TOPOLOGY OF TWO-ROW TYPE SPRINGER FIBERS

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Abstract. It is known that irreducible components of a two-row type Springer fiber are iterated $\mathbb{CP}^1$ bundles. In this paper, we prove that their pair-wise intersections are isomorphic to some irreducible components of other two-row type Springer fibers. In particular, the intersections are iterated $\mathbb{CP}^1$ bundles, as conjectured by Fung ([1]). We relate the irreducible components with lattice paths and give an combinatorial algorithm to determine their intersections. For any two-row type Springer fiber, we prove that its singular locus is equidimensional. We find a bound on the number of components of this singular locus and investigate its topology. We make some speculations here, including that the singular locus is a union of Springer fibers of two-row type.

1. Introduction

Let n be a positive integer, and let N be a nilpotent map $\mathbb{C}^n \to \mathbb{C}^n$, in the sense that $N^b = 0$ for some positive integer b. A *flag* is a chain of subspaces $0 \subset F_1 \subset F_2 \cdots \subset F_n = \mathbb{C}^n$, such that $\dim F_i = i$ for each i. We say that N fixes such a flag if $NF_i \subset F_{i-1}$ for all $1 \leq i \leq n$. The *Springer fiber* of N is the set of flags fixed by N, and is denoted by $\mathcal{B}_N$. It is a projective variety.

The Springer theory says the top homology of $\mathcal{B}_N$ is an irreducible representation of S_n, the n-th symmetric group. This theory has proven to be successful in understanding the representations of S_n. In particular, Kazhdan and Lusztig ([2]) have a topological construction of basis for each irreducible representation. However, except the existence of such distinguished basis, there is only a few papers studying the topology of Springer fibers and that of their singular loci.

Fung has studied the topology of hook type and two-row type Springer fibers (see Definition 2.1), and relates the pair-wise intersections of irreducible components with inner products of Kazdan-Lusztig basis. He proves that each irreducible component is an iterated fiber bundle, which is therefore smooth. For the hook type Springer fiber, he proved that the pair-wise intersections of irreducible components are also iterated fiber bundles. However, his proof strongly depends on the simple characterization of irreducible components of hook type. His method can not be applied to general case, and not even to two-row type. For two-row type Springer fibers, he proves that each irreducible component is an iterated $\mathbb{CP}^1$ bundle, and conjectures that their pair-wise intersections are also iterated $\mathbb{CP}^1$ bundles.

In this paper, we start by proving Fung's conjecture and aim to understand the topology of intersections better. Since Fung has proven that irreducible components are smooth, the union of their pair-wise intersections is equal to the singular locus of the Springer fiber. Our study is focused on the topology of this singular locus, and the main theorem is

Theorem 1.1. *For a two-row type Springer fiber, its singular locus is equidimensional. Each component of the singular locus is isomorphic to some irreducible component of a two-row type Springer fiber. The singular locus is connected if the length of the second row is greater than 1.*

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Based on these observations, we conjecture that the singular locus is a union of two-row type Springer fibers.

The paper is organized as following: We prove the conjecture of Fung in section 3. For any two irreducible components, we will combine the defining conditions for each of them and give a unified set of conditions on the subspaces of flags. This new set of defining conditions is similar to that of an irreducible component. Indeed, we eventually prove that each intersection is isomorphic to some irreducible component of other two-row type Springer fiber. In particular, the intersection is an iterated $\mathbb{CP}^1$ bundle.

In section 4, we classify the case when two irreducible components have a codimension 1 intersection, and prove that the singular locus is equidimensional. We then bound the number of components of the singular locus and prove its connectedness. In section 5, we give two theorems that may be useful in computing intersections, especially in counting the dimensions.

2. Preliminaries

2.1. **Irreducible Components of Springer Fibers.** From the definition, it is clear that $\mathcal{B}_N$ is determined by the conjugacy class of N. So we can always assume N is in its Jordan normal form. Since N is nilpotent, the diagonal of N is zero everywhere. The Jordan blocks of N gives a partition of $n = b_1 + b_2 \cdots b_k$, with $b_1 \geq b_2 \geq \cdots b_k \geq 1$. The *Young Shape* of N is the shape from this partition, i.e. the first row has b_1 boxes, and the second row has b_2 boxes $\cdots$

Definition 2.1. We say that the Springer fiber $\mathcal{B}_N$ is of two-row type if the Young shape of N has two-rows. In particular, we say that the Springer fiber is of type (b, a) if the Young shape of N has two-rows and the length of the rows are b, a respectively, with $b \geq a$.

A *standard tableau* on the Young shape of N is a filling of numbers $1, 2, \cdots, n$ into the boxes, such that the number is decreasing in each row and each column. There is a one to one correspondence between standard tableaux and irreducible components of $\mathcal{B}_N$, which is the following theorem proved by Vargas and Spaltenstein:

Theorem 2.1. *(Vargas [5], Spaltenstein [4]) Let N be a nilpotent map. Then given a standard tableau A on the Young shape of N, we have a locally closed subset $SV(A)$ of the Springer fiber $\mathcal{B}_N$; whose closure $\overline{SV(A)}$ is an irreducible component of $\mathcal{B}_N$: We have a partition $\mathcal{B}_N = \bigcup_A SV(A)$ of the Springer fiber into disjoint locally closed subsets. Thus the number of irreducible components of $\mathcal{B}_N$ is equal to the number of standard tableaux on the Young shape of N. In addition, the components are all of the same dimension. In fact, if the lengths of the columns of the Young shape of N are $a_1, a_2, \cdots, a_k$ then the dimension of each component is*

$$\sum_{i=1}^{k} \frac{a_i(a_i - 1)}{2}. \tag{2.1}$$

This is a general theorem for all type of nilpotent maps N. For Springer fibers of two-row type, Fung gives an explicit defining condition on the irreducible components of $\mathcal{B}_N$. For a standard tableau A, we use the notation $K(A)$ to denote the corresponding component.

Theorem 2.2. *(Fung [1]) Let N be a nilpotent matrix of two-row type, and let A be a standard tableau on the Young shape of N, with the first row being $n, i_{b-1}, \cdots, i_1$. Then $K(A)$ of the Springer fiber $\mathcal{B}_N$ consists of all flags whose subspaces satisfy the following conditions*

$$F_{i-1} \subset F_i \subset N^{-1}(F_{i-1}) \tag{2.2}$$

for all $1 \leq i \leq n$. And if i is on the top row of A and $i-1$ is on the bottom row, then

$$F_i = N^{-1}(F_{i-2}). \tag{2.3}$$

If $i, i-1$ are both on the top row and $F_{i-1} = N^{-r}(F_d)$, with $d > 0$, then

$$F_i = N^{-r-1}(F_{d-1}). \tag{2.4}$$

If $i, i-1$ are both on the top row and $F_{i-1} = N^{-r}(Im\ N^d)$, $a \le d \le b$ then

$$F_i = N^{-r}(Im\ N^{d-1}). \tag{2.5}$$

Example 2.1. The irreducible component corresponding to $\begin{array}{|c|c|c|}\hline 6&5&4\\\hline 3&2&1\\\hline\end{array}$ consists of flags

$$0 \subset F_1 \subset F_2 \subset F_3 \subset N^{-1}(F_2) \subset N^{-2}(F_1) \subset \mathbb{C}^6. \tag{2.6}$$

and $F_1 \subset N^{-1}(0), F_2 \subset N^{-1}(F_1)$.

Seemingly complicated, Fung's theorem says that those subspaces F_i with i in the second row of A is *independent*, in the sense that F_i can be any subspaces such that $F_{i-1} \subset F_i \subset N^{-1}(F_i)$. The choice of such F_i is a $\mathbb{CP}^1$ because F_i/F_{i-1} is a one dimensional subspace of the two dimensional space $N^{-1}(F_i)/F_i$.

Fung's theorem implies that any other subspaces F_j can be uniquely determined. In particular, this implies that the dimension of $K(A)$ is equal to the length of the second row, which agrees with Vargas and Spaltenstein's theorem. In fact, Fung's theorem implies that $K(A)$ is an iterated $\mathbb{CP}^1$ bundle, in the following sense:

Definition 2.2. A space X_1 is an iterated $\mathbb{CP}^1$ bundle if there exists $X_1, X_2, \cdots, X_{n+1} = pt$ and maps $p_1, \cdots, p_n$ such that $p_i : X_i \to \mathbb{CP}^1$ is a fiber bundle with typical fiber X_{i+1}.

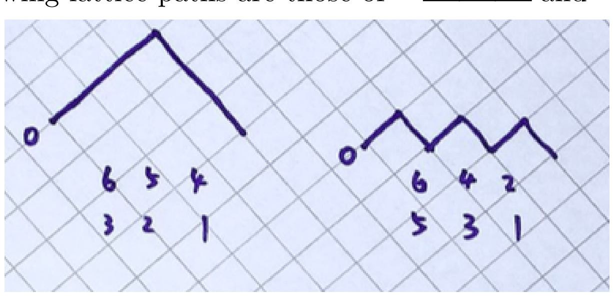
Let $j_1 < j_2 \cdots < j_a$ be the numbers in the second row of A. Then each irreducible component is an iterated $\mathbb{CP}^1$ bundle by projecting to $F_{j_1}, \cdots, F_{j_a}$ successively. In particular, it is smooth. From there and evidence from inner products of Kazhdan-Lusztig basis, *Fung* conjectures that the pair-wise intersections of two irreducible components of $\mathcal{B}_N$ are also iterated $\mathbb{CP}^1$ bundles.

2.2. **Relation with Lattice Paths.** Before we start to give a unified defining condition for intersections of two irreducible components, we relate standard tableaux with lattice paths and restate Theorem 2.2 so that the three situations are unified. In this section, we fix the Young shape to be of (b, a) type.

A lattice path is a path by adjoining consecutive points $(i, h(i))$ such that $|h(i) - h(i-1)| = 1$ for each i. For a standard tableau A, we define a lattice path in the following way:

(1) $h(i) = i, -(b-a) \le i \le 0$.
(2) If i is on the top row of A, then $h(i) = h(i-1) - 1$. If i is in the bottom row, then $h(i) = h(i-1) + 1$.

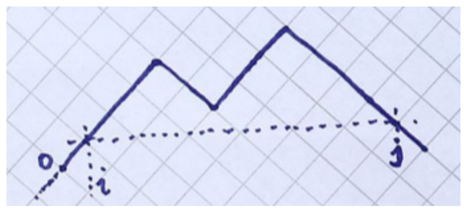
Example 2.2. The following lattice paths are those of $\begin{array}{|c|c|c|}\hline 6&5&4\\\hline 3&2&1\\\hline\end{array}$ and $\begin{array}{|c|c|c|}\hline 6&4&2\\\hline 5&3&1\\\hline\end{array}$ respectively.



We call $h(i)$ the *height* of i. Clearly, $h(i) \geq -(b-a)$ for each i. And for any lattice path, we can get a standard tableau by adding numbers to the Young shape: if $h(i) - h(i-1) = 1$, then add i to the bottom row, otherwise add it to the top row. This is a standard tableau if $h(i) \geq -(b-a)$ for each i.

So there is a one to one correspondence between standard tableaux on Young shape (b, a), and lattice paths starting from 0, with length $b+a$, and always above $-(b-a)$. From now on, we don't distinguish a standard tableau and its corresponding lattice path.

Definition 2.3. For a lattice path L, we say i is *independent* if $h(i) - h(i-1) = 1$. We say j *depends* on i, if the horizontal line from j to the left, intersecting L at i at the first time. (See the picture below).



Theorem 2.2 can be restated as following:

Theorem 2.3. *(Theorem 2.2 restated) Let A be a standard tableau on the Young shape of N, which is of (b, a) type. Let $F_{-i} = Im\ N^{b-i}$. Then the irreducible component $K(A)$ consists of all the flags such that*

(1) $F_{i-1} \subset F_i \subset N^{-1}(F_{i-1})$.
(2) If j depends on i, then $F_j = N^{-d}(F_i)$ with $d = \frac{j-|i|}{2}$.

It is part of the theorem that the defining conditions are also symmetric: if F_j depends on F_i and $F_j = N^{-d}(F_i)$, then $F_i = N^d F_j$.

3. Intersections as Irreducible Components

In this section, we prove that for any two-row type Springer fiber $\mathcal{B}_N$, the pair-wise intersections of its irreducible components are also irreducible components of other two-row type Springer fiber.

3.1. A Unified Condition for Intersections.

In this section, N will be a fixed nilpotent matrix of (b, a) type, $b \geq a$. All the lattice paths mentioned will also correspond to a standard tableau on Young shape (b, a). In other words, they are always above the level $-(b-a)$.

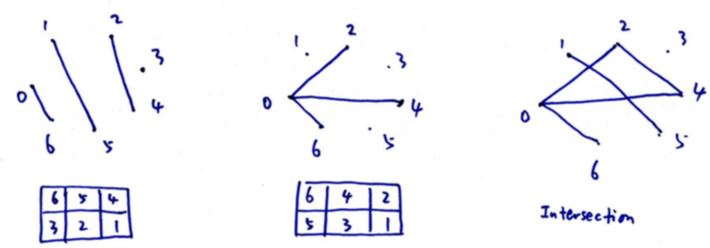
First, we define a graph $G(A)$ which represents all the dependences.

Definition 3.1. For any lattice path A, we define a graph $G(A)$: Its vertices are numbers $-(b-a), -(b-a)+1, \cdots, b+a$. And (i, j) is an edge if j depends on i. (see Definition 2.3)

Similarly, we define a graph for many components:

Definition 3.2. For lattice paths $A_1, \cdots, A_n$, we define $G(A_1, \cdots, A_n) = \bigcup_{i=1}^n G(A_i)$ in the sense that vertices are the same and the set of edges of $G(A_1, A_2, \cdots, A_n)$ are the union of edges of $G(A_i)$.

Example 3.1. The graphs of standard tableaux in Example 2.2 and the graph of their intersection are drawn below.



We imitate Theorem 2.3 and give a set of defining conditions for $\bigcap_{i=1}^{n} K(A_i)$ in terms of $G(A_1, \cdots, A_n)$. Notice that F_i ($i \leq 0$) are fixed spaces in Theorem 2.3. It is thus convenient to distinguish them with independent spaces.

Definition 3.3. For a connected component of G, we say it is positive if any number in that component is positive.

So Theorem 2.3 says that for every positive connected component, the subspaces are uniquely determined by F_i, such that i is the smallest number in that component. The next theorem generalizes this to intersection of two components.

Theorem 3.1. *For any two standard tableaux A, B, let $F_{-i} = Im\ N^{b-i}$ for $b - a \geq i \geq 0$. If any connected component of $G(A, B)$ contains at least two non-positive numbers, then $K(A) \cap K(B) = \emptyset$. Otherwise $K(A) \cap K(B) \neq \emptyset$ and it consists of all the flags satisfying conditions:*

(1) $F_{i-1} \subset F_i \subset N^{-1}(F_{i-1})$, if $i > 0$ and is the smallest number of some connected component. We call these F_i, i independent.

(2) If j and i are in the same connected component, then $F_j = N^{-d}(F_i)$, $d = \frac{|j|-|i|}{2}$.

Proof. First, we prove that (2) is necessary by doing induction on the distance between j and i (in the graph $G(A, B)$). If the distance is 1, then both j, i must be in the same component of graph $G(A)$ or $G(B)$. So (2) holds by Theorem 2.3.

Assume (2) holds when distance is not greater than k. Consider a shortest path between j and i: $(i, i_1, \cdots, i_k, j)$. Clearly i_k and j are in the same connected component in either $G(A)$ or $G(B)$. Therefore $F_j = N^{-\frac{|j|-|i_k|}{2}} F_{i_k}$. By induction assumption, $F_{i_k} = N^{-\frac{|i_k|-|i|}{2}} F_i$. Notice that F_i has dimension $|i|$. If $|i_k| > |i|$, then we have a chain of spaces $F_{i_k} \supset NF_{i_k} \supset \cdots \supset N^{\frac{|i_k|-|i|}{2}} F_{i_k} = F_i$. Since N is of two-row type, ker N is 2 dimensional. So each step in the chain decreases the dimension by 2. Therefore, $N^a N^b F_{i_k} = N^{a+b} F_{i_k}$ if $0 \leq a, b, a + b \leq \frac{|i_k|-|i|}{2}$. This is true for the chain between F_j, F_{i_k}. Thus we always have

$$F_j = N^{-\frac{|j|-|i_k|}{2}} F_{i_k} = N^{-\frac{|j|-|i_k|}{2}} N^{-\frac{|i_k|-|i|}{2}} F_i = N^{-\frac{|j|-|i|}{2}} F_i. \tag{3.1}$$

Notice that each connected component of $G(A)$ has at most one non-positive number. If there exists a connected component of $G(A, B)$ that has at least two non-positive numbers, say $-(b - a) \leq x < y \leq 0$, then $F_x = N^{-\frac{|x|-|y|}{2}} F_y$. This cannot be true because $F_x = Span\{e_1, \cdots, e_{|x|}\}$ and $F_y = Span\{e_1, \cdots, e_{|y|}\}$, $|y| \leq |x| \leq b - a$. Whenever we take the inverse image, there will be basis element e_{b+1}. So $K(A) \cap K(B) = \emptyset$ if any connected component of $G(A, B)$ contains at least two non-positive numbers.

Clearly a flag satisfying both (1) and (2) is contained in $K(A) \cap K(B)$. ■

The theorem can be easily generalized to intersections of many components.

Example 3.2. Consider the intersection of irreducible components $\begin{array}{|c|c|c|}\hline 6&5&4\\\hline 3&2&1\\\hline\end{array}$ and $\begin{array}{|c|c|c|}\hline 6&4&2\\\hline 5&3&1\\\hline\end{array}$. The graph of intersection determines connected components $\{0,2,4,6\}$, $\{1,5\}$, $\{3\}$. So the intersection consists all the flags

$$0 \subset F_1 \subset N^{-1}(0) \subset F_3 \subset N^{-2}(0) \subset N^{-2}(F_1) \subset N^{-3}(0) = \mathbb{C}^6. \tag{3.2}$$

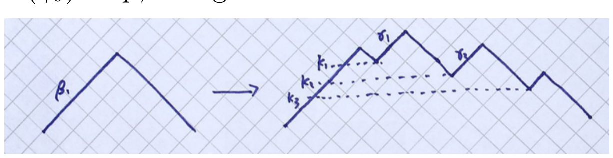
3.2. **Intersections as Irreducible Components.** In this section, we prove that the intersection of two irreducible components, if non-empty, is isomorphic to some irreducible component of other two-row type Springer fiber. We will first construct a lattice path for the intersection and then prove that its corresponding irreducible component is indeed isomorphic to the intersection.

3.2.1. *Construction of lattice Path.* Suppose A, B are two standard tableaux on the Young shape of N. We will construct a new lattice path $P(A,B)$ of shorter length.

First, we can assume $K(A) \cap K(B) \neq \emptyset$. Theorem 3.1 gives a set of defining conditions for $K(A) \cap K(B)$. Let X be the set of independent indices. They form some disjoint intervals. (An interval is a set of numbers $\{i, i+1, \cdots, i+k\}$). Label the intervals from small to big, by $\alpha_1, \cdots, \alpha_l$. Let $(0 <)i_1 < \cdots < i_l$ be the first numbers in each interval. Let l_k be the length of the interval α_k. We construct the lattice path $P(A,B)$ through the following steps:

Step 1 For all i_t, by definition $i_t - 1$ doesn't belong to X. So F_{i_t-1} depends on a subspace with index in X, or it depends on some F_{-i}, $i \geq 0$. We pick out all the numbers i_t, such that $i_t - 1$ depends on subspaces with non-positive indices. Starting from 0, we add a triangle with side length l_t for each such i_t. (It means going upward l_t times then going downward back to the axis.) Note that there is at least one such triangle. The order of triangles doesn't matter. Suppose the intervals in this step are $\beta_1, \beta_2, \cdots, \beta_u$. There is an obvious correspondence between the upward endings and elements in $\beta_1, \beta_2, \cdots, \beta_u$.

Step 2 Suppose there are remaining intervals. Consider those i_t such that $i_t - 1$ depends on numbers in β_1. If exists, we can let Y be the set of all such i_t. For $i_t \in Y$, let j_t be the number in β_1 that $i_t - 1$ depends on. We order the elements of X such that the corresponding j_t is decreasing. Suppose the corresponding intervals for elements in Y, are $\gamma_1, \gamma_2, \cdots, \gamma_v$, with the order we just defined. Suppose the corresponding j_t are $k_1 \geq k_2 \geq \cdots \geq k_v$. Then we extend the triangle of β_1: starting from the peak, go down until it reaches the same height as k_1, (using the correspondence defined in the previous step), then goes up by $l(\gamma_1)$ steps, then goes down until reach the same height as k_2, and then go up by $l(\gamma_2)$ steps... Eventually, it goes up $l(\gamma_v)$ step, and goes back to the axis.



If $X = \emptyset$, do nothing to the triangle. Do the same for all other β_i in the first step. Notice that after this step, each interval appeared in the lattice path except those β_i, is a left leg of some peak, and after the peak, it will go down to a place lower than where it starts.

If none of the $i_t - 1$ depends on intervals β_i $(i \leq l)$, then there are no remaining intervals. So the lattice path is done.

Step 3 If there are remaining intervals, then those $i_t - 1$ cannot depends on intervals in the first step. Consider those $i_t - 1$ that depends on intervals in the second step. For each interval γ in the second step, repeat the process before.

Clearly eventually we can exhaust all the intervals and get a lattice path $P(A, B)$. $P(A, B)$ starts from (0, 0), and ends at $(2|X|, 0)$. It is always above the axis.

3.2.2. *Proof of Isomorphism.* The proof is based on the observation that up to isomorphism, $K(A) \cap K(B)$ is determined by defining conditions for those F_{i_t-1}. And since the lattice path we constructed has the same dependence for those, the independent spaces only differ by span of standard basis.

Theorem 3.2. *$K(A) \cap K(B)$ is isomorphic to the irreducible component corresponding to $P(A, B)$.*

Proof. Let $m = |X|$. Let N' be a $2m \times 2m$ nilpotent matrix. It has two Jordan blocks, each is $m \times m$. The construction of $P(A, B)$ gives an obvious map from X to the upward ends of $P(A, B)$. Let f be this map. We order the elements of X such that their image under f is increasing order: $x_1, \cdots, x_m$. Let $y_i = f(x_i)$. From the construction, $y_1 = 1$ and we know that if $y_i - 1$ depends on y_k, then $x_i - 1$ depends on x_k. If $y_i - 1$ depends on 0, then $x_i - 1$ depends on a non-positive number.

Without loss of generalization, we assume only $y_1 - 1$ depends on 0.

$K(A) \cap K(B)$ is an iterated $\mathbb{CP}^1$ bundle, by first projecting to F_{x_1}, and the fiber projecting to $F_{x_2} \cdots$ (for each flag F)Let $K(A, B)$ be the irreducible component corresponding to $P(A, B)$. It is also an iterated $\mathbb{CP}^1$ bundle, by first projecting to G_{y_1}, then the fiber projecting to $G_{y_2} \cdots$ (for each flag G)

In $K(A) \cap K(B)$, F_{x_1-1} is a known space, spanned by standard basis elements, because it is $N^{-d}(\text{Im } N^k)$, for some d, k. It must be of the form $\text{Span}\{e_1, e_2, \cdots, e_{d+k}, e_{b+1}, e_{b+2}, \cdots, e_{b+d}\}$.

If $s \geq 0$, consider the operator $D(s) : \mathbb{C}^{2m} \to \mathbb{C}^{\infty}$, such that for $1 \leq i \leq m$, $e_i \mapsto e_{d+k+s+i}$. For $m < i \leq 2m$, $e_i \mapsto e_{b+d+s+i}$.

Consider the operator $E(s)$ on subspaces of $\mathbb{C}^{2m}$:

$$\mathbb{C}^{2m} \supset V \mapsto \text{Span}\{e_1, e_2, \cdots, e_{d+k+s}, e_{b+1}, e_{b+2}, \cdots, e_{b+d+s}\} \oplus D(s)V \tag{3.3}$$

If $-m < s < 0$, we define $F(s) : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$, such that for $-s < i \leq m$ or $m - s < i \leq 2m$, $e_i \mapsto e_{i+s}$, and $e_i \mapsto 0$ for all other i. Define $E(s)$:

$$\mathbb{C}^{2m} \supset V \mapsto \text{Span}\{e_1, e_2, \cdots, e_{d+k}, e_{b+1}, e_{b+2}, \cdots, e_{b+d}\} \oplus D(0)F(s)V \tag{3.4}$$

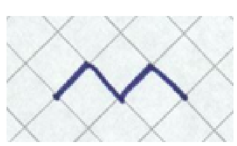
From the construction, $x_i - y_i - (2d + k)$ is an even number. Let $s_i = \frac{1}{2}(x_i - y_i - 2d - k)$. Consider the map

$$(G_{y_1}, \cdots, G_{y_m}) \to (E(s_1)G_{y_1}, \cdots, E(s_m)G_{y_m}). \tag{3.5}$$

It gives an isomorphism between allowable choices $(G_{y_1}, \cdots, G_{y_m})$ and $(F_{x_1}, \cdots, F_{x_m})$. This gives an isomorphism between the two iterated $\mathbb{CP}^1$ bundle. ∎

Corollary 3.3. *For any two standard tableaux A, B, the intersection $K(A) \cap K(B)$ is an iterated $\mathbb{CP}^1$ bundle. And the dimension of this intersection is the number of positive connected component of $G(A, B)$.*

Example 3.3. Consider Example 3.2. Here $X = \{1, 3\}$. And 2 is in the connected component $\{0, 2, 4, 6\}$. So the lattice path for the intersection is completed after step 1. It is drawn below. This is $\mathbb{CP}^1 \times \mathbb{CP}^1$, which is isomorphic to the intersection according to the description given by (3.2).



3.3. **Finitude of Irreducible Components.** Generically, there can be infinitely many iterated $\mathbb{CP}^1$ bundles of fixed dimension. The problem is that the dimension of each component is equal to the length of the second row, but the length of the first row can be arbitrary.

For $\dim K(A) = 2$, Fung ([1]) and Lorist ([3]) show that $K(A)$ is either $\mathbb{CP}^1 \times \mathbb{CP}^1$, or is a nontrivial $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$ which comes from the line bundle $O(2)$. We will generalize this statement.

Theorem 3.4. *For any positive integer n, there are only finitely many n dimensional irreducible components of springer fibers of two-row type.*

Proof. Let A be a standard tableau of two-row type. Assume $\dim K(A) = n$. Then the second row of the Young shape has length n.

Consider the lattice path of A. We pick a set of non-negative numbers $X = \{i_1, i_2, \cdots, i_k\}$, such that

$$h(i_t) = \min_{0 \le j \le i_t} h(j) \tag{3.6}$$

$$h(i_t + 1) = h(i_t) + 1. \tag{3.7}$$

Clearly $X \neq \emptyset$. For each element $i_t \in X$, let $j_t > i_t$ be the first index such that $h(j_t) = h(i_t)$. Such j_t exist because $h(n)$ is lowest.

From the definition of j_t, $j_t - i_t$ is an even integer. And by definition of i_t, F_{i_t} is $N^{-d}(F_{-l})$ for some non-negative integer l. Also $F_{j_t} = N^{-d-d_t}(F_{-l})$, where $d_t = \frac{j_t - i_t}{2}$. From the lattice path and Theorem 2.3, the part of the flag $F_{i_t} \subset F_{i_t+1} \subset \cdots \subset F_{j_t}$ is independent of other subspaces, because for any $s \in (i_t, j_t)$, it is either free or depends on a subspace with index in $[i_t, j_t]$. This means $K(A)$ can be written as product of $M \times L$ for some L.

Let M be the variety of all the chains $F_{i_t} \subset F_{i_t+1} \subset \cdots \subset F_{j_t}$ satisfying the defining conditions for A. We claim that M is isomorphic to a irreducible component with Young shape (d_t, d_t). The part of lattice path from i_t to j_t can be seen as a lattice path from 0 to $2d_t$, and always above the axis. Let B be the corresponding standard tableau, with two-row Young shape(d_t, d_t). Let N' be the nilpotent matrix with 2 Jordan blocks, each block is $d_t \times d_t$.

Consider the map $\phi : M \to K(B)$, by taking quotients over F_{i_t}, i.e.

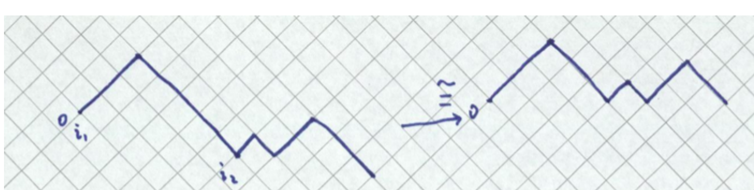
$$\phi : F_{i_t} \subset F_{i_t+1} \subset \cdots \subset F_{j_t} \mapsto 0 \subset F_{i_t+1}/F_{i_t} \subset \cdots \subset F_{j_t}/F_{i_t}. \tag{3.8}$$

Then the defining conditions $NF_j \subset F_{j-1}$ becomes $N'\phi(F)_{j-i_t} \subset \phi(F)_{j-i_t-1}$. And $F_i = N^{-d}(F_j)$ becomes $\phi(F)_{i-i_t} = N'^{-d}(\phi(F)_{j-j_t})$. So M and $K(B)$ are isomorphic.

According to the previous discussion, $K(A)$ is a product of iterated $\mathbb{P}^1$ bundles. Each factor is isomorphic to some irreducible component, also with lattice path always above the axis. So there are finitely many possibilities. ■

Corollary 3.5. *The number of different irreducible components of two-row type, with fixed dimension n is not greater than the number of lattice paths from 0 to $2n$ that never cross the axis. Therefore, the number of different irreducible component is at most $\frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.*

Proof. Theorem 3.4 essentially gives an isomorphism between any (b, a) type irreducible component and (a, a) type irreducible component. (See the picture below)



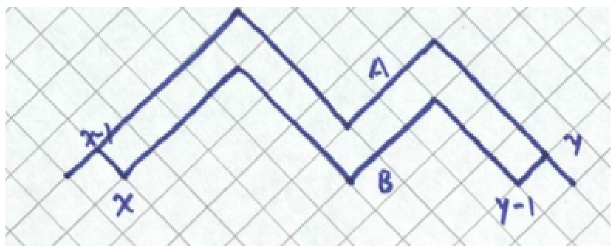
■

Since we have shown that intersections are isomorphic to irreducible components, Theorem 3.4 also shows the finitude of intersections of fixed dimension.

4. Properties of the Singular Locus

For a two-row type Springer fiber, we know that each of its irreducible components is an iterated $\mathbb{CP}^1$ bundle. In particular, each component is smooth. Thus the singular locus of the Springer fiber is just the union of intersections of components. In this section, we analyze the topology of this singular locus. We will assume the Springer fiber is of (b, a) type, unless otherwise mentioned.

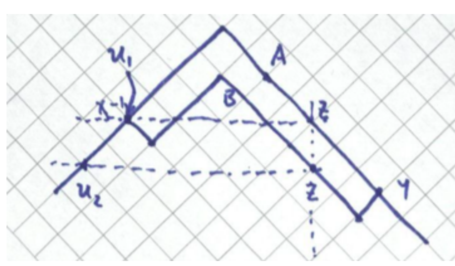
4.1. **Codimension 1 Intersections.** We will prove that the singular locus is equidimensional. In this context, it means any intersection of two irreducible components is contained in a codimension 1 intersection. We first classify the case when codim $K(A) \cap K(B) = 1$. Let $S(A)$ be the set of numbers in the bottom row of A. Then $S(A)$ is exactly the set of independent indices of $G(A)$. According to Theorem 3.3, $\dim K(A) \cap K(B) \leq |S(A) \cap S(B)|$. Therefore, we must have $|S(A) \cap S(B)| = a - 1$. If we draw the lattice paths of A, B, their lattice path must differ by a band of length 1. (See picture below.)



There are two elements in $S(A) \cup S(B) - S(A) \cap S(B)$. Suppose the two elements are x, y, and $x \in A, y \in B$. Then we say that the band is *balanced* if $x - 1, y$ have the same height in the lattice path of A, and the part of the lattice path between $x - 1, y$ is always above the height of $x - 1$. Equivalently, this is to say that $x, y - 1$ have the same height in the lattice path of B, and the part of lattice path of B between $x, y - 1$ is always above the level of x.

Theorem 4.1. *$K(A) \cap K(B)$ has codimension 1 if and only if their lattice paths differ by a balanced band.*

Proof. Suppose the lattice paths of A, B differ by a balanced band. Assume the four ends of the band is $x - 1, x, y - 1, y$, and A is on the top. It is clear from the definition that the only difference between $G(A), G(B)$ is the dependence of x, y. Suppose x depends on u in B. Then the $G(A, B)$ is obtained by adding a line between x, u in $G(A)$. Both x, u are the smallest numbers in the corresponding equivalence classes (u may not be positive). So the number of connected components is decreased by 1.



On the other hand, suppose codim $K(A) \cap K(B) = 1$. Then the lattice path must differ by a band. We need to show that it is balanced. If the lattice path of A between $x - 1, y$ is not always above the level of $x-1$. (See the picture above.) On the lattice path of A, consider the first point to the right of $x-1$ that is lower than $h(x-1)$. Suppose this is at position $z \in [x+1, y]$. The segment from $z-1$ to z is downward. Suppose z depends on u_1 in A, and u_2 in B, then $u_1, u_2 \leq x-1$, and u_1, u_2 are in different connected components of A. However, in $G(A,B)$, there are edges $(z,u_1),(z,u_2)$. If both u_1, u_2 are non-positive, then $K(A)\cap K(B)=\emptyset$. Otherwise, u_1, u_2 are in the same connected component. In $G(A,B)$, comparing to $G(A)$, the connected components of x and that of one of u_1, u_2 are combined intro others. So the number of positive connected components will decrease by at least 2. This means codim $K(A)\cap K(B)\geq 2$.

If the the path of A between $x-1, y$ is not always above the level of y. The paths coincide at and after y. Since they eventually goes to the lowest point $(b+a, -(b-a))$. There is a first time after y, that both path reach the height of $y-1$ in B. Suppose this is k. Suppose k depends on u_1, u_2 in A, B respectively. Then by assumption $x-1 \leq u_1 < u_2$. Therefore $u_2 > x$, since $u_2 - u_1$ is even. In $G(A,B)$, comparing to $G(A)$ the connected components of u_2, x of A disappears. So codim $K(A)\cap K(B)\geq 2$. ■

Example 4.1. Compare the lattice paths in Example 2.2, we already known that their intersection is $\mathbb{CP}^1\times\mathbb{CP}^1$, which is of codimenion 1.

On the other hand, consider the intersection of $\begin{array}{|c|c|c|}\hline 6&5&4\\\hline 3&2&1\\\hline\end{array}$ and $\begin{array}{|c|c|c|}\hline 6&5&2\\\hline 4&3&1\\\hline\end{array}$. Although the second row has two numbers in common, their intersection is

$$0\subset F_1\subset N^{-1}(0)\subset N^{-1}(F_1)\subset N^{-2}(0)\subset N^{-2}(F_1)\subset \mathbb{C}^6. \tag{4.1}$$

So it is one dimensional, and therefore has codimension 2.

For convenience, we define the following graph G for a Springer fiber.

Definition 4.1. For a Springer fiber of type (b,a), we define a graph $G(b,a)$: The vertices are irreducible components of the Springer fiber. For two components A, B, there is an directed edge $A\to B$ if codim $K(A)\cap K(B)=1$ and lattice path of A is above that of B.

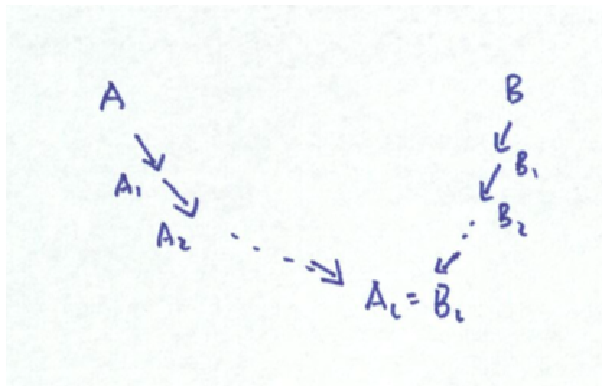
4.2. **Equidimensionality of the Singular Locus.** We use G instead of $G(b,a)$, whenever there is no ambiguity. We will see that G also gives information on intersection of any two different components. First, we have

Theorem 4.2. *For any irreducible components $K(A), K(B)$, there is a chain $C_0\to C_1\to\cdots C_t\leftarrow C_{t+1}\cdots\leftarrow C_k$, such that $C_0=K(A), C_k=K(B)$, and $K(A)\cap K(B)=C_0\cap C_1\cap\cdots C_k$.*

Proof. Consider the pair (A,B). Find the first point x where lattice paths of A,B diverge. Suppose A goes up after x, and B goes down. Suppose y is the first point after x such that its height is the same as that of x, in A. We construct a lattice path A_1. A_1 agrees with A before x, and goes down after x, then imitate the graph of A, until $y-1$. Then go up one step and close the band. After y, A_1 agrees with A. (See the picture below.)

From the construction, $A\to A_1$. Notice that the graph of intersection $G(A,A_1)$ is obtained from $G(A)$, by adding a line adjoining $x+1$ and u, which is the smallest number in the connected components of $x+1$, in $G(B)$. For $G(A,B)$, $(x+1,u)$ is one of the lines needed to be added on $G(A)$. So $G(A,A_1)$ is a subgraph of $G(A,B)$. Thus $K(A)\cap K(B)\subset K(A)\cap K(A_1)$. In particular, $K(A)\cap K(B)\subset K(A_1)\cap K(B)$.

Consider the new pair $(A_1, B_1=B)$. Continue this procedure above. The first time these two lattice paths diverge is greater than x. Each time we replace either A or B by another lattice path. The intersection of this pair is always larger. Eventually, the two path coincide. The chain is obtained by first taking all different A's in the pair, then going backward and take all the different B's. (See the picture below.)



■

Example 4.2. Consider the intersection of

6	5	4
3	2	1

and

6	5	2
4	3	1

. The chain is drawn in the following picture.

Corollary 4.3. *G is connected.*

Corollary 4.4. *The singular locus of the any Springer fiber of two-row type is equidimensional.*

Proof. For any intersection $K(A) \cap K(B)$, take a chain as in the theorem. We know $K(A) \cap K(B) \subset C_0 \cap C_1$. So $K(A) \cap K(B)$ is contained in a codimension 1 intersection. This implies that the singular locus is equidimensional. ■

4.3. **Number of Components of the Singular Locus.** We will see that G also gives a bound on the number of components of the singular locus.

Lemma 4.1. *For each vertex A of G, the number of edges going outward from A is $\deg_+(A) \leq a$.*

Proof. Any balanced band below A must start from a point x, where $(x, x+1)$ is an upward segment in the path of A. Clearly, for any such x, there is a unique balanced band. Because it must end at the first point after x that has the same height as x (on the path of A). ■

Theorem 4.5. *Each edge in G stands for a unique codimension 1 intersection.*

Proof. We need to prove that two edges cannot represent the same intersection. In fact, if $K(A) \cap K(B) = K(C) \cap K(D)$ and both have codimension 1. If A, B, C, D are different. Then $K(A) \cap K(B) \cap K(C) \cap K(D) = K(A) \cap K(B)$. So $K(A) \cap K(B) = K(A) \cap K(C)$. So it suffice to prove the case when $D = A$.

Case 1 $A \to B, A \to C$. Consider first point x where A, B are different, and the first point y where A, B are different. Since $B \neq C$, $x \neq y$. Consider the graph $G(A, B, C)$, comparing to $G(A)$, the connected component of x, y disappears. So codim $K(A) \cap K(B) \cap K(C) = 2$. Contradiction!

Case 2 $A \to B, C \to A$. Suppose the band of A, B starts and ends at u, v respectively, and the band of C, A starts and ends at x, y respectively. Then the graph $G(A, B)$, comparing t$G(A)$, loses a connected component of $u+1$. Similarly, the graph of $K(A) \cap K(C)$ loses a connected component of y. So we must have $y = u + 1$. (See the picture below.)

Suppose y depends on c in C, and b in B, then A, B, C coincide at c, b. And c, b has different height, which implies $c \neq b$ and are in different connected component in $G(A)$.

If both c, d are non-positive, then $K(A) \cap K(B) \cap K(C) = \emptyset$, contradiction! Otherwise $G(A, B, C)$ comparing to $G(A)$, loses two of the three connected components corresponding to c, b, y. Contradiction!

Case 3 $B \to A, C \to A$. This is almost the same as in Case 1, because above A, for any y, there is at most one balanced band ending at y.

■

Corollary 4.6. *The number of components of the singular locus is the number of edges of G. So the number of components is at most $a \cdot$ #standard tableaux of (b, a) type.*

4.4. **Connectedness of the Singular Locus.** In this section, we prove the connectedness of the Springer fiber and its singular locus.

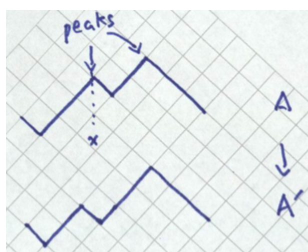
Theorem 4.7. *Any Springer fiber of two-row type is connected.*

Proof. This can be seen from the connectedness of G. ■

Theorem 4.8. *For a Springer fiber of type (b, a), with $a > 1$, its singular locus is connected.*

Proof. Assume $a > 1$. Since the singular locus is equidimensional, we only need to show that the codimension 1 intersections are connected to each other, maybe through a chain. According to the proof of Theorem 4.5, any two edges $A \to B, A \to C$, has a codimension 2 intersection $K(A) \cap K(B) \cap K(C)$. So $K(A) \cap K(B) \cap K(C) \neq \emptyset$. This means the intersections $K(A) \cap K(B)$ and $K(A) \cap K(C)$ are connected to each other.

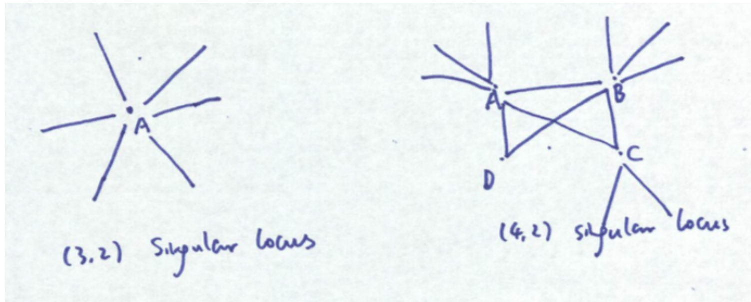
We call x a *peak* of A, if $(x-1, x)$ is upward, and $(x, x+1)$ is downward. Then for any peak x, we can always "delete" it by making $(x-1, x)$ downward, and $(x, x+1)$ upward. Suppose this new lattice path is A', then $A \to A'$.



Now for any edge $A \to B$, from the above observation, it is connected with some edge $A \to A'$, where A' is obtained by deleting a peak in the band of A, B. Suppose the peak deleted is x. If A' is not the lowest path (see the picture below), then we can always find a peak of A', delete it and get $A' \to A''$. According to the proof of Theorem 4.5, $K(A) \cap K(A') \cap K(A'')$ is possible to be empty if and only if the new peak is $x + 1$. However, unless, $a = 1$, we can always find a peak not equal to $x + 1$.

So every codimension 1 intersection is connected (through a chain) with some intersection $A \to B$, where B is the lowest path. And again from the proof of Theorem 4.5, any $A \to B, C \to B$ has non-empty intersection. So the singular locus is connected if $a > 1$. ■

Example 4.3. The singular locus of (3, 2) type Springer fiber and (4, 2) type Springer fiber is drawn below. Each line stands for a $\mathbb{CP}^1$ and labeled points are intersections of some two irreducible components. The intersections of these $\mathbb{CP}^1$ are exactly the common points they have in the following picture.



Conjecture 4.1. *Based on the connectedness and equidimensionality, and the analysis of components of the singular locus, we conjecture that the singular locus of the Springer fiber of (b, a) type is covered by Springer fibers of $(b-1, a-1)$ type. Recall that we already proved each component of the singular locus is an irreducible component of the Springer fiber of $(b-1, a-1)$ type. So it remains find for each codimension 1 intersection, a set of codimension 1 intersections such that their union is a $(b-1, a-1)$ Springer fiber.*

Example 4.4. The $(b, 1)$ Springer fiber is a chain of b $\mathbb{CP}^1$'s, such that only consecutive $\mathbb{CP}^1$'s has non-empty intersection: a point. As seen from Example 4.3, the singular locus of (3, 2) Springer fiber can be written as a union of length 2 chains. The singular locus of (4, 2) Springer fiber can be written as a union of length 3 chains.

5. Simplifying the Intersection

5.1. **G and Dimensions of Intersections.** The main idea is that dimensions of any intersection can be read from the graph G. In fact, if the intersection is non-empty, we will see that the codimension is the distance between the two components, in graph G.

Theorem 5.1. *If $K(A) \cap K(B) \neq \emptyset$, then there is a chain of irreducible components $C_0, C_1, \ldots, C_k$ such that*

(1) $C_0 = K(A), C_k = K(B)$ and $K(A) \cap K(B) = C_0 \cap C_1 \cap \cdots C_k$.
(2) codim $C_i \cap C_{i+1} \cap \cdots C_{i+r} = r$, for any $r \geq 0$ and $0 \leq i \leq i+r \leq k$.
(3) $C_i \cap C_{i+r} = C_i \cap C_{i+1} \cap \cdots C_{i+r}$, for any $r \geq 0$ and $0 \leq i \leq i+r \leq k$.

In particular, the length of the chain is the codimension of $K(A) \cap K(B)$.

Eventually, we will show that the chain constructed in Theorem 4.2 is one that satisfies all the conditions. Before proving the general case, consider the situation when $a = b$. So the lattice path of any standard tableau is actually a Dyck path, with length $2b$. Let l be the vertical line at position b. Let R represent the operation of taking the reflection with respect to l. For any Dyck path A, clearly RA is also a Dyck path of length $2b$.

Theorem 5.2. *For (b, b) type Springer fiber, and two different Dyck paths A, B.*

(1) $K(A) \cap K(B) \neq \emptyset$.
(2) The length of the chain constructed in the Theorem 4.2 for A, B is codim $K(A) \cap K(B)$.

Proof. The first statement is obvious, because there is only one non-positive vertex in $G(A, B)$. Thus each connected component has at most one non-positive integer.

Using the algorithm in Theorem 4.2, we have a chain $C_0 \to C_1 \to \cdots C_t \leftarrow C_{t+1} \cdots \leftarrow C_k$ for RA, RB. That is $C_0 = RA, C_k = RB$. Let $D_i = RC_i$. Then we have chain $D : D_0 \to D_1 \to \cdots D_t \leftarrow D_{t+1} \cdots \leftarrow D_k$, and $D_0 = A, D_k = B$.

We claim that $k = \text{codim } K(A) \cap K(B)$. Use induction on the length of D. If the length of D is 0, 1, (2) is satisfied. Let $l(D)$ denote the length of D.

Suppose we have proven the case for $l(D) = k$. Then suppose $l(D) = k + 1$. According to our construction, either $D_1, D_2, \cdots, D_{k+1}$ comes from the algorithm for D_1, D_{k+1} or $D_0, D_1 \cdots, D_k$ comes from D_0, D_k. Without loss of generalization, assume we are in the former case. Suppose D_1, D_{k+1} agree on part $[x, 2b]$ but not at $x - 1$. Then all $D_1, D_2, \cdots, D_{k+1}$ agree on part $[x, 2b]$. And the band of D_0, D_1 ends at some $y \in [x + 1, 2b]$. $(y - 1, y)$ must be an upward segment in D_1.

Consider the graph $G(D_1, D_2, \cdots, D_{k+1})$, comparing to $G(D_1)$, since $D_1, D_2, \cdots, D_{k+1}$ agree on part $[x, 2b]$, any connected components with least number in $[x, 2b]$ is not changed. By intersecting this with D_0, the graph is added by a segment between y and some other smaller number. So the number of connected component is decreased by 1. By induction, codim $D_0 \cap D_1 \cdots \cap D_{k+1} =$ codim $D_1 \cap D_2 \cdots D_{k+1} + 1 = k + 1$.

Notice that in the (b, b) type springer fiber, dependence of subspaces are symmetric: In previous sections, we labeled each connected component using the least number in each of them, and call these number independent. We can equivalently label them using the largest number in each component. And the set of free spaces are F_i, for i is the largest number in its connected component. Other spaces are determined uniquely from these spaces, by taking the images of N for properly many times. The space $F_{2b} = \mathbb{C}^{2n}$.

In this manner, the above analysis for D now is equivalent to saying that the chain C constructed for A, B in Theorem 4.2 also has the property that codim $C_0 \cap C_1 \cdots \cap C_k = k$. ■

In general, if the Springer fiber is of type (b, a), we add an upward line of length $b - a$ adjoining 0, to make any lattice path a Dyck path of length $2b$. (See the picture below.) Let this map be f: irreducible components of type $(b, a) \to$ irreducible components of type (b, b). Below is an example of the effect of f.

For any type (b, a) lattice path A, the graph $G(fA)$ is obtained from $G(A)$ by adding each number $b-a$. So the graphs of intersections also have this correspondence. In particular, if $K(A) \cap K(B) \neq \emptyset$, as components of type (b, a), then each connected component contains at most one non-positive number between $-(b - a), 0$. So in the graph for $K(fA) \cap K(fB)$, each $0, 1, 2, \cdots, b - a$ stands for a connected components. The number of positive connected components in $G(A, B)$ is the number of connected components greater than $b - a$, in the graph for $G(fA, fB)$. Therefore,

Theorem 5.3. *If $K(A) \cap K(B) \neq \emptyset$, then codim $K(A) \cap K(B) =$ codim $K(fA) \cap K(fB)$.*

Notice that if $A \to B$, then $fA \to fB$. So $G(b, a)$ is a subgraph of $G(b, b)$. Also f respects the algorithm we defined in Theorem 4.2. Therefore

Theorem 5.4. *In (b, a) type Springer fiber, if $K(A) \cap K(B) \neq \emptyset$, then the length of the chain constructed in Theorem 4.2 is codim $K(A) \cap K(B)$.*

Before proving the second condition of the main theorem, we have the following lemma:

Lemma 5.1. *If $C_0, C_1, \cdots, C_{l+1}$ is a chain of lattice paths, such that codim $K(C_i) \cap K(C_{i+1}) = 1$ for $i > 0$, then codim $C_0 \cap C_1 \cdots \cap C_{l+1} \leq$ codim $C_0 \cap C_1 + l$.*

Proof. We prove by induction. The lemma is correct when $l = 0$. Suppose it is true for $l = k$. Then for a chain $C_0, \cdots, C_{k+2}$. We know by induction codim $C_0 \cap \cdots C_{k+1} \leq$ codim $C_0 \cap C_1 + k$. By intersecting C_{k+2}, since codim $K(C_{k+1}) \cap K(C_{k+2}) = 1$, the graph $G(C_0, \cdots, C_{k+1})$ is added by one edge. So the number of positive connected components is at most decreased by 1. So

$$\text{codim } C_0 \cap \cdots C_k \cap C_{k+1} \leq \text{codim } C_0 \cap \cdots \cap C_k + 1 \leq \text{codim } C_0 \cap C_1 + k + 1. \quad (5.1)$$

■

Now we are able to prove the main theorem.

Proof. Let $C_0, \cdots, C_k$ be the chain for A, B. We only need to check the conditions 2, 3. We already know that codim $C_0 \cap \cdots C_k = k$. Notice that by Theorem 4.2,

$$C_0 \cap C_k = C_0 \cap C_1 \cdots \cap C_k. \tag{5.2}$$

So we have

$$\bigcap_{j\in[0,k]} C_j = \bigcap_{j\in[0,i]\cup[i+r,k]} C_j. \tag{5.3}$$

Using Lemma 5.1 we have

$$k = \text{codim} \bigcap_{j\in[0,k]} C_j = \text{codim} \bigcap_{j\in[0,i]\cup[i+r,k]} C_j \le \text{codim } C_i \cap C_{i+r} + (k-r). \tag{5.4}$$

So codim $C_i \cap C_{i+r} \ge r$. But since $C_i \cap C_{i+r} \supset C_i \cap C_{i+1} \cdots \cap C_{i+r}$, the lemma says

$$\text{codim } C_i \cap C_j \le \text{codim } C_i \cap C_{i+1} \cdots \cap C_{i+r} \le r. \tag{5.5}$$

So every equality is hold. In particular, the second and third condition of the main theorem is hold. ■

In fact, the main theorem also implies that k is the distance between A, B in graph G. And the chain constructed from Theorem 4.2 is one of the shortest path from A to B.

Theorem 5.5. *For two irreducible components A, B of the (b, a) type Springer fiber, if $A \cap B \neq \emptyset$, then codim $A \cap B = dist(A, B)$ in G.*

Proof. Since $G(b, a)$ is a subgraph of $G(b, b)$, we only need to prove the theorem for (b, b) type Springer fiber. In this context, any two irreducible components have non-empty intersection. Suppose $A = C_0, C_1, \cdots, C_l = B$ is a chain in $G(b, b)$ (not necessarily directional), then from Lemma 5.1, and $C_0 \cap C_l \supset C_0 \cap C_1 \cap \cdots \cap C_l$, we have

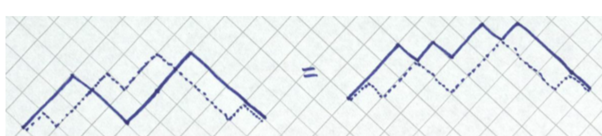
$$\text{codim } C_0 \cap C_l = \text{codim } C_0 \cap C_1 \cap \cdots \cap C_l \le l \tag{5.6}$$

By Theorem 5.1, there exists a chain of length codim $C_0 \cap C_l$. So the distance between $C_0 = A, C_l = B$ is exactly codim $A \cap B$. ■

5.2. **Simplification of Lattice Paths.** In general, when we consider the intersection of two irreducible components, their lattice path can cross each other many times. In this section, we prove that we can replace the two lattice paths by another pair, such that one is above is other.

For two lattice paths A, B, let $A + B$ represent the union of line segments of A, B, including repetition. Let $t(A, B)$ be the upper boundary $A + B$, and $b(A, B)$ be the lower boundary. Then $t(A, B), b(A, B)$ are lattice paths and $t(A, B) + b(A, B) = A + B$.

Theorem 5.6. *For any two lattice path A, B, $K(A) \cap K(B) = K(t(A, B)) \cap K(b(A, B))$. (see the picture below)*



Proof. Use induction on the area between $t(A, B)$ and $b(A, B)$. If the area is 1, then $\{t(A, B), b(A, B)\} = \{A, B\}$. The theorem obviously holds. Suppose we have proven the case for area not greater than k.

Suppose A, B are lattice paths such that the area between $t(A, B), b(A, B)$ is $k + 1$. Then the area is separated by some parts lying in consecutive intersections of A, B. Let C be the left most part. C lies in between x, y, which are intersections of A, B. (See the picture below.) Suppose in this part, the top boundary is from A, and the bottom boundary is from B.

If x is lower than y, then we have a band below A that ends at y, and completely lying in C. If x is higher than or at the same level of y, then there is a band below A starting at x and completely lying in C. In each case, we have a $A \to A'$, such that the band of A, A' is contained in C.

From the construction, $b(A', B) = b(A, B)$. And $A \cap B = A \cap A' \cap B, t(A, B) \cap b(A, B) = t(A, B) \cap t(A', B) \cap b(A', B)$. The band between A and A', $t(A, B)$ and $t(A', B)$ are the same. Comparing to $G(A, B)$, $G(A, A', B)$ has one more segment. Comparing to $G(t(A', B), b(A', B))$, $G(t(A, B), t(A', B), b(A', B))$ also has one more segment. The two additional segments are the same.

By induction, $K(A') \cap K(B) = K(t(A', B)) \cap K(b(A', B))$, so $G(A, B)$ and $G(t(A', B), b(A', B))$ determine the same connected components. So after adding a same segment, the new connected components are also the same. Therefore $K(A) \cap K(B) = K(t(A, B)) \cap K(b(A, B))$. ■

Corollary 5.7. *For any k lattice paths $A_1, A_2, \cdots, A_k$. There exists $B_1, B_2, \cdots, B_k$, such that*

(1) $A_1 + A_2 \cdots + A_n = B_1 + B_2 \cdots + B_n$

(2) B_i *is above* B_{i+1}, *for* $1 \leq i \leq k - 1$.

(3) $\bigcap_{i=1}^{k} K(A_i) = \bigcap_{i=1}^{k} K(B_i)$.

Conjecture 5.1. *Based on the simplification above and (3) in Theorem 5.1, we conjecture that the intersection of any number of components is an intersection of two components. It suffices to prove that any intersection of three irreducible components is equal to the intersection of some two irreducible components.*

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