-homology (so, in particular, it is a homotopy equivalence since both Y and $S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m})$ are $K(m)$-local). If a homotopy equivalence is zero on $K(n)$-homology, then $K(n)_*(Y \wedge S/(p^{i_0}, v_1^{i_1}, \cdots, v_m^{i_m}))$ must be zero, as desired. □

2.4 Landweber exactness

We warn the reader beforehand that we will not pay attention to the grading. Let M be a ring with a formal group law. According to Quillen's theorem, this is equivalent to a ring map $MU_* \to M$.

Question 2.4.1. When can this map be lifted to a map of spectra out of MU?

Landweber's theorem is an answer to this question. The most natural way to define a homology theory, given the map $MU_* \to M$, is via the functor $X \mapsto MU_*(X) \otimes_{MU_*} M$. This satisfies all the Eilenberg-Steenrod axioms — *except* the axiom that cofiber sequences go to long exact sequences. Since MU is a homology theory, the failure of this functor to be a homology theory can be thought of as the failure of M to be a flat MU_*-module. If M is a flat MU_*-module, this functor certainly is represented by a homology theory. In general, this condition is too strict, and there are not many interesting examples of homology theories stemming this way. To give a simpler condition, we must get into the inner workings of MU.

Without loss of generality, let us assume that M has a p-typical formal group law defined over it, so that there is a ring map $BP_* \to M$.

Theorem 2.4.2. *Let M be as above. Then the functor $X \mapsto BP_*(X) \otimes_{BP_*} M$ is a homology theory iff for all n, the sequence $(p, v_1, \cdots, v_n)$ is a regular sequence in M (in other words, v_k is a non-zero-divisor in $M/(p, v_1, \cdots, v_{k-1})$).*

Before we give the (short) proof, let us recall the Landweber filtration theorem.

Lemma 2.4.3 (Landweber filtration theorem)**.** *Any BP_*BP-comodule M which is finitely presented as a BP_*-module has a filtration $0 = M_k \subset \cdots \subset M_1 \subset M$ where M_j/M_{j+1} is isomorphic as a BP_*BP-comodule to (a shift of) BP_*/I_{n_j}.*

Proof of Theorem 2.4.2. Suppose that $(p, v_1, \cdots, v_n)$ forms a regular sequence in M. Then tensoring with M preserves the short exact sequences $0 \to BP_*/I_n \xrightarrow{\cdot v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0$. In particular, $\mathrm{Tor}_1^{BP_*}(BP_*/I_n, M) \simeq 0$ for all n. By Lemma 2.4.3, we find that $\mathrm{Tor}_1^{BP_*}(N, M) \simeq 0$ for every BP_*BP-comodule N which is finitely presented as a BP_*-module. If X is a finite complex, then $BP_*(X)$ is such a comodule, so $\mathrm{Tor}_1^{BP_*}(BP_*(X), M) \simeq 0$. It follows immediately that the functor $X \mapsto BP_*(X) \otimes_{BP_*} M$ defines a homology theory on finite spectra; since every spectrum is a filtered colimit of finite spectra, the result follows. We will not prove the converse. $\square$

There is an analogue (a corollary, in fact) of Theorem 2.4.2 for MU (both integrally and p-locally):

Theorem 2.4.4 (Landweber exact functor theorem)**.** *Let M be an MU_*-module. Then the functor $X \mapsto MU_*(X) \otimes_{MU_*} M$ is a homology theory iff for every prime p and integer n, multiplication by v_n is monic on $M/(p, v_1, \cdots, v_{n-1})$.*

If E is a complex oriented cohomology theory such that the induced formal group law on E_* is Landweber exact, then the map $MU \to E$ induces an isomorphism $MU_*(X) \otimes_{MU_*} E_* \to E_*(X)$ for all spectra X. Indeed, both sides are homology theories by Theorem 2.4.4, so it suffices to prove that the map is an isomorphism when X is the sphere spectrum. In this case, the result is obvious.

To apply Theorems 2.4.2 and 2.4.4, we only need to know that multiplication by v_n is injective modulo $(p, v_1, \cdots, v_{n-1})$, so the discrepancy discussed after Definition 2.3.5 is not relevant.

Warning 2.4.5. The module $K(n)_* = v_n^{-1} BP_*/(p, v_1, \cdots, v_{n-1}, v_{n+1}, \cdots)$ is the graded ring associated to the complex-oriented cohomology theory $K(n)$ given by Morava K-theory (discussed above) which is *not* Landweber exact.

The fact that complex K-theory can be recovered from its formal group law should not be very surprising, particularly in light of the following observation.

Example 2.4.6. The module $E(n)_* = v_n^{-1} BP_*/(p, v_1, \cdots, v_{n-1})$ is Landweber exact; the associated homotopy commutative ring spectrum $E(n)$ is precisely the Johnson-Wilson theory constructed earlier. We will reiterate here that this spectrum might depend on the choice of the elements $v_1, \cdots, v_{n-1}, v_n$; see Proposition 2.4.9. If $n = 1$, then the spectrum $E(1)$ admits the structure of an $\mathbf{E}_\infty$-ring, and can be identified (up to extension of scalars) with the Adams summand of p-adic K-theory. The primary question answered by Theorem 1.0.1 is the existence of an $\mathbf{E}_\infty$-structure on $E(2)$ at all primes p.

Conjecture 2.4.7. Let E and E' be two homotopy commutative complex-oriented ring spectra such that there are choices of $v_1^E, \cdots, v_n^E$ and $v_1^{E'}, \cdots, v_n^{E'}$ such that the maps $\mathbf{Z}_{(p)}[v_1^E, \cdots, v_n^E] \subseteq BP_* \to E_*$ and $\mathbf{Z}_{(p)}[v_1^{E'}, \cdots, v_n^{E'}] \subseteq$

$BP_* \to E'_*$ send v_n^E and $v_n^{E'}$ to invertible elements and such that the induced map $\mathbf{Z}_{(p)}[v_1^E, \cdots, v_{n-1}^E, (v_n^E)^{\pm 1}] \to E_*$ and $\mathbf{Z}_{(p)}[v_1^{E'}, \cdots, v_{n-1}^{E'}, (v_n^{E'})^{\pm 1}] \to E'_*$ are isomorphisms. Then there is an equivalence between the spectra underlying E and E' (i.e., they are the same "additively").

The analogous statement for $BP\langle n \rangle$ is true, by [AL17]:

Theorem 2.4.8. *Let E and E' be two homotopy commutative complex-oriented ring spectra such that there are choices of $v_1^E, \cdots, v_n^E$ and $v_1^{E'}, \cdots, v_n^{E'}$ such that the maps $\mathbf{Z}_{(p)}[v_1^E, \cdots, v_n^E] \subseteq BP_* \to E_*$ and $\mathbf{Z}_{(p)}[v_1^{E'}, \cdots, v_n^{E'}] \subseteq BP_* \to E'_*$ are isomorphisms. Then there is an equivalence between the spectra underlying E and E'.*

Recall that the spectrum $E(2)$ has homotopy groups $\mathbf{Z}_{(p)}[v_1, v_2^{\pm 1}]$. However, there are multiple different choices of the elements v_1 and v_2, and if one wants to construct an $\mathbf{E}_\infty$-ring structure "on the spectrum $E(2)$" (note that, by Conjecture 2.4.7, the spectrum $E(n)$ is expected to be unique, just as with $BP\langle n \rangle_p^\wedge$), then we would at least need to know that any two choices of v_1 and v_2 give the same homotopy commutative ring structure. However, this is false, as proved in [Str10]:

Proposition 2.4.9 (Strickland)**.** *The $BP\langle 2 \rangle$'s (and $E(2)$'s) associated to the Hazewinkel generators and the Araki generators (see [Rav86, Theorem A2.2.3]) are* not *equivalent as homotopy commutative ring spectra.*

This illustrates why some care is required in making Theorem 1.0.1 precise. We will study this problem in more detail below. However, note that if $p > 2$, then the spectra $K(n)$ admit the structure of homotopy commutative rings, but they *never* admit the structure of an $\mathbf{E}_\infty$-ring (in fact, even the structure of an $\mathbf{E}_2$-ring). Indeed, suppose that $K(n)$ did admit the structure of an $\mathbf{E}_2$-ring. Since $p = 0$ in $\pi_0 K(n) \cong \mathbf{F}_p$, the Hopkins-Mahowald theorem expressing $\mathbf{F}_p$ as the initial $\mathbf{E}_2$-algebra with $p = 0$ would show that $K(n)$ is an $\mathbf{F}_p$-algebra. In particular, $K(n) \wedge \mathbf{F}_p$ would be nonzero. This, however, is impossible, since this ring would carry an isomorphism between the additive formal group and a formal group of height n.

2.5 Morava E-theory

We will now use Theorem 2.4.4 to construct *Morava E-theory*, denoted E_n. (This is not the same as the Johnson-Wilson theory $E(n)$.) The formal group law over $\pi_* E_n$ will have an explicit algebro-geometric interpretation. We begin with an analysis of the local structure of formal group laws, following [Lur10]. Fix a perfect field k of characteristic p and a formal group $\mathbf{G}_0$ over k.

Definition 2.5.1. Let R be a local Artinian ring with a map $k \to R/\mathfrak{m}$ (so the maximal ideal $\mathfrak{m}$ of R is nilpotent, and each quotient $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a finite-dimensional k-vector space). A deformation of $\mathbf{G}_0$ is a formal group $\mathbf{G}$ over R along with an isomorphism $\mathbf{G} \otimes_R R/\mathfrak{m} \cong \mathbf{G}_0 \otimes_k R/\mathfrak{m}$. Let $\mathrm{Def}(R)$ denote the groupoid of deformations (up to isomorphism) of $\mathbf{G}$.

Theorem 2.5.2 (Lubin-Tate)**.** *Let R be any local Artinian ring with residue field k. Then $\mathrm{Def}(R)$ is discrete, and there is a complete local ring $\mathscr{O}_{\mathrm{LT}(k,\mathbf{G}_0)}$ such that $\{$local ring maps $\mathscr{O}_{\mathrm{LT}(k,\mathbf{G}_0)} \to R\} \to \mathrm{Def}(R)$ is a bijection.*

By Theorem 2.5.2, there is a universal deformation $\mathbf{G}$ of $\mathbf{G}_0$ living over $\mathrm{Spf}\, \mathscr{O}_{\mathrm{LT}(k,\mathbf{G}_0)} =: \mathrm{LT}(k, \mathbf{G}_0)$. We can choose a coordinate x on $\mathbf{G}$ to obtain a formal group law $\mathbf{G}_f$ such that its p-series is $[p]_{\mathbf{G}_f}(x) = px +_{\mathbf{G}_f} u_1 x^p +_{\mathbf{G}_f} \cdots +_{\mathbf{G}_f} u_{n-1} x^{p^{n-1}} +_{\mathbf{G}_f} x^{p^n}$, where n denotes the height of $\mathbf{G}_0$. This defines an isomorphism $\mathscr{O}_{\mathrm{LT}(k,\mathbf{G}_0)} \cong \mathrm{W}(k)[[u_1, \cdots, u_{n-1}]]$ which is noncanonical since it depends on a choice of coordinate on $\mathbf{G}$. parametrizes universal deformations. Applying the Landweber exact functor theorem, we get a cohomology theory $E(k, \mathbf{G}_0)$ for which $\pi_* E(k, \mathbf{G}_0) \cong \mathscr{O}_{\mathrm{LT}(k,\mathbf{G}_0)}[u^{\pm 1}]$ with $|u| = 2$.

Definition 2.5.3. The even periodic homotopy commutative ring spectrum $E(k, \mathbf{G}_0)$ is called *Morava E-theory.*

For example, when $n = 1$ and $\mathbf{G} = \widehat{\mathbf{G}_m}$ is the usual multiplicative formal group, this is exactly p-adically complete K-theory.

Let $\mathbf{G}_0$ be a formal group law of height n over $k = \mathbf{F}_{p^n}$. Then every automorphism of $\mathbf{G}_0$ gives rise to an automorphism of the discrete groupoid $\mathrm{Def}(R)$; by Theorem 2.5.2, this begets an action of $\mathrm{Aut}(\mathbf{G}_0)$ on $\pi_* E_n$. This action lifts to the level of spectra: every element of $\mathrm{Aut}(\mathbf{G}_0)$ gives rise to an automorphism of E_n. These automorphisms are very well-behaved; in fact, we have:

Theorem 2.5.4 (Goerss-Hopkins-Miller)**.** *The spectrum $E(k, \mathbf{G}_0)$ admits the structure an $\mathbf{E}_\infty$-ring; moreover, this $\mathbf{E}_\infty$-ring structure is unique if k*

is algebraically closed. The space $\mathrm{Aut}_{\mathbf{E}_\infty}(E(k, \mathbf{G}_0))$ *of* $\mathbf{E}_\infty$*-ring maps is discrete, and there is a continuous isomorphism between* $\mathrm{Aut}_{\mathbf{E}_\infty}(E(k, \mathbf{G}_0))$ *and* $\mathrm{Aut}(\mathbf{G}_0)$.

In the proof of Theorem 3.3.2 (which is crucial for Theorem 1.0.1), we will rely heavily upon this moduli-theoretic interpration. The group $\Gamma := \mathrm{Aut}(\mathbf{G}_0)$ is a profinite p-adic Lie group, as can be proved using Dieudonné theory; it is called the *Morava stabilizer group*.

Remark 2.5.5. Note that E is complex-oriented by construction — however, it is not known whether there is an complex orientation $MU \to E$ which is an $\mathbf{E}_\infty$-ring map.

3 p-divisible groups and power operations

3.1 p-divisible groups

The theory of p-divisible groups was originally introduced by Tate in [Tat67]. We will recall some of the basic theory. Good references for this material include Tate's original paper, as well as [Mes72, Goe09], the latter of which we will be closely following. Fix a prime p, and assume that p is topologically nilpotent.

Definition 3.1.1. Let R be a $\mathbf{Z}_p$-algebra. A p-divisible group $\mathbf{G}$ of height n over R is an *fppf* sheaf of abelian groups on R such that

1. the map $[p^k] : \mathbf{G} \to \mathbf{G}$ is a surjection in the *fppf* topology;
2. the kernel $\mathbf{G}[p^k]$ of $[p^k]$ is a finite flat group scheme of constant rank p^{nk}; and
3. $\mathbf{G}$ is the colimit of $\mathbf{G}[p^k]$.

A morphism between p-divisible groups is simply a homomorphism of *fppf* sheaves.

The first and second conditions are equivalent to the seemingly weaker requirements that $[p] : \mathbf{G} \to \mathbf{G}$ is a surjection in the *fppf* topology, and that $\mathbf{G}[p]$ is a finite flat group scheme (of constant rank p^n).

There are numerous facts about p-divisible groups which we will utilize below. Let $1 \in \mathbf{G}$ denote the identity section, and let $\mathbf{G}^\circ$ be the completion at 1. This is a formal group, and will generally be referred to as the formal group associated to $\mathbf{G}$. The *dimension* of $\mathbf{G}$ will be defined to be the dimension of $\mathbf{G}^\circ$. Note that $\mathbf{G}^\circ$ is not necessarily 1-dimensional. It is a theorem of Messing's from [Mes72] that this is the *fppf* sheaf defined by $\mathbf{G}^\circ(A) = \ker(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}}))$ if p is nilpotent in R. Moreover, the quotient $\mathbf{G}/\mathbf{G}^\circ =: \mathbf{G}^{\mathrm{et}}$ is an étale group scheme over R, so we obtain a short exact sequence $0 \to \mathbf{G}^\circ \to \mathbf{G} \to \mathbf{G}^{\mathrm{et}} \to 0$. Say that a p-divisible group $\mathbf{G}$ is connected if $\mathbf{G}^\circ = \mathbf{G}$. By [Tat67, Proposition 1], there is an equivalence of categories between connected p-divisible groups and p-divisible formal groups (i.e., formal groups for which the multiplication by $[p]$ map is an isogeny) over a complete Noetherian local ring of mixed characteristic $(0, p)$.

p-divisible groups are very natural objects to consider. For example, if A is an abelian variety over a $\mathbf{Z}_p$-algebra R of dimension g, then $A[p^\infty] = \operatorname{colim} A[p^n]$ is a p-divisible group over R of dimension g and height $2g$. If E is an elliptic curve (i.e., an abelian variety of dimension 1) over R, then $E[p^\infty]$ is connected if and only if E is supersingular. If the p-divisible group $E[p^\infty]$ has a nontrivial étale component, then it is ordinary, and the connected component $E[p^\infty]^\circ$ is the formal group associated to E (i.e., the relative Picard scheme $\mathrm{Pic}^0_{E/\operatorname{Spec} R}$).

Since homotopy theory is concerned with formal groups of dimension 1, the simplest entrance of the theory of p-divisible groups into homotopy theory comes from spectra associated to (certain) elliptic curves. The example of elliptic curves also illustrates an important fact about p-divisible groups which is simply false for formal groups (and makes them a lot more convenient to work with): the height of a p-divisible group remains constant under base change. Let us give two examples showing that the height of a formal group can vary (it can only decrease) after base change.

Example 3.1.2. Let E be the elliptic curve defined by the Weierstrass form $y^2 = x(x-1)(x+2)$ over $\mathbf{Z}[1/6]$. It is easy to check (by computing the Hasse invariant, or by glancing at [Har77, Example 4.23.6]) that E is supersingular when $p = 23$ (so its associated formal group has height 2), but that for $5 \leq p \leq 73$ and $p \neq 23$, this curve is ordinary (so its associated formal group has height 1).

Example 3.1.3. Another example (more relevant, perhaps) to our discussion below, is the following. Let $(\mathbf{G}_0)_f$ be a formal group law over $\mathbf{F}_{p^2}$, and let $\mathbf{G}_f$ be its universal deformation over $\mathrm{W}(\mathbf{F}_{p^2})[[u_1]]$. This has p-series $[p]_{\mathbf{G}_f}(x) = px +_{\mathbf{G}_f} u_1 x^p +_{\mathbf{G}_f} x^{p^2}$. If we base change via the map $\mathrm{W}(\mathbf{F}_{p^2})[[u_1]] \to \mathbf{F}_{p^2}[[u_1]] \hookrightarrow \mathbf{F}_{p^2}((u_1))$, then we find that the p-series is $u \cdot x^p +_{\mathbf{G}_f} x^{p^2}$, where u is a unit in $\mathbf{F}_{p^2}((u_1))$. In particular, the height of the formal group is now 1.

A deformation of a connected p-divisible group need not remain connected, which implies that the height of the connected component of a deformation can decrease. For this reason, it is more natural to study p-divisible groups deforming a (formal) connected p-divisible group, rather than *connected* p-divisible groups deforming a connected p-divisible group.

Warning 3.1.4. From this point onwards, we will only consider 1-dimensional p-divisible groups (i.e., p-divisible groups whose associated connected component is a formal group of dimension 1).

Then, one has the following result (see [BL10, Theorem 7.1.3] and [Goe09, Remark 4.12]):

Theorem 3.1.5. *Let $\mathbf{G}_0$ be a p-divisible group over a perfect field k of finite height n. Consider the moduli problem sending a complete local Noetherian ring $(R, \mathfrak{m})$ to the groupoid of p-divisible groups $\mathbf{G}$ over R along with an isomorphism $\iota : \mathbf{G} \otimes_R R/\mathfrak{m} \xrightarrow{\cong} \mathbf{G}_0 \otimes_k R/\mathfrak{m}$, where morphisms are isomorphisms of p-divisible groups restricting to the identity on $\mathbf{G}_0$. Then this moduli problem is represented by an affine scheme $\mathrm{LT}(\mathbf{G}_0)$, which is noncanonically isomorphic to $\mathrm{Spf}\,\mathrm{W}(k)[[u_1, \cdots, u_{n-1}]]$.*

Remark 3.1.6. In particular, a deformation of a p-divisible group of height n and formal height n need not have formal height n. This will be used a great deal in the next subsection.

The primary relationship between p-divisible groups and homotopy theory can be phrased as follows (see [BL10, Theorem 8.1.4] and [Lur18]; in the latter reference, the result is not phrased like this, but it is not hard to extrapolate from there).

Theorem 3.1.7 (Lurie). *Let X be a locally Noetherian separated Deligne-Mumford stack over* $\operatorname{Spec} \mathbf{Z}_p$*, and let* $\mathbf{G} \to X$ *be a p-divisible group over X of constant height n and (formal) dimension* 1*. Suppose that there exists an étale cover $\pi : Y \to X$ by a scheme Y such that for every point in $x \in Y_p^\wedge$, the induced map $Y_x^\wedge \to \mathrm{LT}_{(\pi^*\mathbf{G})_x}$ classifying the deformation $(\pi^*\mathbf{G})|_{Y_x^\wedge}$ of $(\pi^*\mathbf{G})_x$ is an isomorphism of formal schemes. Then there exists a sheaf of $\mathbf{E}_\infty$-rings $\mathscr{O}_{\mathscr{X}}$ on the étale site of $X_p^\wedge$ such that*

- *for every formal affine étale open $f : \operatorname{Spf} R \to X_p^\wedge$, the $\mathbf{E}_\infty$-ring $\mathscr{O}_{\mathscr{X}}(\operatorname{Spf} R)$ is an even periodic $\mathbf{E}_\infty$-ring with $\pi_0 \mathscr{O}_{\mathscr{X}}(\operatorname{Spf} R) = R$; and*
- *there is an isomorphism (natural in the map f) between the formal group associated to $\mathscr{O}_{\mathscr{X}}(\operatorname{Spf} R)$ and $f^*\mathbf{G}$.*

Remark 3.1.8. A simpler way to state the theorem is: let $\mathscr{M}_{\mathrm{BT}}(n)$ be the moduli stack of p-divisible groups of height n, and suppose that X is a locally Noetherian separated Deligne-Mumford stack over $\operatorname{Spec} \mathbf{Z}_p$. Let $f : X \to \mathscr{M}_{\mathrm{BT}}(n)$ classify a p-divisible group of height n over X. If f is formally étale, then there is a sheaf of $\mathbf{E}_\infty$-rings on the étale site of $X_p^\wedge$ satisfying the conditions of Theorem 3.1.7. We warn the reader that one needs to be a lot more careful when stating the result in this way: for one, the stack $\mathscr{M}_{\mathrm{BT}}(n)$ is not an algebraic stack in the *fpqc* topology; rather, it is a pro-Artin stack. (Note that the moduli of p-torsion finite flat group schemes is not a tame stack: the automorphism group of the étale group scheme $\underline{(\mathbf{Z}/p^k)^n}$ of rank p^{nk} is $\mathrm{GL}_n(\mathbf{Z}/p^k)$, which has order divisible by p.)

3.2 Constructing power operations

Let E be an $\mathbf{E}_\infty$-ring spectrum. The product of cocycles in E-cohomology is generally not strictly commutative. Power operations provide a concrete method to measure this discrepancy. In this section, we review the requisite background from [AHS04].

Let X be a space. Consider a cohomology class $\alpha \in E^0(X) = [\Sigma^\infty_+ X, E]$, represented by a map $\alpha : \Sigma^\infty_+ X \to E$. The nth power of α is represented by a map $\alpha^n : (\Sigma^\infty_+ X)^{\wedge n} \to E^{\wedge n} \to E$, where the latter map is the multiplication on E. Since E is an $\mathbf{E}_\infty$-ring, the map $E^{\wedge n} \to E$ factors through $(\Sigma^\infty_+ E\Sigma_n \wedge$

$E^{\wedge n})/\Sigma_n = (E^{\wedge n})_{h\Sigma_n}$. It follows that the map α^n in turn factorizes as a map

$$\widetilde{\alpha^n} : (\Sigma^\infty_+(E\Sigma_n \times X^{\times n}))/\Sigma_n \to (E^{\wedge n})_{h\Sigma_n} \to E.$$

Since X is a space, it is in particular an $\mathbf{E}_\infty$-coalgebra (via the diagonal map $\Sigma^\infty_+\Delta : \Sigma^\infty_+ X \to \Sigma^\infty_+ X^{\times n}$), so we obtain a composite map

$$\widetilde{\alpha^n} : \Sigma^\infty_+ E\Sigma_n/\Sigma_n \wedge \Sigma^\infty_+ X \simeq \Sigma^\infty_+(B\Sigma_n \times X) \to (E^{\wedge n})_{h\Sigma_n} \to E.$$

This defines a map $P_n : E^0(X) \to E^0(B\Sigma_n \times X)$, which is easily seen to be multiplicative. Note that if the multiplication on E was strictly commutative, so that the multiplication map factored through $E^{\wedge n}/\Sigma_n$, the map P_n would simply be the nth power map.

Example 3.2.1. One can extend the above construction to a map $P_n : E^*(X) \to E^*(B\Sigma_n \times X)$. We illustrate this concretely with the case when E is the Eilenberg-Maclane spectrum $\mathbf{F}_2$. Let $n = 2$. Suppose X is an arbitrary space, and consider the composite $P_2 : \mathrm{H}^*(X; \mathbf{F}_2) \to \mathrm{H}^*(B\Sigma_2 \times X; \mathbf{F}_2) \cong \mathrm{H}^*(B\Sigma_2; \mathbf{F}_2) \otimes_{\mathbf{F}_2} \mathrm{H}^*(X; \mathbf{F}_2)$. Since $B\Sigma_2 \simeq \mathbf{R}P^\infty$ and $\mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2) \cong \mathbf{F}_2[t]$, this is a map $P_2 : \mathrm{H}^*(X; \mathbf{F}_2) \to \mathrm{H}^*(X; \mathbf{F}_2)[t]$. If $\alpha \in \mathrm{H}^*(X; \mathbf{F}_2)$, then $P_2(\alpha) = \sum_{i\geq 0} f_i(\alpha)t^i$, and one can prove that $f_i(\alpha) = \mathrm{Sq}^i(\alpha)$.

The map P_n is *not* additive. Since the space X will be irrelevant in the sequel, we will restrict to the case $X = *$. Even in this case, the map $P_n : E^0(*) \to E^0(B\Sigma_n)$ is not additive. Indeed, one can prove the following result.

Lemma 3.2.2. *Let* Tr_k *denote the map* $E^0(B\Sigma_k) \otimes_{E^0} E^0(B\Sigma_{n-k}) \to E^0(B\Sigma_k \times B\Sigma_{n-k}) \to E^0(B\Sigma_n)$ *induced by the transfer map along the inclusion* $\Sigma_k \times \Sigma_{n-k} \to \Sigma_n$ *on* E*-cohomology. Then*

$$P_n(x+y) = P_n(x) + P_n(y) + \sum_{1\leq k\leq n-1} \mathrm{Tr}_k(P_k(x) \otimes P_{n-k}(y)).$$

To obtain a ring map, we therefore need to quotient out $E^0(B\Sigma_n)$ by the image of Tr_k for $1 \leq k \leq n-1$. In other words, if I denotes the ideal generated by the image of Tr_k as k ranges between 1 and $n-1$ (inclusive), then the map $\psi^n : E^0 \to E^0(B\Sigma_n)/\mathrm{Tr}$ is a ring map. For p prime, the composite $E^0 \to E^0(B\Sigma_n)/\operatorname{Tr} \to E^0/n$ is precisely the pth power map, so the map ψ^p is a lift of the pth power map.

Let us now specialize to the case when E is an even-periodic $\mathbf{E}_\infty$-ring such that E^0 is a p-torsion free complete local ring with maximal ideal $\mathfrak{m}$ such that $p \in \mathfrak{m}$, and such that if $\mathbf{G}$ is the formal group on E^0, then $\mathbf{G} \otimes_{E^0} E^0/\mathfrak{m}$ has finite constant height. There is a fiber sequence $\mathbf{Z} \xrightarrow{p^n} \mathbf{Z} \to \mathbf{Z}/p^n$ of Eilenberg-Maclane spectra, which gives a fiber sequence $S^1 \to BC_{p^n} \to \mathbf{C}P^\infty$ after rotating and delooping. The associated Gysin sequence shows that $E^0(BC_{p^n}) \cong E^0[[t]]/[p^n](t)$, where t is a complex orientation of E (corresponding to a choice of coordinate on the universal deformation $\mathbf{G}$). In particular, $\operatorname{Spec} E^0(BC_{p^n})$ is precisely the group scheme $\mathbf{G}[p^n]$ of p^n-torsion in the universal deformation $\mathbf{G}$.

One then obtains the following result (see [AHS04, Theorem 3.25]):

Theorem 3.2.3 (Ando-Hopkins-Strickland, Rezk)**.** *Let* $\psi^*\mathbf{G}$ *denote the induced formal group over* $\operatorname{Spf} E^0(B\Sigma_{p^n})/\operatorname{Tr}$*, and let* $\operatorname{Spec} E^0(BC_{p^n}) \subseteq \psi^*\mathbf{G}$ *denote the canonical subgroup of rank* p^n*. Then* $(\operatorname{Spf} E^0(B\Sigma_{p^n})/\operatorname{Tr}, \operatorname{Spec} E^0(BC_{p^n}))$ *is universal among pairs* (A, C) *of* E^0*-algebras* $f : E^0 \to A$ *and cyclic subgroups* $C \subseteq f^*\mathbf{G}$ *of rank* p^n*.*

In particular, any such pair (A, C) gives rise to an E^0-algebra map $\widetilde{f} : E^0(B\Sigma_{p^n})/\operatorname{Tr} \to A$ such that the map $f^*\mathbf{G} \to \widetilde{f}^*\psi^*\mathbf{G}$ of formal groups is an isogeny of degree p with kernel C.

Remark 3.2.4. It is interesting and useful to consider the case of height 1. Indeed, in this case, there is exactly *one* subgroup scheme of rank p^n, given by $\mathbf{G}[p^n] \subseteq \mathbf{G}$. It follows from Theorem 3.2.3 that there is a canonical isomorphism $E^0(B\Sigma_{p^n})/\operatorname{Tr} \cong E^0$, and the resulting map ψ^{p^n} is just a ring endomorphism $E^0 \to E^0$. We will denote by $\theta : E^0 \to E^0$ the map of sets defined by $\theta(x) = (\psi^p(x) - x^p)/p$.

Example 3.2.5. Let $E = K_p$ be p-adic K-theory, so that the associated formal group does indeed have height 1. Then the map ψ^{p^n} is the p^nth Adams operation.

In the case of Morava E-theory, this result is even simpler to state. Recall that if k is a perfect field of characteristic p (we did not need this assumption for Theorem 3.2.3) and $\mathbf{G}_0$ is a formal group of height n over k, then we get an

$\mathbf{E}_\infty$-ring $E = E(k, \mathbf{G}_0)$ functorially in the pair $(k, \mathbf{G}_0)$ such that $\pi_0 E(k, \mathbf{G}_0)$ is the universal ring representing deformations of $\mathbf{G}_0$. Let $\mathbf{G}$ denote the universal formal group over $\pi_0 E(k, \mathbf{G}_0)$. Then:

Corollary 3.2.6. *The affine formal scheme* $\mathrm{LT}_{\Gamma_0(p^n)} = \operatorname{Spf} E^0(B\Sigma_{p^n})/\operatorname{Tr}$ *represents the moduli problem sending an Artinian local* $\mathrm{W}(k)$*-algebra* $(A, \mathfrak{m}_A)$ *to the groupoid (which is* a posteriori *a set) of formal groups* $\mathbf{G}'$ *over* A *along with an isomorphism* $\iota : \mathbf{G} \otimes_A A/\mathfrak{m}_A \xrightarrow{\simeq} \mathbf{G}_0 \otimes_k A/\mathfrak{m}_A$ *as well as a cyclic subgroup scheme* C *of rank* p.

Remark 3.2.7. The notation $\mathrm{LT}_{\Gamma_0(p^n)}$ is not an accident: if $\mathbf{G}_0$ is the completion at the identity section of a supersingular curve C over $\mathbf{F}_{p^2}$, for instance, then $\mathrm{LT}_{\Gamma_0(p^n)}$ is the completion of the Artin stack $\mathscr{M}_{\Gamma_0(p^n)}$ of (smooth, but one could equally well work with generalized) elliptic curves with a $\Gamma_0(p^n)$-structure (i.e., a cyclic subgroup of order p^n) at the geometric point defined by C.

Remark 3.2.8. This interpretation of $E^0(B\Sigma_{p^n})/\operatorname{Tr}$ is quite helpful; it gives a far more conceptual (and purely algebraic!) construction of power operations on Morava E-theory, for instance. To explain this construction, note that the scheme $\mathrm{LT}_{\Gamma_0(p^n)}$ has a universal formal group $\mathbf{G}'$ defined over it, along with a canonical subgroup C of rank p^n. This defines two maps $\pi_1, \pi_2 : \mathrm{LT}_{\Gamma_0(p^n)} \to \mathrm{LT}$, roughly given on the level of moduli problems by $\pi_1(\mathbf{G}', C) = \mathbf{G}'$ (which is just the universal deformation $\mathbf{G}$ itself), and $\pi_2(\mathbf{G}', C) = \mathbf{G}'/C$ (the quotient of $\mathbf{G}'$ by the subgroup C). On the level of coefficient rings, the first map is simply the map $E^0 \to E^0(B\Sigma_{p^n})/\operatorname{Tr}$ given by the projection $B\Sigma_{p^n} \to *$, while the second is precisely the power operation ψ^{p^n} (as can be deduced from Theorem 3.2.3).

Remark 3.2.9. The scheme $\mathrm{LT}_{\Gamma_0(p^n)}$ admits an Atkin-Lehner involution (see, e.g., [KM85, Example 11.3.1] for the discussion in the case of elliptic curves, which passes through to the setting of p-divisible groups with a little bit of work): the universal deformation $\mathbf{G}$ and its subgroup C of rank p^n is sent to the deformation $\mathbf{G}/C$ with subgroup $\mathbf{G}[p^n]/C$. One must still check that $\mathbf{G}/C$ is a deformation of $\mathbf{G}_0$, but this follows from the discussion in [AHS04, Section 12.3].

Warning 3.2.10. Although the mapping spectrum $\underline{\mathrm{Map}}(\Sigma^\infty_+ B\Sigma_{p^n}, E)$ is an $\mathbf{E}_\infty$-E-algebra, it is not true that the E^0-algebra $E^0(\overline{B\Sigma_{p^n}})/\operatorname{Tr}$ can in general be realized as the π_0 of an $\mathbf{E}_\infty$-E-algebra. This is proved as [Dev17, Theorem 3.5] for the formal affine scheme $\mathrm{LT}_{\Gamma_1(p)}$, which is the formal affine scheme $\operatorname{Spf} E^0(BC_p)/\operatorname{Tr}$. To observe this, note that the discussion prior to Theorem 3.2.3 shows that $\operatorname{Spec} E^0(BC_p)$ may be regarded as the p-torsion $\mathbf{G}[p]$ of the universal deformation $\mathbf{G}$.

It is not hard to come up with a moduli-theoretic interpretation for points of $\mathbf{G}[p]$: an A-point of $\mathbf{G}[p]$ is precisely a formal group over A deforming $\mathbf{G}_0$ along with a p-torsion point. This, however, does not prohibit the choice of the identity as the p-torsion point. (The $\Gamma_1(p)$-moduli problem precisely restricts away from this degenerate case.) Nevertheless, the stable transfer map $\Sigma^\infty_+ BC_p \to \Sigma^\infty_+ * = S^0$ induces a map $E^0 \to E^0(BC_p)$, and the ideal generated by its image is the transfer ideal Tr; quotienting out by this ideal precisely kills the choice of the identity. It follows that the formal scheme $\operatorname{Spf} E^0(BC_p)/\operatorname{Tr}$ is precisely the moduli problem whose A-points are formal groups over A deforming $\mathbf{G}_0$ with a chosen point of order exactly p, which is precisely the moduli problem $\mathrm{LT}_{\Gamma_1(p)}$.

3.3 Transchromatic compatibility

Let E denote a Morava E-theory at height n, so there is a ring map $\psi^p : E^0 \to E^0(B\Sigma_p)/\operatorname{Tr}$. Let $m < n$, so $L_{K(m)}E$ is a $K(m)$-local $\mathbf{E}_\infty$-ring. Choose a (noncanonical) isomorphism $\pi_* E \cong \mathrm{W}(k)[[u_1, \cdots, u_{n-1}]][u^{\pm 1}]$. It follows from Proposition 2.3.8 that $\pi_0 L_{K(m)}E$ is the degree 0 component of the graded ring $(u_m^{-1}\mathrm{W}(k)[[u_1, \cdots, u_{n-1}]])^\wedge_{(p,u_1,\cdots,u_{m-1})}$. Let $\mathbf{G}^{=m}$ denote the formal group of height m defined over $\pi_0 L_{K(m)}E$ coming from its complex orientation. The goal of this section is to prove Theorem 3.3.2, which is a generalization of [LN12, Proposition 8.1] to higher chromatic heights. Since the discussion is a little technical, we recommend that the casual reader only look at Corollary 3.3.4 (which is simply [LN12, Proposition 8.1]) and the preceding paragraph before moving on, since that is the only result which will be used in the proof of Theorem 1.0.1.

The construction described in Section 3.2 gives a map $\pi_0 L_{K(m)}E \to$

$(L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$. This begets a natural commutative square

$$\begin{array}{ccc} E^0 & \xrightarrow{\psi_n^p} & E^0(B\Sigma_p)/\operatorname{Tr} \\ \downarrow f & & \downarrow \widetilde{f} \\ \pi_0 L_{K(m)}E & \xrightarrow[\psi_m^p]{} & (L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr} \end{array}$$

Note that the maps f and $\widetilde{f}$ are not decorated with m; we hope that the dependence on m will be evident in what follows. In this section, we will prove Theorem 3.3.2, which dictates the (purely algebraic) construction of a map $\widetilde{\psi_n^p}$ such that $\widetilde{\psi_n^p} = \psi_m^p$.

In order to do so, we need to be able to relate the p-divisible group defined over $\operatorname{Spf}(L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$ with the p-divisible group defined over $\operatorname{Spf} E^0(B\Sigma_p)/\operatorname{Tr}$. We begin by noting that the work in [Sta13, Section 2.1] implies the following result.

Lemma 3.3.1. *There is an exact sequence*

$$0 \to \mathbf{G}^{=m} \to f^*\mathbf{G} \to (f^*\mathbf{G})^{\mathrm{et}} \to 0$$

which exhibits $\mathbf{G}^{=m}$ as the connected component of the p-divisible group $f^*\mathbf{G}$ over $\operatorname{Spf} \pi_0 L_{K(m)}E$.

Note that $f^*\mathbf{G}$ is of height n as a p-divisible group, but its connected component is of height m.

The $\mathbf{E}_\infty$-ring $L_{K(m)}E$ satisfies the conditions of Theorem 3.2.3, so $\operatorname{Spf}(L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$ has a universal formal group (i.e., connected p-divisible group) $\mathbf{G}^{=m}_{\Gamma_0(p)}$ defined over it, which has a canonical subgroup scheme of rank p. We will denote the underlying formal group of the universal deformation with a subgroup scheme of rank p defined over $\mathrm{LT}_{\Gamma_0(p)}$ by $\mathbf{G}_{\Gamma_0(p)}$.

Let $i_m : \pi_0 L_{K(m)}E \to (L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$ denote the map induced by the projection $B\Sigma_p \to *$; this is simply the inclusion. This induces a map, also denoted i_m, after taking Spf. By construction, there is an isomorphism between $i_m^*\mathbf{G}^{=m}$ and $\mathbf{G}^{=m}_{\Gamma_0(p)}$. Let $i_n : E^0 \to E^0(B\Sigma_p)/\operatorname{Tr}$ denote the map induced by the projection $B\Sigma_p \to *$ (again, this is just the inclusion). There is a commutative diagram

$$\begin{array}{ccc} E^0 & \xrightarrow{i_n} & E^0(B\Sigma_p)/\operatorname{Tr} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widetilde{f}} \\ \pi_0 L_{K(m)}E & \xrightarrow[i_m]{} & (L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}, \end{array}$$

which gives natural isomorphisms $i_m^* f^* \cong \widetilde{f}^* i_n^*$. In particular, we learn that

$$i_m^* f^* \mathbf{G} \cong \widetilde{f}^* i_n^* \mathbf{G} = \widetilde{f}^* \mathbf{G}_{\Gamma_0(p)}.$$

The morphisms i_n and i_m are flat, so the pullback functors i_n^* and i_m^* are exact (on *fppf* sheaves). It follows that the exact sequence of Lemma 3.3.1 pulls back to an exact sequence

$$0 \to \mathbf{G}_{\Gamma_0(p)}^{=m} \to \widetilde{f}^* \mathbf{G}_{\Gamma_0(p)} \to (\widetilde{f}^* \mathbf{G}_{\Gamma_0(p)})^{\mathrm{et}} \to 0$$

where $(\widetilde{f}^* \mathbf{G}_{\Gamma_0(p)})^{\mathrm{et}}$ is defined to be $i_m^*(f^*\mathbf{G})^{\mathrm{et}}$. This exact sequence exhibits $\mathbf{G}_{\Gamma_0(p)}^{=m}$ as the connected component of the p-divisible group $\widetilde{f}^* \mathbf{G}_{\Gamma_0(p)}$. Let $\iota : \mathbf{G}_{\Gamma_0(p)}^{=m} \to \widetilde{f}^* \mathbf{G}_{\Gamma_0(p)}$ denote the inclusion of this connected component.

Let $C^{=m} \subseteq \mathbf{G}_{\Gamma_0(p)}^{=m}$ denote the universal subgroup scheme of order p (which exists because of Theorem 3.2.3). Then the map $\iota : \mathbf{G}_{\Gamma_0(p)}^{=m} \to \widetilde{f}^* \mathbf{G}_{\Gamma_0(p)}$ defines a subgroup $\iota(C^{=m}) \subseteq \widetilde{f}^* \mathbf{G}_{\Gamma_0(p)}$ of order p. By construction, the quotient $\widetilde{f}^* \mathbf{G}_{\Gamma_0(p)}/\iota(C^{=m})$ is another deformation (with formal height m) of the connected p-divisible group $\mathbf{G}_0$ (of height n) over the perfect field k. If the formal affine scheme $\operatorname{Spf} \pi_0 L_{K(m)}E$ classified the moduli problem of p-divisible groups of height n and formal height m which deform $\mathbf{G}_0$, then we would obtain a map $\widetilde{\psi_n^p} : \pi_0 L_{K(m)}E \to (L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$ classifying the deformation $\widetilde{f}^* \mathbf{G}_{\Gamma_0(p)}/\iota(C^{=m})$. Unfortunately, this is not quite true — the ring $\pi_0 L_{K(m)}E$ classifies a more sophisticated moduli problem [MPS15]; nevertheless, it is possible to show using [MPS15, Theorem 36] that one can indeed construct a map $\widetilde{\psi_n^p}$ as above[3].

[3] We provide a brief sketch, using the tools from [MPS15]. The subgroup $C^{=m} \subseteq \mathbf{G}_{\Gamma_0(p)}^{=m}$ is the pullback $\widetilde{f}^* C$ of the universal subgroup scheme $C \subseteq \mathbf{G}_{\Gamma_0(p)}$ (on $\mathrm{LT}_{\Gamma_0(p)}$) of rank p. There is a map Spp $(L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr} \to \operatorname{Spf}(L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$, which induces a

The main result regarding the map $\widetilde{\psi_n^p}$ is the following.

Theorem 3.3.2. *The maps* $\widetilde{\psi_n^p}, \psi_m^p : \pi_0 L_{K(m)}E \to (L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$ *constructed above are the same.*

Proof. We argue as in [LN12, Proposition 8.1]. As discussed in Remark 3.2.8, the universal property of the map $\psi_m^p : \pi_0 L_{K(m)}E \to (L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$ is that it classifies the universal deformation $\widetilde{f}^*\mathbf{G}_{\Gamma_0(p)}/\iota(C^{=m})$; this is precisely the characterization of the map $\widetilde{\psi_n^p}$. □

Theorem 3.3.2 is a compatibility between power operations of different heights.

Remark 3.3.3. As with Theorem 3.2.3, it is possible to prove Theorem 3.3.2 in the case when E is an even-periodic $\mathbf{E}_\infty$-ring such that E^0 is a p-torsion free complete local ring with maximal ideal $\mathfrak{m}$ such that $p \in \mathfrak{m}$, and such that if $\mathbf{G}$ is the formal group on E^0, then $\mathbf{G} \otimes_{E^0} E^0/\mathfrak{m}$ has finite constant height. This can also be generalized to a result involving (the global sections of) those derived stacks arising from Theorem 3.1.7; however, we will not do so here. Since the arguments provided above are very general, the key tool is writing down a precise relationship between $X_{\Gamma_0(p)} = X \times_{\mathscr{M}_{\mathrm{BT}}(n)} \mathscr{M}_{\mathrm{BT}}(n)_{\Gamma_0(p)}$ and $\Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})^0(B\Sigma_p)/\operatorname{Tr}$. We note that shadows of this result have already appeared in the literature: for instance, the work in [AHR10] relies heavily on a related result.

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formal group, also denoted $\widetilde{f}^*\mathbf{G}_{\Gamma_0(p)}/\iota(C^{=m})$ over Spp $(L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$ (by [MPS15, Lemma 20] when $m = 1$ and $k = 0$). It follows that there is a commuting diagram

$$\begin{array}{ccccc} \mathbf{G}_0 & \longrightarrow & \mathbf{G}_{\Gamma_0(p)}/C & \longleftarrow & \widetilde{f}^*\mathbf{G}_{\Gamma_0(p)}/\iota(C^{=m}) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spf} E^0(B\Sigma_p)/\operatorname{Tr} & \underset{\widetilde{f}}{\longleftarrow} & \operatorname{Spp} (L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}. \end{array}$$

By [MPS15, Definition 35 and Theorem 36], this defines a point in the moduli problem defined by the 1-staged pipe ring $E^0 \to \pi_0 L_{K(m)}E$, and in particular (e.g. by the restatement of [MPS15, Theorem 36] as [MPS15, Main theorem]) defines a ring map $\widetilde{\psi_n^p} : \pi_0 L_{K(m)}E \to (L_{K(m)}E)^0(B\Sigma_p)/\operatorname{Tr}$, as desired. More care is required to make this precise, but we will not do so here.

For the sake of concreteness, let us specialize to the case $m = 1$, and let us pick an isomorphism $E^0 \cong \mathrm{W}(\mathbf{F}_{p^2})[[u_1]]$ (this is the same as picking a coordinate on $\mathbf{G}$). Then, we get a diagram

$$\begin{array}{ccccc} E^0 \cong \mathrm{W}(\mathbf{F}_{p^2})[[u_1]] & \xrightarrow{\psi_2^p} & E^0(B\Sigma_p)/\operatorname{Tr} & & \\ \big\downarrow f & & \big\downarrow \widetilde{f} & & \\ \pi_0 L_{K(1)}E \cong \mathrm{W}(\mathbf{F}_{p^2})((u_1))^\wedge_p & \xrightarrow[\psi_1^p]{} & (L_{K(1)}E)^0(B\Sigma_p)/\operatorname{Tr} & \xrightarrow[\cong]{} & \pi_0 L_{K(1)}E \end{array}$$

The canonical isomorphism on the bottom line comes from Remark 3.2.4. The ring $E^0(B\Sigma_p)/\operatorname{Tr}$ is isomorphic to $\mathrm{W}(\mathbf{F}_{p^2})[[u_1]][d]/w(d)$, where $w(d)$ is a monic polynomial with coefficients in $\mathrm{W}(\mathbf{F}_{p^2})[[u_1]]$ since $E^0(B\Sigma_p)$ is free of finite rank over E^0 (by [Str98, Theorem 3.2]). Solutions of $w(d)$ give rise to subgroups of order p, so again by the discussion in Remark 3.2.4, we learn that there is a unique solution to $w(d)$ in the ring $\mathrm{W}(\mathbf{F}_{p^2})((u_1))^\wedge_p$. The map $\widetilde{\psi_2^p}$ defined above is then just the canonical extension of $\psi_2^p : \mathrm{W}(\mathbf{F}_{p^2})[[u_1]] \to \mathrm{W}(\mathbf{F}_{p^2})[[u_1]][d]/w(d)$ to a ring map $\mathrm{W}(\mathbf{F}_{p^2})((u_1))^\wedge_p \to \mathrm{W}(\mathbf{F}_{p^2})((u_1))^\wedge_p$. We can then conclude the following result, which is stated as [LN12, Proposition 8.1], from Theorem 3.3.2:

Corollary 3.3.4. *The maps* $\widetilde{\psi_2^p}, \psi_1^p : \mathrm{W}(\mathbf{F}_{p^2})((u_1))^\wedge_p \to \mathrm{W}(\mathbf{F}_{p^2})((u_1))^\wedge_p$ *are the same.*

4 Constructing $E(2)$

4.1 Realization problems

To fix the issue brought up by Proposition 2.4.9, we will take a hint from the statement of Conjecture 2.4.7, and make the following definition (also see [HM17, Definition 3.14]):

Definition 4.1.1. An $E(n)$-realization problem $(E_*, \mathbf{G}_f)$ is a graded ring E_* along with a formal group law $\mathbf{G}_f$ classified by a map $MU_* \to E_*$, such that if $f : BP_* \to E_*$ is the p-typification of its formal group law, then there is a choice of indecomposables $v_1, \cdots, v_n \in BP_*$ with $|v_i| = 2(p^i - 1)$, such that the image of v_n under f is invertible, and such that the induced map $\mathbf{Z}_{(p)}[v_1, \cdots, v_{n-1}, v_n^{\pm 1}] \to E_*$ is an isomorphism.

If E is p-completed, then we will often abuse terminology by saying that $(E_*, \mathbf{G}_f)$ is a (p-complete) $E(n)$-realization problem if the induced map $\mathbf{Z}_{(p)}[v_1, \cdots, v_{n-1}, v_n^{\pm 1}]_p^\wedge \to E_*$ is an isomorphism.

Definition 4.1.2. A solution to an $E(n)$-realization problem $(E_*, \mathbf{G}_f)$ is a complex-oriented $\mathbf{E}_\infty$-ring A such that there is an isomorphism $\pi_* A \to E_*$ which sends the formal group law of A to the formal group law $\mathbf{G}_f$ over E_*.

4.2 The proof of Theorem 1.0.1

In this section, we will begin the proof of Theorem 1.0.1 using the tools described above. We will restrict ourselves to the case when $p > 2$, although it is possible to remove this restriction. Most of the arguments are very similar to those provided in [LN12, Sections 5 and 6]. In order to apply Goerss-Hopkins obstruction theory, we will need to specify the homotopy groups of R; after choosing Zhu's $\mathscr{P}_n$-model for E (which picks out the element u_1), we will define R to be a complex-oriented homotopy commutative ring spectrum with homotopy groups $R_* = \mathrm{W}(\mathbf{F}_{p^2})[u_1][u^{\pm 1}]$ such that the complex orientation $MU_* \to E_*$ factors through R_*. It is not immediately evident that the homotopy commutative ring spectrum R exists; however, since the map $MU_* \to R_*$ is flat, the Landweber exact functor theorem provides us with such a homotopy commutative ring spectrum R.

We will delay the proofs of the following two technical results to the next two sections.

Lemma 4.2.1. *Let $F = L_{K(1)} L_{K(2)} R = L_{K(1)} E$, and let $T_* \subseteq F_*$ denote the subring of F_* given by the image of the monomorphism $\pi_* L_{K(1)} R \to \pi_* L_{K(1)} L_{K(2)} R$. Then the degree zero component T_0 of T_* is closed under the power operation θ if and only if there is an $\mathbf{E}_\infty$-ring structure on the homotopy commutative ring spectrum $L_{K(1)} R$ such that $G = \mathbf{F}_{p^2}^\times \rtimes \mathrm{Gal}(\mathbf{F}_{p^2}/\mathbf{F}_p)$ acts on $L_{K(1)} R$ by $\mathbf{E}_\infty$-ring maps, and such that there is an $\mathbf{E}_\infty$-ring map $L_{K(1)} R \to F$ which is equivariant for the action of G.*

Lemma 4.2.2. *The subring $\mathrm{W}(\mathbf{F}_{p^2})(u_1)_p^\wedge$ is closed under the power operation $\widetilde{\psi_2^p} = \psi_1^p : \mathrm{W}(\mathbf{F}_{p^2})((u_1))_p^\wedge \to \mathrm{W}(\mathbf{F}_{p^2})((u_1))_p^\wedge$.*

We will now proceed to the proofs of Theorem 1.0.1 and Corollary 1.0.2 modulo Lemma 4.2.2 and Proposition 4.2.1.

Proof of Theorem 1.0.1. By Lemma 4.2.2 and Lemma 4.2.1, we conclude that there is an $\mathbf{E}_\infty$-ring $L_{K(1)}R$ with an action of G by $\mathbf{E}_\infty$-ring maps, along with G-equivariant $\mathbf{E}_\infty$-ring maps $L_{K(1)}R \to L_{K(1)}L_{K(2)}R \leftarrow L_{K(2)}R = E$. Let $\widetilde{R}$ be the $\mathbf{E}_\infty$-ring obtained as the pullback

$$\begin{array}{ccc} \widetilde{R} & \longrightarrow & L_{K(2)}R \\ \downarrow & & \downarrow \\ L_{K(1)}R & \longrightarrow & L_{K(1)}L_{K(2)}R. \end{array}$$

We check that the $\mathbf{E}_\infty$-ring $\widetilde{R}$ satisfies the conditions of Theorem 1.0.1.

<u>Part 1:</u> There is a long exact sequence

$$\begin{aligned} 0 = \pi_{2n+1}L_{K(1)}L_{K(2)}R \to \pi_{2n}\widetilde{R} \to \pi_{2n}L_{K(1)}R \oplus \pi_{2n}L_{K(2)}R \\ \to \pi_{2n}L_{K(1)}L_{K(2)}R \to \pi_{2n-1}\widetilde{R} \to \pi_{2n-1}L_{K(1)}L_{K(2)}R = 0. \end{aligned}$$

By Proposition 2.3.8, there are isomorphisms

$$\begin{aligned} \pi_*L_{K(2)}R &\cong \mathrm{W}(\mathbf{F}_{p^2})[[u_1]][u^{\pm 1}], \\ \pi_*L_{K(1)}L_{K(2)}R &\cong \mathrm{W}(\mathbf{F}_{p^2})((u_1))^\wedge_p[u^{\pm 1}] \\ \pi_*L_{K(1)}R &\cong \mathrm{W}(\mathbf{F}_{p^2})(u_1)^\wedge_p[u^{\pm 1}]. \end{aligned}$$

It follows that the outer two terms in the above long exact sequence are zero, since $L_{K(1)}L_{K(2)}R$ is concentrated in even degrees. We also find that every element of $\pi_{2n}L_{K(1)}L_{K(2)}R = \mathrm{W}(\mathbf{F}_{p^2})((u_1))^\wedge_p\{u^n\}$ is a sum of elements in (the images of) $\pi_{2n}L_{K(1)}R = \mathrm{W}(\mathbf{F}_{p^2})(u_1)^\wedge_p\{u^n\}$ and $\pi_{2n}L_{K(2)}R = \mathrm{W}(\mathbf{F}_{p^2})[[u_1]]\{u^n\}$. In particular, $\pi_{2n-1}\widetilde{R} = 0$, and $\pi_{2n}\widetilde{R} = \mathrm{W}(\mathbf{F}_{p^2})[u_1]\{u^n\}$. As a ring, we have $\pi_*\widetilde{R} = \mathrm{W}(\mathbf{F}_{p^2})[u_1][u^{\pm 1}]$, so we obtain the desired $\mathbf{E}_\infty$-ring R of Theorem 1.0.1 as the $\mathbf{E}_\infty$-ring $\widetilde{R}$. This establishes part (1) of Theorem 1.0.1.

<u>Part 2:</u> It remains to prove part (2). To do so, we need to understand the action of G on $\pi_*\widetilde{R}$. Since there is a G-equivariant inclusion of $\pi_*\widetilde{R}$ into $\pi_*L_{K(2)}R = \pi_*E$, it will suffice to understand the G-action on $\pi_*E \cong$

$W(\mathbf{F}_{p^2})[[u_1]][u^{\pm 1}]$. Let $\sigma \in \mathrm{Gal}(\mathbf{F}_{p^2}/\mathbf{F}_p)$ and $\zeta \in \mathbf{F}_{p^2}^{\times}$ be generators. Then σ acts trivially on u_1 and u, and acts by Galois conjugation on $W(\mathbf{F}_{p^2})$. We will show that ζ sends $u_1 \mapsto \zeta^{p-1}u_1$ and $u \mapsto \zeta^{-1}u$. We follow [Hen07, Equation (8), p. 32]. Let $\mathbf{G}_f$ denote the formal group law of Zhu's preferred $\mathscr{P}_n$-model for E. If g is an element of the Morava stabilizer group, then according to [DH95], the action of g is determined as follows. Lift g to an element $\widetilde{g} \in (\pi_0 E)[[x]]$, so that there is a unique continuous ring map $g_* : \pi_0 E \to \pi_0 E$ and an isomorphism $h \in (\pi_0 E)[[x]]$ between $g_* \mathbf{G}_f$ and $F(x,y) = \widetilde{g}^{-1}\mathbf{G}_f(\widetilde{g}(x), \widetilde{g}(y))$. The action of g on the element u (note that in [Hen07], the element u lives in degree -2, so his u is our u^{-1}) is given by $g_*(u) = (\widetilde{g}'(0)h'(0))^{-1}u$. Therefore, if g is the Teichmüller lift of an element $a \in \mathbf{F}_{p^2}$, then

$$[p]_F(x) = a^{-1}[p]_{\mathbf{G}_f}(ax) = a^{-1}(p(ax) +_{\mathbf{G}_f} u_1(ax)^p +_{\mathbf{G}_f} (ax)^{p^2}) = px +_F a^{p-1}u_1 x^p +_F x^{p^2},$$

so that $h(x) = x$, $g_*(u_1) = a^{p-1}u_1$, and $g_*(u) = a^{-1}u$. The desired result follows by setting $a = \zeta$.

There is a descent/homotopy fixed points spectral sequence converging to $\pi_* R^{hG}$ with E_2-page $\mathrm{H}^*(G; \pi_* R)$. Since G has order $2(p^2-1)$, which is coprime to p, this spectral sequence collapses (there are no higher cohomology groups), and we conclude that $\pi_* R^{hG} \cong (\pi_* R)^G$. This ring is isomorphic to $\mathbf{Z}_p[u_1 u^{p-1}, u^{\pm(p^2-1)}]$. The complex orientation of R factors through R^{hG}, so there is an induced formal group law $\mathbf{G}'_f$ on $(\pi_* R)^G$. Let v_1 and v_2 be the Hazewinkel generators coming from the p-typification of the formal group law $\mathbf{G}'_f$. Then the $\mathbf{E}_\infty$-ring R^{hG} is a solution to (the p-completion of) the $E(2)$-realization problem $(\mathbf{Z}_p[u_1 u^{p-1}, u^{\pm(p^2-1)}], \mathbf{G}'_f)$ in sense of Definition 4.1.1. □

As a consequence, we have:

Proof of Corollary 1.0.2. This is the usual method of reconstructing a p-local $\mathbf{E}_\infty$-ring from its p-completion. Indeed, the formal group law $\mathbf{G}'_f$ over $\mathbf{Z}_p[u_1 u^{p-1}, u^{\pm(p^2-1)}]$ is base changed up from a formal group law, also denoted $\mathbf{G}'_f$, over $\mathbf{Z}_{(p)}[u_1 u^{p-1}, u^{\pm(p^2-1)}]$. Let $E(2)_{\mathbf{Q}}$ denote the free $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra on a generator in degree $2(p-1)$ and an invertible generator in degree $2(p^2-1)$. This has homotopy groups given by $\mathbf{Q}[u_1 u^{p-1}, u^{\pm(p^2-1)}]$. Rational stable homotopy theory begets a natural map $E(2)_{\mathbf{Q}} \to L_{\mathbf{Q}} R^{hG}$ of $\mathbf{E}_\infty$-rings, so we define an $\mathbf{E}_\infty$-ring $E(2)$ via the pullback square

$$\begin{array}{ccc} E(2) & \longrightarrow & R^{hG} \\ \downarrow & & \downarrow \\ E(2)_{\mathbf{Q}} & \longrightarrow & L_{\mathbf{Q}} R^{hG}. \end{array}$$

Arguing as in Part 1 of the proof of Theorem 1.0.1, we conclude that $E(2)$ is a solution to the (p-local) $E(2)$-realization problem $(\mathbf{Z}_{(p)}[u_1 u^{p-1}, u^{\pm(p^2-1)}], \mathbf{G}'_f)$.

□

4.3 The proof of Lemma 4.2.1

The proof of Lemma 4.2.1 will need some results from [LN12], summarized in the following theorem.

Theorem 4.3.1 (Lawson-Naumann). *1. Let F be a complex oriented homotopy commutative ring spectrum such that F_* is p-torsion free. Then there is a natural isomorphism $(K_* \otimes_{MUP_*} MUP_* MUP \otimes_{MUP_*} F_*)^\wedge_p \to K^\vee_* F$.*

2. *Let T is a p-adically complete ring, with a formal group law classified by a map $MUP_0 \to T$ such that the induced formal group law on T/p is of exact height 1. Let $V = (K_0 \otimes_{MUP_0} MUP_0 MUP \otimes_{MUP_0} T)^\wedge_p$. Then the map $T \to V$ is an ind-Galois extension with Galois group $\mathbf{Z}_p^\times$.*

3. *Let A_* be an even periodic graded p-adic θ-algebra which is concentrated in even degrees such that*

   - *A_0 is the p-adic completion of an ind-étale extension of $(A_0)^{\mathbf{Z}_p^\times}$,*
   - *$(A_0)^{\mathbf{Z}_p^\times}$ is the p-adic completion of a smooth $\mathbf{Z}_p$-algebra, and*
   - *the continuous cohomology $\mathrm{H}^s_c(\mathbf{Z}_p^\times, A_*[t])$ vanishes for $s > 0$ and all $t \in \mathbf{Z}$.*

   Then there exists a $K(1)$-local $\mathbf{E}_\infty$-ring spectrum R, unique up to weak equivalence, such that $K^\vee_* R \cong A_*$ as θ-algebras (under the power operation structure provided by Remark 3.2.4).

4. *Let E and E' be two $K(1)$-local even periodic complex oriented $\mathbf{E}_\infty$-rings such that E_0 and E'_0 are p-torsion free, and such that E_0 is the p-adic completion of a smooth $\mathbf{Z}_p$-algebra. Then the map*

$$\pi_0 \mathrm{Map}_{\mathbf{E}_\infty}(E, E') \to \mathrm{Hom}_{\theta\text{-}alg}(K_0^\vee E, K_0^\vee E')$$

is bijective. Moreover, if G is a finite group of order prime to p, then any action of G on E in the homotopy category of $\mathbf{E}_\infty$-ring spectra lifts uniquely (up to weak equivalence) to an action of G on E in the category of $\mathbf{E}_\infty$-ring spectra. In this case, if E and E' as above are equipped with actions of G in the category of $\mathbf{E}_\infty$-ring spectra, the natural map

$$\pi_0 \mathrm{Map}^G_{\mathbf{E}_\infty}(E, E') \to \mathrm{Hom}^G_{\theta\text{-}alg}(K_0^\vee E, K_0^\vee E')$$

is an isomorphism.

Proof. In order, citations for these results are [LN12, Lemma 5.4], [LN12, Lemma 5.5.1], [LN12, Corollary 5.16], and [LN12, Proposition 5.18 and Corollary 5.19]. □

Proof of Lemma 4.2.1. Using Theorem 4.3.1.3 and Theorem 4.3.1.4, we will construct an $\mathbf{E}_\infty$-ring structure on the homotopy commutative ring spectrum $L_{K(1)}R$ and a G-equivariant $\mathbf{E}_\infty$-ring map $L_{K(1)}R \to F$. Let $T = L_{K(1)}R$. The orientation $MU_* \to F_*$ factors through T_*, so let $\mathbf{G}$ denote the induced connected p-divisible group over T_*. Note that this is an abuse of notation, since $\mathbf{G}$ generally denotes the connected p-divisible group given by the universal deformation; however, this notation will only persist until the end of the proof.

We begin by showing that the completed K-homology of T satisfies the conditions of Theorem 4.3.1.3. Define graded rings $A_* = (K_* \otimes_{MUP_*} MUP_*MUP \otimes_{MUP_*} T_*)^\wedge_p$ and $B_* = (K_* \otimes_{MUP_*} MUP_*MUP \otimes_{MUP_*} F_*)^\wedge_p$. There is an action of G on each of these rings, given by the associated action on T_* and F_*. Moreover, there is an action of the group $\mathbf{Z}_p^\times$ by the Adams operations. By Theorem 4.3.1.1, we can identify $A_* \cong K_*^\vee T$ and $B_* \cong K_*^\vee F$, equivariantly for the actions of G and $\mathbf{Z}_p^\times$. The subring B_0 is closed under the θ-operation, by Remark 3.2.4.

Using Theorem 4.3.1.2, we find that the maps $T_0 \to A_0$ and $F_0 \to B_0$ are ind-Galois extensions with Galois group $\mathbf{Z}_p^\times$. It follows that A_* is the

p-adic completion of a smooth $\mathbf{Z}_p$-algebra T_* whose degree 0 component is closed under the θ operation, and that the continuous cohomology $\mathrm{H}^s_c(\mathbf{Z}_p^\times; A_*)$ vanishes for $s > 0$. Therefore, Theorem 4.3.1.3 begets a unique (complex oriented) $K(1)$-local $\mathbf{E}_\infty$-ring spectrum $\widetilde{T}$ such that $K_0^\vee \widetilde{T} \simeq A_0$ as θ-algebras. We also have an isomorphism $\pi_* \widetilde{T} \cong T_*$, and the formal group law of $\widetilde{T}$ is given by the map $MU_* \to T_*$. It follows that there is an equivalence of homotopy commutative ring spectra between T and the $\mathbf{E}_\infty$-ring $\widetilde{T}$ regarded as a homotopy commutative ring. We will therefore drop the notation $\widetilde{T}$, and simply write T for this $\mathbf{E}_\infty$-ring.

We will now use Theorem 4.3.1.4 to show that there is a G-equivariant $\mathbf{E}_\infty$-ring map $T \to F$. We will use the fact that T_0 is closed under θ to show that the subring $A_0 \subseteq B_0$ is closed under the θ-operation. Indeed, A_0 is the universal p-adically complete T_0-algebra $f : T_0 \to R$ equipped with an isomorphism $\widehat{\mathbf{G}_m} \to f^*\mathbf{G}$. By Theorem 3.2.3, the map $\psi : B_0 = \pi_0 L_{K(1)}(K_p \wedge F) \to B_0$ gives a composite isogeny $\widehat{\mathbf{G}_m} \xrightarrow{\cong} \mathbf{G} \to \psi^*\mathbf{G}$ of degree p over B_0. Since $\widehat{\mathbf{G}_m}$ has only one subgroup of order p, this isogeny factors as $\widehat{\mathbf{G}_m} \xrightarrow{[p]} \widehat{\mathbf{G}_m} \xrightarrow{\cong} \psi^*\mathbf{G}$, so the universal property of A_0 shows that the map $\psi : B_0 \to B_0$ restricts to a map $\psi : A_0 \to A_0$, which is a lift of Frobenius (again by the universal property). Observe that this map commutes with the action of G.

Theorem 4.3.1.4 tells us that there is a unique map of $\mathbf{E}_\infty$-rings $T \to F$ lifting the map $A_* \to B_*$ of θ-algebras. Moreover, since $|G| = 2(p^2 - 1)$, and p was assumed to be odd, there is a unique lift of the action of G to an action by $\mathbf{E}_\infty$-ring maps such that the map $T \to F$ is G-equivariant, as desired. □

4.4 The proof of Lemma 4.2.2

We will now provide the proof of Lemma 4.2.2, which fills in the final gap in the proof of Theorem 1.0.1. In order to do so, we will need to study power operations on E_2. The computation of these power operations was conducted by Zhu in [Zhu12, Zhu15a, Zhu15b], so we will summarize his results, and provide a proof of Lemma 4.2.2 at the end of this section. Let $n > 3$ be an integer such that $\gcd(n, p) = 1$; then, the moduli problem $\mathscr{M}_{\Gamma_1(n)}$ (over $\operatorname{Spec} \mathbf{Z}[1/n]$, but this is redundant if we work over $\mathbf{Z}_p$) of smooth elliptic curves with a $\Gamma_1(n)$-level structure (i.e., a chosen point of order n) is affine. A $\mathscr{P}_n$-model for E (which will always denote Morava E-theory at height 2) is (roughly) a supersingular elliptic curve $E_0/\mathbf{F}_{p^2}$ along with a deformation of the connected p-divisible group $E_0[p^\infty]$ and a choice of coordinate on this deformation. The choice of a $\mathscr{P}_n$-model for E gives an isomorphism $E^0 \cong \mathrm{W}(\mathbf{F}_{p^2})[[u_1]]$ (as described in [Zhu15b, Example 2.6] in the case $n = 4$ and $p = 5$). This data allows for explicit computations of power operations, as we will now show.

Let $\mathscr{M}_{\Gamma_1(n;p)}$ be the moduli problem over $\mathbf{Z}[1/n]$ for (smooth) elliptic curves equipped with simultaneous $\Gamma_1(n)$-level structures and $\Gamma_0(p)$-Drinfel'd level structures. Then, the ring $E^0(B\Sigma_p)/\operatorname{Tr}$, which is isomorphic to $\mathrm{W}(\mathbf{F}_{p^2})[[u_1]][d]/w(d)$ for some monic polynomial of degree $p+1$ (see the discussion above Corollary 3.3.4) can be thought of as the (global sections of the) completion of $\mathscr{M}_{\Gamma_1(n;p)}$ at the supersingular point (given by the particular $\mathscr{P}_n$-model). Therefore, we can think of the elements t and d (now thought of as an element of $\mathrm{W}(\mathbf{F}_{p^2})[[u_1]][d]/w(d)$) as modular forms (simultaneously for $\Gamma_1(N)$ and $\Gamma_0(p)$). Then, one obtains [Zhu15a, Theorem 1.2]:

Theorem 4.4.1 (Zhu)**.** *The ring* $E^0(B\Sigma_p)/\operatorname{Tr} \cong \mathrm{W}(\mathbf{F}_{p^2})[[u_1]][d]/w(d)$ *is determined by the monic polynomial*

$$\begin{aligned} w(d) &= (d-p)(d+(-1)^p)^p - (u_1 - p^2 + (-1)^p)d \\ &= d^{p+1} - u_1 d + (-1)^{p+1}p + \sum_{k=2}^{p} (-1)^{p(p-k+1)} \left(\binom{p}{k-1} + (-1)^{p+1} p \binom{p}{k} \right) d^k. \end{aligned}$$

We will denote by w_k the coefficient of d^k.

Let ψ denote ψ_2^p. The Atkin-Lehner involution (see Remark 3.2.9) on $\Gamma_0(p)$ sends u_1 and d to other modular forms $\widetilde{u_1}$ and $\widetilde{d}$. Then, by the discussion in Remark 3.2.8 and Remark 3.2.9, we find that $\psi(u_1) = \widetilde{u_1}$. As is evident from Theorem 4.4.1, the only term in $w(d)$ whose coefficient involves u_1 is the coefficient of d (and in fact this coefficient is just $-u_1$); all other coefficients lie in $p\mathbf{Z}$. Therefore, we may write

$$0 = w(d) = d^{p+1} + w_p d^p + \cdots + w_2 d^2 - u_1 d + (-1)^{p+1}p,$$

with $w_i \in p\mathbf{Z}$. By applying the Atkin-Lehner involution, it follows that

$$0 = w(\widetilde{d}) = \widetilde{d}^{p+1} + w_p \widetilde{d}^p + \cdots + w_2 \widetilde{d}^2 - \widetilde{u_1}\widetilde{d} + (-1)^{p+1}p,$$

By definition of the Atkin-Lehner involution, we have $\widetilde{d}d = (-1)^{p+1}p$. Therefore,

$$\widetilde{u_1} = \widetilde{d}^p + w_p\widetilde{d}^{p-1} + \cdots + w_2\widetilde{d} + d.$$

We conclude that

$$\widetilde{d} = \frac{(-1)^{p+1}p - w(d)}{d} = d^p + w_p d^{p-1} + \cdots + w_2 d - u_1.$$

Substituting this formula for $\widetilde{d}$, we find the following result, stated as [Zhu15a, Theorem 1.6]:

Theorem 4.4.2 (Zhu)**.** *We have*

$$\psi(u_1) = \widetilde{u_1} = d + \sum_{k=2}^{p+1} w_k \left(\sum_{k=2}^{p+1} w_k d^{k-1} - u_1 \right)^{k-1},$$

where, recall, that w_k is the coefficient of d^k in $w(d)$.

We now provide the proof of Lemma 4.2.2.

Proof of Lemma 4.2.2. It suffices to check that $\psi(u_1^{-1})$ has only finitely many positive powers of u_1, and that the negative powers of u_1 have coefficients which p-adically converge to zero. As mentioned above Corollary 3.3.4, the polynomial $w(d)$ has a unique solution, denoted α, in the ring $\mathrm{W}(\mathbf{F}_{p^2})((u_1))^{\wedge}_p$. Since $\widetilde{\alpha}\alpha = (-1)^{p+1}p$, we know that α is divisible by p. We define $c = \frac{1}{p}\alpha u_1$. Then the relation $w(\alpha) = 0$ translates into the relation

$$c = (-1)^{p+1} + \sum_{k=2}^{p+1} w_k p^{k-1} c^k u_1^{-k}.$$

The displayed equation is a well-defined recurrence relation since the only powers of c appearing on the right hand side are divisible by c^2. This can be solved recursively, and it is evident from this relation that the coefficients of u_1^{-k} will p-adically converge to zero as k goes to ∞. The same therefore holds for α. Since

$$\psi(u_1) = \alpha + \sum_{k=2}^{p+1} w_k \left(\sum_{k=2}^{p+1} w_k \alpha^{k-1} - u_1 \right)^{k-1},$$

we learn that $\psi(u_1)$ has only finitely many positive powers of u_1. It follows from this that $\psi(u_1^{-1}) = \psi(u_1)^{-1}$ also has only finitely many positive powers of u_1, and that the negative powers of u_1 in $\psi(u_1^{-1})$ have coefficients which p-adically converge to zero. We conclude that $\widetilde{\psi_2^p}$ (in the notation of Theorem 3.3.2) preserves the subring $\mathrm{W}(\mathbf{F}_{p^2})(u_1)_p^\wedge \subseteq \mathrm{W}(\mathbf{F}_{p^2})((u_1))_p^\wedge$. The desired result now follows from Corollary 3.3.4. □

Remark 4.4.3. Taking a hint from the discussion in Warning 3.2.10, we find that the data of the power operation is essentially contained in the p-torsion group scheme $\mathbf{G}[p]$ along with its structure map to $\operatorname{Spf} \pi_0 E$. In [Lur17, Lur18], Lurie developed the theory of spectral formal groups and spectral p-divisible groups. The (oriented) spectral formal group associated to E is precisely $\operatorname{Spf} \underline{\mathrm{Map}}(\Sigma^\infty_+ \mathbf{C}P^\infty, E)$. Since Morava E-theory is $K(n)$-local, this may be regarded as a connected spectral p-divisible group (see [Lur18, Section 4.6]); we will denote this spectral p-divisible group by $\widetilde{\mathbf{G}}$. Then $\widetilde{\mathbf{G}}[p] = \mathrm{Hom}(C_p, \widetilde{\mathbf{G}})$ is a spectral refinement of the classical p-torsion group scheme $\mathbf{G}[p]$. As explained in [Lur09, Section 3], the structure sheaf $\mathscr{O}_{\mathbf{G}[p]}$ should be regarded as the genuine C_p-equivariant version of E (if $E = K_p$, so that $\widetilde{\mathbf{G}} = \mu_{p^\infty}$, this is just a restatement of the classical Atiyah-Segal completion theorem for C_p-equivariant K-theory). It is possible to use this perspective to describe an alternative approach to understanding the θ-closure condition. However, we will not describe this approach here, since we have been unable to gain any mileage out of this perspective (the major reason being that there is no geometric interpretation yet of the cocycles in Morava E-theory).

4.5 Open questions

We conclude this paper with an incomplete list of questions left open.

Question 4.5.1. Does the additive structure of R depend on the choice of $\mathscr{P}_n$-model? Is there a choice of coordinate on E for which it is *not* possible to construct such an $\mathbf{E}_\infty$-ring R?

Question 4.5.2. In light of Corollary 1.0.2, we can ask: does there exist a solution to some $E(n)$-realization problem for $n > 2$?

Remark 4.5.3. In [Law10, Example 5.4], Lawson constructed an $\mathbf{E}_\infty$-ring realizing the $K(2) \vee \cdots \vee K(n)$-localization of the height n analogue of the $\mathbf{E}_\infty$-ring R constructed above (along with its $G = \mathbf{F}_{p^n}^\times \rtimes \mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p)$-action by $\mathbf{E}_\infty$-ring maps). In order to construct $L_{K(1)\vee K(2)\vee\cdots\vee K(n)}R$ as an $\mathbf{E}_\infty$-ring, it should be possible to argue as in Theorem 1.0.1 (which, as mentioned before, is just the argument in [LN12, Sections 5 and 6]) and use Theorem 3.3.2 to conclude, *if* one has some control over power operations for Morava E-theory at height n (in particular, one would need an analogue of Lemma 4.2.2). In particular, one would only need to know that the operation $\widetilde{\psi_n^p} = \psi_1^p : \pi_0 L_{K(1)} L_{K(2)\vee\cdots\vee K(n)} E_n \to \pi_0 L_{K(1)} L_{K(2)\vee\cdots\vee K(n)}$ stabilizes the subring $\pi_0 L_{K(1)} R = \mathrm{W}(\mathbf{F}_{p^2})[u_1, \cdots, u_{n-1}][u_1^{-1}]_p^\wedge$; in other words, one essentially needs to show that the coefficient of $\widetilde{\psi_n^p}(u_1^{-1})$ contains only finitely many positive powers of the u_i's, and that the p-adic valuations of the coefficients of terms involving u_1^{-1} go to zero. (One should be careful to choose a good coordinate; see also Question 4.5.1.) It might be possible to coax some information out of Weinstein's work on the Lubin-Tate tower in [Wei16], although we have not explored this direction yet. However, with the pessimism stemming from the results in [Law17, Sen17], we expect that the above question will admit a *negative* answer.

Question 4.5.4. Does the particular $\mathbf{E}_\infty$-ring $E(2)$ constructed above arise as the localization of some $BP\langle 2\rangle$? Preliminary computations at the primes 3 and 5 suggest that the above question admits a negative answer. A related question: is there an $\mathbf{E}_\infty$-ring B with an action by $G = \mathbf{F}_{p^2}^\times \rtimes \mathrm{Gal}(\mathbf{F}_{p^2}/\mathbf{F}_p)$ such that $B^{hG} = BP\langle 2\rangle$ and $u^{-1}B \simeq R$?

Question 4.5.5. Is the space of $\mathbf{E}_\infty$-automorphisms of R discrete? If so, is it the Morava stabilizer group? What is the moduli-theoretic interpretation associated to the $\mathbf{E}_\infty$-ring R?

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