Generalizations of the Szemerédi-Trotter Theorem

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Abstract

In this paper, we generalize the Szemerédi-Trotter theorem, a fundamental result of incidence geometry in the plane, to flags in higher dimensions. In particular, we employ a stronger version of the polynomial cell decomposition technique, which has recently shown to be a powerful tool, to generalize the Szemerédi-Trotter Theorem to an upper bound for the number of incidences of complete flags in $\mathbb{R}^n$ (i.e. amongst sets of points, lines, planes, etc.). We also consider variants of this problem in three dimensions, such as the incidences of points and light-like lines, as well as the incidences of points, lines, and planes, where the number of points and planes on each line is restricted. Finally, we explore a group theoretic interpretation of flags, which leads us to new incidence problems.

1 Introduction

In [SzT], Endre Szemerédi and William Trotter proved a tight upper bound on the number of incidences between a set of points and lines in the plane, i.e. the set of point-line pairs so that the point lies on the line. In particular, they proved the following theorem.

Theorem 1 (Szemerédi-Trotter). *Let P be a set of points in $\mathbb{R}^2$ and L a set of lines in $\mathbb{R}^2$. Then the number of incidences between P and L satisfies*

$$I(P,L) \lesssim |P|^{\frac{2}{3}}|L|^{\frac{2}{3}} + |P| + |L|.$$

Note that here we write $f(x) \lesssim g(x)$ to mean $f(x) \in O(g(x))$.

In this paper, we principally employ the polynomial cell decomposition technique to generalize this theorem of Szemerédi and Trotter. The use of polynomials in incidence geometry has been shown to be particularly effective in recent years. In [D], Dvir proved the finite field Kakeya conjecture using the polynomial method. In [GK1], Guth and Katz solved the joints problem in three dimensions using the polynomial method. Kaplan, Sharir, and Shustin in [KSS] and Quilodrán in [Q] independently extended this to n dimensions, also using the polynomial method. The main idea in many of these proofs is to choose a polynomial of controlled degree which vanishes on several points or several lines. Then facts from algebraic geometry can be used to prove statements in incidence geometry.

Polynomial cell decomposition is a more specific method of using polynomials. Given some points of interest, the idea of the technique is to choose a polynomial of small degree whose zero set will divide the plane into connected components, each of which will have about the same number of points. We may then analyze the incidence geometry of the smaller sets of points in each cell (often times by induction) and use algebraic geometry to determine any properties about the points which may have ended up in the zero set. A precursor to this method was introduced by Clarkson, Edelsbrunner, Guibas, Sharir, and Welzl in [CEGSW], but the most notable use of it is probably by Guth and Katz in [GK2] to solve the Erdős distinct distances problem. Then they used a theorem which allowed them only to divide points, but we will employ a stronger theorem due to Guth and Zahl in [GZ] which will allow us to divide all i-planes.

In Section 2, we will give some precise definitions and preliminaries from algebraic geometry and incidence geometry which will be needed to prove our theorems. In Section 3, we will generalize the result of Szemerédi and Trotter to higher dimensions. More specifically, given sets S_i of i-planes in $\mathbb{R}^n$ (i.e. S_0 is a set of points, S_1 is a set of lines, etc.), we consider the number of incidences $I(S_0, \ldots, S_{n-1})$, i.e. how many ways can we pick a point, which is contained in a line, which is contained in a plane, and so on, to form a complete flag. Ultimately, we will give the following upper bound for the number of such incidences in $\mathbb{R}^n$.

Theorem 2. *Let S_i be a set of i-planes in $\mathbb{R}^n$ for $0 \leq i \leq n-1$. Then*

$$I(S_0, \ldots, S_{n-1}) \lesssim \sum_{(a_0,\ldots,a_{n-1}) \in \left\{0, \frac{2}{3}, 1\right\}^n} \prod_{i=0}^{n-1} |S_i|^{a_i},$$

where the ordered n-tuples in the sum are such that

- *no three consecutive a_i's are nonzero,*
- *every 1 is succeeded and followed by 0's (if possible),*
- *every $\frac{2}{3}$ is either succeeded or followed by another $\frac{2}{3}$,*
- *every 0 is either succeeded or followed by a nonzero a_i.*

The first two conditions on the n-tuples are necessary but the second two are only to rule out terms which would be dominated anyways. The n-tuples will have the form of pairs of consecutive $\frac{2}{3}$'s or solitary 1's separated by one or two 0's. For instance, the statement for the case of $n = 4$ looks like

$$\begin{aligned} I(S_0, S_1, S_2, S_3) \lesssim |S_0|^{\frac{2}{3}}|S_1|^{\frac{2}{3}}|S_3| + |S_0||S_2|^{\frac{2}{3}}|S_3|^{\frac{2}{3}} + |S_0||S_2| \\ + |S_0||S_3| + |S_1||S_3| + |S_1|^{\frac{2}{3}}|S_2|^{\frac{2}{3}}. \end{aligned}$$

We will also give examples to show that the bound in Theorem 2 is tight.

In Section 4, we will consider two variants of the incidence problem in $\mathbb{R}^3$. We will actually see that the optimal examples for flags in $\mathbb{R}^n$ will look very similar to the optimal examples in $\mathbb{R}^2$. Therefore, we will consider incidence problems with conditions which will rule out these examples. In particular, we first consider the incidences of points and light-like lines (lines which are parallel to some fixed double cone). We give a proof using polynomial cell decomposition of an upper bound which was gotten in [EKS], and also comment on how the theorem applies to other incidence problems. Next we also consider the incidences of N points, N lines, and N planes in $\mathbb{R}^3$ but where there are only $N^{\frac{1}{2}}$ points on each line and $N^{\frac{1}{2}}$ planes through each line. Here we will get an upper bound of $O(N^{\frac{3}{2}} \log N)$ incidences, and we will give two examples that have $\Theta(N^{\frac{3}{2}})$ incidences to show the bound is almost tight.

In Section 5, we will generalize our notion of flags to other groups, and consider similar incidence problems in those groups. In particular, we address the cases of $O(2,2)$ and $Sp(4)$. The incidence problem in $O(2,2)$ will simply reduce to incidences on a hyperboloid. The incidence problem in $Sp(4)$ on the other hand will inspire a new incidence problem about points and Legendrian lines in $\mathbb{R}^3$, i.e. lines which are orthogonal to some vector field at every point. In this problem, we will give an upper bound of $O(|P|^{\frac{3}{4}}|L|^{\frac{1}{2}}+|P|^{\frac{1}{2}}|L|^{\frac{3}{4}}+|P|+|L|)$ and we will explore a grid-type example which will show that it is very difficult to find a non-trivial arrangement of points and Legendrian lines.

2 Definitions and Preliminaries

We begin by defining incidences in full generality.

Definition. Let $S_1, \dots, S_n$ be sets, and let the expression $i(s_1, \dots, s_n)$ be equal to 1 if $s_i \subset s_{i+1}$ for all $1 \le i \le n-1$ and equal to 0 otherwise. Then the number of incidences amongst the sets $S_1, \dots, S_n$ is

$$I(S_1, \dots, S_n) := \sum_{(s_1,\dots,s_n)\in S_1\times\cdots\times S_n} i(s_1, \dots, s_n).$$

Throughout this paper, we will be bounding the number of incidences of certain sets asymptotically. To clarify notation, we write $f \lesssim g$ to mean $f \in O(g)$. In other words, $f(x_1, \dots, x_k) \lesssim g(x_1, \dots, x_k)$ if there exists some positive C so that

$$f(x_1, \dots, x_k) \le Cg(x_1, \dots, x_k)$$

for all $x_1, \dots, x_k$ in the domain of f and g. Similarly, we write $f \approx g$ to mean $f \in \Theta(g)$, or in other words that there exist positive C_1, C_2 so that

$$C_1 g(x_1, \dots, x_k) \le f(x_1, \dots, x_k) \le C_2 g(x_1, \dots, x_k)$$

for all $x_1, \dots, x_k$ in the domain of f and g. If statements are made in $\mathbb{R}^n$, these constants may depend on n. In other words, the statements are made separately about each dimension.

In general, we wish for the bounds we find to be tight. Precisely, this means that if we have sets $\mathfrak{S}_1, \dots, \mathfrak{S}_n$ and some bound

$$I(S_1, \dots, S_n) \lesssim f(|S_1|, \dots, |S_n|)$$

for all $S_1 \subset \mathfrak{S}_1, \dots, S_n \subset \mathfrak{S}_n$, then we say this bound is tight if

$$M(k_1, \dots, k_n) := \sup_{S_i \subset \mathfrak{S}_i, |S_i| = k_i} I(S_1, \dots, S_n) \in \Theta(f(k_1, \dots, k_n)).$$

Now we move on to define some terms from algebraic geometry, as well as introduce preliminaries which will be necessary to prove our results.

Definition. For a polynomial $Q \in \mathbb{R}[x_1, \dots, x_n]$, the zero set of Q is

$$Z(Q) = \{x \in \mathbb{R}^n \mid Q(x) = 0\}.$$

Definition. For a polynomial $Q \in \mathbb{R}[x_1, \dots, x_n]$, a point $x \in Z(Q)$ is a critical point if $\nabla Q(x) = 0$.

Definition. For a polynomial $Q \in \mathbb{R}[x_1, \dots, x_n]$, a line $l \subset Z(Q)$ is a critical line if every point on l is a critical point.

Definition. For a polynomial $Q \in \mathbb{R}[x_1, \dots, x_n]$, a non-critical point $x \in Z(Q)$ is flat if $\mathbf{u}^T H(x)\mathbf{v} = 0$ for all $\mathbf{u}, \mathbf{v}$ in the tangent space of $Z(Q)$ at x. (Here $H(x)$ denotes the Hessian matrix.)

Definition. For a polynomial $Q \in \mathbb{R}[x_1, \dots, x_n]$, a non-critical line $l \subset Z(Q)$ is a flat line if every point on l is critical or flat.

Now we introduce the theorem which is the basis of the polynomial cell decomposition technique.

Theorem 3 ([GZ]). *If S is a set of i-planes in $\mathbb{R}^n$ and d is any degree, then there is a nonzero polynomial Q of degree at most d so that $\mathbb{R}^n \setminus Z(Q)$ is the union of $O(d^n)$ connected components, each of which intersects $O\left(\frac{|S|}{d^{n-i}}\right)$ i-planes of S.*

Up until recently, Theorem 3 was only known for $i = 0$, i.e. for separating points. With this stronger theorem, our cell decomposition is more powerful. In particular, we have the following corollary.

Corollary 4. *If S_i is a set of i-planes in $\mathbb{R}^n$ for $0 \le i \le n-1$ and d is any degree, then there is a nonzero polynomial Q of degree $O(d)$ so that $\mathbb{R}^n \setminus Z(Q)$ is the union of $O(d^n)$ connected components, each of which intersects $O\left(\frac{|S_i|}{d^{n-i}}\right)$ i-planes of S_i for all i.*

Proof. By Theorem 3, for all i, we find a polynomial P_i of degree at most d which partitions $\mathbb{R}^n$ into $O(d^n)$ connected components, each of which intersects $O\left(\frac{|S_i|}{d^{n-i}}\right)$ i-planes of S_i. Let $Q = P_0 \cdots P_{n-1}$ so that Q has degree at most nd. Since Q has degree $O(d)$, there can be at most $O(d^n)$ connected components of $\mathbb{R}^n \setminus Z(Q)$ (Theorem A.1 in [SoT]). Moreover these components are subsets of the components of $\mathbb{R}^n \setminus Z(P_i)$, and so these components can intersect at most $O\left(\frac{|S_i|}{d^{n-i}}\right)$ i-planes of S_i. □

3 Incidences of Flags in $\mathbb{R}^n$

In this section we will generalize the Szemerédi-Trotter Theorem. Throughout this section, S_i will denote a finite set of i-planes (in some $\mathbb{R}^n$) and we will be considering $I(S_0, \ldots, S_{n-1})$, eventually arriving at a proof of Theorem 2. Our principal tool in proving this will be polynomial cell decomposition, in particular Corollary 4. This technique has proven to be very powerful. First we will show how it can be used to prove the Szemerédi-Trotter Theorem.

Proof of Theorem 1. First we establish the simpler bound

$$I(P, L) \lesssim \min(|P|^{\frac{1}{2}}|L| + |P|, |P||L|^{\frac{1}{2}} + |L|).$$

Let P' be the set of points which lie on two or more lines. Note that $I(P \setminus P', L) \le |P|$ since each point can only give one incidence. Now it remains to bound $I(P', L)$. For any $p \in P'$, let L_p be the set of lines in L which lie on p. Then by Cauchy-Schwarz, we have

$$I(P', L) = \sum_{p \in P'} |L_p| \le |P'|^{\frac{1}{2}} \left(\sum_{p \in P'} |L_p|^2 \right)^{\frac{1}{2}} \approx |P'|^{\frac{1}{2}} \left(\sum_{p \in P'} \binom{|L_p|}{2} \right)^{\frac{1}{2}}.$$

Now since any pair of lines lies on at most point, we have

$$|P'|^{\frac{1}{2}} \left(\sum_{p \in P'} \binom{|L_p|}{2} \right)^{\frac{1}{2}} \le |P'|^{\frac{1}{2}} \binom{|L|}{2}^{\frac{1}{2}} \approx |P'|^{\frac{1}{2}} |L|.$$

Thus we have $I(P,L) \lesssim |P|^{\frac{1}{2}}|L| + |P|$. The other part of the bound follows similarly.

Now we fix some d to be chosen later and employ Corollary 4. We get a nonzero polynomial Q of degree $O(d)$ which divides the plane into $O(d^2)$ connected components each of which has $O\left(\frac{|P|}{d^2}\right)$ points and intersects $O\left(\frac{|L|}{d}\right)$ lines. First we will bound the incidences in each cell and then we will bound the incidences in the boundary $Z(Q)$. In other words, for cell C_i, let P_i be the points in C_i and L_i be the set of lines which intersect it. Moreover let P_0 be the points in $Z(Q)$ and L_0 be the lines contained entirely in $Z(Q)$. Then we have

$$I(P,L) = \sum_i I(P_i, L_i) + I(P_0, L \setminus L_0) + I(P_0, L_0).$$

Now using our simpler bound, we have

$$\begin{aligned}\sum_i I(P_i, L_i) &\lesssim \sum_i \left(|P_i||L_i|^{\frac{1}{2}} + |L_i|\right) \lesssim \sum_i \left(\frac{|P||L|^{\frac{1}{2}}}{d^{\frac{5}{2}}} + \frac{|L|}{d}\right) \\ &\lesssim \frac{|P||L|^{\frac{1}{2}}}{d^{\frac{1}{2}}} + d|L|.\end{aligned}$$

Now note that any line which is not contained in $Z(Q)$ can only intersect it $\deg Q$ times, for otherwise Q would vanish at more than $\deg Q$ points on the line, and so would vanish everywhere on it. Therefore we have

$$I(P_0, L \setminus L_0) \lesssim d|L|.$$

Finally $Z(Q)$ can only contain $\deg Q$ lines, for otherwise any line intersects $Z(Q)$ more than $\deg Q$ times and so Q must vanish on every line, meaning $Q = 0$. Then by our simpler bound, we get

$$I(P_0, L_0) \lesssim |P_0|^{\frac{1}{2}}|L_0| + |P_0| \lesssim d|P|^{\frac{1}{2}} + |P|.$$

Combining all of these we now have

$$I(P,L) \lesssim \frac{|P||L|^{\frac{1}{2}}}{d^{\frac{1}{2}}} + d|L| + d|P|^{\frac{1}{2}} + |P|.$$

Now if $|P| \geq |L|^2$, our simpler bound gives $I(P,L) \lesssim |P|$. Otherwise the $d|P|^{\frac{1}{2}}$ term drops out of our bound. Again if $|L| \geq |P|^2$, our simpler bound gives $I(P,L) \lesssim |L|$. Otherwise we may choose $d \approx \frac{|P|^{\frac{2}{3}}}{|L|^{\frac{1}{3}}}$ and we get

$$I(P,L) \lesssim |P|^{\frac{2}{3}}|L|^{\frac{2}{3}} + |P|.$$

$\square$

Now we give examples to show that the bound in the Szemerédi-Trotter Theorem is tight. When arranging P and L to maximize incidences, note that $|P|$ incidences is easily attained by placing all points on one of the lines, and $|L|$ incidences is easily attained by having all lines intersect at one point. It is then only necessary to give an example which gives $\Theta(|P|^{\frac{2}{3}}|L|^{\frac{2}{3}})$ incidences. We do this by fixing $m, k > 0$ and taking P to be the grid of points $\{1, \ldots, k\} \times \{1, \ldots, 4mk\}$. We then choose L to be the lines of the form $y = ax + b$ where $a \in \{1, \ldots, 2m\}$ and $b \in \{1, \ldots, 2mk\}$. Note that every line intersects k points, so we have $|P| = 4mk^2$ and $|L| = 4m^2k$ with

$$I(P,L) = 4m^2k^2 = \frac{1}{\sqrt[3]{4}}|P|^{\frac{2}{3}}|L|^{\frac{2}{3}}.$$

So now we have shown that for all $p, l > 0$, that $M(p,l) \geq p$ and $M(p,l) \geq l$, as well as $M(p,l) \geq \frac{1}{\sqrt[3]{4}}p^{\frac{2}{3}}l^{\frac{2}{3}}$ for all p, l which can be written in the form $p = 4mk^2$ and $l = 4m^2k$ for some integer m, k. In order to formally show optimality, we must show this for all p, l and combine these three equations. Note that for any $p, l \geq 4$, we make take $m = \left\lfloor \frac{l^{\frac{2}{3}}}{\sqrt[3]{4}p^{\frac{1}{3}}} \right\rfloor$ and $k = \left\lfloor \frac{p^{\frac{2}{3}}}{\sqrt[3]{4}l^{\frac{1}{3}}} \right\rfloor$ and our above example would give

$$I(P,L) = 4\left\lfloor \frac{l^{\frac{2}{3}}}{\sqrt[3]{4}p^{\frac{1}{3}}} \right\rfloor^2 \left\lfloor \frac{p^{\frac{2}{3}}}{\sqrt[3]{4}l^{\frac{1}{3}}} \right\rfloor^2 \geq \left\lfloor \frac{1}{\sqrt[3]{4}}p^{\frac{2}{3}}l^{\frac{2}{3}} \right\rfloor \geq \frac{1}{\sqrt[3]{4}}p^{\frac{2}{3}}l^{\frac{2}{3}} - 1.$$

We can then say that $M(p,l) \geq Cp^{\frac{2}{3}}l^{\frac{2}{3}}$ for some $C < 1$ (which we choose large enough to deal with the cases of $p \leq 3$ or $l \leq 3$ also). Then combining our three equations, we have $M(p,l) \geq \frac{C}{3}(p^{\frac{2}{3}}l^{\frac{2}{3}} + p + l)$ so our bound is indeed optimal.

Now we move on to higher dimensional cases. First we will present a lemma about how bounds in one dimension can be raised to higher dimensions.

Lemma 5. *Suppose that*

$$I(S_{i_1}, \ldots, S_{i_k}) \lesssim f(|S_{i_1}|, \ldots, |S_{i_k}|),$$

for all $S_{i_1}, \ldots, S_{i_k}$ in $\mathbb{R}^n$. Then for all $m \geq j \geq 0$,

$$I(S_{i_1+j}, \ldots, S_{i_k+j}) \lesssim f(|S_{i_1+j}|, \ldots, |S_{i_k+j}|),$$

for all $S_{i_1+j}, \ldots, S_{i_k+j}$ in $\mathbb{R}^{n+m}$.

Proof. Suppose that the bound in the hypothesis is true and that we are given sets $S_{i_1+j}, \ldots, S_{i_k+j}$ in $\mathbb{R}^{n+m}$. First we choose a random $(n+m-j)$-plane P in $\mathbb{R}^{n+m}$, in particular one which is not parallel to or contains any of the elements of $S_{i_1+j}, \ldots, S_{i_k+j}$ (which is possible since all these sets are finite). Then note that the intersection of any i-plane in $\mathbb{R}^{n+m}$ with P will be an $(i-j)$-plane (supposing that $i > j$). Moreover, if a p-plane is contained in a q-plane, then the intersection of the p-plane with P will be contained in the intersection of the q-plane with P. Therefore if $S'_i = \{s \cap P | s \in S_{i+j}\}$, then we have

$$I(S'_{i_1}, \ldots, S'_{i_k}) = I(S_{i_1+j}, \ldots, S_{i_k+j}).$$

Identifying P with $\mathbb{R}^{n+m-j}$, we see that we have reduced the problem to an incidence of i-planes in $\mathbb{R}^{n+m-j}$.

Now choose a random n-plane Q in P, in particular one so that the projection of every element of $S'_{i_1}, \ldots, S'_{i_k}$ onto Q has the same dimension as the original element. Again, if a p-plane was contained in a q-plane, the projection of the p-plane will still be contained in the projection of the q-plane. Therefore if $S''_i = \{\text{proj}_Q(s) | s \in S'_i\}$, then

$$I(S''_{i_1}, \ldots, S''_{i_k}) = I(S'_{i_1}, \ldots, S'_{i_k}).$$

Now identifying Q with $\mathbb{R}^n$, we get $I(S''_{i_1}, \ldots, S''_{i_k}) \lesssim f(|S''_{i_1}|, \ldots, |S''_{i_k}|)$. Then since $|S''_i| = |S_{i+j}|$, we have

$$I(S_{i_1+j}, \ldots, S_{i_k+j}) \lesssim f(|S_{i_1+j}|, \ldots, |S_{i_k+j}|).$$

□

Now we prove our theorem for three dimensions, which simply follows from the Szemerédi-Trotter Theorem.

Theorem 6. *Let P be a set of points in $\mathbb{R}^3$, L a set of lines in $\mathbb{R}^3$, and S a set of planes in $\mathbb{R}^3$. Then*

$$I(P, L, S) \lesssim |P|^{\frac{2}{3}}|L|^{\frac{2}{3}} + |L|^{\frac{2}{3}}|S|^{\frac{2}{3}} + |P||S| + |L|.$$

Proof. Let us first consider the lines in L on which at most one point of P lies. Since each line has at most one point, the number of incidences contributed by these lines is at most the number of incidences between L and S, which is $O(|L|^{\frac{2}{3}}|S|^{\frac{2}{3}} + |L| + |S|)$ by Lemma 5 and Szemerédi-Trotter.

Now consider the lines in L which lie in at most one plane of S. Again, the number of incidences contributed by these lines is at most the number of incidences between P and L, which is $O(|P|^{\frac{2}{3}}|L|^{\frac{2}{3}} + |P| + |L|)$.

Finally, this leaves us with L_0, the set of lines on which lie at least two points of P and which lie on at least two planes of S. For any $l \in L_0$, let P_l be the set of points which lie on l and S_l be the set of planes which l lies on. By Cauchy-Schwarz, the number of incidences contributed by the lines of L_0 is then at most

$$\sum_{l \in L_0} |P_l||S_l| \leq \left(\sum_{l \in L_0} |P_l|^2\right)^{\frac{1}{2}} \left(\sum_{l \in L_0} |S_l|^2\right)^{\frac{1}{2}} \approx \left(\sum_{l \in L_0} \binom{|P_l|}{2}\right)^{\frac{1}{2}} \left(\sum_{l \in L_0} \binom{|S_l|}{2}\right)^{\frac{1}{2}}.$$

Now the first sum can be interpretted as the number of pairs of points on each line of L_0. But any pair of points can only lie on one line, and so this sum is necessarily less than the total number of pairs of points, $\binom{|P|}{2}$. Similarly, since any two planes can only intersect in one line, the second sum must be less than $\binom{|S|}{2}$. Therefore, the number of incidences that L_0 contributes is at most (up to a constant multiple)

$$\binom{|P|}{2}^{\frac{1}{2}} \binom{|S|}{2}^{\frac{1}{2}} \approx (|P|^2)^{\frac{1}{2}}(|S|^2)^{\frac{1}{2}} = |P||S|.$$

Combining these three results then gives

$$I(P, L, S) \lesssim |P|^{\frac{2}{3}}|L|^{\frac{2}{3}} + |L|^{\frac{2}{3}}|S|^{\frac{2}{3}} + |P||S| + |L|.$$

□

The bound in Theorem 6 is tight. For the $|P|^{\frac{2}{3}}|L|^{\frac{2}{3}}$ term, we can simply use the grid example for two-dimensions in the $z = 0$ plane (which we choose to be in S). Therefore $M(p,l,s) \geq Cp^{\frac{2}{3}}l^{\frac{2}{3}}$ (where C is the constant in the two-dimensional case). We can extend this example from points and lines to lines and planes by putting $(0,0,-1) \in P$ and taking L to be all the lines through $(0,0,-1)$ and one of the points in our grid. Similarly we take S to be all the planes through $(0,0,-1)$ and all the lines of our grid. Then we have $M(p,l,s) \geq Cl^{\frac{2}{3}}s^{\frac{2}{3}}$. We can also get the $|P||S|$ term by simply choosing a line in L and putting all our points on this line and all our planes through this line. This gives $M(p,l,s) \geq ps$. Finally, we can simply have all our lines be coplanar and concurrent to give $M(p,l,s) \geq l$. Therefore $M(p,l,s) \geq \frac{C}{4}(p^{\frac{2}{3}}l^{\frac{2}{3}} + l^{\frac{2}{3}}s^{\frac{2}{3}} + ps + l)$.

Now we move on to proving Theorem 2, the result for general $\mathbb{R}^n$.

Proof of Theorem 2. Our argument first follows by strong induction over n. The cases of $n = 1, 2, 3$ have been dealt with separately. First we will present the inductive argument for $n \geq 6$. Therefore throughout this argument we assume the statement to be true for $\mathbb{R}^1, \ldots, \mathbb{R}^{n-1}$. Afterwards, we will show how the argument can be altered for the cases of $n = 4, 5$. We must also argue by strong induction over $|S_0|$. As a result we will deal more explicitly with constants. In the base case of $|S_0| = 1$, we note that $I(S_0, \ldots, S_{n-1})$ is simply bounded by $I(S_1, \ldots, S_{n-1})$. By Lemma 5 and the inductive hypothesis over n, this reduces to the formula for the case of $n-1$, which is clearly less than the formula for n.

Now we deal with the inductive case, and so we assume our statement is true for all cases where there are less than $|S_0|$ points. We will do this by polynomial cell decomposition. In particular, we will choose a polynomial which evenly divides our points, lines, planes, etc. Within each of these cells, we will use induction to show that it simply reduces to smaller cases. We must pay careful attention to the constant in the statement of the theorem here because we should have the same constant in the statement at the end of our inductive step. In the boundary, we will divide the incidences into cases, in each of which, the dependence on one of our $|S_i|$'s will drop out. Therefore, the incidences in the boundary can simply be bounded by the sum of the incidence bounds for partial flags (which we can get using Lemma 5 and our inductive hypothesis over n). Since the sum in our statement is essentially just the sum of these incidence bounds, we will be done.

Now we fix some d to be chosen later and employ Corollary 4. We get a polynomial Q of degree at most $k_{deg}(n)d$ (where $k_{deg}(n)$ is a constant which may depend on n). Moreover, $\mathbb{R}^n \setminus Z(Q)$ has at most $k_{cell}(n)d^n$ components, each of which contains no more than $k_{plane}(n)\frac{|S_i|}{d^{n-i}}$ i-planes. Our inductive hypothesis (over $|S_0|$) then tells us that the number of incidences in each of these cells can be no more than

$$k_{inc}(n) \cdot \sum_{(a_0,\ldots,a_{n-1})} \prod_{i=0}^{n-1} \left(k_{plane}(n) \frac{|S_i|}{d^{n-i}} \right)^{a_i},$$

where the sum is taken over all n-tuples as specified in the statement. Note that it can easily be checked that as long as $n \geq 6$, given the restrictions on the a_i's, the power of d in the denominator of each term is greater than or equal to $n+1$. Therefore, summing over all $k_{cell}(n)d^n$ of our cells, we have that the total number of incidences outside $Z(Q)$ is bounded by

$$\frac{k_{inc}(n)K_1(n)}{d} \cdot \sum_{(a_0,\ldots,a_{n-1})} \prod_{i=0}^{n-1} |S_i|^{a_i},$$

where $K_1(n)$ is some constant (which is gotten by multiplying $k_{cell}(n)$ by some power of $k_{plane}(n)$).

Now we move on to bound the incidences in $Z(Q)$. We divide these into cases. Namely, we consider the incidences where the i-plane is contained in $Z(Q)$ but the $(i+1)$-plane is not. If we consider these cases for all $0 \leq i \leq n-1$, we will have covered all the cases. Note that if an $(i+1)$-plane is not contained in $Z(Q)$, it cannot contain more than $k_{deg}(n)d$ i-planes which are. This means that we can bound the number of incidences where the i-plane is contained in $Z(Q)$ but the $(i+1)$-plane is not by

$$k_{deg}(n)d \cdot I(S_0,\ldots,S_{i-1},S_{i+1},\ldots,S_{n-1}).$$

In other words, the total number of incidences in $Z(Q)$ is simply bounded by $k_{deg}(n)d$ times the sum of the incidences over all partial flags (where we only remove one of the S_i's). Note that we have

$$I(S_0,\ldots,S_{i-1},S_{i+1},\ldots,S_{n-1}) \leq I(S_0,\ldots,S_{i-1})I(S_{i+1},\ldots,S_{n-1}).$$

Then Lemma 5 and our inductive hypothesis imply that this is less than

$$\left(k_{inc}(i) \sum_{(a_0,\ldots,a_{i-1})} \prod_{j=0}^{i-1} |S_j|^{a_j} \right) \left(k_{inc}(n-i-1) \sum_{(a_{i+1},\ldots,a_{n-1})} \prod_{j=i+1}^{n-1} |S_j|^{a_j} \right).$$

Since the i-tuples and $(n-i-1)$-tuples in these sums follow the restrictions given in the statement, it can be seen that the product of any pair of terms from these sums will also abide by the first two restrictions. In particular there can be no three consecutive nonzero a_j because the only entry which was added was $a_i = 0$. Also every 1 will still be succeeded and followed by 0's (since they were before and $a_i = 0$). Any terms which do not abide by the second two restrictions will simply be suprised, i.e. any n-tuple with $a_j = \frac{2}{3}$ but breaks the third restriction or $a_j = 0$ but breaks the fourth restriction will be suppressed by the same n-tuple where $a_j = 1$. Therefore, summing over all i, we have that the total number of incidences in $Z(Q)$ is less than

$$k_{deg}(n)d \cdot \sum_{i=0}^{n-1} I(S_0, \ldots, S_{i-1}) I(S_{i+1}, \ldots, S_{n-1}) \leq K_2(n)d \cdot \sum_{(a_0,\ldots,a_{n-1})} \prod_{i=0}^{n-1} |S_i|^{a_i},$$

where $K_2(n)$ is some constant (which is gotten by multiplying $k_{deg}(n)$, the highest value of $k_{inc}(i)k_{inc}(n-i-1)$, and whatever factor was necessary to suppress the extraneous terms in our sum). Adding our bound for the incidences inside $Z(Q)$ and outside $Z(Q)$, we get

$$I(S_0, \ldots, S_{n-1}) \leq \left(\frac{k_{inc}(n)K_1(n)}{d} + K_2(n)d \right) \sum_{(a_0,\ldots,a_{n-1})} \prod_{i=0}^{n-1} |S_i|^{a_i},$$

where again the sum is taken over n-tuples as specified in the statement. Note that $K_1(n)$ and $K_2(n)$ are constants which we have no control over. However we may still choose d and $k_{inc}(n)$ large enough (but constant) so that

$$\frac{k_{inc}(n)K_1(n)}{d} + K_2(n)d \leq k_{inc}(n),$$

and then we have

$$I(S_0, \ldots, S_{n-1}) \leq k_{inc}(n) \cdot \sum_{(a_0,\ldots,a_{n-1})} \prod_{i=0}^{n-1} |S_i|^{a_i}$$

and our induction for $n \geq 6$ is complete.

Now we show how the above inductive argument can be altered for the case of $n = 4$. First note that the above argument fails for $n = 4$ because of the $|S_1|^{\frac{2}{3}}|S_2|^{\frac{2}{3}}$ and $|S_1||S_3|$ terms in the bound. When summing over the incidences in each cell by induction, these terms will result in terms like $d^4 \cdot \left(\frac{|S_1|}{d^3}\right)^{\frac{2}{3}} \left(\frac{|S_2|}{d^2}\right)^{\frac{2}{3}} = d^{\frac{2}{3}}|S_1|^{\frac{2}{3}}|S_2|^{\frac{2}{3}}$ and $d^4 \cdot \frac{|S_1|}{d^3} \cdot \frac{|S_3|}{d} = |S_1||S_3|$ which do not have a power of d in their denominators. Then we cannot pick a value of d which is sufficiently large so to make the coefficient of $k_{inc}(4)$ less than 1. We will fix this by simple casework. When our planes have few 3-planes through them, our problem will reduce to incidences of points, lines, and planes. Similarly when our lines have few points on them, our problem will reduce to incidences of lines, planes, and 3-planes. When neither of these are true, we will show that the two terms which were troubling become negligible in our bound.

More specifically, let S_1' be the set of lines with less than or equal to $d^{\frac{5}{2}}$ points of $|S_0|$ on them, and let $S_1'' = S_1 \setminus S_1'$. Similarly let S_2' be the set of planes which lie on less than or equal to $d^{\frac{5}{2}}$ 3-planes of $|S_3|$, and let $S_2'' = S_2 \setminus S_2'$. Note that

$$I(S_0, S_1, S_2, S_3) \le I(S_0, S_1', S_2, S_3) + I(S_0, S_1, S_2', S_3) + I(S_0, S_1'', S_2'', S_3).$$

Now since each line of S_1' has at most $d^{\frac{5}{2}}$ points, we have

$$I(S_0, S_1', S_2, S_3) \le d^{\frac{5}{2}} \cdot I(S_1', S_2, S_3) \le d^{\frac{5}{2}} \cdot I(S_1, S_2, S_3).$$

This incidence term is that of a partial flag, and so as discussed before, is clearly less than our desired bound. Similarly we can argue that

$$I(S_0, S_1, S_2', S_3) \le d^{\frac{5}{2}} \cdot \sum_{(a_0,\dots,a_{n-1})} \prod_{i=0}^{n-1} |S_i|^{a_i}$$

because each plane of S_2' lies on at most $d^{\frac{5}{2}}$ 3-planes.

Now we are left to bound the last term, and we will do this by the same inductive argument we had for $n \ge 6$, but by first showing that the two troubling terms are neglible. For any line $l \in S_1''$, let P_l be the set of points of S_0 which lie on l, and note that

$$|S_0|^2 \ge 2\binom{|S_0|}{2} \ge 2\sum_{l \in S_1''} \binom{|P_l|}{2} \ge 2\sum_{l \in S_1''} \binom{d^{\frac{5}{2}}}{2} = 2|S_1''|\binom{d^{\frac{5}{2}}}{2} > \frac{d^5}{2}|S_1''|$$

(provided that $d \ge 2$). Note that this second step follows from the fact that any pair of points can lie on at most one line. Similarly, we have

$|S_3|^2 > \frac{d^5}{2}|S_2''|$ since any pair of 3-planes intersect in at most one line. Now as in our argument for $n \geq 6$, we may find a polynomial Q whose zero set divides S_0, S_1, S_2, S_3 evenly. Then as before, the number of incidences (of S_0, S_1'', S_2'', and S_3) outside $Z(Q)$ will be less than or equal to

$$k_{inc}(4)K_1(4)\left(\frac{|S_0||S_2''|^{\frac{2}{3}}|S_3|^{\frac{2}{3}}}{d^2} + \frac{|S_0|^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_3|}{d^{\frac{5}{3}}} + \frac{|S_0||S_2''|}{d^2}\right.$$
$$\left. + \frac{|S_0||S_3|}{d} + |S_1''||S_3| + d^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_2''|^{\frac{2}{3}}\right).$$

Note however that if $|S_0| \geq |S_3|$, we have

$$\frac{|S_0|^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_3|}{d^{\frac{5}{3}}} \geq \frac{|S_1''|^{\frac{2}{3}}|S_3|^{\frac{5}{3}}}{d^{\frac{5}{3}}} > \frac{d^{\frac{5}{3}}}{2^{\frac{2}{3}}}|S_1''|^{\frac{2}{3}}|S_2''|^{\frac{2}{3}}|S_3|^{\frac{1}{3}} > d^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_2''|^{\frac{2}{3}}$$

(provided that $d \geq 2$). Otherwise if $|S_3| > |S_0|$, we have

$$\frac{|S_0||S_2''|^{\frac{2}{3}}|S_3|^{\frac{2}{3}}}{d^2} > \frac{|S_0|^{\frac{5}{3}}|S_2''|^{\frac{2}{3}}}{d^2} > \frac{d^{\frac{4}{3}}}{2^{\frac{2}{3}}}|S_0|^{\frac{1}{3}}|S_1''|^{\frac{2}{3}}|S_2''|^{\frac{2}{3}} \geq d^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_2''|^{\frac{2}{3}}$$

(provided that $d \geq 2$). So in all cases,

$$\frac{|S_0||S_2''|^{\frac{2}{3}}|S_3|^{\frac{2}{3}}}{d^2} + \frac{|S_0|^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_3|}{d^{\frac{5}{3}}} > d^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_2''|^{\frac{2}{3}}.$$

Finally, we also have that

$$\frac{|S_0|^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_3|}{d^{\frac{5}{3}}} > \frac{|S_1''||S_3|}{2}.$$

Therefore, the number of incidences (of S_0, S_1'', S_2'', and S_3) outside $Z(Q)$ is less than

$$4k_{inc}(4)K_1(4)\left(\frac{|S_0||S_2''|^{\frac{2}{3}}|S_3|^{\frac{2}{3}}}{d^2} + \frac{|S_0|^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_3|}{d^{\frac{5}{3}}} + \frac{|S_0||S_2''|}{d^2} + \frac{|S_0||S_3|}{d}\right).$$

Now we have the same bound for the incidences inside $Z(Q)$ as we did in the case of $n \geq 6$, so combining everything we get

$$I(S_0, S_1, S_2, S_3) \leq \left(2d^{\frac{5}{2}} + \frac{4k_{inc}(4)K_1(4)}{d} + K_2(4)d\right) \cdot \sum_{(a_0,\ldots,a_{n-1})} \prod_{i=0}^{n-1} |S_i|^{a_i}.$$

Again we can choose d and $k_{inc}(4)$ so large (but constant) that

$$2d^{\frac{5}{2}} + \frac{4k_{inc}(4)K_1(4)}{d} + K_2(4)d \leq k_{inc}(4),$$

and our induction for $n = 4$ will be done.

A similar argument can be made for $n = 5$, letting S_1' be the lines of S_1 which have less than or equal to d^3 points of S_0 on them, and $S_1'' = S_1 \setminus S_1'$. (The only troubling term is $|S_1||S_4|$, but we can show that $\frac{|S_0|^{\frac{2}{3}}|S_1''|^{\frac{2}{3}}|S_4|}{d^2} > \frac{|S_1''||S_4|}{2}$.) □

Now we give examples to show that the bound in Theorem 2 is tight. First we must note that the grid example in two dimensions can be lifted to higher dimensions. In other words, we can arrange any S_i and S_{i+1} in a grid-like example in any $\mathbb{R}^n$ (for $0 \leq i \leq n-2$) to achieve $\Theta(|S_i|^{\frac{2}{3}}|S_{i+1}|^{\frac{2}{3}})$ incidences. This can be done by fixing a two-dimensional grid on some 2-plane Q in $\mathbb{R}^n$ and choosing an $(i-1)$-plane that it does not intersect. Then we let S_i be the set of i-planes which contain Q and a point in our grid, and we let S_{i+1} be the set of $(i+1)$-planes which contain Q and a line in our grid. Then every point-line incidence in our two-dimensional grid gives an incidence between an i-plane and $(i+1)$-plane in our new grid, so we have $\Theta(|S_i|^{\frac{2}{3}}|S_{i+1}|^{\frac{2}{3}})$ incidences. Moreover, note that all the elements of S_i and S_{i+1} intersect in one $(i-1)$-plane and can be contained in one $(i+2)$-plane (since this construction can be done in $\mathbb{R}^{i+2}$).

Returning to the bound in Theorem 2, we see that the ordered n-tuples which give the exponents of each term essentially consist of pairs of consecutive $\frac{2}{3}$'s or solitary 1's separated by one or two 0's. Therefore for any term, we can construct a tight example as follows. If $a_i = a_{i+1} = \frac{2}{3}$, then we construct a grid out of S_i and S_{i+1} as specified above. Moreover, we choose S_{i-1} to include the $(i-1)$-plane in which they all intersect, and we choose S_{i+2} to include the $(i+2)$-plane in which the grid is contained. If $a_i = 1$, the we take S_i to be a set of i-planes all of which intersect in an $(i-1)$-plane and are contained in an $(i+1)$-plane. Moreover, we choose S_{i-1} to contain this $(i-1)$-plane and S_{i+1} to contain this $(i+1)$-plane. Because our pairs of $\frac{2}{3}$'s and solitary 1's are all separated by 0's this construction is possible, and it does indeed give a tight example for the given term. Since this is possible for all terms, as before, the bound is tight.

Note also that Theorem 2 gives a tight bound for not only complete flags but also for partial flags, by restricting $a_i = 0$ for the i-planes which do not appear. It can be seen that the upper bound works by simply bounding the number of incidences by products of incidences of complete flags. For instance

$$I(S_0, S_1, S_3, S_4, S_5, S_7, S_8) \lesssim I(S_0, S_1) I(S_3, S_4, S_5) I(S_7, S_8).$$

Then the formula will give the proper bound. Also the bound is still tight by the examples presented above.

4 Variants of the Incidence Problem

In this section, we consider two variants of the classical incidence problem discussed in the previous section. The results of the previous section suggest that the answer to the classical incidence problem in n dimensions in some sense reduces to the answer in two dimensions, since all the optimal examples are simply constructed from the two-dimensional example. Therefore, we consider here two variants of the problem in three dimensions for which the examples in the last section fail.

First, we consider a problem studied by Sharir and Welzl in [SW], namely the incidences between points and light-like lines, i.e. lines which are parallel to some fixed double cone (such as $z^2 = x^2 + y^2$). In [SW], they gave an example with $\Theta(|P|^{\frac{2}{3}}|L|^{\frac{1}{2}})$ incidences. They also proved the number of incidences was $O(|P|^{\frac{3}{4}}|L|^{\frac{1}{2}} \log |P| + |P| + |L|)$ and $O(|P|^{\frac{4}{7}}|L|^{\frac{5}{7}} + |P| + |L|)$. In [EKS], this second bound was lowered to $O(|P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P| + |L|)$. Here we use polynomial cell decomposition and follow a similar argument to the proof of Theorem 2.1 in Lecture 19 of [G], to give a different proof of this second bound in a more general context.

Theorem 7. *Let b be a constant, and let P be a set of points and L a set of lines in $\mathbb{R}^3$ so that every point has at most b coplanar lines through it. Then*

$$I(P, L) \lesssim |P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P| + |L|.$$

(Note that the constant in the above inequality may depend on b.)

Proof. We will prove this by induction over the number of lines $|L|$. For the base case, we may simply choose a large enough constant, so we show the inductive step here. In particular, assuming the theorem applies in all cases with fewer lines, we are trying to prove

$$I(P,L) \le k\left(|P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P| + |L|\right)$$

for some fixed k which is independent of $|L|$ (and is also the constant in the inductive hypothesis).

Now fix some d to be chosen later and apply Corollary 4. Let Q be a polynomial of degree at most d so that $\mathbb{R}^3 \setminus Z(Q)$ has at most $O(d^3)$ components, each of which contains no more than $O\left(\frac{|P|}{d^3}\right)$ points and intersects no more than $O\left(\frac{|L|}{d^2}\right)$ lines. Note that we make take Q to be square free (as repeated factors do not change $Z(Q)$).

We will divide P and L and consider incidences in the various cases. Let P_{cell} be the points of P in $\mathbb{R}^3 \setminus Z(Q)$, P_{alg} be the points of P in $Z(Q)$, and P_{crit} be the points of P_{alg} which are also critical points of Q. Now let L_{cell} be the lines of L not contained in $Z(Q)$, L_{alg} be the lines of L which are contained in $Z(Q)$, and L_{crit} be the lines of L_{alg} which are also critical lines of Q (i.e. all points on it are critical). Then we have

$$I(P,L) = I(P_{cell}, L_{cell}) + I(P_{alg}, L_{cell}) + I(P_{alg}, L_{alg}),$$

and we may further write

$$I(P_{alg}, L_{alg}) = I(P_{crit}, L_{crit}) + I(P_{crit}, L_{alg} \setminus L_{crit}) + I(P_{alg} \setminus P_{crit}, L_{alg}).$$

We will get bounds for all of these terms, except $I(P_{crit}, L_{crit})$ which we will bound by induction. First by Szemerédi-Trotter, the number of incidences in a cell with $|P_i|$ points and $|L_i|$ lines is bounded by (up to constant factor)

$$|P_i|^{\frac{2}{3}}|L_i|^{\frac{2}{3}} + |P_i| + |L_i| \le \left(\frac{|P|}{d^3}\right)^{\frac{2}{3}}\left(\frac{|L|}{d^2}\right)^{\frac{2}{3}} + |P_i| + |L_i|,$$

Summing over all $O(d^3)$ cells, we get

$$I(P_{cell}, L_{cell}) \lesssim \frac{|P|^{\frac{2}{3}}|L|^{\frac{2}{3}}}{d^{\frac{1}{3}}} + |P_{cell}| + d|L_{cell}|.$$

Now any line which is not contained in $Z(Q)$ can intersect it at most d times, so

$$I(P_{alg}, L_{cell}) \lesssim d|L_{cell}|.$$

Also note that any non-critical line in $Z(Q)$ can contain at most $d-1$ critical points (because ∇Q has degree $d-1$) so

$$I(P_{crit}, L_{alg} \setminus L_{crit}) \lesssim d|L_{alg} \setminus L_{crit}|.$$

Finally, note that if a point of $Z(Q)$ lies on $b+1$ lines, then these lines cannot be coplanar and so have directions which span $\mathbb{R}^3$. Since Q vanishes on all of them, then the directional derivative vanishes at the point in all of these directions, and so in every direction. Thus ∇Q vanishes at the point, so it is a critical point. Therefore we have

$$I(P_{alg} \setminus P_{crit}, L_{alg}) \leq b|P_{alg} \setminus P_{crit}|.$$

So now we have

$$I(P, L) - I(P_{crit}, L_{crit}) \lesssim \frac{|P|^{\frac{2}{3}}|L|^{\frac{2}{3}}}{d^{\frac{1}{3}}} + |P \setminus P_{crit}| + d|L|.$$

If $|L| \geq |P|^2$, then Szemerédi-Trotter gives $I(P, L) \lesssim |L|$. Otherwise, we may choose $d = \left\lfloor \min\left(\frac{|P|^{\frac{1}{2}}}{|L|^{\frac{1}{4}}}, \frac{|L|^{\frac{1}{2}}}{4\sqrt{2}}\right) \right\rfloor$. If the former value is chosen, we have

$$I(P, L) - I(P_{crit}, L_{crit}) \lesssim |P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P \setminus P_{crit}|.$$

If the latter value is chosen, we have

$$I(P, L) - I(P_{crit}, L_{crit}) \lesssim |P|^{\frac{2}{3}}|L|^{\frac{1}{2}} + |P \setminus P_{crit}| + |L|^{\frac{3}{2}}.$$

But since $\frac{|P|}{|L|^{\frac{1}{2}}} \geq \frac{|L|}{32}$, then $|P|^{\frac{2}{3}}|L|^{\frac{1}{2}} \lesssim |P|$ and $|L|^{\frac{3}{2}} \lesssim |P|$. So in any case we have

$$I(P, L) \leq C\left(|P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P \setminus P_{crit}| + |L|\right) + I(P_{crit}, L_{crit}).$$

for some constant C.

Now since Q is square free, there are at most $2d^2$ critical lines in $Z(Q)$ (by Bezout's theorem applied to Q and ∇Q; see Proposition 3 in [EKS]), and since $2d^2 \leq \frac{L}{16}$, then we can employ our inductive hypothesis. In particular this says that

$$I(P_{crit}, L_{crit}) \leq k\left(\frac{|P|^{\frac{1}{2}}|L|^{\frac{3}{4}}}{8} + |P_{crit}| + \frac{|L|}{16}\right).$$

So then we have

$$I(P,L) \le \left(C + \frac{k}{8}\right)|P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + \left(C + \frac{k}{16}\right)|L| + C|P \setminus P_{crit}| + k|P_{crit}|.$$

Now if we choose k so large that $C + \frac{k}{8} \le k$, then we will have

$$I(P,L) \le k(|P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P| + |L|),$$

and our inductive step will be complete. $\square$

We can see that the above theorem applies for light-like lines, in particular when $b = 2$. This is because the double cone centered at any point can only intersect a plane in at most two lines (since the double cone has degree 2). The theorem can also apply when we restrict the direction of the lines in different ways. In general we can take some algebraic curve in $\mathbb{RP}^2$ and restrict the direction of our lines to this curve. Then so long as we require that the curve has bounded degree, then only a bounded number of lines through any point can be coplanar (since a curve of degree d in $\mathbb{RP}^2$ can only intersect a line in d points).

Now we move on to our second variant of the three-dimensional incidence problem. Here we consider the incidences of flags when we are given N points, N lines, and N planes. However we wish to exclude the case where all the points lie on one line, which all the planes also intersect in. Therefore, we apply the following restriction.

Theorem 8. *Let P be a set of N points in $\mathbb{R}^3$, L be a set of N lines in $\mathbb{R}^3$, and S be a set of N planes in $\mathbb{R}^3$ so that no more than $N^{\frac{1}{2}}$ points lie on any line and no more than $N^{\frac{1}{2}}$ planes go through any line. Then*

$$I(P,L,S) \lesssim N^{\frac{3}{2}} \log N.$$

Proof. First at the cost of a factor of $\log N$, we will reduce the problem to one where we may assume that all lines have approximately the same number of points and planes on them. For all $l \in L$, the P_l denote the set of points on l and S_l denote the set of planes through l. Then define

$$L_{ij} := \{l \in L \mid 2^i \le |P_l| < 2^{i+1}, 2^j \le |S_l| < 2^{j+1}\}.$$

Since $|P_l|, |S_l| \le \sqrt{N}$ for all l, then there are $\Theta(\log N)$ sets L_{ij}. Therefore we have

$$I(P,L,S) = \sum_{i,j} I(P, L_{ij}, S) \lesssim \log N \cdot \max_{i,j} I(P, L_{ij}, S)$$

Now we need only prove that $I(P, L_{ij}, S) \lesssim N^{\frac{3}{2}}$ for all L_{ij} and we will be done.

Now let us fix i and j. Note that we have

$$I(P, L_{ij}, S) = \sum_{l \in L_{ij}} |P_l||S_l| \leq 2^{i+1} 2^{j+1} |L_{ij}|,$$

so we will aim to show $2^{i+1}2^{j+1}|L_{ij}| \lesssim N^{\frac{3}{2}}$. We will now reference Theorem 2.1 in Lecture 19 of [G], which states that if we have a set of points P and lines L in $\mathbb{R}^3$ so that no plane contains more than B lines, then

$$I(P, L) \lesssim |P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P|^{\frac{2}{3}}|L|^{\frac{1}{3}}B^{\frac{1}{3}} + |P| + |L|.$$

Now note that if there exists some plane with more than $\frac{N}{2^j}$ lines of L_{ij} on it, then since each line of L_{ij} has 2^j planes of S through it (all of which must be distinct now since these lines already share a different plane), then we would have more than N planes in S. Therefore, we can see that no plane contains more than $\frac{N}{2^j}$ lines of L_{ij}. Now applying the above theorem to P and L_{ij}, we get

$$2^i|L_{ij}| \leq I(P, L_{ij}) \lesssim N^{\frac{1}{2}}|L_{ij}|^{\frac{3}{4}} + N|L_{ij}|^{\frac{1}{3}}2^{-\frac{j}{3}} + N + |L_{ij}|.$$

This gives us three cases, namely when each term dominates in this sum (note that the last term cannot dominate, since it is less than the left-hand side). We can also make a duality argument so that the last inequality also applies when the roles of points and planes are flipped. In other words, we also have

$$2^j|L_{ij}| \lesssim N^{\frac{1}{2}}|L_{ij}|^{\frac{3}{4}} + N|L_{ij}|^{\frac{1}{3}}2^{-\frac{i}{3}} + N.$$

To sum up, we have that one of (a), (b), or (c) is true, and one of (i), (ii), or (iii) is true.

(a) $2^i|L_{ij}|^{\frac{1}{4}} \lesssim N^{\frac{1}{2}}$,

(b) $2^i 2^{\frac{j}{3}}|L_{ij}|^{\frac{2}{3}} \lesssim N$,

(c) $2^i|L_{ij}| \lesssim N$,

(i) $2^j|L_{ij}|^{\frac{1}{4}} \lesssim N^{\frac{1}{2}}$,

(ii) $2^{\frac{i}{3}} 2^j|L_{ij}|^{\frac{2}{3}} \lesssim N$,

(iii) $2^j|L_{ij}| \lesssim N$,

Now if (c) is true, then since $2^j \leq N^{\frac{1}{2}}$, we have $2^{i+1}2^{j+1}|L_{ij}| \lesssim N^{\frac{3}{2}}$, and we are done. Similarly, if (iii) is true we are done. If (b) and (ii) are true, then multiplying the two inequalities and taking the $\frac{3}{4}$ power gives our desired statement. Since $|L_{ij}| \leq N$, we may instead write (a) as $2^i |L_{ij}|^{\frac{1}{2}} \lesssim N^{\frac{3}{4}}$, and so a similar thing for (i). Then if (a) and (i) are true, multiplying them gives the desired bound. If (a) and (ii) are true, raising (a) to the $\frac{2}{3}$ power and multiplying them gives the desired bound (and similarly for (b) and (i)). $\square$

The bound in Theorem 8 is almost tight (except for the $\log N$ factor). We can give a simple example which has $\Theta(N^{\frac{3}{2}})$ incidences. Take $N^{\frac{1}{2}}$ parallel lines, each with $N^{\frac{1}{2}}$ points on them and $N^{\frac{1}{2}}$ planes through them. Then we have N points, $N^{\frac{1}{2}}$ lines, and N planes with $N^{\frac{3}{2}}$ incidences.

To show that our restrictions has allowed for interesting optimal examples, we will also give a grid example with $\Theta(N^{\frac{3}{2}})$ incidences. Take P to be an $N^{\frac{1}{3}} \times N^{\frac{1}{3}} \times N^{\frac{1}{3}}$ grid of points. For each point in the center $\frac{N^{\frac{1}{3}}}{2} \times \frac{N^{\frac{1}{3}}}{2} \times \frac{N^{\frac{1}{3}}}{2}$ grid, consider the $N^{\frac{1}{12}} \times N^{\frac{1}{12}} \times N^{\frac{1}{12}}$ cube around this point. We then pick L by connecting every point in this cube to our center point with a line. To avoid having too many lines that connect the center point to two other points, we will only connect points which are outside a $\frac{N^{\frac{1}{12}}}{2} \times \frac{N^{\frac{1}{12}}}{2} \times \frac{N^{\frac{1}{12}}}{2}$ cube. Then each point has $\Theta(N^{\frac{1}{4}})$ lines through it. There are N points and each line goes through about $\frac{N^{\frac{1}{3}}}{N^{\frac{1}{12}}} \approx N^{\frac{1}{4}}$ points, so we have a total of about $\frac{N^{\frac{1}{4}} \cdot N}{N^{\frac{1}{4}}} \approx N$ lines.

Now through each point there are $\Theta(N^{\frac{1}{4}})$ lines. Every pair of these spans a plane, so we will choose S to be these planes. This gives $\Theta(N^{\frac{1}{2}})$ planes through each point. (Note that not too many pairs span the same line, so we have not overcounted.) We can see that the intersection of any of these planes with the grid gives approximately a $N^{\frac{1}{4}} \times N^{\frac{1}{4}}$ grid (since any line in the plane goes through $\Theta(N^{\frac{1}{4}})$ points.) Therefore each plane has $\Theta(N^{\frac{1}{2}})$ points and there are N points, so we must have $\frac{N^{\frac{1}{2}} \cdot N}{N^{\frac{1}{2}}} \approx N$ planes.

Thus our example has $\Theta(N)$ points, $\Theta(N)$ lines, and $\Theta(N)$ planes. Moreover because of how we chose L and S, every line has $\Theta(N^{\frac{1}{4}}) \in O(N^{\frac{1}{2}})$ points on it and $\Theta(N^{\frac{1}{4}}) \in O(N^{\frac{1}{2}})$ planes through it. Finally, this means we have approximately $N^{\frac{1}{4}} \cdot N \cdot N^{\frac{1}{4}} \approx N^{\frac{3}{2}}$ incidences.

5 Incidences in Terms of Other Groups

The problem presented in Section 3 can be viewed as a problem about flags in $GL(n+1)$. In particular, if we consider the problem in $\mathbb{RP}^n$ rather than $\mathbb{R}^n$ (which are equivalent since it only adds a single $(n-1)$-plane), then a flag becomes nested subspaces of $\mathbb{R}^{n+1}$ as a vector space. Then the stabilizer of any flag is the set $T(n+1)$ of all $(n+1)\times(n+1)$ invertible upper triangular matrices. The set of flags forms an orbit under this group action by $GL(n+1)$, and so we may identify it with $GL(n+1)/T(n+1)$.

Similarly, we may consider other Lie groups and by taking the quotient with a certain subgroup, we can get a set of flags. Here we will only look at it in contexts which can be realized in terms of vector spaces. For instance, a flag in the group $O(2,2)$ will be a series of nested subspaces of $\mathbb{R}^4$ so that some fixed symmetric form of signature (2,2) vanishes on all the proper subspaces. This is only possible for one- and two-dimensional subspaces. We wish to identify $\mathbb{R}^4$ with $\mathbb{RP}^3$, so this will correspond to point-line pairs in $\mathbb{R}^3$. If we simply choose the form to have the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

then a vector (a,b,c,d) has length 0 with respect to the form if and only if $a^2+b^2=c^2+d^2$. If we identify $\mathbb{R}^4$ with $\mathbb{RP}^3$, i.e. we take $(x,y,z)=(\frac{a}{d},\frac{b}{d},\frac{c}{d})$, then we have $x^2+y^2-z^2=1$. In other words, the only allowed points lie on a hyperboloid. Therefore this is simply an incidence problem on a hyperboloid, but here there are only two lines through any point. Therefore, in the group $O(2,2)$, we have $I(P,L)\lesssim|P|$.

Now we consider the group $Sp(4)$. Here flags correspond to nested subspaces of $\mathbb{R}^4$ so that some fixed symplectic form (non-degenerate skew-symmetric form) vanishes on all the proper subspaces. Again this will only be possible for one- and two-dimensional subspaces, so this will correspond to point-line pairs in $\mathbb{R}^3$. By the definition of a symplectic form, any vector has length 0 with respect to the form. Therefore the form vanishes on all one-dimensional subspaces, and so all points in $\mathbb{R}^3$ are allowed. If we take our form to have the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

then it evaluates to zero on (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2) if and only if $a_1b_2 + c_1d_2 = a_2b_1 + c_2d_1$. Again identifying $\mathbb{R}^4$ with $\mathbb{RP}^3$ we get, $x_1y_2 + z_1 = x_2y_1 + z_2$. This means two points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on an allowed line if they satisfy the above condition. In other words, through any point (x_1, y_1, z_1), the allowed lines through that point lie on a plane with normal vector $(-y_1, x_1, 1)$.

We will thus reword the incidence problem in $Sp(4)$ as follows. We call a line Legendrian if it is orthogonal to the vector field $(-y, x, 1)$ at every point. Note that if a line is orthogonal to the field at one point, it can easily be shown that it is orthogonal everywhere. Then given a set P of points in $\mathbb{R}^3$ and a set L of Legendrian lines in $\mathbb{R}^3$, we ask how large can $I(P, L)$ be. By a similar argument to the proof of Theorem 7, we can get the following upper bound.

Theorem 9. *Let P be a set of points and L a set of Legendrian lines in $\mathbb{R}^3$. Then*

$$I(P, L) \lesssim |P|^{\frac{3}{4}}|L|^{\frac{1}{2}} + |P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P| + |L|.$$

Proof. Much like in Theorem 7, we will prove this by induction over $|L|$. We fix some d to be chosen later and apply Corollary 4. Let Q be a polynomial of degree at most d which partitions $\mathbb{R}^3$ into $O(d^3)$ cells each of which has no more than $O\left(\frac{|P|}{d^3}\right)$ points and intersects no more than $O\left(\frac{|L|}{d^2}\right)$ lines. Again we may take Q to be square-free.

Let P_{cell}, L_{cell}, P_{alg}, and L_{alg} be as they were in the proof of Theorem 7. In addtion let P_{flat} be the points of P_{alg} which are flat, let L_{flat} be the lines of L_{alg} which are flat, and L_{plane} be the lines of L_{alg} which lie in a plane contained in $Z(Q)$. Now we have

$$\begin{aligned} I(P, L) = I(P_{cell}, L_{cell}) + I(P_{alg}, L_{cell}) + I(P_{flat}, L_{flat} \setminus L_{plane}) + I(P_{flat}, L_{plane}) \\ + I(P_{flat}, L_{alg} \setminus L_{flat}) + I(P_{alg} \setminus P_{flat}, L_{alg}). \end{aligned}$$

Again we will bound each of these terms except for $I(P_{flat}, L_{flat} \setminus L_{plane})$, which we will bound by induction. As in Theorem 7, we get

$$I(P_{cell}, L_{cell}) \lesssim \frac{|P|^{\frac{2}{3}}|L|^{\frac{2}{3}}}{d^{\frac{1}{3}}} + |P| + d|L|$$

and

$$I(P_{alg}, L_{cell}) \lesssim d|L|.$$

It is shown in [EKS] (Proposition 6) that the set of flat points can be characterized as zero sets of three polynomials of degree less than d. Therefore there can be no more than $3d$ flat points on any non-flat line and so

$$I(P_{flat}, L_{alg} \setminus L_{flat}) \lesssim d|L|.$$

Now if a point $x \in Z(Q)$ lies on three lines of L_{alg}, these must all be coplanar since they are all orthogonal to the vector field at x. Then since x lies on three coplanar lines on which Q vanishes, x must be flat (see Proposition 6 in [EKS]). Therefore we have

$$I(P_{alg} \setminus P_{flat}, L_{alg}) \leq 2|P|.$$

Finally, we note that if we restrict our lines to a plane, there is only one point with multiple Legendrian lines through it. This is because at such a point, the vector field must be orthogonal to the plane (so that every line in the plane is orthogonal to the vector field). The vector field can only be parallel at two points which have the same z-coordinate though, and this would require that our plane be parallel to the z-axis, but then the vector field is never orthogonal to it.

Now this single point can contribute $|L|$ incidences, and all the other points on the plane together can contribute $|P|$ incidences. Therefore the incidences on any given plane are bounded by $|P| + |L|$. Since $Z(Q)$ can contain at most d planes, we have

$$I(P_{flat}, L_{plane}) \lesssim d|P| + d|L|.$$

So now we have

$$I(P, L) - I(P_{flat}, L_{flat} \setminus L_{plane}) \lesssim \frac{|P|^{\frac{2}{3}}|L|^{\frac{2}{3}}}{d^{\frac{1}{3}}} + d|P| + d|L|.$$

Similarly to the proof of Theorem 7, if $|L|^2 \geq |P| \geq |L|$, we may choose $d \approx \frac{|L|^{\frac{1}{2}}}{|P|^{\frac{1}{4}}}$, and if $|L|^{\frac{1}{2}} \leq |P| \leq |L|$, then we choose $d \approx \frac{|P|^{\frac{1}{2}}}{|L|^{\frac{1}{4}}}$ (being sure to have $d \lesssim |L|^{\frac{1}{2}}$). Then combining our result in these two cases, we will arrive at

$$I(P,L) - I(P_{flat}, L_{flat} \setminus L_{plane}) \lesssim |P|^{\frac{3}{4}}|L|^{\frac{1}{2}} + |P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P| + |L|.$$

Let $S = \frac{Q}{\prod q_i}$ where q_i are the linear factors of Q. Note that all the lines in $L_{flat} \setminus L_{plane}$ are flat lines of S. Now since S is square-free, has no linear factors, and has degree at most d, it can have at most $3d^2$ flat lines (see Proposition 8 in [EKS]). Therefore $L_{flat} \setminus L_{plane}$ has fewer lines than L and we may apply our inductive hypothesis to bound $I(P_{flat}, L_{flat} \setminus L_{plane})$ as in the proof of Theorem 7. This will leave us with

$$I(P,L) \lesssim |P|^{\frac{3}{4}}|L|^{\frac{1}{2}} + |P|^{\frac{1}{2}}|L|^{\frac{3}{4}} + |P| + |L|.$$

□

We lack any non-trivial examples for this incidence problem to give a lower bound for $M(P,L)$. It is worth examining a grid example for this problem though. In other incidence problems, our best examples come from grids. In this problem though, so long as our points are arranged in a cubical grid, any choice of lines gives a trivial bound.

Consider the case where P is an $N \times N \times N$ grid of points centered at the origin. As in the our grid example for flags in $\mathbb{R}^3$, we fix some k and around each point (x,y,z) look in a $k \times k \times k$ cube. (This is the best strategy because we will maximize incidences by having approximately the same number on each line.) We cannot connect the center point to every point in this $k \times k \times k$ cube though, since our line must be orthogonal to the vector field $(-y, x, 1)$. In particular, if we take the plane orthogonal to this vector, the grid points which lie on it will also form a rectangular grid, which is spanned by the vectors $(x, y, 0)$ and $(y, -x, x^2 + y^2)$. Since we only want points in a $k \times k \times k$ cube, this rectangular grid should have a total of approximately $\frac{k}{\max(|x|,|y|)} \cdot \frac{k}{x^2+y^2}$ points. So to get a total count of the incidences, we must take

$$\sum_{-\frac{N}{2} \leq x,y,z \leq \frac{N}{2}} \frac{k^2}{\max(|x|,|y|)(x^2+y^2)} \approx Nk^2 \sum_{y=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{|y|} \sum_{x=-|y|}^{|y|} \frac{1}{x^2+y^2}$$

As $N \to \infty$, this sum converges to a constant (since the inside sum will converge to the reciprocal of some polynomial in y and so the outside sum will converge by the integral test). In other words, for large N, we will have $\Theta(Nk^2)$ incidences, and since each line has $\frac{N}{k}$ points, we must have $|L| \approx k^3$. Since $|P| \approx N^3$, then this means we have $\Theta(|P|^{\frac{1}{3}}|L|^{\frac{2}{3}})$ incidences. This term is always dominated by $|P| + |L|$ though, so this example is indeed trivial.

6 Conclusion and Future Research

In this paper, we employed a stronger version of the polynomial cell decomposition technique to further results in incidence geometry. We successfully attained a tight bound for the incidences of flags in $\mathbb{R}^n$, as well as an almost tight bound on a three-dimensional variant of this problem. A tight bound for incidences between points and light-like lines remains to be found though. There are also new but related incidence problems to be explored, such as a variant of the light-like problem for different curves in $\mathbb{RP}^2$ or higher-dimensional analogs of our three-dimensional variant.

The most intriguing problem to see a solution for would be the incidence problem of points and Legendrian lines. Moreover, it would be interesting to see how results about points and Legendrian lines could extend to higher dimensions, i.e. explore the incidences of flags in $Sp(2n)$ or even $O(n,n)$. Finally, we can generalize our problem to certain Lie group as follows.

Let G be a split simple real Lie group of rank k. Let B be a Borel subgroup of G, and let $P_1, \ldots, P_k$ be the maximal parabolic subgroups which contain B. Then G/B corresponds to the set of complete flags, and G/P_i corresponds to the set of the i-th subspaces in those flags. We thus have a natural map $f : G/B \to G/P_1 \times \cdots \times G/P_k$. Then we ask, given finite subsets $S_i \subset G/P_i$, how large can the preimage $f^{-1}(S_1, \ldots, S_k)$ be in terms of $|S_1|, \ldots, |S_k|$.

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