On a $\mathbb{Z}_4$-Symmetric Operator over a Lemniscatic Elliptic Curve

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Abstract

The theory of algebraically integrable systems is deeply connected to various areas in mathematical physics and representation theory. For example, Prof. Etingof proved that a certain class of algebraically integrable differential operators on an elliptic curve with twofold or threefold symmetry corresponds to critical points of the classical crystallographic elliptic Calogero-Moser potential. In this paper, we extend these results by expressing an analogous class of differential operators with fourfold symmetry over a lemniscatic elliptic curve E in terms of trivial monodromy relations. We show that if the operators have eight poles, algebraically integrable operators correspond to critical points of a potential function of the form $\psi = p^4 + A(z_0)p^2 + B(z_0)p + C(z_0)$, where A, B and C are meromorphic functions on E.

1 Introduction

A differential operator $L = \partial^n + a_2(z)\partial^{n-2} + a_3(z)\partial^{n-3} + \cdots + a_n(z)$ is *algebraically integrable* iff there exists a nonzero differential operator M of relatively prime order that commutes with L. The notion of an algebraically integrable system is related to certain explicit solutions related to the Korteweg-de-Vries equation (see [1] and [4]). It is known that if L is algebraically integrable, then the coefficients $a_i(z)$ are meromorphic on the complex plane and their poles have order at most i (see Theorem 2.1 in [1]). For example, algebraically integrable operators can be constructed with coefficients doubly periodic in $\{1, \tau\}$, where $\tau \notin \mathbb{R}$. These coefficients can be identified with rational functions on an elliptic curve $y^2 = 4x^3 - g_2x - g_3$. It turns out that in this special case, algebraic integrability is equivalent to the existence of a basis of solutions

$$\psi(z) = e^{\beta z} \prod_{i=1}^{m} \frac{\theta(z - \alpha_i, \tau)}{\theta(z - \beta_i, \tau)}$$

of the differential equation $L\psi = \lambda\psi$, for a suitable choice of $\beta, m, \alpha_i, \beta_i$ for all but finitely many λ. Here $\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2} z^n$. It is also equivalent to trivial monodromy of the equation $L\psi = \lambda\psi$ at all poles of the equation (see Theorem 2.5 in [1]). From this definition, it is easy to prove that any algebraically integrable doubly periodic second-order differential operator with one pole must be of the form $\partial^2 - m(m+1)\wp(z,\tau)$.

Etingof and Rains considered in [1] operators of this form that are invariant under symmetries of an elliptic curve. There are four such symmetries (where $w_r = \exp 2\pi i/r$):

1. Twofold symmetry, on any elliptic curve. The action is $-\mathbf{1} : (x,y) \mapsto (x,-y)$.
2. Threefold symmetry, on an equianharmonic curve $y^2 = 4x^3 - g_3$. The action is $\mathbf{w_3^j} : (x,y) \mapsto (w_3^j x, y)$.
3. Fourfold symmetry, on a lemniscatic curve $y^2 = 4x^3 - g_2 x$. The action is $\mathbf{w_4^j} : (x,y) \mapsto (w_4^{2j} x, w_4^j y)$.
4. Sixfold symmetry, on an equianharmonic curve $y^2 = 4x^3 - g_3$. The action is $\mathbf{w_6^j} : (x,y) \mapsto (w_6^{2j} x, w_6^{3j} y)$.

In particular, he found a correspondence between algebraically integrable second-order operators with twofold symmetry and critical points of the Inozemtsev potential. Namely, if L has poles at the fixed points $W_0 = 0, W_1 = 1/2, W_2 = \tau/2, W_3 = (1+\tau)/2$ of the $\mathbb{Z}_2$ action with arbitrary indices $-m_j, m_{j+1}$ and poles at other points $\pm z_0, ..., \pm z_{N-1}$ with indices $-1, 2$, L is algebraically integrable iff the parameters correspond to critical points of:

$$\sum_{i=0}^{3} \sum_{j=0}^{N-1} \left(m_i + \frac{1}{2} \right)^2 \wp(z_j - W_i) + \sum_{0 \le k \ne j \le N-1} (\wp(z_j - z_k) + \wp(z_j + z_k))$$

Likewise, if L is a third-order operator on an equianharmonic curve with threefold symmetry, with poles of arbitrary indices at the fixed points of the $\mathbb{Z}_3$ action as well as on orbits of other points with indices $-1, 1, 3$, L is algebraically integrable iff the parameters correspond to critical points of the classical crystallographic elliptic Calogero-Moser potential (see Proposition 4.7 in [1], and also [2]).

It is thus natural to consider fourth-order operators on a lemniscatic curve $y^2 = 4x^3 - g_2 x$ with fourfold symmetry, with poles of arbitrary indices at the fixed points of the $\mathbb{Z}_4$ action as well as on orbits of other points $z_0, z_1, \cdots, z_{N-1}$ (such that $z_i^4 / z_j^4 \ne 1$) with indices $-1, 1, 2, 4$. With the added constraint that the indices at the fixed points with stabilizer 2 are of the form $\{a, a+2, 1-a, 3-a\}$, algebraically integrable operators of this form are conjectured in [1] to correspond to critical points of the classical crystallographic elliptic Calogero-Moser Hamiltonian for the group $\mathbb{Z}_4$.

In this paper, we take the first step towards formulating such a theory by classifying algebraically integrable operators if this form when $N = 1$.

2 Trivial monodromy relations

Consider a fourth-order Fuchsian differential equation $Lf = \lambda f$, where $L = \partial^4 + a\partial^2 + b\partial + c$ and a, b, c are meromorphic functions on the complex plane. Fix a point $z_0 \in \mathbb{C}$ and expand

a, b and c in Laurent series:

$$a = \sum_{j=-2}^{\infty} a_j (z - z_0)^j$$

$$b = \sum_{j=-3}^{\infty} b_j (z - z_0)^j$$

$$c = \sum_{j=-4}^{\infty} c_j (z - z_0)^j$$

Furthermore, suppose that $a_{-2} = -4, b_{-3} = 8, c_{-4} = -8$. This is to ensure that the indices are $-1, 1, 2, 4$. For the monodromy of this equation to be trivial at z_0, there must exist a basis of solutions $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ analytic in a punctured neighborhood of z_0.

Theorem 1: The equation $Lf = \lambda f$ has trivial monodromy at z_0 iff:
1) $a_1 = -c_{-1}$
2) $c_{-2} = -\frac{1}{4}(c_{-3}^2 + 8a_0)$
3) $4b_0 - c_{-3}c_{-2} + 4c_{-1} = 0$
4) $-32a_3 + 16b_2 + 8a_2c_{-3} - 4b_1c_{-3} + c_{-3}^2 c_{-1} - 16c_1 = 0$
5) $a_{-1} = 0$
6) $b_{-1} = 0$
7) $b_{-2} - c_{-3} = 0$
Proof: This is a routine application of Frobenius' method best done with a computer algebra system such as Mathematica.

Corollary 1: The equation $Lf = \lambda f$ has trivial monodromy at z_0 iff it has a basis of meromorphic solutions $\{\phi_1, \phi_2, \phi_3, \phi_4\}$. Furthermore, the orders of the poles of ϕ_j at z_0 are $-1, 1, 2, 4$ respectively.

3 Elliptic Functions

Here I will primarily deal with the elliptic curve $E : y^2 = 4x^3 - 4x$, and its associated $\wp$-function. Recall that the $\wp$-function is doubly periodic with periods ρ and $i\rho$, where

$$\rho = \frac{1}{\sqrt{8\pi}} \Gamma(1/4)^2$$

Consider the fourth-order differential operator $L = \partial^4 + a\partial^2 + b\partial + c$, where each a, b, c are meromorphic on E such that L has foufold symmetry, with poles of arbitrary indices at the fixed points of the $\mathbb{Z}_4$ action as well as on the other four points $z_0, iz_0, -z_0, -iz_0$ with indices $-1, 1, 2, 4$. The fixed points shall be denoted $\eta_0 = 0, \eta_1 = i\rho/2, \eta_3 = \rho/2, \eta_2 = (1 + i)\rho/2$. Given that $L = \partial^4 + a\partial^2 + b\partial + c$ is invariant under the action of $\mathbb{Z}_4$ on the elliptic curve, it follows that:

$$a(z) = a_F(z) + \sum_\omega (-4\wp(z - \omega z_0))$$

$$b(z) = b_F(z) + \sum_\omega (-4\wp'(z - \omega z_0) + p\omega^{-1}\wp(z - \omega z_0))$$

$$c(z) = c_F(z) + \sum_\omega (-\frac{4}{3}\wp''(z - \omega z_0) + \frac{p}{2}\omega^{-1}\wp'(z - \omega z_0) + \gamma\omega^2\wp(z - \omega z_0) - \alpha\omega\zeta_e(z - \omega z_0))$$

where ω ranges over fourth roots of unity, and:

$$a_F(z) = \sum_\ell \wp(z - \eta_\ell) f_\ell$$

$$b_F(z) = \sum_\ell \wp'(z - \eta_\ell) g_\ell$$

$$c_F(z) = \sum_\ell \wp''(z - \eta_\ell) h_\ell$$

The symmetry implies that $f_1 - f_3 = g_1 - g_3 = h_1 - h_3 = 0$. Here ζ_e and σ_e are the Weierstrass zeta and sigma functions. Consider the Laurent expansions of a, b, c around the pole z_0:

$$a(z) = \sum_j (z - z_0)^j a_j, b(z) = \sum_j (z - z_0)^j b_j, c(z) = \sum_j (z - z_0)^j c_j$$

Then algebraic integrability is equivalent to these four relations, obtained from trivial monodromy by solving the differential equation $L\psi = \lambda\psi$ with the Frobenius method, as seen before:

1) $a_1 = -c_{-1}$
2) $c_{-2} = -\frac{1}{4}(c_{-3}^2 + 8a_0)$
3) $4b_0 - c_{-3}c_{-2} + 4c_{-1} = 0$
4) $-32a_3 + 16b_2 + 8a_2c_{-3} - 4b_1c_{-3} + c_{-3}^2c_{-1} - 16c_1 = 0$

The first two relations define γ and α:

$$\gamma = -\frac{p^2}{4} - 2(a_F(z_0) - 4\sum_{\omega \neq \omega^4 = 1} \wp((1 - \omega)z_0))$$

$$\alpha = a_F'(z_0) - 4\sum_{\omega \neq \omega^4 = 1} \wp'((1 - \omega)z_0)$$

The other two relations narrow the set of possible values for z_0 and p.

4 Computations

4.1 Differential and difference operators

From now on, $x = \wp(z), y = \wp'(z), X = \wp(z_0), Y = \wp'(z_0)$. For $(a, b) \in \mathbb{R}^2$ such that $b^2 = 4a^3 - 4a$, let:

$$E(a,b) = \frac{-4a^3 + 4a^2x + 4ax^2 + b^2 - 4x}{4(a-x)^2} + y\frac{b}{2(a-x)^2}$$

So if $\wp(w) = a$ and $\wp'(w) = b$, then $E(a, b) = \wp(z - w)$. This identity is well-known and can be proven by Liouville's theorem. The main differential operator I use in this paper corresponds to differentiation on the lattice of the elliptic curve, and is given by:

$$D(p(x) + yq(x)) = \left(-2 + 6x^2\right) q(x) + 4\left(-x + x^3\right) q'(x) + yp'(x)$$

4.2 The fixed points

Identifying the point z with (x, y), the three fixed points of our $\mathbb{Z}_4$-action (disregarding the point at infinity) are $(1, 0)$, $(0, 0)$, and $(-1, 0)$. These are also the half-periods of the curve. Hence we have the following:

$$a_F = f(0)x + f(1)(E(1,0) + E(-1,0)) + f(2)E(0,0)$$
$$b_F = D(g(0)x + g(1)(E(1,0) + E(-1,0)) + g(2)E(0,0))$$
$$c_F = D(D(h(0)x + h(1)(E(1,0) + E(-1,0)) + h(2)E(0,0)))$$

However, E at the fixed points evaluates to something especially simple:

$$E(-1,0) = \frac{1-x}{1+x}$$
$$E(0,0) = -\frac{1}{x}$$
$$E(1,0) = \frac{x+1}{x-1}$$

Therefore,

$$a_F = \frac{f(0)x^4 - f(0)x^2 + 4f(1)x^2 - f(2)x^2 + f(2)}{(x-1)x(x+1)}$$

$$b_F = \frac{y\left(g(0)x^6 - 2g(0)x^4 - 4g(1)x^4 + g(2)x^4 + g(0)x^2 - 4g(1)x^2 - 2g(2)x^2 + g(2)\right)}{(x-1)^2x^2(x+1)^2}$$

$$c_F = \frac{2\left(3h(0)x^8 - 7h(0)x^6 + 4h(1)x^6 - h(2)x^6 + 5h(0)x^4 + 40h(1)x^4 + 5h(2)x^4 - h(0)x^2 + 4h(1)x^2 - 7h(2)x^2 + 3h(2)\right)}{(x-1)^2x^2(x+1)^2}$$

Similarly, evaluating a, b and c give the following:

$a = \frac{1}{(x-1)x(x+1)(x-X)^2(x+X)^2}(f(0)x^8 - 2f(0)x^6X^2 - f(0)x^6 + 4f(1)x^6 - f(2)x^6 + f(0)x^4X^4 + 2f(0)x^4X^2 - 8f(1)x^4X^2 + 2f(2)x^4X^2 + f(2)x^4 - f(0)x^2X^4 + 4f(1)x^2X^4 - f(2)x^2X^4 - 2f(2)x^2X^2 + f(2)X^4 - 48x^6X^2 + 16x^6 - 16x^4X^4 + 96x^4X^2 - 16x^4 + 16x^2X^4 - 48x^2X^2)$

$b = (y(g(0)x^{12} - 3g(0)x^{10}X^2 - 2g(0)x^{10} - 4g(1)x^{10} + g(2)x^{10} + 3g(0)x^8X^4 + 6g(0)x^8X^2 + 12g(1)x^8X^2 - 3g(2)x^8X^2 + g(0)x^8 - 4g(1)x^8 - 2g(2)x^8 - g(0)x^6X^6 - 6g(0)x^6X^4 - 12g(1)x^6X^4 + 3g(2)x^6X^4 - 3g(0)x^6X^2 + 12g(1)x^6X^2 + 6g(2)x^6X^2 + g(2)x^6 + 2g(0)x^4X^6 + 4g(1)x^4X^6 - g(2)x^4X^6 + 3g(0)x^4X^4 - 12g(1)x^4X^4 - 6g(2)x^4X^4 - 3g(2)x^4X^2 - g(0)x^2X^6 + 4g(1)x^2X^6 + 2g(2)x^2X^6 + 3g(2)x^2X^4 - g(2)X^6 + 48x^{10}X^2 - 16x^{10} + 192x^8X^4 - 288x^8X^2 + 32x^8 + 16x^6X^6 - 432x^6X^4 + 432x^6X^2 - 16x^6 - 32x^4X^6 + 288x^4X^4 - 192x^4X^2 + 16x^2X^6 - 48x^2X^4))/((x-1)^2x^2(x+1)^2(x-X)^3(x+X)^3) + \frac{2pyY(x^2+X^2)}{(x-X)^2(x+X)^2}$

$c = \frac{1}{3(x-1)^2x^2(x+1)^2(x-X)^4(x+X)^4}2(9h(0)x^{16} - 48X^2x^{14} - 3pYx^{14} + 6X\gamma x^{14} - 36X^2h(0)x^{14} - 21h(0)x^{14} + 12h(1)x^{14} - 3h(2)x^{14} + 16x^{14} - 1488X^4x^{12} + 1376X^2x^{12} - 33pX^2Yx^{12} + 15pYx^{12} + 3XY\alpha x^{12} + 6X^3\gamma x^{12} - 30X\gamma x^{12} + 54X^4h(0)x^{12} + 84X^2h(0)x^{12} + 15h(0)x^{12} - 48X^2h(1)x^{12} + 120h(1)x^{12} + 12X^2h(2)x^{12} + 15h(2)x^{12} - 80x^{12} - 1488X^6x^{10} + 6528X^4x^{10} - 4096X^2x^{10} + 27pX^4Yx^{10} + 93pX^2Yx^{10} - 21pYx^{10} - 9X^3Y\alpha x^{10} - 6XY\alpha x^{10} - 30X^5\gamma x^{10} + 18X^3\gamma x^{10} + 42X\gamma x^{10} - 36X^6h(0)x^{10} - 126X^4h(0)x^{10} - 60X^2h(0)x^{10} - 3h(0)x^{10} + 72X^4h(1)x^{10} - 480X^2h(1)x^{10} + 12h(1)x^{10} - 18X^4h(2)x^{10} - 60X^2h(2)x^{10} - 21h(2)x^{10} + 112x^{10} - 48X^8x^8 + 4256X^6x^8 - 10080X^4x^8 + 4256X^2x^8 + 9pX^6Yx^8 - 87pX^4Yx^8 - 87pX^2Yx^8 + 9pYx^8 + 9X^5Y\alpha x^8 + 18X^3Y\alpha x^8 + 3XY\alpha x^8 + 18X^7\gamma x^8 + 54X^5\gamma x^8 - 54X^3\gamma x^8 - 18X\gamma x^8 + 9X^8h(0)x^8 + 84X^6h(0)x^8 + 90X^4h(0)x^8 + 12X^2h(0)x^8 - 48X^6h(1)x^8 + 720X^4h(1)x^8 - 48X^2h(1)x^8 + 12X^6h(2)x^8 + 90X^4h(2)x^8 + 84X^2h(2)x^8 + 9h(2)x^8 - 48x^8 + 112X^8x^6 - 4096X^6x^6 + 6528X^4x^6 - 1488X^2x^6 - 21pX^6Yx^6 + 93pX^4Yx^6 + 27pX^2Yx^6 - 3X^7Y\alpha x^6 - 18X^5Y\alpha x^6 - 9X^3Y\alpha x^6 - 42X^7\gamma x^6 - 18X^5\gamma x^6 + 30X^3\gamma x^6 - 21X^8h(0)x^6 - 60X^6h(0)x^6 - 18X^4h(0)x^6 + 12X^8h(1)x^6 - 480X^6h(1)x^6 + 72X^4h(1)x^6 - 3X^8h(2)x^6 - 60X^6h(2)x^6 - 126X^4h(2)x^6 - 36X^2h(2)x^6 - 80X^8x^4 + 1376X^6x^4 - 1488X^4x^4 + 15pX^6Yx^4 - 33pX^4Yx^4 + 6X^7Y\alpha x^4 + 9X^5Y\alpha x^4 + 30X^7\gamma x^4 - 6X^5\gamma x^4 + 15X^8h(0)x^4 + 12X^6h(0)x^4 + 120X^8h(1)x^4 - 48X^6h(1)x^4 + 15X^8h(2)x^4 + 84X^6h(2)x^4 + 54X^4h(2)x^4 + 16X^8x^2 - 48X^6x^2 - 3pX^6Yx^2 - 3X^7Y\alpha x^2 - 6X^7\gamma x^2 - 3X^8h(0)x^2 + 12X^8h(1)x^2 - 21X^8h(2)x^2 - 36X^6h(2)x^2 + 9X^8h(2))$

Evaluating γ and α based on the first two equations gives:

$\alpha + \frac{1}{2X^2(X^2-1)^2}Y(2f(0)X^6 - 4f(0)X^4 - 8f(1)X^4 + 2f(2)X^4 + 2f(0)X^2 - 8f(1)X^2 - 4f(2)X^2 + 2f(2) + 3X^6 + X^4 + X^2 + 3) = 0$

$\gamma + \frac{1}{4}(\frac{8\left(f(0)X^4 - f(0)X^2 + 4f(1)X^2 - f(2)X^2 + f(2) - X^4 - 2X^2 - 1\right)}{X(X^2-1)} + p^2) = 0$

Substituting,

$c = (18X^2h(0)x^{16} - 18h(0)x^{16} - 72X^4x^{14} - 3p^2X^3x^{14} + 176X^2x^{14} + 3p^2Xx^{14} - 24X^4f(0)x^{14} + 24X^2f(0)x^{14} - 96X^2f(1)x^{14} + 24X^2f(2)x^{14} - 24f(2)x^{14} - 72X^4h(0)x^{14} + 30X^2h(0)x^{14} + 42h(0)x^{14} + 24X^2h(1)x^{14} - 24h(1)x^{14} - 6X^2h(2)x^{14} + 6h(2)x^{14} - 8x^{14} - 2988X^6x^{12} - 3p^2X^5x^{12} + 5644X^4x^{12} + 18p^2X^3x^{12} - 3140X^2x^{12} - 15p^2Xx^{12} - 48X^6f(0)x^{12} + 192X^4f(0)x^{12} - 144X^2f(0)x^{12} + 576X^2f(1)x^{12} - 96X^2f(2)x^{12} + 96f(2)x^{12} + 108X^6h(0)x^{12} + 60X^4h(0)x^{12} - 138X^2h(0)x^{12} - 30h(0)x^{12} - 96X^4h(1)x^{12} + 336X^2h(1)x^{12} - 240h(1)x^{12} + 24X^4h(2)x^{12} + 6X^2h(2)x^{12} - 30h(2)x^{12} +$

$4x^{12} - 2988X^8x^{10} + 15p^2X^7x^{10} + 15972X^6x^{10} - 24p^2X^5x^{10} - 20996X^4x^{10} - 12p^2X^3x^{10} + 8956X^2x^{10} + 21p^2Xx^{10} + 192X^8f(0)x^{10} - 288X^6f(0)x^{10} - 120X^4f(0)x^{10} + 216X^2f(0)x^{10} + 192X^6f(1)x^{10} - 768X^4f(1)x^{10} - 864X^2f(1)x^{10} - 48X^6f(2)x^{10} + 96X^4f(2)x^{10} + 72X^2f(2)x^{10} - 120f(2)x^{10} - 72X^8h(0)x^{10} - 180X^6h(0)x^{10} + 132X^4h(0)x^{10} + 114X^2h(0)x^{10} + 6h(0)x^{10} + 144X^6h(1)x^{10} - 1104X^4h(1)x^{10} + 984X^2h(1)x^{10} - 24h(1)x^{10} - 36X^6h(2)x^{10} - 84X^4h(2)x^{10} + 78X^2h(2)x^{10} + 42h(2)x^{10} + 16x^{10} - 132X^{10}x^8 - 9p^2X^9x^8 + 8716X^8x^8 - 18p^2X^7x^8 - 28528X^6x^8 + 54p^2X^5x^8 + 28192X^4x^8 - 18p^2X^3x^8 - 9196X^2x^8 - 9p^2Xx^8 - 144X^{10}f(0)x^8 - 144X^8f(0)x^8 + 624X^6f(0)x^8 - 240X^4f(0)x^8 - 96X^2f(0)x^8 + 1536X^4f(1)x^8 + 384X^2f(1)x^8 + 144X^6f(2)x^8 - 240X^4f(2)x^8 + 48X^2f(2)x^8 + 48f(2)x^8 + 18X^{10}h(0)x^8 + 150X^8h(0)x^8 + 12X^6h(0)x^8 - 156X^4h(0)x^8 - 24X^2h(0)x^8 - 96X^8h(1)x^8 + 1536X^6h(1)x^8 - 1536X^4h(1)x^8 + 96X^2h(1)x^8 + 24X^8h(2)x^8 + 156X^6h(2)x^8 - 12X^4h(2)x^8 - 150X^2h(2)x^8 - 18h(2)x^8 - 12x^8 + 36X^{12}x^6 + 284X^{10}x^6 + 21p^2X^9x^6 - 8632X^8x^6 - 12p^2X^7x^6 + 21200X^6x^6 - 24p^2X^5x^6 - 15612X^4x^6 + 15p^2X^3x^6 + 3204X^2x^6 + 24X^{12}f(0)x^6 + 264X^{10}f(0)x^6 - 288X^8f(0)x^6 - 192X^6f(0)x^6 + 192X^4f(0)x^6 - 96X^{10}f(1)x^6 - 576X^6f(1)x^6 - 768X^4f(1)x^6 + 24X^{10}f(2)x^6 - 72X^8f(2)x^6 - 96X^6f(2)x^6 + 192X^4f(2)x^6 - 48X^2f(2)x^6 - 42X^{10}h(0)x^6 - 78X^8h(0)x^6 + 84X^6h(0)x^6 + 36X^4h(0)x^6 + 24X^{10}h(1)x^6 - 984X^8h(1)x^6 + 1104X^6h(1)x^6 - 144X^4h(1)x^6 - 6X^{10}h(2)x^6 - 114X^8h(2)x^6 - 132X^6h(2)x^6 + 180X^4h(2)x^6 + 72X^2h(2)x^6 - 72X^{12}x^4 - 172X^{10}x^4 - 15p^2X^9x^4 + 3068X^8x^4 + 18p^2X^7x^4 - 5764X^6x^4 - 3p^2X^5x^4 + 2844X^4x^4 - 48X^{12}f(0)x^4 - 96X^{10}f(0)x^4 + 240X^8f(0)x^4 - 96X^6f(0)x^4 + 192X^{10}f(1)x^4 + 384X^6f(1)x^4 - 48X^{10}f(2)x^4 + 144X^8f(2)x^4 - 48X^6f(2)x^4 - 48X^4f(2)x^4 + 30X^{10}h(0)x^4 - 6X^8h(0)x^4 - 24X^6h(0)x^4 + 240X^{10}h(1)x^4 - 336X^8h(1)x^4 + 96X^6h(1)x^4 + 30X^{10}h(2)x^4 + 138X^8h(2)x^4 - 60X^6h(2)x^4 - 108X^4h(2)x^4 + 36X^{12}x^2 + 20X^{10}x^2 + 3p^2X^9x^2 - 164X^8x^2 - 3p^2X^7x^2 + 108X^6x^2 + 24X^{12}f(0)x^2 - 24X^{10}f(0)x^2 - 96X^{10}f(1)x^2 + 24X^{10}f(2)x^2 - 72X^8f(2)x^2 + 48X^6f(2)x^2 - 6X^{10}h(0)x^2 + 6X^8h(0)x^2 + 24X^{10}h(1)x^2 - 24X^8h(1)x^2 - 42X^{10}h(2)x^2 - 30X^8h(2)x^2 + 72X^6h(2)x^2 + 18X^{10}h(2) - 18X^8h(2))/(3(x-1)^2x^2(x+1)^2(x-X)^4(x+X)^4(X^2-1)) - (2p(x^6+12X^2x^4-3x^4+3X^4x^2-12X^2x^2-X^4)Y)/((x-X)^3(x+X)^3)$

The algebraic integrability equations thus reduce to:

$$-\frac{4y\left(x^6(f(0)-g(0))+x^4(-2f(0)-4f(1)+f(2)+2g(0)+4g(1)-g(2))+x^2(f(0)-4f(1)-2f(2)-g(0)+4g(1)+2g(2))+f(2)-g(2)\right)}{x^2(x^2-1)^2} - \frac{p\left((2f(0)-5)x^4-2(f(0)-4f(1)+f(2)-3)x^2+2f(2)-5\right)}{x(x^2-1)} - \frac{p^3}{4} =: A = 0$$

$$-\frac{1}{x^3(x^2-1)^3}2y(3x^{10}(4f(0) - 16g(0) + 32h(0) + 3) - 9x^8(4f(0) - 16g(0) + 32h(0) + 3) + 2x^6(26f(0) - 224f(1) - 14f(2) - 72g(0) + 384g(1) + 24g(2) + 144h(0) - 768h(1) - 48h(2) + 1) - 2x^4(14f(0) + 224f(1) - 26f(2) - 24g(0) - 384g(1) + 72g(2) + 48h(0) + 768h(1) - 144h(2) - 1) - 9x^2(4f(2) - 16g(2) + 32h(2) + 3) + 12f(2) - 48g(2) + 96h(2) + 9) - \frac{1}{x^2(x^2-1)^2}8p(3x^8(f(0) - g(0)) + x^6(-7f(0) + 4f(1) - f(2) + 7g(0) - 4g(1) + g(2)) + 5x^4(f(0) + 8f(1) + f(2) - g(0) - 8g(1) - g(2)) + x^2(-f(0) + 4f(1) - 7f(2) + g(0) - 4g(1) + 7g(2)) + 3(f(2) - g(2))) - \frac{1}{2x^2(x^2-1)^2}p^2y((2f(0) - 5)x^6 + (-4f(0) - 8f(1) + 2f(2) + 9)x^4 + (2f(0) - 8f(1) - 4f(2) + 9)x^2 + 2f(2) - 5) =: B = 0$$

It is easy to check that $D(A) = \partial B/\partial p$, so that these are the set of critical points of a potential function. Solving the differential equation gives the following expression:

$$A = \partial\psi/\partial p$$

$$B = D(\psi)$$

where

$$\psi = \frac{1}{2x^2(x^2-1)^2}(-24f(0)x^8 + 48f(0)x^6 + 8f(0)x^4 - 896f(1)x^4 - 56f(2)x^4 - 32f(0)x^2 + 80f(2)x^2 - 24f(2) + 96g(0)x^8 - 192g(0)x^6 + 96g(0)x^4 + 1536g(1)x^4 + 96g(2)x^4 - 192g(2)x^2 + 96g(2) - 192h(0)x^8 + 384h(0)x^6 - 192h(0)x^4 - 3072h(1)x^4 - 192h(2)x^4 + 384h(2)x^2 - 192h(2) - 18x^8 + 36x^6 - 68x^4 + 36x^2 - 18) + \frac{1}{2x^2(x^2-1)^2}p(-8f(0)x^6y + 16f(0)x^4y + 32f(1)x^4y - 8f(2)x^4y - 8f(0)x^2y + 32f(1)x^2y + 16f(2)x^2y - 8f(2)y + 8g(0)x^6y - 16g(0)x^4y - 32g(1)x^4y + 8g(2)x^4y + 8g(0)x^2y - 32g(1)x^2y - 16g(2)x^2y + 8g(2)y) + \frac{1}{2x^2(x^2-1)^2}p^2(-2f(0)x^7 + 4f(0)x^5 - 8f(1)x^5 + 2f(2)x^5 - 2f(0)x^3 + 8f(1)x^3 - 4f(2)x^3 + 2f(2)x + 5x^7 - 11x^5 + 11x^3 - 5x) - \frac{p^4}{16}$$

5 Future Work and Open Questions

- We would like to find out how many algebraically integrable operators exist for a certain symmetry group. In particular, does this number always coincide with the number of critical points of the classical crystallographic elliptic Calogero-Moser Hamiltonian?

- We conjecture that over a lemniscatic elliptic curve, algebraically integrable operators correspond to critical points of a potential if and only if there is only one orbit of poles that are not fixed points, in which case the potential is not exactly the Calogero-Moser Hamiltonian (see equation 4.3 in [2]). Does the same hold true for operators with sixfold symmetry over an equianharmonic elliptic curve?

- A similar analysis can be done for ℓth-order differential equations with rational coefficients that vanish at infinity, with ℓth-order rotational symmetry, for any fixed value of ℓ. In this case, trivial monodromy is equivalent to algebraic integrability. Is there a simple way to find out for which ℓ does algebraic integrability correspond to Calogero-Moser critical points?

- Is there a physical explanation for why algebraically integrable operators correspond to critical points of the classical crystallographic elliptic Calogero-Moser Hamiltonian for threefold symmetry, and is there a way to generalize it to fourfold symmetry?

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