# On Snowflakes and Pizza: Graph Theoretic Properties of 2D Steiner Solutions

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## Abstract

Steiner's plane cutting problem asks for configurations of lines in the plane that create the maximum number of regions (termed S-solutions). Following previous work, we associate to every S-solution a graph in the plane and establish graph theoretic properties common to these graphs. We also introduce a new technique for analyzing and constructing S-solutions, called the “snowflake transformation". This allows us to construct families of $S_n$-solutions where we can bound the maximum vertex degree between 4 and $n$, construct traceable diagrams with greater ease, and have large maximum independent sets. In future research, we want to generalize this to $n$ dimensions, as well as find necessary conditions for a given graph to be an S-solution.

## Summary

We explore the classic *pizza-cutting problem* from a graph theoretic standpoint by translating our “pizza" to a network of points and lines, with points corresponding to the regions we cut. We seek to find consistent graph theoretic properties among these graphs in order to find which ones are equivalent, and construct specific graphs with particular properties. We conclude by considering the possibility of extending the results of the research to higher dimensions (cutting a tall cake instead of a flat pizza).

# 1 Introduction

What is the greatest number of pieces one can divide a finite circular pizza into with $n$ straight cuts? Known as Steiner's plane-cutting problem, which asks for the maximum number of regions created by $n$ lines in the plane, if we define $\ell_n$ to be the maximum number of pieces for $n$ cuts, then $\ell_n = \frac{n(n+1)}{2} + 1$.

That might be the end of it, but perhaps there is more than meets the eye in this pizza pie. We can transform any sliced pizzas, dubbed $S$-*solutions*, into a graph $G$ by doing the following:

- The vertex set $V$ consists of the pieces of pizza (regions in the plane).
- Two vertices $p, q$ are connected by an edge if and only if their corresponding regions share a side.

Figure 1 [1] shows an example of such a transformation.

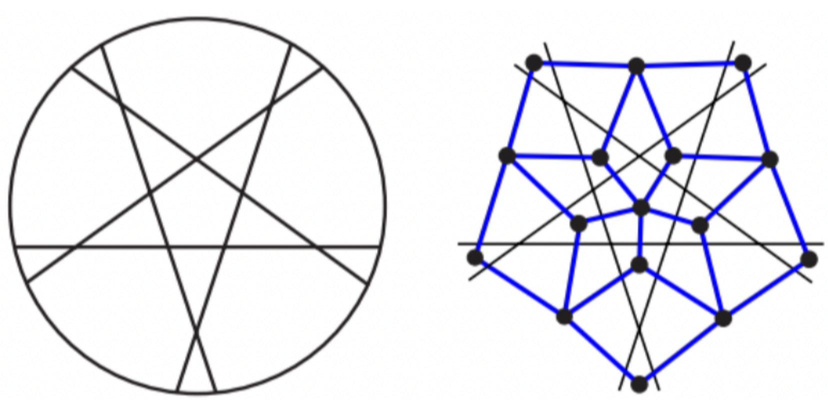

Figure 1: A $S_5$-solution and its graph.

The number of isomorphism classes of graphs of $S$-solutions with $n$ slices is an open problem for $n \geq 10$ [1]. The first nine values follow the sequence A090338 [2, 3]: 1, 1, 1, 1, 6, 43, 922, 38609, 3111341. With increasingly large graphs, isomorphism becomes difficult to check. As a tool for approaching this problem, we investigate and construct $S_n$ solutions with certain graph-theoretic properties that are preserved under isomorphism.

Previously, Steiner's problem was examined through the use of oriented matroids [4, 5]. In our case, we continue the work of Baril and Santos [1], who constructed $S$-solutions with and without Hamiltonian paths, as well as introducing an approach for analyzing $S$-solutions in terms of binary strings. In continuation of their work, we construct $S$-solutions with particular properties.

## 2 Preliminaries

We define some graph theory notations and conventions that we use in the remainder of the paper.

**Definition 2.1.** The *degree* of a vertex $v$ in a graph $G$, denoted $\deg(v)$ is the number of edges incident to it.

We also refer to the degree of a region in an S-solution as the degree of its corresponding vertex in the associated graph.

**Definition 2.2.** For a graph $G$ with vertex set $V$, we define $\Delta(G) = \max(\deg(V(G))$ and $\delta(G) = \min(\deg(V(G))$.

**Definition 2.3.** A graph is *Hamiltonian* if there exists a path that passes through each vertex exactly once. Such a path is called a *Hamiltonian path.*

**Definition 2.4.** A *bipartite* graph is a graph whose vertices can be sorted into exactly two disjoint sets, such that no two vertices in the same set are connected to each other.

**Definition 2.5.** A *maximum independent set* of a graph, denoted $\mathcal{S}$, is a subset (not necessarily unique) of vertices in the graph with maximum cardinality such that no two vertices in $\mathcal{S}$ are connected through an edge.

Now for terms specifically related to S-solutions.

**Definition 2.6.** An $S_n$-solution is an $S$-solution with $n$ slices.

**Definition 2.7.** An S-solution is *traceable* if its associated graph $G$ is Hamiltonian.

Constructing traceable S-solutions has many applications in computing [6], and also poses as a property that can distinguish between S-solutions, as they can be either traceable or nontraceable. [1].

**Definition 2.8.** An *interior region* of an S-solution is a region of the plane bounded entirely by lines, which will take the form of a convex polygon. Regions that are not interior regions are called *border regions*.

Since an $S_n$-solution has $n$ lines, there are $2n$ points on the circle that are the endpoints of these lines, dividing up the circle into $2n$ circular arcs. Each border region is bounded by one of these circular arcs. Thus, an $S_n$-solution has $2n$ border regions and $\ell_n - 2n$ interior regions.

## 3 Snowflake Diagrams

Now we define the notion of "snowflake diagrams", which are a crucial part of analyzing S-solutions.

**Definition 3.1.** A *snowflake diagram* of a $S_n$-solution is a diagram consisting of $n$ concurrent lines whose $2n$ sectors represent the $2n$ border regions of the $S_n$-solution.

Note that if we want to add a new line to an $S_n$ solution to create an $S_{n+1}$ solution by adding a line that avoids the interior regions, we can do so by thinking of the snowflake diagram as a zoomed out copy of the $S_n$ solution where all of its interior regions appear to converge on the point of concurrency. We can then analyze the $S_{n+1}$ solution by retaining

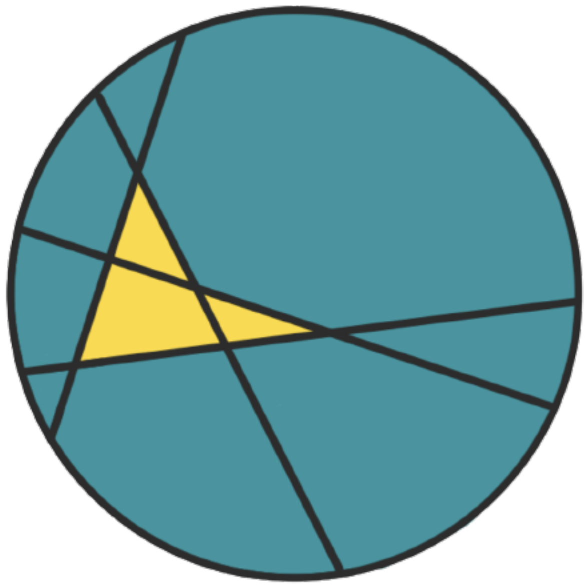

Figure 2: An $S_4$-solution, with border regions in blue and interior regions in yellow.

the features of the interior regions in the $S_n$ solution, and incorporating the new regions in the $S_{n+1}$ solution which will be encoded in the snowflake diagram.

There are many benefits when utilizing snowflake diagrams instead of normal S-solution diagrams. The number of regions to account for is significantly smaller, with $2n$ instead of $\ell_n$, growing linearly instead of quadratically. Adjacencies between border regions are also preserved. When an additional line is drawn, one can see exactly where the new interior regions are created without disturbing any that already exist. It is also easy to verify that the new line actually intersects all of the previously existing lines, unlike in a normal S-solution.

The most important aspect of the border regions that is *not* preserved in a snowflake transformation is their degrees: we remedy this by labelling each of the border regions with their degree.

**Definition 3.2.** Two border regions on a snowflake diagram are *diametrically opposite* if they are bounded by the same two lines.

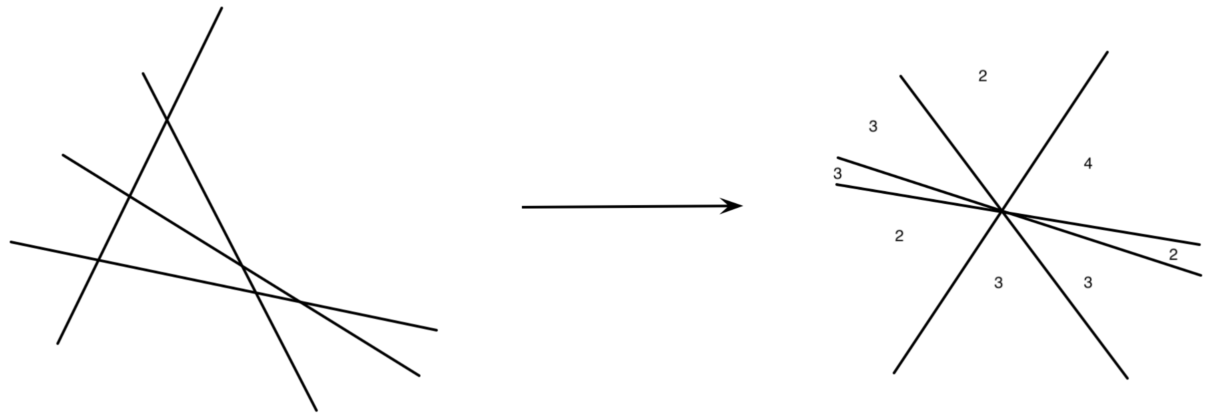


Figure 3: Snowflake transformation for $n = 4$. The degrees of the border regions are labelled.

In a snowflake diagram, two diametrically opposite regions will appear to be vertical angles. They need not have the same vertex degree, however.

# 4 Degrees

The first graph-theoretic property of $S_n$ solutions we will consider is vertex degree.

We refer to regions that are adjacent to $k$ regions as *k-regions*. $G_n$ refers to the graph corresponding to an arbitrary $S_n$-solution.

**Theorem 4.1.** *For all $n \geq 2$, $\delta(G_n) = 2$.*

*Proof.* Note that 2-regions exist only as border regions, as interior $k$-regions are convex $k$-gons for $k \geq 3$. Denote the center of the S-solution circle (or the pizza) as $O$, and denote the furthest intersection point from $O$ between two lines as $K$. Then it follows that the region bounded by those two lines and the circle is a 2-region, as if there were any other boundary, it would contradict our assumption that $K$ is the furthest intersection point from $O$. $\square$

**Theorem 4.2.** *Over all $S_n$-solutions, $\max(\Delta(G_n)) = n$.*

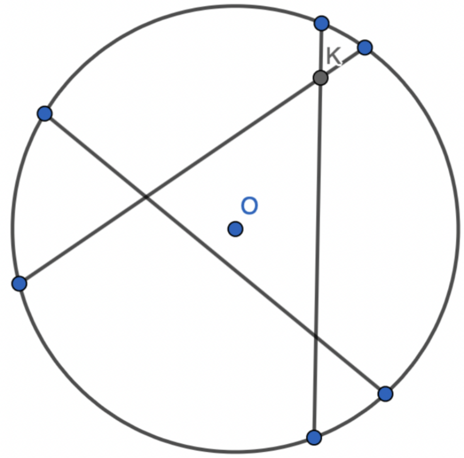


Figure 4: An $S_3$-solution with points defined as in Theorem 4.1.

*Proof.* The maximum possible degree of an $S_n$-solution is $n$, as a region in an $S_n$-solution can have at most $n$ sides since there are $n$ lines. Now we show that this is possible to obtain. Draw a $n$-gon such that no two sides are parallel. Then, extend all of the lines until each pair intersects. The circle that encompasses all of the intersections and regions is our desired pizza, forming the S-solution. □

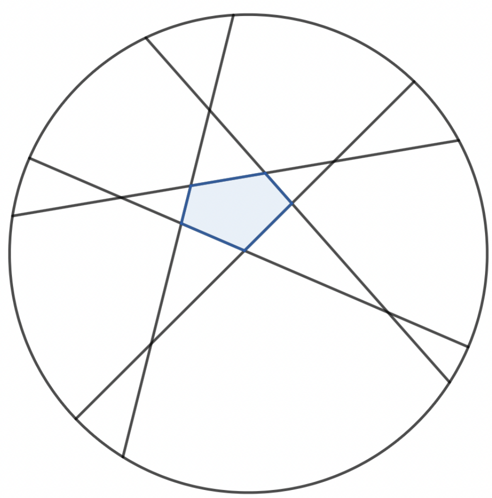

Figure 5: An $S_5$-solution with a 5-region.

**Theorem 4.3.** *Over all $S_n$-solutions with $n \geq 4$,* $\min(\Delta(G_n)) = 4$.

*Proof.* Firstly, note that there is only one isomorphism class for of $S_4$ solutions, which has a single vertex with degree 4. Since new lines do not eliminate any existing adjacencies in the regions, this maximum degree will never decrease. Thus, 4 is the lowest possible maximum degree.

We now construct a family $\mathcal{F}_n$ of $S_n$-solutions with associated graphs $G_n$ such that $\Delta(G_n) = 4$, using an inductive procedure. In our base case $n = 4$, our claim is true.

Now assume as inductive hypothesis that we have some $S_n$-solution with $\Delta(G_n) = 4$, whose snowflake diagram has the following properties:

- Assume that all border 4-regions lie on the same half, where a *half* of a snowflake diagram with $n$ lines is defined as a contiguous set of $n$ regions.
- There exists a 2-region $A$ and a 3-region $B$ that are diametrically opposite.
- Of the two sets of contiguous $n-1$ regions between $A$ and $B$ in the snowflake diagram, one of them does not contain any 4-regions.

We will add a new line to our $S_n$ solution to create an $S_{n+1}$ solution with these same properties.

Draw a new line on the $S_n$-snowflake diagram such that it passes through both $A$ and $B$, but does not pass through the half that the 4-regions are located in. This new line has the following properties (refer to Figure 6):

- It passes through $n+1$ border regions with degrees 2 and 3.
- For each of the $n-1$ border regions it enters and exits, it creates $n-1$ interior regions with degrees 3 or 4, and $n-1$ border regions all with degree 3. This is because all of the regions it passed through had either degrees 2 or 3, by our inductive hypothesis.

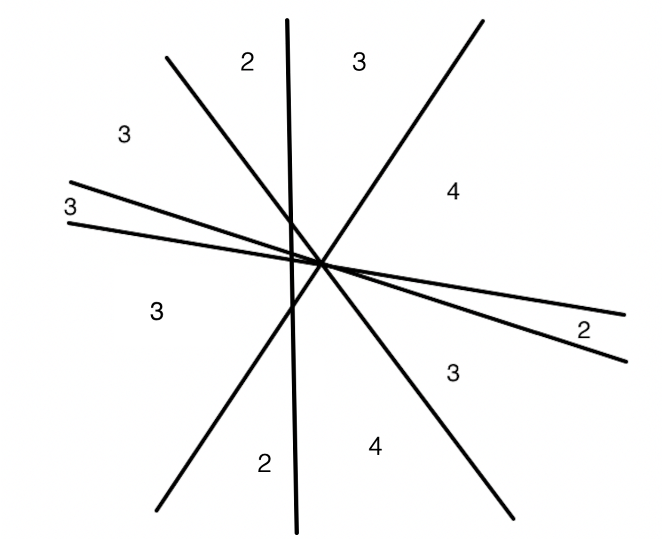


Figure 6: Adding a line to the $S_4$-solution from Figure 3. Note the locations of the newly created regions, and the new 4-region is on the right.

- Because the first and last regions it passes through are $A$ and $B$, it will split them into a 3 and 2-region and a 2 and 4-region respectively. The 3 region lying inside $A$ and the 2 region lying inside $B$ are diametrically opposite.

Note that the new 4-region lies on the same half as the other 4-regions, while the 3-region created from $A$, denoted $B'$, is diametrically opposite to the 2-region created from $B$, denoted $A'$.

Therefore, all of our properties are retained: the 4-regions are contained in a half and there is a diametrically oppositely paired 2 and 3-region with a stretch of no 4-regions between them. Therefore $\Delta(G_{n+1}) = 4$. □

We've successfully bounded the degrees, leading us to discover some other nice S-solution properties.

**Theorem 4.4.** *For any value a such that* $4 \leq a \leq k$*, there exists a graph with maximum degree* $a$*.*

*Proof.* Begin by constructing a graph with maximum degree 4 with $k - a + 4$ lines. With the

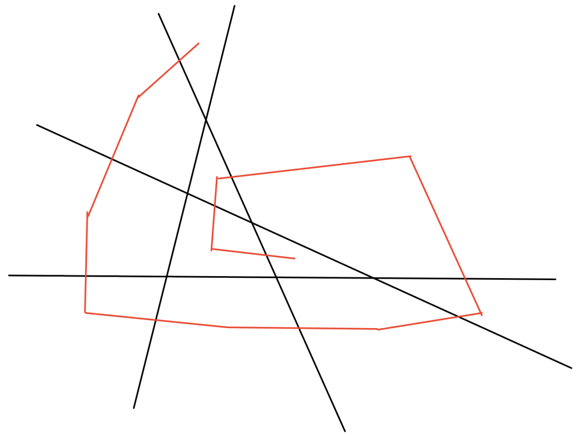

Figure 7: Hamiltonian path on $k = 4$

remaining $a - 4$ lines, draw them in a way such that a corner of the region(s) of maximum degree is/are cut, and as a result the maximum degree of the graph increases by 1 each time, bringing the maximum degree up to $a$, completing the construction. □

**Theorem 4.5.** *The family $\mathcal{F}_n$, constructed in Theorem 4.3, contains traceable S-solutions for all $n$.*

*Proof.* We show this result with induction. Note that there is only one isomorphism class for $k = 4$, and it is traceable, as shown in Figure 7: start from the interior regions and move out, ending in a 2-region on the northwest.

Now we assume that the graph created using Theorem 4.3 for $k - 1$ is traceable, and furthermore that the path always ends in $A$, the 2-region described in Theorem 4.3. In the inductive construction described in Theorem 4.3, the $k$th line $\ell$ is drawn in such a way that everything to one side of $\ell$ may be covered by the Hamiltonian path created for the $k - 1$ solution, just by shifting the path to lie entirely on one side of $\ell$ (see Figure 8). We want to extend this path to a Hamiltonian path; the regions on the other side of $\ell$ are the newly created regions that need to be covered. Since we ended in a 2-region that is about to be cut into a 2 and 3-region, we can continue our Hamiltonian path through the newly created

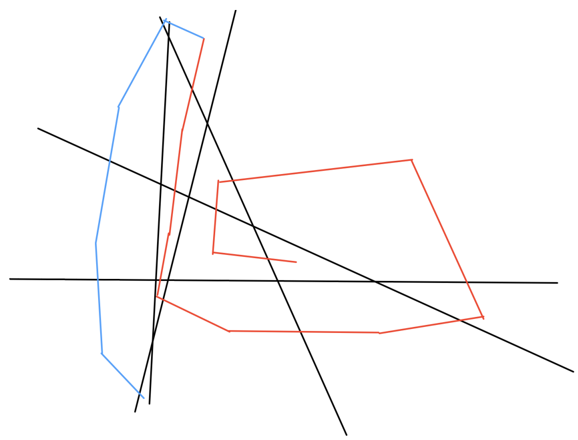

Figure 8: Continuation of Hamiltonian path (in blue)

border regions in a straight line, and end up in the new 2-region, denoted $A'$.

□

Note that this Hamiltonian construction is distinct from the one created previously in Theorem 2[1], as ours satisfies $\Delta(G_n) = 4$.

# 5 Maximum Independent Set

In this section, we construct a family of $S$-solutions with large maximum independent sets. In order to do this, we first establish a property shared by all graphs of $S$-solutions.

**Claim 5.1.** *Graphs corresponding to $S$-solutions are bipartite.*

*Proof.* Following the same binary definition as Baril and Santos [1], we assign every line in an S solution a label from 1 to $n$, and label each region with an $n$-digit binary string, where the $k^{th}$ digit is 0 or 1 if the region is on one side or the other of the line labelled $k$ – which side does not matter so long as they correspond to different digits. Then label vertices in the associated graph black or white depending on whether they correspond to regions with an even and odd number of 1's in their string. □

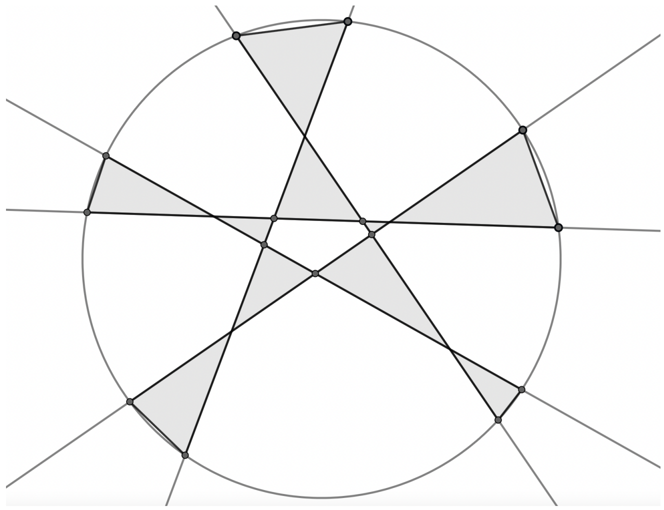

Figure 9: Checkerboarded $n = 5$ S-solution with 10 shaded regions out of 16 total.

**Definition 5.1.** A *checkerboarded* S-solution is an S-solution whose regions are colored black or white such that no two adjacent regions are the same color.

To construct a checkerboarded $S_n$-solution given a checkerboarded $S_{n-1}$-solution, draw a new line that passes through $n$ regions, then invert the color of every region on one side of the line. An example of this is depicted in Figure 9.

Note that for bipartite graphs, the maximum independent set is greater than or equal to the larger of the two disjoint vertex sets. In an S-solution, one can find those two sets by coloring a region in a checkerboard pattern. For Hamiltonian graphs, traverse the Hamiltonian path to obtain a maximum independent set of size $\lceil \frac{\ell_n}{2} \rceil$.

We thus want to investigate S-solutions with partitions that are particularly unbalanced, as these can lead to important properties with regards to cycle length and independent sets [7, 8].

For $n \leq 4$, all of the maximum independent set sizes are exactly $\lceil \frac{\ell_n}{2} \rceil$. Afterwards, though, we are able to deviate from half and half, as shown in Figure 9.

**Lemma 5.2.** *Suppose we have an $S_{n-1}$-solution with bipartite sets $\{B, W\}$, with $|B| > |W|$. We can add a line $\ell$ such that it begins in a white region and intersects only border regions,*

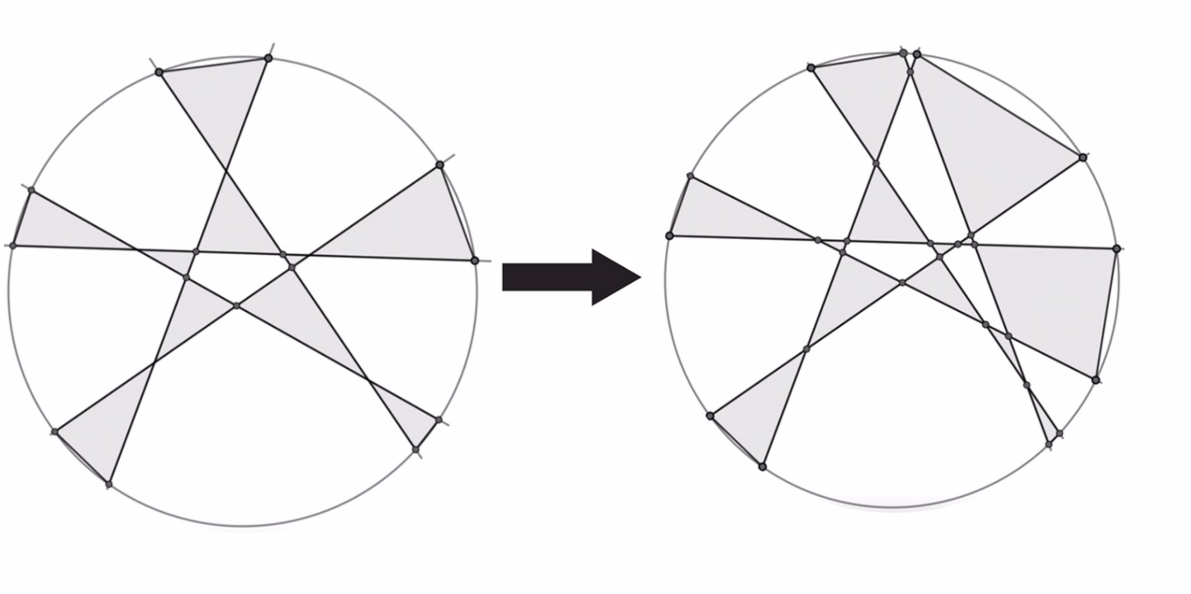

Figure 10: Adding a new line to a $S_5$-solution, gaining three new black regions.

*then invert the side of $\ell$ consisting only of border regions. Then for our new bipartite sets $\{B', W'\}$ we have*

$$|B'| = |B| + \left\lceil \frac{n}{2} \right\rceil .$$

*Proof.* Construct a snowflake diagram for this $S_n$-solution. Note that the $2n$ sectors of a snowflake diagram alternate between black and white since they are all adjacent. Draw $\ell$ such that it begins in a white region: as we invert $n$ regions total and begin in a white region, we will gain $\lceil \frac{n}{2} \rceil$ more black regions, giving us an increasingly unbalanced pair of bipartite sets for increasing $n$. □

We modify the construction given in Lemma 5.2 to increase the number of black regions more quickly.

**Lemma 5.3.** *For even $n$, when we draw a new line $\ell$ on an $n-1$ line checkerboarded snowflake diagram that only passes through border regions, we create a white 2-region bounded by $\ell$.*

*Proof.* Since our new line $\ell$ bounds $n$ adjacent border regions, it will start in a white region

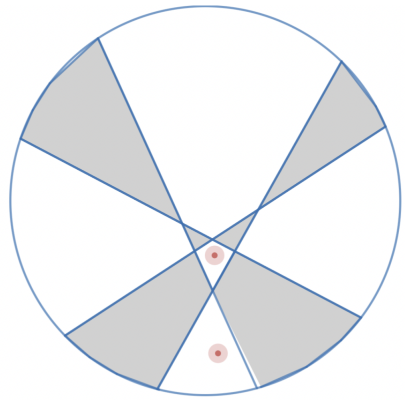

Figure 11: A checkerboarded $S_4$ solution, with dots in a 2-region and the region across from it.

and end in a black region (or vice versa). As seen in Theorem 4.1, drawing a new line this way constructs two 2-regions, in this case of opposite colors – thus one of them will be white.

□

This seemingly simple lemma is essential for achieving gains greater than $\lceil \frac{n}{2} \rceil$ regions per line.

**Lemma 5.4.** *For odd $n$, we can draw a new line $\ell'$ on an $n-1$ line checkerboarded diagram that is **not** a snowflake diagram such that it intersects a single interior region that is across from a white 2-region.*

*Proof.* By Lemma 5.3, the $S_{n-1}$-solution contains a 2-region, one of whose boundary lines separates $n-1$ border regions from the $\ell_{n-1} - (n-1)$ other regions. We define an interior region as being *across* from this 2-region if it contains the vertical angle that the two lines in the 2-region create, as shown in Figure 11. By virtue of being bounded by the same two lines as the 2-region, that interior region is adjacent to the same two border regions as the 2-region.

Like before we can draw a line $\ell$ that intersects only border regions, but here, because both the 2-region and the region across from it are white, we can translate $\ell$ to some line

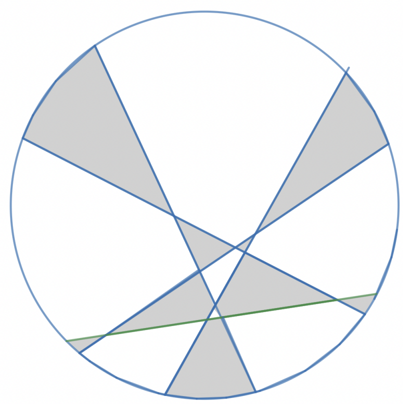

Figure 12: Figure 11 with $\ell$ drawn in green, then re-checkerboarded. Three new black regions are constructed.

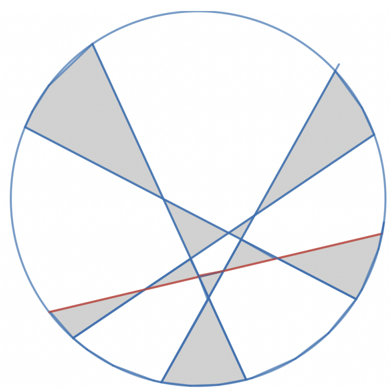

Figure 13: Figure 11 with $\ell'$ shown in red. Note that it only passes through one interior region and doesn't touch the 2-region across from it at all. Four new black regions are constructed.

$\ell'$, whose positioning is described as follows. It passes through the same exterior regions as $\ell$ until $\ell$ passes through the exterior white 2-region $U$. $\ell'$ instead passes through the single white interior region across from $U$, then continues through the remaining border regions that would have been passed through by $\ell$. As a result, this new line $\ell'$ will have only one region on that side, the 2-region, that is not bounded by it.

□

As a result of this construction, we obtain:

**Corollary 1.** *Once we've drawn $\ell'$, by inverting colors on the side of $\ell'$ that contains only* *one interior region, we gain $\lceil \frac{n}{2} \rceil$ black regions, since $\ell'$ will pass through $n$ regions total, and a $+1$ from the starred white border region.*

Note that this procedure we have described is in some sense optimal, because it allows us to cut off a white 2-region, for a gain of 1, without "losing" any black regions, because we only slice into the interior once. If we try to cut off more white 2-regions at once we have to slice further into the interior, and this makes it hard to predict how many black regions will be inadvertently inverted in the process.

**Theorem 5.5.** *There exist $S$-solutions whose bipartite graphs $(B, W)$ have*

$$\max(|B|, |W|) = \frac{\ell_n + 3\lceil \frac{n}{2} \rceil - 5}{2}$$

.

We can combine Lemmas 5.2 and 5.4, as well as Corollary 5.5, to optimize the number of black regions we gain. For $n = 4$ let $(B, W)$ be a checkerboarded diagram of the $S_4$ solution that maximizes the number of black regions. Inductively proceed as follows. For even $n$ we construct $\frac{n}{2}$ new black regions through Lemma 5.2, as well as a white 2-region guaranteed by Lemma 5.3. For odd $n$, we create $\lceil \frac{n}{2} \rceil + 1$ by applying the construction in Lemma 5.4 for a gain of $\lceil \frac{n}{2} \rceil + 1$ by Corollary 5.5. Recursively, this is shown in (1): Let $B$ be the number of black regions at step $n$, and let $B'$ be the number of black regions at step $n + 1$. Then altogether we have

$$|B'| - |B| = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \lceil \frac{n}{2} \rceil + 1 & \text{if } n \text{ is odd.} \end{cases} \quad . \tag{1}$$

Thus we count the number of black regions in our checkerboarded $S_n$-solutions created in this way: Combining this recursive form with our maximum independent sets for $n \leq 4$ and we obtain the following:

$$|B| = \frac{\ell_n + 3\lceil \frac{n}{2} \rceil - 5}{2}.$$

As a corollary we see that the size of the maximal independent set is bounded from below by

$$\max(S_n) \geq \frac{\ell_n + 3\lceil \frac{n}{2} \rceil - 5}{2}.$$

# 6 Future Work

After establishing the graph theoretic properties of S-solutions on a plane, we want to extend these to higher dimensions, with a 3D space being divided by planes (dubbed the cake-slicing problem) or a 4D space being cut by hyperplanes. This may eventually help determine the number of isomorphism classes for S-solutions in $n$ dimensions.

We would also like to either strengthen our lower bound of the cardinality of the maximum independent set or verify that it is in fact the closed form for it.

Lastly, we would like to examine the properties of a given graph and determine whether or not it can be transformed into an S-solution: what sorts of necessary conditions must be satisfied, beyond the ones already delineated?

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