

# STRAIGHT LINE ORTHOGONAL DRAWINGS OF COMPLETE TERNARY TREES

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ABSTRACT. In this paper we study embeddings of complete ternary trees in a 2-D lattice. Suppose we are given an  $n$ -node complete ternary tree  $T_n$  with height  $h$ . We can construct an embedding of  $T_h$  on a 2-D lattice  $L$  so that every node in  $T_h$  corresponds to a vertex in  $L$ , and every edge in  $T_h$  corresponds to a horizontal or vertical path in the lattice. We provide a better upper bound on the area requirement of straight-line orthogonal drawings of ternary trees, namely  $O(n^{1.118})$  area. Further, we consider abstract configurations of ternary trees, where subtrees of level  $h - k$  for an integer  $h < k$  are modeled by disjoint boxes, providing lower bounds on the space requirements for  $T_h$ .

## 1. BACKGROUND AND PRELIMINARIES

Planar graph drawings have been an important field of study, as they lend themselves to applications in circuit design and cartography, among others, where it is convenient to have readable representations of the graphs. An early result in the study of planar graphs states that every planar graph admits a straight-line planar drawing. That is, if a graph can be drawn on the plane without edge crossings with arbitrary shaped curves, then it can be drawn without edge crossings using only straight-lines.

Then, in 1979, Lipton and Tarjan presented a separator theorem for planar graphs. It states that an  $n$ -vertex planar graph can always be partitioned into two disjoint sets, each with no more than  $\frac{2n}{3}$  vertices, with  $O(\sqrt{n})$  edges between them [3]. In fact, this implies that planar graphs are never good expanders. With this in mind, it seems possible that planar graphs can be embedded into 2-dimensional lattices with small area.

In this paper, we will study a class of planar graphs, trees, and their embeddings into lattices.

Next, we formally define and introduce the notation we will use throughout the paper.

**1.1. Planar Graphs and Trees.** A *planar drawing* of a graph  $G(V, E)$  is an embedding into a lattice in which every vertex of  $V$  is mapped to a point in the lattice and each edge in  $E$  is mapped to a path in the lattice such that no distinct edges intersect except at their endpoints. Thus, an equivalent definition for a planar graph is a graph admitting a planar drawing.

The *degree* of a vertex is the number of edges that are incident to it, and the *degree* of a graph  $G(V, E)$  is the maximal degree of its vertex set.

A *tree* is a connected acyclic graph, and a *complete binary tree* (a *complete ternary tree*) is a rooted tree such that each non-leaf node has exactly two (exactly three) children. Complete binary trees are of degree three, and complete ternary trees are of degree four.

A *spine* in a rooted tree  $T$  is a path connecting the root  $r(T)$  to a leaf.

**1.2. Drawing Standards.** An *orthogonal planar drawing* is a planar drawing  $\Gamma$  of a graph  $G(V, E)$  where each node  $v \in V$  is represented by a point in a 2-D lattice, and each edge  $e \in E$  is represented by a path composed by horizontal and vertical segments in the lattice. Clearly, a

graph must have degree at most four in order to admit an orthogonal drawing.

A *straight-line orthogonal planar drawing* is a planar drawing  $\Gamma$  where the path is composed of a single horizontal or vertical line segment. There are classes of drawings, like four-connected graphs, that do not admit a straight-line orthogonal drawings.

In this paper, we will assume that the drawings are all planar.

Qualitative features of optimal graphs include minimal area, and ideal aspect ratio. The *area* of a planar drawing of a graph  $G$  is the area of the smallest rectangular grid, the *bounding box*, that contains the drawing completely. The *aspect ratio* of a planar drawing is the ratio of the width to the height of the bounding box.

## 2. KNOWN RESULTS

In 1981, Les Valiant addresses orthogonal drawings of planar graphs with minimal area. He proves that every  $n$ -node tree with degree at most four can be embedded in  $\Theta(n)$  area [4]. Valiant's algorithm for constructing the embeddings in  $O(n)$  area uses a recursive splitting procedure on separators of the tree, and combines the subtrees using exit tracks with bounded size to glue the components together.

Regarding straight-line orthogonal drawings, Garg and Rusu showed that  $O(n)$  is a tight bound for the area of binary trees [2].

On the other hand, the best-known upper bound for drawing ternary trees is  $O(n^{1.5})$  [1]. Previously, for complete ternary trees, the best-known upper bound is  $O(n^{1.262})$  area, and there are no non-trivial lower bounds. The goal of this paper is to investigate a tight bound on the minimal area of straight-line orthogonal drawings of complete ternary trees.

## 3. ABSTRACT CONFIGURATIONS

In general, finding the minimal area straight-line orthogonal drawing of a complete ternary tree for a given level  $h$  is a difficult problem; it has been shown that the problem is  $\mathcal{NP}$ -hard (GD 2008). Because of this, we can make tree constructions more general by considering *abstract configurations*, where we draw a tree of level  $h$  with boxes representing subtrees of size  $h - k$  (see figure for the abstract configuration Frati uses to prove the upper bound  $O(n^{1.262})$ ).

According to the layout, the abstract configuration will satisfy linear recurrence relations in the width  $W_{h-k}$  and height  $H_{h-k}$  of the form:

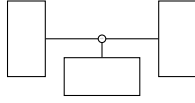


FIGURE 1. Frati's abstract configuration with subtrees of level  $h - 1$

$$\begin{pmatrix} W_h \\ H_h \end{pmatrix} = M_k \begin{pmatrix} W_{h-k} \\ H_{h-k} \end{pmatrix}$$

Let the dominant eigenvalue of  $M_k$  be  $\mu$ . Then,  $W_h = \alpha \cdot \mu^h + \dots$  and  $H_h = \beta \cdot \mu^h + \dots$ . Then, the area of the configuration grows with  $\mu^{2h}$ . With respect to the number of nodes, the area grows with  $\mu^{2O(\log_3 n)} = O(n^{\log_3 \mu^2})$  and the aspect ratio approaches  $\frac{\alpha}{\beta}$ .

Then, in order to have a valid abstract configuration, we must have that the layout of boxes with aspect ratio  $\lambda$  representing the subtrees fit in a box of aspect ratio  $\lambda$  as well. This condition leads to the following definition of abstract configurations:

**Definition 3.1.** An *abstract configuration*  $C(\lambda, k)$  of a complete ternary tree  $T_h$  of level  $h$  is a layout of aspect ratio  $\lambda > 1$  composed of non-overlapping boxes representing subtrees of level  $h - k$  with aspect ratio  $\lambda$ . We normalize the drawing by scaling the subtrees so that the width of the  $h - k$  subtrees is  $\lambda$ , and the height is 1.

With a way to calculate the growth of these abstract configurations, we now define notation regarding the minimal area abstract configurations. Let  $A_k(\lambda)$  be the minimal area abstract configuration for a given aspect ratio  $\lambda$  and subtrees  $h - k$ . Then, let  $\lambda(k)$  be the aspect ratio that minimizes  $A_k(\lambda)/\lambda$  (normalized so the area of the subtrees is 1). Thus,  $A_k(\lambda(k))/\lambda(k) = A_k$  is the minimal area abstract configuration for a fixed  $k$ .

Moreover, if  $A_k$  is the minimal area abstract configuration for a fixed  $k$ , then we construct inductively a family of trees  $T_{hk+\ell}$  where  $\ell$  is in the range  $[0, k - 1]$ , and  $T_{(h+1)k+\ell}$  is constructed from  $T_{hk+\ell}$  using the layout from the abstract configuration  $A_k$  and  $k$  initial trees. Next, we define  $R_k = (A_k)^{\frac{1}{k}}$ , so that the growth of the area of the trees is  $R_k^h$ . Then, we want to estimate  $\inf_k R_k$ . Ideally, it should be as close to 3 (linear growth) as possible.

**3.1. Finding  $A_k$ : Abstract Configurations for Fixed  $k$ .** We now investigate finding  $A_k$  for small values of  $k$ . These provide lower bounds on  $A_k(\lambda)$ . That is, asking for the minimal area configuration  $A_k(\lambda(k))/\lambda(k)$

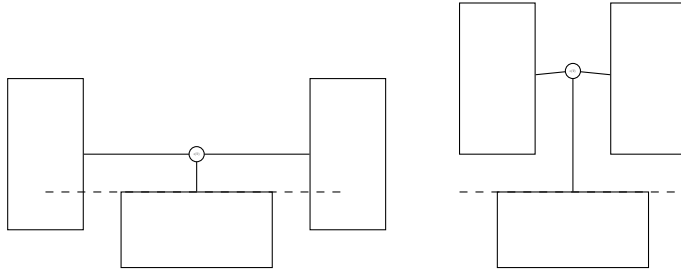


FIGURE 2. Configuration 1 (left) and Configuration 2 (right) with  $\ell$  represented by the dashed line

is equivalent to asking: how optimally can a tree be drawing only using level  $h - k$  size subtrees?

#### 4. $h - 1$ LAYOUTS

The current best-known upper bound  $O(n^{1.262})$  for the minimal area, due to Frati, was based on constructing a level  $h$  tree from level  $h - 1$  subtrees, where the aspect ratio was 2. In our language, Frati found an upper bound by finding  $A_1(2)/2$ .

Let  $k = 1$ . We have the following lower bound on how well an inductive strategy can work with subtrees of size  $h - 1$  by finding  $A_1$ :

**Theorem 4.1.** *An abstract configuration with  $k = 1$  has at least  $O(n^{\log_3 4}) \approx O(n^{1.262})$  area.*

*Proof.* An abstract configuration of level 1 is a configuration  $\Gamma_h$  of a complete ternary tree  $T_h$  of three boxes, each with width  $W_{h-1}$  and height  $H_{h-1}$ . Let the aspect ratio of  $\Gamma_h$  and the three subtrees be  $\lambda$ , so  $W_{h-1} = \lambda H_{h-1}$ .

Assume, without loss of generality, that the width  $W_{h-1}$  is  $\lambda$ , and the height  $H_{h-1}$  is 1. Let  $r(T_h)$  be the root of  $T_h$ . Two of the three subtrees lie on the same straight line passing through  $r(T_h)$ . We orient the tree so that this line is horizontal. Then, the third box is centered on the vertical.

Let  $\ell$  be the horizontal line extending from the top of the third box. There are two configurations:  $\ell$  intersects the two other subtrees, or  $\ell$  lies below them (see Figure 2.)

In the case where  $\ell$  intersects them, the width of  $\Gamma_h$  is  $2 + \lambda$  and the height is at least  $\frac{1}{2}\lambda + 1$ . The area is  $2 + 2\lambda + \frac{1}{2}\lambda^2$ . Enforcing the condition that the aspect ratio of the full box must be  $\lambda$ , we have that  $\lambda = 2$ , and the area is 8.

In the case where  $\ell$  lies below the two boxes, the width  $W_h$  of  $\Gamma_h$  is  $\max(2, \lambda)$  and the height  $H_h$  is  $1 + \lambda$ . If  $2 \leq \lambda$ , there are no values of  $\lambda$

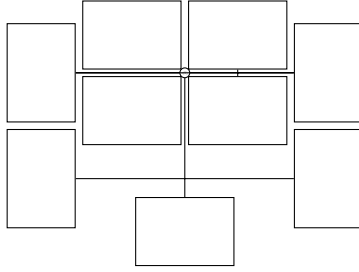


FIGURE 3. Construction from  $h - 2$  subpieces with growth  $O(n^{1.118})$

that produce a tree of the same aspect ratio, and if  $2 > \lambda$ ,  $\lambda = 1$  and the area is 4.

Both of these configurations produce minimal area configurations. Take  $\lambda(k) = 1$ .

Then, the minimal area is  $A_1(\lambda(k))/\lambda(k) = 4$ . Thus, the least area the abstract configuration of level 1 can produce is  $O(n^{\log_3 4}) = O(n^{1.262})$  with aspect ratio 1.  $\square$

## 5. UPPER BOUNDS FROM THE $h - 2$ LAYOUTS

Next, we extend Frati's method and present an algorithm that constructs a tree 2 steps in higher in tree depth and has an area bound  $O(n^{1.118})$ . Let  $\Gamma_h$  be a straight-line orthogonal drawing of a complete ternary tree  $T_h$  with height  $h$ . We inductively assume to have box drawings  $T_{h-2}$  and arrange the boxes without overlapping in a minimal area configuration. Then, we have the following:

**Theorem 5.1.** *An  $n$ -node complete ternary tree  $T_h$  with height  $h$  admits a straight-line orthogonal drawing with area  $O(n^{\log(2+\sqrt{2})^2/\log(3)^2}) \approx O(n^{1.118})$ .*

*Proof.* Construct a drawing  $\Gamma_h$  of the complete ternary tree  $T_h$  by inductively using the construction shows in Figure 3. Let  $W_h, H_h$  be the width and height of the bounding box of  $\Gamma_h$  respectively. By the construction, we have  $W_h = 2W_{h-2} + 2H_{h-2}$  and  $H_h = \max(W_{h-2}, H_{h-2}) + 2H_{h-2}$ .

Assume by inductive hypothesis that  $W_{h-2} > H_{h-2}$ . At the base case this is true:  $W_1 = 2 > H_1 = 1$ . Then, by the hypothesis  $W_h = 2W_{h-2} + 2H_{h-2}$  and  $H_h = W_{h-2} + 2H_{h-2}$ , so  $W_h > H_h$ , and  $H_h = W_{h-2} + 2H_{h-2}$ .

Thus we have the recurrence relation:

$$\begin{pmatrix} W_h \\ H_h \end{pmatrix} = \begin{pmatrix} 22 \\ 12 \end{pmatrix} \begin{pmatrix} W_{h-2} \\ H_{h-2} \end{pmatrix}$$

The dominant eigenvalue in the system is  $\mu = 2 + \sqrt{2}$ . Thus, the area of  $\Gamma_h$  grows as  $O(n^{\log(2+\sqrt{2})^2/\log(3)^2}) \approx O(n^{1.118})$ . We find that the aspect ratio is  $\lambda = \sqrt{2}$ . □

Next, we approach the abstract configuration problem described in Section 3.1 for level  $k = 2$ .

**Lemma 5.2.** *There exists a minimal area abstract configuration of a complete ternary tree  $\Gamma_h$  of level  $h$  that is symmetrical over the vertical line passing through the root node.*

*Proof.* Let  $v$  be the vertical line passing through the root node. Assume that the configuration is not symmetrical over  $v$ . Then, find the area of the right and left half-tree, where the right (or left) half-tree is defined to be the tree containing all of the vertices and edges lying on or to the right (or left) of  $v$ . Choose the minimal area half-tree, say,  $\gamma_h$ . If there is a tie, then choose randomly. We reflect  $\gamma_h$  over  $v$  to obtain a  $v$ -symmetrical complete ternary tree of level  $h$ ,  $\Gamma'_h$ .

If  $A(\Gamma'_h) < A(\Gamma_h)$ , then  $\Gamma_h$  was not area minimal, and we have a contradiction.

Else, if  $A(\Gamma'_h) = A(\Gamma_h)$ , then we have a minimal area symmetrical configuration. □

**Theorem 5.3.** *An abstract configuration of level 2 has at least  $O(n^{\log_9(6+4\sqrt{2})}) \approx O(n^{1.118})$  area.*

*Proof.* Here, we consider abstract configurations of 9 boxes with aspect ratio  $\lambda$ . We aim to find  $A_2$ . Assume, without loss of generality, that the height is 1 and the width is  $\lambda$ . By Lemma 5.2, we only need to consider configurations symmetrical over the root node.

First, we define a 2 main cases (shown in Figure 4) and minimize the area for each subcase. Let  $\ell$  be the leftmost vertical line from the subtree labelled  $R$ , and  $A$  the rightmost leaf.

**Case 1**  $\ell$  intersects  $A$ .

**Case 2**  $\ell$  does not intersect  $A$ .

Next, for each of these cases, we consider the placement of  $B$ , and then compute the value for  $\lambda$  that will ensure that the aspect ratio of the subtrees is equal to the aspect ratio of the configuration. Then, we compute  $A_2(\lambda)$  for each layout and  $\lambda$ .

We enumerate the eight possible configurations in Case 2. The configurations for Case 1 are similar.

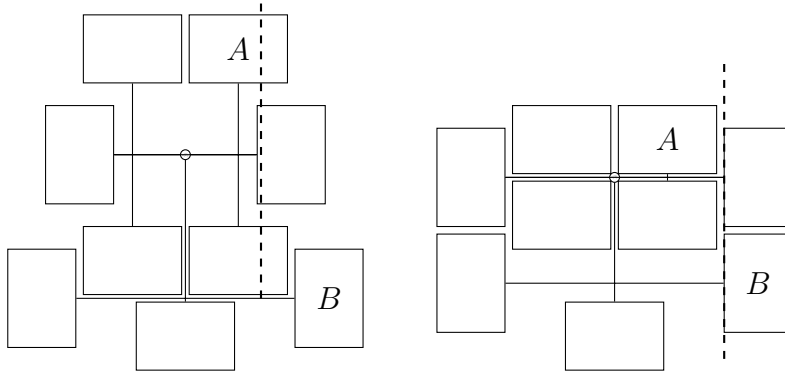


FIGURE 4. 2 cases, with  $\ell$  shown as a dashed line

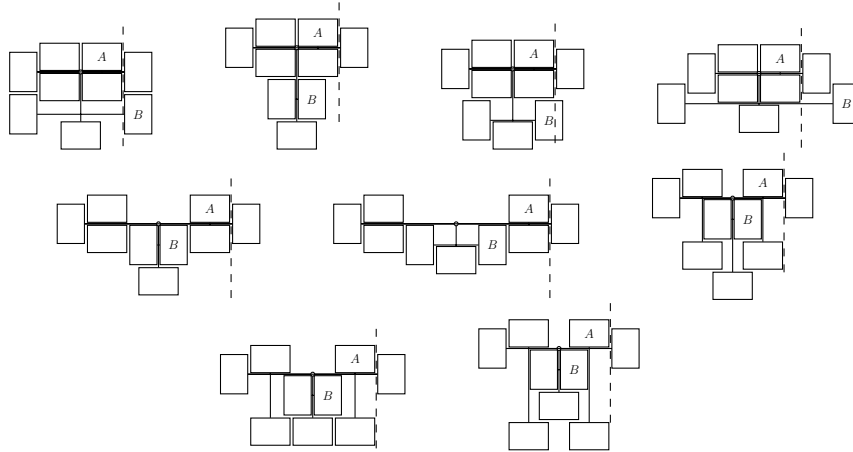


FIGURE 5. Subcases for Case 2

We demonstrate the calculation for the optimal area case, the second configuration in Figure 2. In this case, the height  $H_h = 2 + 2\lambda$  and the width  $W_h = 1 + 2\lambda$ . We minimize and enforce the aspect ratio equivalence between the subtrees and  $\Gamma_h$ . This leads to the following condition:

$$\frac{1}{\lambda} = \frac{2 + 2\lambda}{1 + 2\lambda}$$

Then,  $\lambda = \frac{1}{\sqrt{2}}$ , and the area of the configuration is  $(6 + 4\sqrt{2})\lambda$ , and so the configuration grows a factor of  $6 + 4\sqrt{2}$  at each inductive step. Thus, the area grows as  $O(n^{\log_{33} 6+4\sqrt{2}}) \approx O(n^{1.118})$ .  $\square$



## 6. LIMITATIONS AND FURTHER STRATEGIES

The strategies presented for producing better upper bounds on the area of a complete ternary tree by arranging boxes representing subtrees of size  $h - k$  has a large disadvantage. For any given  $k$ , recursing on the optimal area configuration produces diminishing returns as  $h$  approaches infinity. The subtrees represented by the boxes are each contained in their own bounding boxes, and so each strategy does not benefit from interleaving subtrees contained in the drawings of  $T_{h-k}$  at each recursive step.

Moreover, for  $k > 3$ , it is already difficult to construct increasingly optimal drawings.

**6.1. Relating  $A_k(\lambda)$  and  $A_k(1)$ .** In order to simplify the problem, we can look at the class of abstract configurations for a given aspect ratio  $\lambda$ . For this to be useful, there would need to be some relation between minimal area configurations for different ratios.

We now explore the relation between an abstract configuration of aspect ratio  $\lambda$  and an abstract configuration of aspect ratio 1.

**Theorem 6.1.** *Let  $A_k(\lambda)$  be the minimal area abstract configuration using level  $3^k$  boxes, and  $A_k(1)$  be the minimal area abstract configuration for square pieces. Then we have:*

$$\lambda A_k(\lambda) \geq A_k(1)$$

*Proof.* Take the optimal layout  $A_k(\lambda)$  with pieces of aspect ratio  $\lambda$ . We can stretch the layout vertically by a factor of  $\lambda$ , so that the aspect ratio of the new drawing is 1. Then, inside each box, we can draw a  $1 \times 1$  box. This provides an abstract configuration for  $\lambda = 1$  with area  $C_k(1)$ . Because  $A_k(1)$  is minimal,  $A_k(1) \leq C_k(1)$ . Then,

$$\lambda A_k(\lambda) = C_k(1) \geq A_k(1)$$

□

Next, we bound  $A_h(1)$  from below:

**Theorem 6.2.** *Let  $A_k(\lambda)$  be the minimal area abstract configuration using  $3^k$  boxes, and  $A_k(1)$  be the minimal area abstract configuration for square pieces. Then we have:*

$$A_k(1) \geq \frac{A_k(\lambda)}{\lambda^3}$$

*Proof.* We follow a procedure similar to the proof of Theorem 6.1. Given the minimal area configuration for aspect ratio 1,  $A_k(1)$ , we stretch the lattice by  $\lambda^2$  horizontally and  $\lambda$  vertically. Thus, we have

a layout with aspect ratio  $\lambda$ . Inside each box, we can now draw a box with aspect ratio  $\lambda$ , obtaining a configuration with area  $C_k(\lambda)$ . Then, we have the following bound on the size of the layout.

$$\lambda^3 A_k(1) = C_k(\lambda) \geq A_k(\lambda)$$

□

More generally, we can use the method from Theorem 6.1 and 6.2 to compare trees with different aspect ratio  $\lambda$  and  $\mu$ . We obtain the following:

**Corollary 6.3.** *Let  $A_k(\lambda)$  and  $A_k(\mu)$  be the minimal area abstract configurations for aspect ratio  $\lambda$  and  $\mu$  (with  $\lambda > \mu$ ). Then,*

$$\frac{\lambda}{\mu} A_k(\lambda) \geq A_k(\mu)$$

Combining Theorem 6.1 and 6.2 allows us to compare the growth of optimal trees in the following Corollary:

**Corollary 6.4.** *If the sequence  $\lambda(k)^{1/k} \rightarrow 1$ , then  $\frac{R_k}{(A_k(1))^{1/k}} \rightarrow 1$ .*

*Proof.* Recall that  $\lambda(k)$  is the  $\lambda$  that provides the minimal area configuration for a given  $k$ , and  $R_k = (A_k(\lambda(k))/\lambda(k))^{1/k}$  is the growth factor with respect to the height for a configuration.

We take the bound from Theorem 6.1. Then, dividing by  $\lambda^2$ , we have and raising to the power  $1/k$ :

$$\left(\frac{A_k(\lambda)}{\lambda}\right)^{1/k} \geq \left(\frac{A_k(1)}{\lambda^2}\right)^{1/k}$$

Then, taking  $\lambda = \lambda(k)$ , we have:

$$R_k \geq \left(\frac{A_k(1)}{\lambda(k)^2}\right)^{1/k}$$

$$(1) \quad \frac{R_k}{(A_k(1))^{1/k}} \geq \frac{1}{\lambda(k)^{2/k}}$$

We now show that  $\frac{R_k}{(A_k(1))^{1/k}} \leq 1$ . Taking the bound from Theorem 6.2, we obtain the following bound:

$$(2) \quad \lambda(k)^{2/k} \geq \frac{R_k}{A_k(1)^{1/k}}$$

If  $\lambda(k)^{1/k} \rightarrow 1$ , then combining (1) and (2) gives:

$$\frac{R_k}{A_k(1)^{1/k}} \rightarrow 1$$

□

One observation on the behavior of the growth of  $A_k(\lambda)$  is the following:

**Theorem 6.5.** *Let  $A_{a+b}(\lambda)$  be an abstract configuration with  $3^{a+b}$  boxes and ratio  $\lambda$ . Then,*

$$A_{a+b}(\lambda) \leq A_a(\lambda)A_b(\lambda)$$

*Proof.* Let  $A_a(\lambda)$  be a layout with  $3^a$  boxes, and  $A_b(\lambda)$  be a layout with  $3^b$  boxes. Then, in each box of the  $A_a(\lambda)$  layout, we place a layout identical to that of  $A_b(\lambda)$ . This provides a construction with  $3^a \cdot 3^b$  boxes all of ratio  $\lambda$ , say  $C_{a+b}(\lambda)$ . Clearly,  $A_{a+b}(\lambda) \leq C_{a+b}(\lambda) = A_a(\lambda)A_b(\lambda)$ . □

Thus, if we assume that  $\lambda(k)^{1/k} \rightarrow 1$ , then we can restrict our abstract configuration problem to looking at minimal square configurations, because it has the same order of growth as  $R_k$ . With this approach, the next questions to continue the study would be: how do  $\lambda(k)$  and  $A_k(1)$  behave? We hope that answering these questions will help tighten the bounds for minimal area embeddings for complete ternary trees. We conjecture a tight  $O(n^{1+\epsilon})$  with  $\epsilon$  fixed bound on the minimal area for an  $n$ -node complete ternary trees.

## 7. ACKNOWLEDGEMENTS

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