THE DIAGONAL COHOMOLOGY CLASS OF VERTICAL BUNDLES

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Abstract. Given a manifold M, Milnor and Stasheff studied in [1] the *diagonal cohomology class* $u'' \in H^m(M \times M; \mathbb{Z}/2)$ that describes the orientation of the tangent bundle, and is related to its Stiefel-Whitney Classes. We generalize this concept to fiber bundles $M \to E \to N$ where the fiber and base are manifolds, relate it to the diagonal homology class, study the naturality of the construction, give further characterizations of the class, and compute it for certain examples.

Contents

In [1], Milnor and Stasheff give a method to compute the Stiefel-Whitney classes of tangent bundles to manifolds given its $\mathbb{Z}/2$ cohomology and the actions of Steenrod squares on it, via the Wu class and the diagonal cohomology class. Given a fiber bundle of manifolds $M^m \xrightarrow{i} E \xrightarrow{\pi} N^n$ with M a closed manifold, this method generalizes nicely to compute $w(\xi)$, the Stiefel-Whitney classes of the vertical bundle $\xi : \ker d\pi \downarrow E$, which is a subbundle of TE. We detail it below. We will work in $\mathbb{Z}/2$ coefficients.

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1. Construction of the Diagonal Cohomology Class

Let $D := E \times_N E = \{(e_1, e_2 \in E \times E : \pi(e_1) = \pi(e_2)\}$, and $\Delta : E \to D$ defined as $e \mapsto (e, e)$ is an inclusion, whose resulting normal vector bundle (over $\Delta(E) \cong E$) will henceforth be denoted $\nu\Delta E$. Then $\nu\Delta E \cong \xi$ canonically:

Lemma 1.1. *We have $\nu\Delta E \cong \xi$ are isomorphic as m-plane bundles.*

Proof. For any $e \in E$, a vector $(v_1, v_2) \in T_eE \times_{T_{\pi(e)}N} T_eE \cong T_{(e,e)}D$ is normal to $\Delta(E)$ iff $v_1 + v_2 = 0$. Thus we have an isomorphism of fibers

$$\begin{gathered} \xi_e \to (\nu\Delta(E))_{(e,e)} \\ v \mapsto (-v, v) \end{gathered}$$

which extends to an isomorphism of the vector bundles. □

Remark 1.2. Notice D is a fiber bundle over N with fiber $M \times M$, so for $n \in N$ we denote the fiber by D_n. (Similarly, denote E_n the fiber in E over n).

Since $\Delta(E) \in D$ is closed, by Cor. 11.2 in [1], $H^*(\xi, \xi_0) \cong H^*(\nu\Delta E, \nu\Delta E_0) \cong H^*(D, D - \Delta(E))$. The Thom class u thus corresponds to some $u' \in H^m(D, D - \Delta(E))$, which, by Thm. 11.3 in [1], restricts to the top Stiefel-Whitney class $w^m(\xi) \in H^m(E)$ under Δ^*. (For convenience, we will only use the notation u' to refer to this cohomology class in this section.) The class u' is uniquely characterized as follows.

Lemma 1.3. *u' is the unique class in $H^m(D, D - \Delta(E))$ such that, for any $e \in E$ with $\pi(E) = n$, if $j_e : (E_n, E_n - e) \to (D, D - \Delta(E))$ via $z \mapsto (e, z)$, then $j_e^*(u')$ is the generator of $H^m(E_n, E_n - e) \cong H^m(M, M - *) \cong \mathbb{Z}/2$.*

Proof. Let $L \subset \xi_e$ an open neighborhood of 0, and let

$$\begin{gathered} (L, L - 0) \to (D, D - \Delta(E)) \\ v \mapsto (\mathrm{Exp}(-v), \mathrm{Exp}(v)), \end{gathered}$$

which is well defined for L small enough since both coordinates project to n since $v \in \ker d\pi$. Then u' is uniquely characterized by the fact the the induced map in cohomology maps u' to the generator of $H^m(L, L - 0) = H^m(\xi_e, \xi_e - 0)$. This follows by our construction in Lemma 1.1. But this map is homotopic, via $(v, t) \mapsto (\mathrm{Exp}(-tv), \mathrm{Exp}(v))$, to $v \mapsto (e, \mathrm{Exp}(v))$, which is the composition of

$$\begin{gathered} (L, L - 0) \overset{\mathrm{Exp}}{\to} (E_n, E_n - e) \xrightarrow{j_e} (D, D - \Delta(E)) \\ v \mapsto \mathrm{Exp}(v) \mapsto (e, \mathrm{Exp}(v)), \end{gathered}$$

and thus this composition map induces the same map in cohomology, which will map u' to the generator of $H^m(L, L - 0) \cong H^m(\xi_e, \xi_e - 0)$. Since all these maps are isomorphisms in cohomology except for j_E^*, $j_E^*(u')$ must be the generator of $H^m(E_n, E_n - e)$ since this generator is the unique element of $H^m(E_n, E_n - e)$ that maps to the generator of $H^m(L, L - 0)$. Thus, the proposition is proven. □

We have that u' restricts to a class $\delta \in H^m(D)$ (denoted as u'' in earlier literature). For $a \in H^i(E), b \in H^j(E)$, define the *cross product over N* (of cohomology classes) as $a \times_N b \in H^{i+j}(E \times_N E)$ as $p_1^*(a) \smile p_2^*(b)$, where $p_1, p_2 : E \times_N E \to E$ are the projections. Then, the following computational tool also holds.

Lemma 1.4. *For any $a \in H^*(E)$, we have $(a \times_N 1) \smile \delta = (1 \times_N a) \smile \delta$ (justifying its name), where $1 \in H^0(E)$.*

Proof. Let $N_\epsilon \supset \Delta(E)$ be a tubular neighborhood (and let $j : N_\epsilon \hookrightarrow D$), with N_ϵ homeomorphic to the total space of $\nu\Delta E$. Notice p_1, p_2 coincide on $\Delta(E)$ and thus $j^*(a \times_N 1) = j^*(p_1^*(a)) = j^*(p_2^*(a)) = j^*(1 \times_N a)$. Then $(a \times_N 1) \smile u' = (1 \times_N a) \smile u'$ from the following commutative diagram applied to $a \times_N 1$ and $1 \times_N a$ (where i is the degree of a), and the proposition follows by restricting.

$$\begin{tikzcd} H^i(D) \arrow[r, "j^*"] \arrow[d, "\smile u'"] & H^i(N_\epsilon) \arrow[d, "\smile u'|(N_\epsilon, N_\epsilon - \Delta(E))"] \\ H^{i+m}(D, D - \Delta(E)) \arrow[r, "\cong"] & H^{i+m}(N_\epsilon, N_\epsilon - \Delta(M)) \end{tikzcd}$$

□

Now, if $\phi : H^*(E) \to H^{*+m}(\xi, \xi_0)$ is the Thom isomorphism, then it is well known (e.g. [1, p.91]) that the Stiefel-Whitney class is given by $w(\xi) = \phi^{-1} Sq(u)$. But since ξ and $\nu\Delta E$ are isomorphic and the existence of the tubular neighborhood N_ϵ, $H^*(\xi, \xi_0) \cong H^*(D, D - \Delta(E))$ and under this identification, $Sq^i(u') = (w_i(\xi) \times_N 1) \smile u'$. Restricting to D:

Lemma 1.5. $Sq^i(\delta) = (w_i \times_N 1) \smile \delta$.

Let $\delta_M \in H^m(M \times M)$ be the diagonal cohomology class of M (as a fiber bundle $M \to M \to *$). Suppose $N = \cup_{\alpha \in A} U_\alpha$ is an open contractible cover of the base (where arbitrary nonempty intersections are also contractible), whose existence is a classical fact in Riemannian geometry. It trivializes the fiber bundle. The following commutative diagram:

$$\begin{tikzcd} (D|U_\alpha, D|U_\alpha - \Delta(E|U_\alpha) \arrow[r, hook] & (D, D - \Delta(E)) \\ (E_n, E_n - e) \arrow[u, "j_e"] \arrow[r, equal] & (E_n, E_n - e) \arrow[u, "j_e"] \end{tikzcd}$$

(which holds for any $e \in E, n = \pi(e) \in U_\alpha$), proves that u'_α, the restriction of u' in $D|U_\alpha$, satisfies the conditions of Lemma 1.2, and thus equals the vertical relative diagonal cohomology class of the fiber bundle $M \to E|U_\alpha \to U_\alpha$. Thus, δ restricts to δ_α, the vertical (absolute) diagonal cohomology class of that fiber bundle. Furthermore,

Corollary 1.6. Restricting to a fiber, $H^m(D) \to H^m(M \times M)$ maps $\delta \mapsto \delta_M$.

Remark 1.7. We explain the notation $H^i(D) \to H^i(M \times M)$, since at first it might seem an abuse of notation. For convenience, we explain it instead for the fiber bundle $M \to E \to N$. Let $\iota_{1,2} : E_{n_{1,2}} \to E$ be the inclusions of the fibers of $n_1, n_2 \in N$, and let $\gamma : [0,1] \to N$ be a path with $\gamma(0) = n_1, \gamma(1) = n_2$. By the homotopy lifting property:

$$\begin{tikzcd} E_{n_1} \arrow[r, "\iota_1"] \arrow[d, "\mathrm{in}_0"] & E \arrow[d, "\pi"] \\ E_{n_1} \times [0,1] \arrow[r, "\gamma \circ \mathrm{pr}_2"] \arrow[ur, dashed] & B \end{tikzcd}$$

there exists a map $f : E_{n_1} \to E_{n_2}$ and a homotopy between ι_1 and $\iota_2 \circ f$. Similarly, there exists a map $g : E_{n_2} \to E_{n_1}$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} E_{n_1} \arrow[r, "f"] \arrow[dr, hook, "\iota_1"'] & E_{n_2} \arrow[r, "g"] \arrow[d, hook, "\iota_2"] & E_{n_1} \arrow[dl, hook', "\iota_1"] \\ & E & \end{tikzcd}$$

Thus, the following diagram commutes:

$$\begin{tikzcd} H^i(E) \arrow[r, "\iota_1^*"] \arrow[dr, "\iota_2^*"'] & H^i(E_{n_1}) \arrow[d, shift right, "f^*"'] \\ & H^i(E_{n_2}) \arrow[u, shift right, "g^*"'] \end{tikzcd}$$

(in particular, f and g are homotopy inverses), which explains the well-definedness of the map $H^i(E) \to H^i(M)$. We will use this construction for the fiber bundle $M \times M \hookrightarrow D \to N$ as well.

Example 1.8. If $E = M \times N$ is trivial, we have $\xi = (TM) \times N$, and $D = M \times M \times N$. By the Relative Kunneth formula [2,p.249],

$$\begin{aligned} [H^*(TM, TM_0) \otimes H^*(N)]^m &\cong [H^*(TM \times N, TM_0 \times N)]^m \\ &\cong H^m(M \times M \times N, M \times M \times N - \Delta(M \times N)) \\ &\cong [H^m(M \times M, M \times M - \Delta(M)) \otimes H^*(N)]^m \end{aligned}$$

so we see $u' = u'_m \otimes 1$ under that correspondence (where u' corresponds to the *fiber bundle* $M \to M \to *$). Then, by the Kunneth Theorem, $H^m(M \times M \times N) \cong [H^*(M \times M) \otimes H^*(N)]^m$, and δ corresponds to $\delta_M \otimes 1$.

Example 1.9. If $N = *$ and $M = S^m$, if $\mathcal{G} \in H^m(S^m)$ is the generator, we have $H^m(S^m \times S^m) = (H^0(S^m) \otimes H^m(S^m)) \oplus (H^m(S^m) \otimes H^0(S^m)) \cong (\mathbb{Z}/2)^2$ generated by $1 \times \mathcal{G}, \mathcal{G} \times 1$. By Lemma 1.4 for $a = \mathcal{G}$ we have $(1 \times \mathcal{G}) \smile \delta = (\mathcal{G} \times 1) \smile \delta)$, so $\delta = 0$ or $1 \times \mathcal{G} + \mathcal{G} \times 1$. But $\delta/[S^m] = 1$ is well known [1], where $[S^m]$ is the fundamental homology class. Thus, $\delta \neq 0$, and so $\delta = 1 \times \mathcal{G} + \mathcal{G} \times 1$.

2. Under Poincaré Duality

We have the following theorem.

Theorem 2.1. *If E is compact (i.e. if M, N are compact), the diagonal cohomology class is dual to $\Delta_*[E] \in H_{m+n}(D)$ under Poincaré Duality, where $[E]$ is the fundamental homology class of E.*

Proof. Let $\rho : \nu \downarrow E$ be the normal bundle of $\Delta : E \hookrightarrow D$. We have the following commutative diagram, where all arrows are isomorphisms:

$$\begin{tikzcd} H^0(E) \arrow[r] \arrow[d] & H^m(\nu, \nu_0) \arrow[r] & H^m_c(\nu) \\ H^0(\nu) \arrow[ur, "-\smile u"] & & \end{tikzcd}$$

where $u \in H^m(\nu, \nu_0)$ is the Thom class and H^*_c denotes compactly supported cohomology. (We have $H^m(\nu, \nu_0) = H^m(Th(\nu)) = H^m(v^*) = H^m_c(\nu)$ where $v^* = Th(\nu)$

is the one-point compactification of ν, in this case equal to the Thom space $Th(\nu)$ of ν since E is compact). There is another isomorphism $H^0(E) \to H^m_c(\nu)$: the one Poincare-dual to $H_{m+n}(E) \to H_{m+n}(\nu)$ induced by the inclusion of the zero section. It is an isomorphism since $E \simeq \nu$. Since all groups are isomorphic to $\mathbb{Z}/2$, these maps $H^0(E) \to H^m_c(\nu)$ are the same.

We have the following diagram, that commutes excepts possibly for the dashed arrows.

$$\begin{tikzcd}
 & H^m(\nu, \nu_0) \arrow[d, "\cong"] & H^m(D, D - \Delta(E)) \arrow[l, "\cong"'] \arrow[r] & H^m(D) \arrow[dd, dashed, "\cong"] \\
H^0(E) \arrow[ur, "\cong"] \arrow[r, "\cong"] \arrow[d, "\cong"] & H^m_c(\nu) \arrow[urr, dashed, "f^*"] \arrow[d, "\cong"] & & \\
H_{m+n}(E) \arrow[r, "\cong"] & H_{m+n}(\nu) \arrow[rr] & & H_{m+n}(D)
\end{tikzcd}$$

Remark 2.2. The map $H^m_c(\nu) \to H^m(D)$ is as follows. It is known $H^m_c(\nu) = H^m(\nu^*)$, the cohomology of the one-point compactification. But the one-point compactification behaves functorially with regards to open embeddings, thus $i : \nu \to D$ gives us $f : D = D^* \to \nu^*$. The map is $H^m_c(\nu) = H^m(\nu^*) \xrightarrow{f^*} H^m(D)$.

(Note maps from cohomology to homology are simply applying Poincaré Duality). Following the upper-right path from $H^m(\nu, \nu_0)$ to $H_{m+n}(D)$ gives the dual to δ, and the lower-left path gives $\Delta_*[E]$, since the composition of the two lowest arrows is simply induced by the inclusion $E \hookrightarrow \nu \hookrightarrow D$, thinking of ν as a tubular neighborhood in D. Thus, proving the diagram is commutative even with the dashed arrows is enough to prove the Theorem.

For the commutativity with f^*, we note $D - \Delta(E) \subset D - \nu$ (viewed inside D) is a deformation retract. Also, by the relative cohomology long exact sequence, $H^m(\nu^*, *) = H^m(\nu^*)$ since $m > 0$, for $* \in \nu^*$ the added point. Thus we have the following commutative diagram:

$$\begin{tikzcd}
H^m(D, D - \Delta(E)) \arrow[r, equal] & H^m(D, D - \nu) \arrow[dr] & \\
H^m_c(\nu) \arrow[r, equal] & H^m(\nu^*, *) \arrow[u] \arrow[r] & H^m(D)
\end{tikzcd}$$

as desired. Finally, we verify

$$\begin{tikzcd}
H^m_c(\nu) \arrow[r, "f^*"] \arrow[d] & H^m(D) \arrow[d] \\
H_{m+n}(\nu) \arrow[r] & H_{m+n}(D)
\end{tikzcd}$$

commutes. This follows from the commutativty of the following diagram, where the upper half commutes by the naturality of Poincaré Duality, the bottom half commutes because the corresponding diagram of maps of spaces commutes, and $H_{m+n}(\nu^*) \cong H_{m+n}(\nu)$ because both are isomorphic to $H^m(\nu^*)$ under Poincaré

Duality.

$$\begin{tikzcd}
H^m(D) \arrow[d, "\cong"] & H^m(\nu^*) \arrow[l] \arrow[d, "\cong"] \\
H_{m+n}(D) \arrow[r, "\cong\, f_*"] & H_{m+n}(\nu^*) \arrow[d, leftrightarrow, "\cong"'] \\
& H_{m+n}(\nu) \arrow[ul, "\cong"]
\end{tikzcd}$$

□

Example 2.3. As in Example 1.8, let $M \to M \times N \to N$ be trivial. Then $\delta = \delta_M \times 1 \in [H^*(M \times M) \otimes H^*(N)]^m = H^m(M \times M \times N)$. We have $[M \times N] \in H_{m+n}(M \times N) = H_m(M) \otimes H_n(N)$ corresponds to $[M] \times [N]$. Under $\Delta : M \times N \to M \times M \times N$ this maps to $(\Delta_{M*}[M]) \times [N]$ (where $\Delta_M : M \to M \times M$). This is dual to δ as $\Delta_{M*}[M]$ is dual to δ_M and $[N]$ is dual to 1, as expected.

3. Naturality of the diagonal cohomology class

3.1. **Base-wise Naturality.** The diagonal cohomology class is natural under pullbacks, in the following sense:

Theorem 3.1. *The class δ is natural under pullback. In other words, let $\pi : E \to N$ be a fiber bundle with fiber M and diagonal cohomology class δ. Let $f : N' \to N$ be any map and $E := f^*E$ and $D' := f^*D$ viewed as a fiber bundle over $h : D \to N$. Let δ' be the diagonal cohomology class of $\pi' : E' \to N'$. Then $D' = E' \times_{N'} E'$ and under the map $D' \to D$, δ pulls back to δ'.*

Proof. Using coordinate notation for pullbacks, we have

$$\begin{aligned}
D' &= \{(n', d) \in N' \times D : f(n') = h(d)\} \\
&= \{(n', e_1, e_2) \in N \times E \times E : f(n') = \pi(e_1) = \pi(e_2)\} \\
&= \{((n', e_1), (n', e_2)) \in E' \times E' : f(n') = \pi(e_1) = \pi(e_2)\} \\
&= \{(e_1', e_2') \in E' \times E' : \pi'(e_1) = \pi'(e_2)\} = E' \times_{N'} E'.
\end{aligned}$$

Choose any metric over D and pull it back to D', and let ν, ν' denote the normal bundles of the inclusions $\Delta : E \hookrightarrow D, \Delta' : E' \hookrightarrow D'$. Notice $\xi' := (\ker d\pi' \downarrow E')$ is the pullback of ξ under $g : E' \to E$, i.e. $\xi' = g^*\xi$. This is clear since both are the bundles tangent to the fibers, and g respects fibers. By naturality of the Thom Class, as in [3, Th. 10.28], we get that if $u \in H^m(\xi, \xi_0), u' \in H^m(\xi', \xi_0')$ are the respective Thom classes, then u' is the pullback of u under the map in cohomology induced by $\zeta : (\xi', \xi_0') \to (\xi, \xi_0)$, i.e. $u' = \zeta^*(u)$. Further, since

$$\begin{tikzcd}
E \arrow[r, hook, "\Delta"] & D \\
E' \arrow[u] \arrow[r, hook, "\Delta'"] & D' \arrow[u]
\end{tikzcd}$$

commutes, there's a map $\nu' \to \nu$. By our construction in Lemma 1.1, the diagram

$$\begin{tikzcd} (\xi, \xi_0) \arrow[d, "\cong"] & (\xi', \xi_0') \arrow[l, "\zeta"'] \arrow[d, "\cong"] \\ (\nu, \nu_0) & (\nu', \nu_0') \arrow[l] \end{tikzcd}$$

commutes (clearly the complement of the zero section maps to the complement of the zero section on all these maps, so they're well-defined), and thus so does

$$\begin{tikzcd} H^m(\xi, \xi_0) \arrow[r, "\zeta^*"] \arrow[d, leftrightarrow, "\cong"'] & H^m(\xi', \xi_0') \arrow[d, leftrightarrow, "\cong"'] \\ H^m(\nu, \nu_0) \arrow[r] & H^m(\nu', \nu_0') \end{tikzcd}$$

Notice the map $j : D' \to D$ creates a map of pairs $(D', D' - \Delta'(E')) \to (D, D - \Delta(E))$, since:

$$j(n', e_1, e_2) \in \Delta(E) \Leftrightarrow e_1 = e_2 \Leftrightarrow (n', e_1, e_2) \in \Delta'(E').$$

Further, for a fixed ϵ, let $\nu(\epsilon)$ denote the ϵ-neighborhood of the zero section under the given metric, as in [1], and similarly for ν'. Then, since the metric pulls back, the map $\nu' \to \nu$ creates a map of triples $(v', v_0', v' - v'(\epsilon)) \to (v, v_0, v - v(\epsilon))$. Thus, by naturality of excision, and since the metric pulls back, for small enough $\epsilon > 0$ we have the following commutative diagram:

$$\begin{tikzcd} H^m(D, D - \Delta(E)) \arrow[r] \arrow[d, "Exp^* \cong"] & H^m(D', D' - \Delta'(E')) \arrow[d, "Exp^* \cong"] \\ H^m(\nu(\epsilon), \nu(\epsilon)_0) \arrow[r] \arrow[d, "\text{exc.} \cong"] & H^m(\nu'(\epsilon), \nu'(\epsilon)_0) \arrow[d, "\text{exc.} \cong"] \\ H^m(\nu, \nu_0) \arrow[r] & H^m(\nu', \nu_0') \end{tikzcd}$$

Combining it with the above commutative diagram and with $u' = \zeta^*(u)$, the lemma is proven. $\square$

Remark 3.2. Notice that Example 1.8 follows immediately from naturality, with the map $N \to *$ pulling back $M \to *$ to $M \times N \to N$.

3.2. **Fiber-wise Naturality.** There is a very weak analog to Theorem 3.1 for changes of fibers, whose proof relies chiefly on Theorem 2.1

Lemma 3.3. *Assume the map of fiber bundles*

$$\begin{tikzcd} M' \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d, "g"] & N \arrow[d, equal] \\ M \arrow[r] & E \arrow[r] & N \end{tikzcd}$$

satisfies $g_*[E'] = [E]$. *Further, assume* $h : D' \to D$ *satisfies that* $H_{m+n}(D') \xrightarrow{h_*} H_{m+n}(D)$ *is injective and* $h_*[D'] = [D]$. *Then* $h^*(\delta) = \delta'$.

Proof. Let $\Delta : E \to D, \Delta' : E' \to D'$ denote the diagonal maps. Notice by the assumptions and by the commutativity of

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, hook, "\Delta'"] & E \arrow[d, hook, "\Delta"] \\ D' \arrow[r, "h"] & D \end{tikzcd}$$

that $h_*(\Delta'_*[E']) = \Delta_*[E]$. By the assumptions, Poincaré Duality is natural with respect to h, i.e. the following diagram commutes:

$$\begin{tikzcd} H_{m+n}(D') \arrow[r, "h_*"] & H_{m+n}(D) \\ H^m(D') \arrow[u, "\cong"] & H^m(D) \arrow[l, "h^*"] \arrow[u, "\cong"] \end{tikzcd}$$

where the upward arrows are Poincaré Duality. By Theorem 2.1 and by injectivity of h_* in the diagram, the lemma is proven. □

4. An Application Of The Eilenberg-Moore Spectral Sequence

The Eilenberg-Moore Spectral Sequence allows us to compute the cohomology of D. Indeed, we have the following result: [4,p.48]

Theorem 4.1. *Given a fiber bundle $M \to E \to N$, there exists a functorial second-quadrant cohomology spectral sequence (E, d) such that:*

- $E_r \Rightarrow H^*(D)$ *converges.*
- $E_2^{p,q} = \mathrm{Tor}_{p,q}^{H^*(N)}\big(H^*(E), H^*(E)\big)$, *where the second index of* Tor *indicates the degree.*
- *The edge homomorphism*

$$H^*(E) \otimes_{H^*(N)} H^*(E) = E_2^{0,*} \twoheadrightarrow E_\infty^{0,*} \hookrightarrow H^*(D)$$

coincides with the obvious map $H^*(E) \otimes_{H^*(N)} H^*(E) \to H^*(D)$.

Here we think of $H^*(E)$ as an $H^*(N)$ module: if $a \in H^*(E), b \in H^*(N)$ then $ba := \pi^*(b) \smile a$. Assume $\pi^* : H^i(N) \to H^i(E)$ is an isomorphism for $i \leq m$. Then $H^*(E) \cong H^*(N)$ as $H^*(N)$-modules, in dimensions $\leq m$. Thus, all the $\mathrm{Tor}_{p,q}$ terms vanish when $p > 0, q \leq m$. Thus, by the theorem, the obvious map $[H^*(E) \otimes_{H^*(N)} H^*(E)]^m \to H^m(D)$ is a surjection. Thus:

Corollary 4.2. If $\pi^* : H^i(N) \to H^i(E)$ is an isomorphism for $i \leq m$, then there exist $a_1, \dots, a_r, b_1, \dots, b_r \in H^*(E)$ with $|a_i| + |b_i| = m$ for all i, such that

$$\delta = \sum_{i=1}^{r} a_i \times_N b_i.$$

The conditions of the corollary can be verified, in some cases, using the Serre Spectral Sequence for $M \to E \to N$. Indeed, if $M = S^m$, we have the E_2 page:

$$\begin{tikzcd}
H^0(N) \arrow[ddddrrrr, "d_{m+1}"] & H^1(N) \arrow[ddddrrrr, "d_{m+1}"] & \cdots & H^m(N) & H^{m+1}(N) & H^{m+2}(N) \\
\vdots & \vdots & & & & \vdots \\
\vdots & \vdots & & & & \vdots \\
H^0(N) & H^1(N) & \cdots & H^m(N) & H^{m+1}(N) & H^{m+2}(N)
\end{tikzcd}$$

where the upper row has coefficients in $H^m(S^m) = \mathbb{Z}/2$ and the lower row has coefficients in $H^0(S^m) = \mathbb{Z}/2$. Thus $\pi^* : H^i(N) \to H^i(E)$ is an isomorphism for $i \leq m-1$. This also holds for $i = m$ if $H^0(N; H^m(M)) \to H^m(N; H^0(M))$ is injective. However, this is unfortunately a rather hard condition to verify.

5. Spherical Fiber Bundles

In this section we assume M is topologically a sphere. We have the Serre Spectral sequence of $M \times M \to D \to N$ (illustrated for $m = 3$):

$$\begin{tikzcd}
H^0(N) \arrow[dddrrrr, "d_{m+1}"] & H^1(N) \arrow[dddrrrr, "d_{m+1}"] & \cdots & H^m(N) & H^{m+1}(N) & H^{m+2}(N) \\
\vdots & \vdots & & & & \vdots \\
\vdots & \vdots & & & & \vdots \\
H^0(N) \arrow[dddrrrr, "d_{m+1}"] & H^1(N) \arrow[dddrrrr, "d_{m+1}"] & \cdots & H^m(N) & H^{m+1}(N) & H^{m+2}(N) \\
\vdots & \vdots & & & & \vdots \\
\vdots & \vdots & & & & \vdots \\
H^0(N) & H^1(N) & \cdots & H^m(N) & H^{m+1}(N) & H^{m+2}(N)
\end{tikzcd}$$

depicting the E_2 page, where the upper row has coefficients in $H^{2m}(S^m \times S^m) = \mathbb{Z}/2$, the middle row has coefficients in $H^m(S^m \times S^m) = \mathbb{Z}/2^2$, and the lower row has coefficients in $H^0(S^m \times S^m) = \mathbb{Z}/2$. There will be differentials in E_{m+1} and in E_{2m+1}. (Notice, for instance, that if $n < 2m+1$ then all the differentials in E_{2m+1} will be 0, since $H^k(N) = 0$ for all $k \geq 2m+1$). In particular, we have

$$H^m(D) \cong H^m(N) \oplus \ker\big(H^0(N; H^m(M \times M)) \xrightarrow{d_{m+1}} H^{m+1}(N)\big)$$

Since $H^0(N; H^m(M \times M)) \cong H^m(M \times M)$, we have that the fiber-restriction map $H^m(D) \to H^m(M \times M)$ mapping $\delta \mapsto \delta_M$ simply ignores the $H^m(N)$ coordinate and includes the kernel into $H^m(M \times M)$. But in Example 1.9 we computed $\delta_M = 1 \times \mathcal{G} + \mathcal{G} \times 1$ where $\mathcal{G} \in H^m(S^m)$ is the generator. Thus, under the identification (4.1), we have:

Lemma 5.1. *Under the identification (4.1), we have $\delta = (\delta^N, 1 \times \mathcal{G} + \mathcal{G} \times 1)$, for some $\delta^N \in H^m(N)$, denoted the* base-wise diagonal cohomology class*. In particular, if $H^m(N) = 0$, then $\delta = 1 \times \mathcal{G} + \mathcal{G} \times 1$.*

Remark 5.2. Notice also that if the differential $H^0(N; H^m(M \times M)) \xrightarrow{d_{m+1}} H^{m+1}(N)$ is injective, then δ^N completely determines δ. However, this condition is hard to verify, since differentials in spectral sequences are generally hard to compute. Nevertheless, Lemma 4.1 implies that $1 \times \mathcal{G} + \mathcal{G} \times 1$ must be in its kernel.

Given a real m-plane bundle $\tau : F' \downarrow N$, recall its *sphere-ification* $Sph(\tau)$ denotes the sphere bundle where all the fibers $\mathbb{R}^m$ of τ have been replaced with their one-point compactification, S^m. We state the main conjecture of the paper:

Conjecture 5.3. The base-wise diagonal cohomology class of $S^m \to Sph(\tau) \to N$ equals the top dimensional Stiefel Whitney class $w_m(\tau) \in H^m(N)$.

For trivial τ, this holds because both classes are 0, as is implied by Example 1.8. Notice that δ^N does satisfy the normalization and naturality axioms in the definition of Stiefel-Whitney classes. Indeed, the normalization axiom for the Infinite Mobius band vector bundle is verified in Section 6, and the naturality axiom is verified in the following result:

Lemma 5.4. *The base-wise diagonal cohomology class is natural. In other words, let $f : N' \to N$ and $\tau : (E \downarrow N)$ be an m-plane bundle, with pullback $f^*\tau$. Let $\delta^{N'}, \delta^N$ denote the base-wise diagonal cohomology classes of $Sph(\tau')$ and $Sph(\tau)$. Then $f^*(\delta^N) = \delta^{N'}$.*

Proof. Clearly $f^*(Sph(\tau)) = Sph(\tau')$, and the map $Sph(\tau') \to Sph(\tau)$ maps the ∞-point of each fiber to the ∞ point of its target fiber. Let $D := Sph(\tau) \times_N Sph(\tau)$ and define D' analogously. We have a map $h : D' \to D$ which, by Theorem 3.1, satisfies $h^*(\delta) = \delta'$, where δ, δ' are the diagonal cohomology classes of $Sph(\tau), Sph(\tau')$, respectively. By functoriality of the Serre Spectral Sequence, the map of fiber bundles

$$\begin{tikzcd}
S^m \times S^m \arrow[r] \arrow[d, equal] & D' \arrow[r] \arrow[d, "h"] & N' \arrow[d, "f"] \\
S^m \times S^m \arrow[r] & D \arrow[r] & N
\end{tikzcd}$$

induces a map in the respective E_2 pages which respects convergence to the E^∞ page:

$$\begin{tikzcd}
E_2^{m,0} = H^m(N) \arrow[rrr, Rightarrow] \arrow[ddd, "f^*"] & & & E_\infty^{m,0} \arrow[ddd] \\
& E_2^{0,m} \arrow[r, Rightarrow] \arrow[d] & E_\infty^{0,m} \arrow[d] & \\
& E_2'^{0,m} \arrow[r, Rightarrow] & E_\infty'^{0,m} & \\
E_2'^{m,0} = H^m(N') \arrow[rrr, Rightarrow] & & & E_\infty'^{m,0}
\end{tikzcd}$$

such that

$$h^* : H^m(D) = E_\infty^{m,0} \oplus E_\infty^{0,m} \to E_\infty'^{m,0} \oplus E_\infty'^{0,m} = H^m(D')$$

is the direct sum of the two maps in the diagram. Thus, $h^*(\delta) = \delta'$ immediately implies $f^*(\delta^N) = \delta^{N'}$, as desired. $\square$

5.1. **An Application of the Gysin Sequence.** Spherical fiber bundles are interesting also in one final scenario. Choose a projection p_i from either of $p_1, p_2 : D \to E$. Notice $M \to D \xrightarrow{p_i} E$ is a fiber bundle. Assume $\pi_1(E)$ acts trivially on $H^*(M)$. This holds, for instance, if $m > 1$ and N is simply connected since the homotopy long exact sequence

$$\cdots \to \pi_1(M) \to \pi_1(E) \to \pi_1(N) \to \cdots$$

implies E is also simply-connected. Then we can construct the (long exact) Gysin sequence:

$$\cdots \to H^k(E) \xrightarrow{p_i^*} H^k(D) \xrightarrow{p_{i!}} H^{k-m}(E) \xrightarrow{c\smile -} H^{k+1}(E) \to \cdots$$

where $p_{i!}$ is the Poincaré-dual to $p_{i*} : H_{2m+n-k}(D) \to H_{2m+n-k}(E)$, and the map $H^{k-m}(E) \to H^{k+1}(E)$ is simply cupping with some fixed class $c \in H^{m+1}(E)$. At $k = m$ we have $\delta \mapsto 1 \mapsto c = 0$ by exactness: indeed, $p_{i!}(\delta) = 1$ because of the following commutative diagram:

$$\begin{tikzcd}
H^m(D) \arrow[r, "p_{i!}"] \arrow[d, "\cong"] & H^0(E) \arrow[d, "\cong"] \\
H_{m+n}(D) \arrow[r, "p_{i*}"] & H_{m+n}(E)
\end{tikzcd}
\qquad
\begin{tikzcd}
\delta \arrow[r, mapsto] \arrow[d, mapsto] & 1 \arrow[d, mapsto] \\
\Delta_*[E] \arrow[r, mapsto] & [E]
\end{tikzcd}$$

by Theorem 2.1. Thus, the Gysin Sequence separates into short exact sequences, as follows:

Lemma 5.5. *Assume $M = S^m$ with $m > 1$ and N is simply connected. For $i = 1, 2$ and for all k we have a short exact sequence (which doesn't split naturally)*

$$0 \to H^k(E) \xrightarrow{p_i^*} H^k(D) \xrightarrow{p_{i!}} H^{k-m}(E) \to 0$$

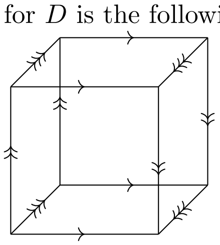
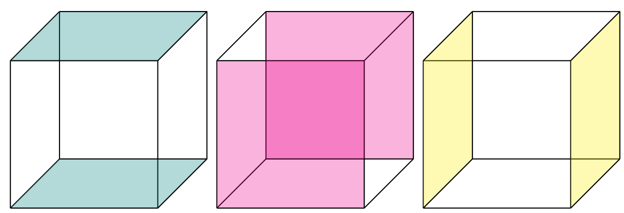
*In particular, δ **does not** come from a class in $H^m(E)$.*

6. A Nontrivial Example: The Klein Bottle

Take $M = S^1, N = S^1$, and E to be the Klein Bottle. It is well-known that the cohomology of the E is $H^*(E) = \mathbb{Z}/2[x, y]/\langle x^3, y^2, x^2y\rangle$. Before computing $H^*(D)$, we compute $\delta_* \in H^1(S^1 \times S^1)$ to aid us in computations. Given $H^*(S^1 \times S^1) = \mathbb{Z}/2[z, w]/\langle z^2, w^2\rangle$. By Lemma 1.4 for a the generator of $H^1(S^1)$ we have $z \smile \delta_* = (a \times 1) \smile \delta_* = (1 \times a) \smile \delta_* = w \smile \delta_*$ which gives $\delta_* = z + w$ or 0. But $\delta_*/\mu_{[S^1]} = 1$ is well-known ([1]), thus $\delta_* \neq 0$ so $\delta_* = z + w$.

6.1. **Computing the Diagonal Cohomology Class.**

The CW-complex diagram for D is the following:


with:


where the face identifications for the first two (top and front) faces is via projection, and for the third we rotate one face π radians. We'll call the top, front, and side faces T, F, S (corresponding to their colors teal, fucshia, and sunset). We also label the arrows with one, two, and three heads a, b, c, for convenience. Notice T, F are Klein Bottles and S is a torus. By abuse of notation, a, b, c will also denote the corresponding 1-cells and T, F, S the corresponding 2-cells. The cellular cochain complex for D is:

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2_a \oplus \mathbb{Z}/2_b \oplus \mathbb{Z}/2_c \to \mathbb{Z}/2_S \oplus \mathbb{Z}/2_T \oplus \mathbb{Z}/2_F \to \mathbb{Z}/2 \to 0$$

where the generator of $\mathbb{Z}/2_i$ represents the cocycle taking $i \mapsto 1$. The first and third maps are clearly 0, and the middle map is

$$(\alpha, \beta, \delta) \mapsto (0, 2\delta, 2\beta)$$

which with $\mathbb{Z}/2$ coefficients is also 0. Thus

- $H^0(D) = \mathbb{Z}/2$ with basis 1.
- $H^1(D) = \mathbb{Z}/2^3$ with basis $[S]^*, [T]^*, [F]^*$.
- $H^2(D) = \mathbb{Z}/2^3$ with basis $[a]^*, [b]^*, [c]^*$.
- $H^3(D) = \mathbb{Z}/2$, and higher cohomology groups are 0.

Where $[i]^*$ denotes the cohomology class corresponding to $[i]$ under Poincaré Duality. To compute the multiplicative structure of $H^*(D)$, we use the fact that $[i]^* \smile [j]^* = [i \cap j]^*$. For a 1-cell i we denote $I = [i]^*$ and for a 2-cell J we denote $j = [J]^*$. We easily get

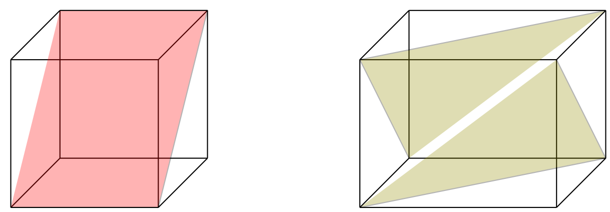
- $s^2 = 0, t^2 = c, f^2 = b, st = c, fs = b, tf = a$, which immediately implies
- $Bs = Cs = Bf = Ct = 0$
- $As = At = Af = Bt = Cf \neq 0$.

Thus $H^*(D) = \mathbb{Z}/2[t, f, s]/\langle s^2, t^3, f^3, t^2f^2, t^2fs, f^2ts\rangle$.

Given $M \times M \overset{\cong}{\to} D_n \subset D$ for some $n \in N$ we get $H^1(D) \to H^1(M \times M)$ which by Corollary 1.6 maps $\delta \mapsto \delta_*$. But $t \mapsto z, f \mapsto w, s \mapsto 0$ and thus $\delta = t + f + s$ or $t + f$. Further, if $x \in H^1(E)$ with $x^2 \neq 0$ then by Lemma

1.4, $T \smile \delta = (x \times_N 1) \smile \delta = (1 \times_N x) \smile \delta = Y \smile \delta$. If $\delta = t + f$ then $t \smile \delta = t^2 + tf = C + A \neq B + A = f^2 + tf = f \smile \delta$, a contradiction. Thus $\boxed{\delta = t + f + s}$. So $B + C = \delta^2 = Sq^1(\delta) = (w_i(\xi) \times_N 1) \smile (t + f + s)$, from which easily follows $\boxed{w_1(\xi) = y} \in H^1(E)$, with $y^2 = 0$.

6.2. **Under Poincaré Duality.** By Theorem 2.1, $\delta = t + f + s$ is dual to $\Delta_*[E]$, which is represented by the following submanifold on the left:



Rather unexpectedly, this is not to be confused with the manifold on the right, which represents $t + f$. Indeed, these representations agree with cup products:

$$\begin{aligned} t(t+f+s) &= c + a + c = a & \qquad t(t+f) &= c + a \\ f(t+f+s) &= a + b + b = a & \qquad f(t+f) &= a + b \\ s(t+f+s) &= c + b & \qquad s(t+f) &= c + b \end{aligned}$$

7. Further Research

We describe a few possible prospects looking forward.

1. A proof (or counterexample) to Conjecture 4.3. The Axiom of the Whitney Sum Formula could potentially be verified: the main obstacle seems to be computing the cohomology of D which corresponds to $S^{m_1+m_2} \to Sph(\xi \oplus \tau) \to N$, where $\xi, \tau \downarrow N$ are m_1,m_2-plane bundles, respectively.

2. A stronger version of Lemma 3.3. If the map $M' \to M$ is either a covering map or a degree 1 map, we hypothesize the lemma holds without the need for further conditions.

3. A version of Corollary 4.2 for $N = *$ is described in [1,Ch. 11]. We hypothesize there is a general statement analog to this one, which does not on such strong assumptions as the Corollary requires.

4. The formula $w(TE) = w(\xi) \smile \pi^*(w(TN))$, together with the fact that $i^*\xi = TM$ for $i : M \hookrightarrow E$, could be related to δ via the computational Lemmas in Section 1. Indeed, let $p_{1,2} : D \to E, q_{1,2} : E \times E \to E, r_{1,2} : M \times M \to M, s_{1,2} : N \times N \to N$ be projections, and $\delta_N \in H^n(N \times N), \delta_M = \delta_*, \delta_E \in H^{m+n}(E \times E)$ be diagonal cohomology classes. Let $h = \pi \circ p_1 = \pi \circ p_2$, and $\Delta_X : X \to X \times X$ denote the diagonal map for any space X. Then,

Conjecture 7.1. The identity

$$\delta_E|_D = \delta \smile h^*(\Delta_N^*(\delta_N))$$

holds.

Assuming the identity holds, applying Sq gives

$$\begin{aligned} Sq(\delta_E|_D) &= Sq(\delta_E)|_D \\ &= (q_1^*(w(TE)) \smile \delta_E)|_D \\ &= \Big(q_1^*(w(\xi)\pi^*(w(TN))) \smile \delta_E\Big)|_D \\ &= \Big(q_1^*(w(\xi)\pi^*(w(TN)))\Big)|_D \smile \delta_E|_D \\ &= p_1^*(w(\xi)\pi^*(w(TN))) \smile \delta \smile h^*(\Delta_N^*(\delta_N)) \\ &= \Big(p_1^*(w(\xi)) \smile \delta\Big) \smile \Big(h^* \circ \Delta_N^* \circ s_1^*(w(TN)) \smile h^* \circ \Delta_N^*(\delta_N)\Big) \\ &= Sq(\delta) \smile h^* \circ \Delta_N^*(Sq(\delta_N)) \\ &= Sq(\delta \smile h^* \circ \Delta_N^*(\delta_N)), \end{aligned}$$

which agrees with Lemma 1.5. This computation is null, but provides a heuristic argument for the validity of the conjecture.

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