Investigation on the Johnson-Leader-Russell Question for Square Posets

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Abstract

We study a problem proposed by Johnson, Leader, and Russell. Given positive integers n and k, we aim to find the maximum number of maximal chains in a subset with size k of a square poset $\mathfrak{P} = \{1, 2, ..., n\}^2$. Kittipassorn made progress on this problem by solving a stronger case of which the number of elements in each level is also given. With his work, we find the exact solution for $1 \leq k \leq 3n - 2$. For general k, we find that the optimal configuration is given by a 1-Lipschitz function, i.e. the difference between the number of elements in two consecutive levels is at most 1.

Summary

We study the following problem: given an $n \times n$ grid and an integer k, among all the configurations of selecting k points in the grid, what is the maximum number of paths going from one end to the other that can only pass through the selected points? We solve the problem for small values of k. For general k, we find a constraint for a configuration to have a maximum number of paths: the difference between the number of elements in adjacent levels is at most 1.

1 Introduction

Given the proportion of size of a subset in the power set of $\{1, 2, ..., n\}$, Johnson, Leader, and Russell [1] solved asymptotically the maximum number of maximal chains in such a subset. At the end of the paper, they considered a variant of the problem of which the poset is $\mathfrak{P} = \{1, 2, ..., n\}^2$. They asked the following question, with $P(T)$ denoting the number of maximal chains in T.

Question 1.1 (Question 9 [1])
Given an integer k with $0 \leq k \leq n^2$. Let T be a subset of the poset $\mathfrak{P}$. What is

$$\max_{T:|T|=k} P(T)?$$

That is, we choose k elements in the poset $\mathfrak{P}$ to form a subset such that aiming for the greatest number of maximal chains. We can represent the poset $\mathfrak{P} = \{1, 2, ..., n\}^2$ by an $n \times n$ square grid. And thus a chain can be represented as a line connecting the elements in $\mathfrak{P}$.

For example, as in Figure 1, we select the blue elements. The red line shows one of the maximal chains in $T \subseteq \mathfrak{P}$.

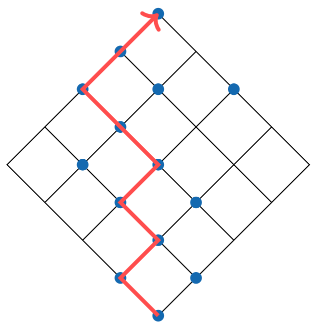

Figure 1: A maximal chain in $T \subset \mathfrak{P}$

Therefore, an simpler yet equivalent statement to the problem is as follows: given an $n \times n$ grid and an integer k, among all configurations of selecting k points in the grid, what is the maximum number of paths going from one end to the other that only passes through the selected points?

In this paper, we aim to answer Question 1.1 and give the optimal configuration of points. In contrast to Johnson, Leader, and Russell's paper of finding the asymptotic solution, we aim to find the exact value of the maximization $\max P(T)$.

In Section 2.2, we review on some progress made by Kittipassorn [3]. He considered a strong variant of the problem: given the number of points in each level $r_1, r_2, \ldots, r_{2n-1}$ instead of the total number of points in the grid k. He gave the unique optimal configuration. Therefore, we only need to consider Kittipassorn's configuration for any n and k.

Our paper gives the maximum number of maximal chains and the optimal configuration for the case of $k \leq 3n-2$. We also find an upper bound of $P(T)$ for general k. At last, we find a constraint to partition k into levels: the difference between number of points in adjacent levels is at most 1, i.e. it is given by a 1-Lipschitz function. We then summarize our results with a pseudo-code to compute the maximum $\max P(T)$ over all configuration T such that $|T| = k$.

2 Background

2.1 Preliminaries

We first review on some terminology and definitions from poset theory. For reference, we refer to Richard P Stanley's *Enumerative Combinatorics* [2].

Let n be a positive integer. We write $[n]$ to denote the set $\{1, 2, ..., n\}$. Consider the poset $(\mathfrak{P}, \succeq)$ where

$$\mathfrak{P} := [n]^2 = \{(i,j) : i, j \in [n], \}$$

and relation $\succeq$ is defined by $(i,j) \succeq (i', j')$ if $i \geq i'$ and $j \geq j'$.

Now we introduce some definitions that will be frequently used in this paper.

A **chain** in $\mathfrak{P}$ is a subset of $\mathfrak{P}$ in which any two elements are comparable. A **maximal chain** in $\mathfrak{P}$ is a chain in $\mathfrak{P}$ with $2n-1$ elements.

We can partition the poset $\mathfrak{P}$ into **levels** by

$$\mathfrak{P} = \bigsqcup_{d=1}^{2n-1} L_d$$

where $L_d = \{(i,j) : i + j = d + 1\}$. Notice that in the standard convention, the elements in the d-th level are exactly the elements with rank $d-1$. A maximal chain is thus a chain with exactly one element from each level L_d, where $d = 1, 2, \ldots, 2n-1$.

2.2 Kittipassorn's configuration

Teeradej Kittipassorn [3] considered a stronger case in which the number of elements in *each level* of T is also given, namely $r_1, r_2, ..., r_{2n-1}$. More precisely, we have the following variant of Johnson-Leader-Russell question: given $r_1, r_2, \ldots, r_{2n-1}$, what is $\max P(T)$, where the maximization is over all configurations T such that $r_i = |T \cap L_i|$, for all $1 \leq i \leq 2n-1$? This has been solved by Kittipassorn. In the following, we will describe his solution to the problem.

Given $r_1, r_2, \ldots, r_{2n-1}$, Kittipassorn considered the following configuration $T^*(r_1, r_2, \ldots, r_{2n-1})$:

$$T^*(r_1, r_2, \ldots, r_{2n-1}) := \bigcup_{h=1}^{2n-1} \left\{ \left(\frac{h+1}{2} + t, \frac{h+1}{2} - t \right) : t = \alpha_r, \alpha_{h+1}, ..., \beta_r \right\},$$

where for each $h = 1, 2, ..., 2n-1$, the numbers α_h and β_h are unique real numbers such that

$$\alpha_h + \beta_h \in \{0, 1\}, \qquad \beta_h - \alpha_h + 1 = k_h, \qquad \text{and} \qquad h + 2\alpha_h \text{ is an odd integer.}$$

Another way to describe the configuration $T^*(r_1, r_2, \ldots, r_{2n-1})$ is that it is the unique configuration satisfying the following conditions:

1. For each level, all the points are condensed in the middle.

2. If we have to break the left-right symmetry, all the extra points are put on the right.

In this paper, we call such a configuration *Kittipassorn's configuration*. We will use a shorthand notation $P(r_1, r_2, \ldots, r_{2n-1}) = P(T^*(r_1, r_2, \ldots, r_{2n-1}))$. The following example demonstrates how we form the Kittipassorn's configuration.

Example 2.1

Figure 2 shows Kittipassorn's configuration $T^*(1, 1, 2, 3, 1, 2, 1)$ when given $n = 4$ and $r_1 = 1$, $r_2 = 1$, $r_3 = 2$, $r_4 = 3$, $r_5 = 1$, $r_6 = 2$, and $r_7 = 1$ with the extra points (colored red) are put on the right.

The number of maximal chains in this configuration is 6, i.e. $P(T^*(1, 1, 2, 3, 1, 2)) = 6$.

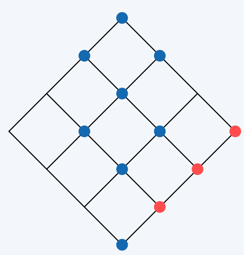

Figure 2: $T^*(1, 1, 2, 3, 1, 2, 1)$

Kittipassorn [3] proves that such configuration has the greatest number of maximal chains with given numbers of elements in each level $r_1, r_2, ..., r_{2n-1}$. We phrase this as the following lemma, which solves the variant question by providing the unique configuration with the maximum number of maximal chains.

Lemma 2.2 (Kittipassorn's lemma [3])
Suppose that non-negative $r_1, r_2, \ldots, r_{2n-1}$ are given. Then, we have

$$\max_{T\,:\,\forall i,\,|T\cap L_i|=r_i} P(T) = P(r_1, r_2, \ldots, r_{2n-1}).$$

With Kittipassorn's lemma, in order to compute the maximum

$$\max_{T\,:\,|T|=k} P(T)$$

over all configurations T such that $|T| = k$, it suffices to consider only the configurations which are Kittipassorn's configurations.

Moreover, Kittipassorn [3] also proposed two conjectures about the original Johnson-Leader-Russell problem of square posets. One of them is a strong conjecture which gives the solution to the original problem, and the other one is a weaker version of it.

Before introducing the conjectures, let us first introduce some notation. Previously, we partition $\mathfrak{P}$ into $2n-1$ parts $L_1, L_2, ..., L_{2n-1}$ by the vertical positions of elements. Now we partition $\mathfrak{P}$ by the horizontal positions. For each $d = -(n-1), ..., n-1$, we can partition the $\mathfrak{P}$ into **columns** by

$$C_d := \{(i, j) \in \mathfrak{P} : j - i = d\}.$$

For convenience, let us call a configuration $T \subseteq \mathfrak{P}$ **optimal** if

$$P(T) = \max_{S\,:\,|S|=|T|} P(S).$$

Now we can start introducing Kittipassorn's conjectures. The first conjecture suggests that we should first fill the points in the middle columns.

Conjecture 2.3 (Kittipassorn's weak conjecture [3])
There exists a sequence of sets $U_{2n-1} \subseteq U_{2n} \subseteq \cdots \subseteq U_{n^2} = \mathfrak{P}$ such that for each i, the difference $U_{i+1} - U_i$ is a singleton, and the set U_i is an optimal configuration, such that for each $t = 1, 2, \ldots n-1$,

$$U_{n+2(n-1)+2(n-2)+\cdots+2t} = \bigcup_{-(n-t)\leq d\leq n-t} C_d,$$

and

$$U_{n+2(n-1)+2(n-2)+\cdots+2(t+1)+t} = \bigcup_{-(n-t)\leq d\leq n-t} C_d.$$

Kittipassorn also proposed a stronger conjecture, which gives a conjectural answer to the original Johnson-Leader-Russell problem, for all n and $1 \leq k \leq n^2$.

Conjecture 2.4 (Kittipassorn's strong conjecture [3])
Let n be a positive integer. We define the sequence $T_1, T_2, \ldots, T_{n^2}$ of Kittipassorn's configurations by adding one point at a time so that each T_i has exactly i points. To add points from T_1 to T_{n^2}, we fill the columns in the following order:

$$C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots, C_n, C_{-n},$$

and in each column, we fill the points from bottom to top (see an example as in Figure 3). Given $1 \leq k \leq n^2$, we have

$$\max_{T\,:\,|T|=k} P(T) = P(T_k).$$

For example, Figure 3 shows the order of adding the points from T_1 to T_{16} according to Conjecture 2.4 when $n = 4$. For example, the configuration T_{11} contains the 11 points labeled 1 to 11.

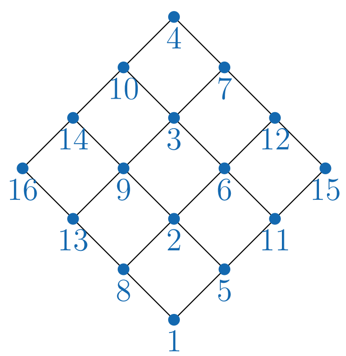

Figure 3: The order of points added from T_1 to T_{16} when $n = 4$

Notice that Kittipassorn's strong conjecture implies the following formula for $2n-1 \leq k \leq 3n-2$:

$$\max_{T\ :\ |T|=k} P(T) = 2^{k-2n+1}.$$

It also implies the following formula for $3n-1 \leq k \leq 4n-4$:

$$\max_{T:|T|=k} P(T) = 2^{4n-k-4} F_{2k-6n+7}$$

where F_i denotes the i-th Fibonacci number. Recall that the Fibonacci numbers are given by $F_0 = 1$, $F_1 = 1$, and for $i \geq 2$, we have $F_i := F_{i-1} + F_{i-2}$.

It was observed by Tanya Khovanova that for the case $k = 3n + c$ for a fixed integer $c \geq -1$, the maximization $\max_{T:|T|=k} P(T)$ appears to double whenever n is increased by 1, for $n \geq c+4$. This "doubling" phenomenon can be explained in view of Kittipassorn's strong conjecture. Thus we give the following remark.

Remark 2.5
Let $c \geq -1$ be a fixed integer. The above observation implies the following conjectural formula. For all integers $n \geq c+4$,

$$\max_{\substack{T \subseteq [n]^2 \\ |T|=3n+c}} P(T) = 2^{n-c-4} F_{2c+7}.$$

Notice that when $k = 3n + c$, we have a "doubling" phenomenon. If $n \geq c+4$, then when n is increased by 1, we have that $\max P(T)$ is doubled.

In this paper, we aim to to prove the two Kittipassorn's conjectures. In Subsection 3.1, we verify Conjecture 2.4 for $1 \leq k \leq 3n-2$. Moreover, in Subsection 4.2, we verify that Conjecture 2.4 is indeed true for all $3n-1 \leq k \leq n^2$ and $n = 1, 2, ..., 6$ with a computer program.

3 Main Results

3.1 The case $2n-1 \leq k \leq 3n-2$

To begin our investigation, we start off by some small values of k. With the notice that every maximal chains contains exactly one element in each level, we have $k \geq 2n-1$; otherwise there exist no maximal chains. And when $k = 2n-1$, there is at most one maximal chain.

In Subsection 2.2, we find that Kittipassorn's configuration implies the explicit form of the greatest number of maximal chains when $k \leq 3n-2$:

Proposition 3.1
If $2n-1 \leq k \leq 3n-2$, then

$$\max_{T:|T|=k} P(T) = 2^{k-2n+1}.$$

Proof. Consider any configuration C with k elements. Let $r_i = |C \cap L_i|$, i.e. the number of elements in the i-th level, we have $r_1 + r_2 + \cdots + r_{2n-1} = k$.

Notice that $P(C) \leq r_1 r_2 \cdots r_{2n-1}$. First, if some $r_i = 0$, then $P(C) = 0$. Therefore, assume $r_i \geq 1$ for all $i = 1, 2, \ldots, 2n-1$, we have $P(C) \leq r_1 r_2 \cdots r_{2n-1}$. And as r_i are non-negative integers, $r_1 r_2 \cdots r_{2n-1}$ attains its maximum when there are exactly $k-2n+1$ 2's and $4n-k-2$ 1's by AM-GM inequality. Thus,

$$P(C) \leq 2^{k-2n+1} \cdot 1^{4n-k-2} = 2^{k-2n+1}$$

as required.

For the construction, the optimal configuration has exactly one element in odd order of levels and at most two elements in even order of levels. Also, the optimal configuration is a Kittipassorn's configuration. Notice that total number of elements in such a configuration is at least $2n-1$ and at most $3n-2$. □

Example 3.2

When $n = 5$ and $k = 11$, Figure 4 gives a construction that attains the upper bound $P(T) = 2^{11-2\times5+1} = 4$.

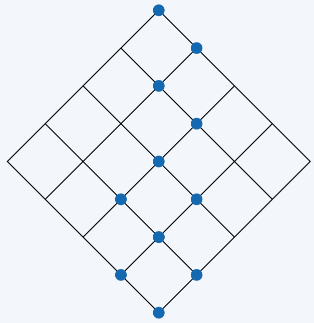

Figure 4: A configuration T such that $P(T) = 4$

3.2 Investigation on general k

In this section, we try to generalize the previous results to other values of k. The first proposition we give considers the upper bound of $P(T)$ for all $0 \leq k \leq n^2$.

Proposition 3.3

For $0 \leq k \leq n^2$, we have

$$P(T) \leq 2^{k-2n+1}$$

where $|T| = k$.

Proof. First consider the case when $0 \leq k \leq 2n - 2$. We cannot arrange the elements such that every level has at least one element. Therefore, $P(T) = 0$.

Now for the case when $2n - 1 \leq k \leq n^2$, notice that each element in the i-the level can form at most two maximal chains with elements in the $(i + 1)$-th level (either left or right). Thus the upper bound of $P(T)$ is 2^{k-2n+1}. □

However, notice that the equality only holds when $2n - 1 \leq k \leq 3n - 2$. Also, the bound is not a sharp bound for large k because the r_i elements in the i-th level can only form maximal chains with at most $r_i + 1$ elements in the next level.

This gives us the idea to consider the number of elements in consecutive levels. An intuitive idea is that no points should be "wasted". In other words, if there are more than $r_i + 1$ elements in the $(i + 1)$-th level, then some elements are not in any maximal chain in Kittipassorn's configuration. Similarly, elements are "wasted" if there are less than $r_i - 1$ elements in the $(i - 1)$-th level.

Therefore, removing the "wasted" points will not affect the number of maximal chains in a configuration. On the other hand, as k is fixed, the removed points should be put in other places to improve the configuration. In the following lemma, we develop an algorithm to construct the new position of the removed points such that the number of maximal chains is increased.

Lemma 3.4
Let $T \subsetneq \mathfrak{P}$ be a configuration such that $P(T) > 0$. Then there exists $v \in \mathfrak{P} - T$ such that

$$P(T) < P(T \cup \{v\}).$$

Proof. As $P(T) > 0$, there exists a maximal chain $\mathfrak{m} \subseteq T$. On the other hand, as $T \neq \mathfrak{P}$, there exists $v \in \mathfrak{P} - T$. As $v \notin \mathfrak{m}$, there are two cases: $\mathfrak{m}$ is to the left of v or $\mathfrak{m}$ is to the right of v.

First consider the case when $\mathfrak{m}$ is to the left of v (as in Figure 5).

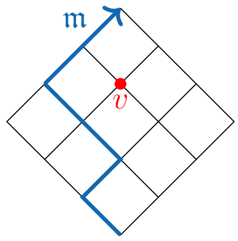

Figure 5: $\mathfrak{m}$ to the left of v

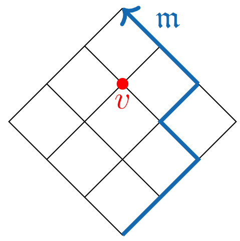

Figure 6: $\mathfrak{m}$ is to the right of v

Thus the set $U := \{\text{maximal chains } \mathfrak{m} \subseteq T : \mathfrak{m} \text{ is to the left of } v\}$ is non-empty.

Define a function $\mathbf{Area} : U \to \mathbb{Z}_{\geq 0}$ which maps a maximal chain to the number of points in $\mathfrak{P}$ on its left. Since L is a non-empty finite set, the image $\mathbf{Area}(U)$ is a finite non-empty subset of $\mathbb{Z}_{\geq 0}$. Let $B := \max(\mathbf{Area}(U))$. Then there exists a maximal chain $\mathfrak{m}^* \in L$ such that $\mathbf{Area}(\mathfrak{m}^*) = B$.

Notice that $\mathfrak{m}^*$ cannot be the right boundary of P because v is on its right. Hence, there exist $v_1, v_2, v_3 \in \mathfrak{m}^*$ as on Figure 7.

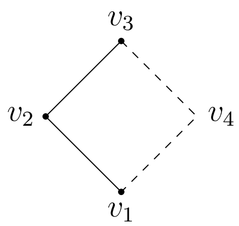

Figure 7: The existence of $v_1, v_2, v_3 \in \mathfrak{m}^*$

Now we want to show that $v_4 \notin T$. Suppose, for the sake of contradiction, that $v_4 \in T$, then $v_4 \neq v$ because $v \notin T$. This means that there exists a maximal chain $\mathfrak{m}^{**} = (\mathfrak{m}^* - \{v_2\}) \cap \{v_4\}$ which is also to the left of v. Thus $\mathfrak{m}^{**} \in U$. However, $\mathbf{Area}(\mathfrak{m}^{**}) = B + 1$, which contradicts to the maximality of $\mathbf{Area}(\mathfrak{m}^*)$. Therefore, we have $v_4 \notin T$ and $P(T \cup \{v_4\}) > P(T)$.

The other case where $\mathfrak{m}$ is to the right of v is proven similarly. And this completes the proof. □

Here we introduce some shorthand notations. Recall that given $r_1, r_2, \ldots, r_{2n-1}$, we have defined a unique Kittipassorn's configuration, denoted by $T^*(r_1, r_2, \ldots, r_{2n-1})$. Now for any $f : \{1, 2, \ldots, 2n-1\} \to \mathbb{Z}_{\geq 0}$ such that $0 \leq f(i) \leq |L_i|$, we define $T^*(f)$ to be

$$T^*(f) := T^*(f(1), f(2), ..., f(2n-1).$$

We also define $P(f)$ to be

$$P(f) := P(T^*(f)).$$

Then we introduce the following theorem by considering r_i in consecutive levels. But before that, we give a definition of 1-Lipschitzness: a function $f : \mathbb{Z} \to \mathbb{R}$ is said to be 1-Lipschitz if for all $n \in \mathbb{Z}$, we have

$$|f(n+1) - f(n)| \leq 1.$$

Theorem 3.5

For given k such that $2n - 1 \leq k \leq n^2$,

$$\max_{T:|T|=k} P(T) = \max_{\sum f = k} P(f)$$

where the maximization of $P(f)$ is over all 1-Lipschitz functions $f : \{1, 2, ..., 2n-1\} \to \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{2n-1} f(i) = k$.

Proof. First notice that $\max P(T) \geq \max P(f)$ can be shown by the fact that the set of all $T^*(f)$ is a subset of the set of all T with $|T| = k$.

Now for $\max P(T) \leq \max P(f)$. Assume that $P(T) > 0$, otherwise $\max P(T) = \max P(f) = 0$. In Kittipassorn's configuration, we can see that if the function is not 1-Lipchitz. Then there exists

$v \in T$ that is not in any maximal chain. Hence we can remove it without changing $P(T)$, i.e.

$$P(T - \{v\}) = P(T).$$

After removing v, we have $T \subsetneq P$. By Lemma 3.4, there exists $u \in P - T$ such that $P(T \cup \{u\}) > P(T)$. Also the algorithm of selecting such u in Lemma 3.4 ensures that the new configuration is given by a 1-Lipschitz function. □

The following example shows how we implement the algorithm in Lemma 3.4 to improve a configuration T so that it is 1-Lipschitz.

Example 3.6

Given $n = 4$ and $k = 12$, consider a configuration as in Figure 8. Notice that the red point v is not in any of the maximal chains. Hence we can remove it without affecting the value of $P(T)$. Figure 9 shows the maximal chain $\mathfrak{m}^*$ with the greatest **Area**. By the algorithm in Lemma 3.4, we pick $u \notin T$. Notice that selecting u increases the number of maximal chains. Now we have the configuration in Figure 10 is given by a 1-Lipschitz function and $|T|$ is unchanged.

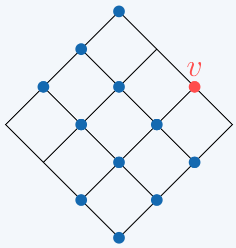

Figure 8: A configuration with an “wasted” point v

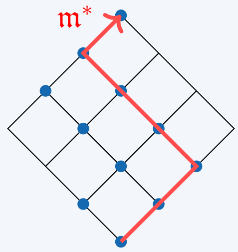

Figure 9: The maximal chain $\mathfrak{m}^*$ with the greatest **Area**

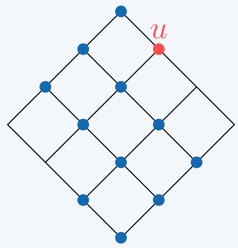

Figure 10: The improved configuration

4 Computational Results

4.1 Pseudo-code

Using the results obtained above, we can develop a much more efficient computer program to compute $\max P(T)$ than exhausting the all the possibilities. By Lemma 2.2 and Theorem 3.5, we reduce the runtime of the program by considering the following three constraints:

1. The sum of numbers of elements in the levels is k, i.e. $r_1 + r_2 + \cdots + r_{2n-1} = k$.
2. For all $2 \leq i \leq 2n-1$, $|r_i - r_{i-1}| \leq 1$.
3. The configuration has to be Kittipassorn's configuration.

Before showing the pseudo-code, we will first explain the idea of the program. From line 1 to 33, a function `chain` is defined by inputting n and $r_1, r_2, ..., r_{2n-1}$ and outputting $P(r_1, r_2, ..., r_n)$. Recall that $P(r_1, r_2, ..., r_n)$ is defined in Section 2.2. From line 34 to 51, we partition the given k into $2n-1$ parts: $r_1, r_2, \ldots, r_{2n-1}$ given by a 1-Lipschitz function (checked by a Boolean variable `bo`), then try out all possible Kittipassorn's configurations to find the optimal configuration.

Notice that each point is indexed by (i, j), where i is the order of level and j is the order counting from the left. For example, a 4×4 poset is indexed as:

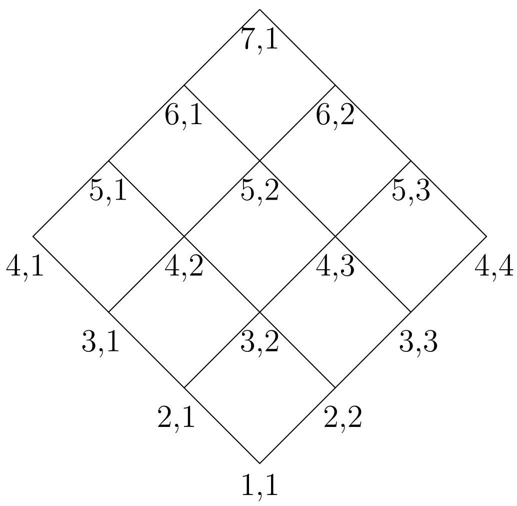

Figure 11: The indexes of points in a 4×4 poset

```
FUNCTION chain (positive integer n, vector (r[1], r[2],..., r[2n-1]))
     a: 2D array [0..(2n-1),0..n] of non-negative integers
     b: 2D array [0..(2n-1),0..n] of boolean
     z: positive integer

     for i from 1 to 2n-1 do
          if r[i] != 0 then
            if i <= n then x := ceil(i/2)
            else x := ceil((2n-i)/2
            if (i mod 2 = 0) then x := x+1
            if (r[i] mod 2 = 0) and (i mod 2 = 0) then
                z := ceil(r[i]/2)
            else z := floor(r[i]/2)+1
            y:=x-1

            for j from 1 to z do
              b[i][x] := TRUE
              x := x+1
            for j from 1 to (r[i]-z) do
              b[i][y] := TRUE
                y := y-1
          else b[i][j] := FALSE

     a[1][1] := 1
     for i from 1 to 2n-1 do
          for j from 1 to n do
          if b[i][j] then
              if i<=n then
               a[i][j] := a[i-1][j-1] +  a[i-1][j]
            else a[i][j] := a[i-1][j] + a[i-1][j+1]
       else a[i][j] := 0
     RETURN a[2n-1][1]

MAIN
     INPUT positive integers n, k
     max, count: non-negative integer
     bo: boolean
```

```

  bo := TRUE
  r[1] := 1
  r[2n-1] := 1
  Partition k-2 into 2n-3 parts: r[2], r[3], ..., r[2n-2]
  for each partition do
      for each i from 2 to 2n-1,
          if abs(r[i]-r[i-1])>1 then
             bo := FALSE
     if bo then
         count := chain(n,r[1], r[2], ..., r[2n-1])
     if max < count then max := count

  OUTPUT max
```

4.2 Numerical results

The computational result of $\max_{|T|=k} P(T)$ when $3 \le n \le 6$ and $3n-1 \le k \le n^2$ is as shown in the following tables. However, this program is not the same as the pseudo-code in Subection 4.1. It is a brute-force program that does not use any result in this paper. Notice that the results supports the conjectural formulae in Conjecture 2.4 and Remark 2.5.

n	k	$\max P(T)$
3	8	5
	9	6
4	11	10
	12	13
	13	15
	14	18
	15	19
	16	20
5	14	20
	15	26

n	k	$\max P(T)$
5	16	34
	17	39
	18	45
	19	54
	20	57
	21	61
	22	64
	23	68
	24	69
	25	70

n	k	$\max P(T)$
6	17	40
	18	52
	19	68
	20	89
	21	102
	22	117
	23	135
	24	162
	25	171
	26	183

n	k	$\max P(T)$
6	27	197
	28	206
	29	218
	30	232
	31	236
	32	241
	33	245
	34	250
	35	251
	36	252

Table 1: $\max\limits_{|T|=k} P(T)$ when $3 \le n \le 6$ and $3n-1 \le k \le n^2$

5 Conclusion & Future work

We gave the maximum number of maximal chains and the construction of optimal configuration for $1 \le k \le 3n-2$. We also found an upper bound of $P(T)$ for general k. At last, we find the difference between number of elements in consecutive levels in optimal configurations at most 1 for all $1 \le k \le n^2$, i.e. it is given by a 1-Lipschitz function. We then summarize our results with a pseudo-code to effectively compute $\max P(T)$ over all configurations T such that $|T| = k$.

One possible direction for future work is to prove the Kittipassorn's Conjectures 2.3 and 2.4. Another direction is to take a closer look at the numerical values of $\max_{T:|T|=k} P(T)$ and propose some conjectural formulae. One can also try to prove the conjecture in Remark 2.5.

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