Properties of Pólya's Circular Symmetrization

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Abstract

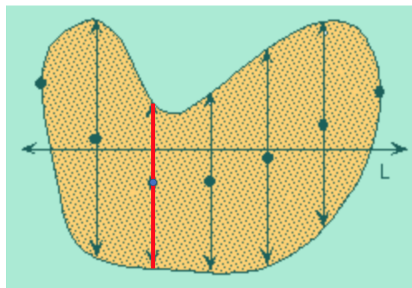
The isoperimetric inequality of a region $\Omega \subset \mathbb{R}^n$ is an inequality comparing $\mathcal{L}^{n-1}(\partial\Omega)$ to $\mathcal{L}^n(\Omega)$. Many symmetrization methods have been developed to prove this, including Steiner $S_L(\Omega)$ and circular $\text{Circ}(\Omega)$ symmetrizations. The underlying idea in these symmetrizations is that they preserve volume but reduce perimeter or surface area, which can be shown using calculus of variations. We present an elementary proof without using calculus of variations on perimeter reduction of a region in $\mathbb{R}^2$ after circular symmetrization. We then show that the diameter of a region entirely to the right of the y-axis does not increase after circular symmetrization if the intersection of the region with an arc of radius r is either a single arc or empty for all $r > 0$.

1 Introduction

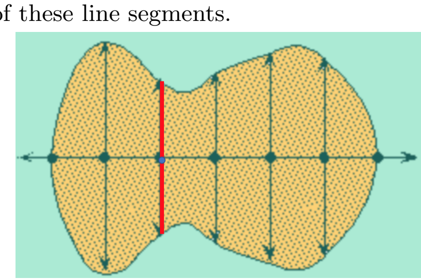
The isoperimetric problem in $\mathbb{R}^2$ asks for the largest area of a subset of $\mathbb{R}^2$ with a fixed perimeter. The roots of the isoperimetric problem can be traced back to Ancient Greece [8]. Pappus of Alexandria knew that this shape is a circle, but the first rigorous proof of this result was obtained in the 19th century. In $\mathbb{R}^2$, if a simple closed curve γ of length L encloses an area of A, then $4\pi A \leq L^2$. Equality holds if and only if γ is a circle. This can be generalized to $\mathbb{R}^n$; namely, from Fusco, Maggi, and Pratelli [1], if $E \subset \mathbb{R}^n$ with $n > 2$, then $n\mathcal{L}^n(E)^{\frac{n-1}{n}}\mathcal{L}^n(\mathbb{B}^n(1))^{\frac{1}{n}} \leq \mathcal{L}^{n-1}(\partial E)$. Here $\mathcal{L}^n(E)$ denotes the n-dimensional volume of E and $\mathbb{B}^n(1)$ denotes the n-ball of radius 1. Yau [9] generalized the isoperimetic inequality to Riemann manifolds, and later, Yau, Li, and Schoen [2] generalized it for minimal surfaces.

Jakob Steiner [6] was a Swiss mathematician who developed Steiner symmetrization to show that the circle is the solution to the isoperimetric problem. Let $\Omega \subset \mathbb{R}^2$ and $x = (x_1, x_2) \in \mathbb{R}^2$. Let $L = \{(x_1, 0) : x_1 \in \mathbb{R}\} \subset \mathbb{R}^2$ be a line. Let $\Omega_{x_1} = \{x_2 \in \mathbb{R} : (x_1, x_2) \in \Omega, x_1 \in \mathbb{R}\}$, and $E_{x_1} = \{x_1 \in \mathbb{R} : |\Omega_{x_1}| > 0\}$. Then the Steiner symmetrization of Ω about the line L is denoted by $S_L(\Omega)$ and is defined as

$$S_L(\Omega) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in E_{x_1}, |x_2| \leq |\Omega_{x_1}|/2\}.$$



(a) A region in $\mathbb{R}^2$ and a line L, with several line segments contained in the region. The blue dots represent the center points of these line segments.



(b) The image of the region after Steiner symmetrization. Notice that the center points of the line segments lie on L.

Figure 1: The process of Steiner symmetrization of a region in $\mathbb{R}^2$.

The Steiner symmetrization of a region preserves its area, but the perimeter of the shape does not increase. For more information on Steiner symmetrization, we refer the reader to [7]. A similar process known as circular symmetrization, developed by George Pólya and Gábor Szegö [3], also preserves the area of a shape while not increasing its perimeter. Suppose $\Omega \subset \mathbb{C}$. The circular symmetrization of Ω is denoted as $\text{Circ}(\Omega)$. The arc of radius t in Ω is defined as $\Omega_t = \{\theta \in [0, 2\pi] : te^{i\theta} \in \Omega\}$. If $|\Omega_t|/t \geq 0$, then

$$\text{Circ}(\Omega) \cap \{|z| = t\} = \left\{ te^{i\theta} : |\theta| \leq \frac{|\Omega_t|}{2t} \right\}.$$

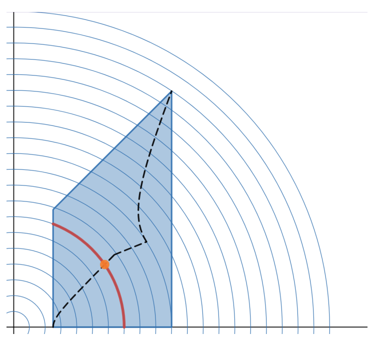
Also, $0, \infty \in \Omega$ if and only if $0, \infty \in \text{Circ}(\Omega)$.

The proof of perimeter reduction of a region in $\mathbb{R}^2$ after circular symmetrization uses calculus of variations. We present an elementary proof without using calculus of variations that the perimeter of the image of a region after circular symmetrization does not increase. We also show that the diameter, the largest distance between two points in a region, does not increase if the region is entirely contained to the right of the y-axis and if the region is composed of single arcs for $r > 0$.

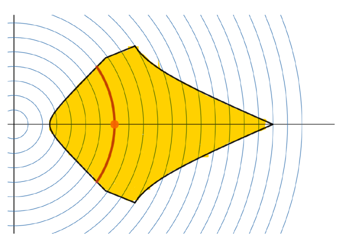
We define the following notation that we use for the rest of the paper. This provides a simple way to denote an arc of radius r by using the subtended angle of the arc to the center of the circle.

Given a radius $r > 0$ and angles $\varphi \in (0, \pi)$ and $\psi \in (-\pi, \pi)$, $S_{r,\psi,\varphi}$ denotes the circular arc of radius r with aperture φ whose center makes an angle ψ to the x-axis. Also, $S_r(0)$ denotes the circle of radius r. Thus, we say

$$S_{r,\psi,\varphi} = \{(r\cos(t), r\sin(t)) : \psi - \varphi \leq t \leq \psi + \varphi\}.$$



(a) A trapezoid with arcs of different radii. The orange dot denotes the center of the red arc. The dotted line represents the centers of every arc contained entirely in the trapezoid.



(b) The image of the trapezoid after circular symmetrization. Notice that the center of the red arc indicated by the orange dot lies on the x-axis.

Figure 2: The process of circular symmetrization on a trapzeoid.

2 An Elementary Proof of Perimeter Reduction

In this section, we present an elementary proof of perimeter reduction for regions with one arc of intersection after circular symmetrization, formally stated in the following theorem.

Theorem 2.1. *Let* $\Omega \subset \mathbb{R}^2$ *be a closed and connected region given by the continuous parametrizations* $\vec{x}(r) = \langle x(r), y(r) \rangle$ *and* $\vec{u}(r) = \langle u(r), v(r) \rangle$ *such that* $y(r) \geq v(r)$ *for all* $r \in [r_0, r_f]$*, where* r *is the radius from the origin. Then* $\mathcal{L}^1(\partial\Omega) \geq \mathcal{L}^1(\partial\,\mathrm{Circ}(\Omega))$*.*

We provide the set-up of the proof. Then, we show a sequence of lemmas that reduces the problem, showing sufficient and necessary conditions to prove the general theorem of perimeter reduction for regions with multiple arcs of intersections after circular symmetrization. This is formally stated in the following theorem.

Theorem 2.2. *Let* $\Omega \subset \mathbb{R}^2$ *be a closed and connected region. Then* $\mathcal{L}^1(\partial\Omega) \geq \mathcal{L}^1(\partial\,\mathrm{Circ}(\Omega))$*.*

2.1 Set-up

Consider an annular region R between two arcs with radii r_1 and r_2, respectively, each centered at the origin with respective central angles φ_1 and φ_2 to the x-axis. Suppose that the center points of these arcs make angles ψ_1 and ψ_2 to the x-axis. This is shown in Figure 3. Without loss of generality, assume $\psi_1 = 0$. Thus, we have the arcs $S_{r_1,0,\varphi_1}$ and S_{r_2,ψ_2,φ_2}. Let m and l be the lengths of the top and bottom line segments as shown in Figure 4.

To obtain the circular symmetrization of R, we rotate S_{r_2,ψ_2,φ_2} along $S_{r_2}(0)$ such that $\psi_2 = 0$. Let m' and l' be the lengths of the line segments after circular symmetrization. The circular symmetrization of R, denoted by $\mathrm{Circ}(R)$, is shown in Figure 4.

We find the lengths of m, l, m', and l' using the Law of Cosines. We have

$$m = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\varphi_1 - \varphi_2 + \psi_2)},$$

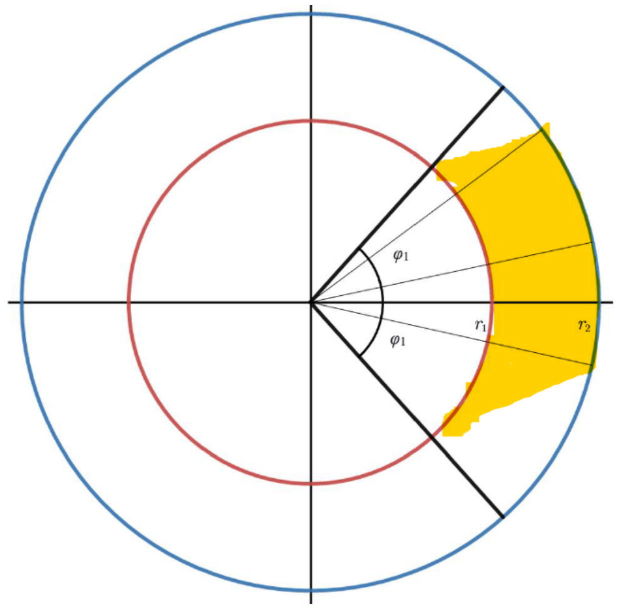
$$l = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\varphi_1 - \varphi_2 - \psi_2)},$$

$$m' = l' = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\varphi_1 - \varphi_2)}.$$

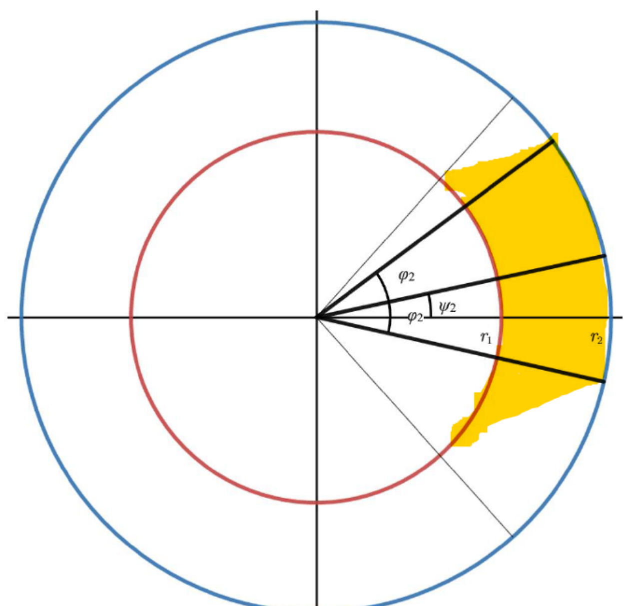
The lengths of the arcs bounding R do not change. Thus, it is sufficient and necessary to show that the sum of the lengths of the line segments decrease. That is,

$$m + l \geq m' + l'. \tag{1}$$

For simplicity, we let $\alpha = r_1^2 + r_2^2$, $\beta = 2r_1r_2$, and $\gamma = \varphi_1 - \varphi_2$ for the rest of the paper.

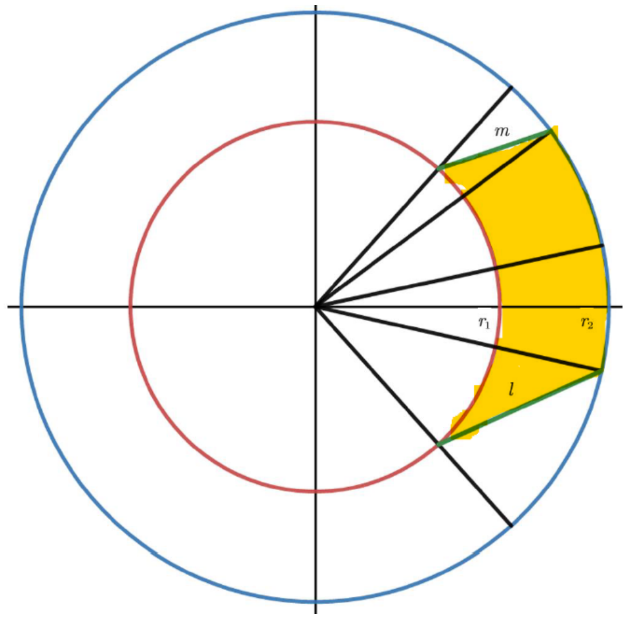


(a) The red arc of radius r_1 has an aperture φ_1, meaning it opens at an angle φ_1 in each direction from the x-axis.

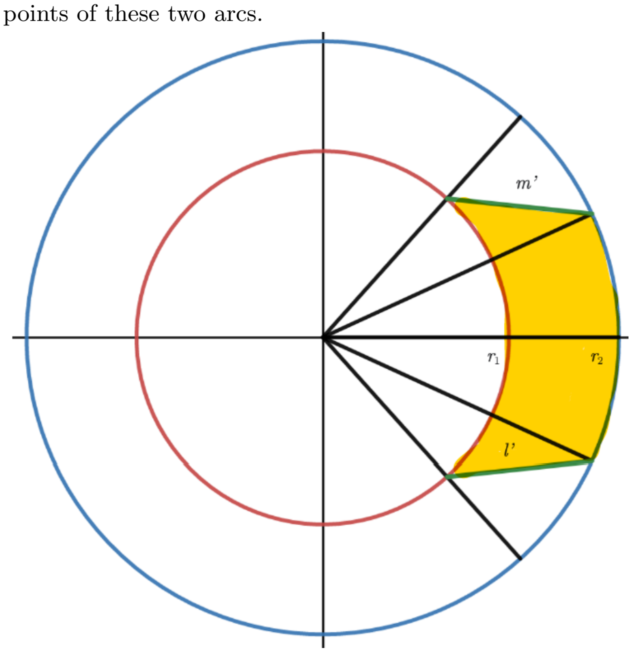


(b) The central black line segment is at an angle ψ_2 from the x-axis. The blue arc of radius r_2 has aperture φ_2.

Figure 3: The set-up for the region R between two arcs.



(a) The green line segments are formed by the endpoints of these two arcs.



(b) The second arc is now centered on the x-axis. The new lengths of the line segments are m' and l'.

Figure 4: The transformation after circular symmetrization.

2.2 Reducing the Problem

Now we present a lemma that is equivalent to Inequality (1).

Lemma 2.3. *Inequality* (1) *is equivalent to*

$$4\cos(\gamma)\left(\frac{\alpha}{\beta}-\cos(\gamma)\right)\geq\left(1-\cos^2(\gamma)\right)\left(1+\cos(\psi_2)\right). \tag{2}$$

Proof. Squaring both sides of Inequality (1) yields in

$$m^2+2ml+l^2\geq(m')^2+2m'l'+(l')^2.$$

We notice that $m^2+l^2\geq(m')^2+(l')^2$ because

$$2\alpha-2\beta\cos(\gamma)\cos(\psi_2)\geq2\alpha-2\beta\cos(\gamma).$$

Then, we have $m^2+l^2-(m')^2-(l')^2=2m'l'-2ml\geq0$. Expanding both sides with some rearrangements of terms, we have

$$\begin{aligned}&\alpha-\beta\cos(\gamma)\left(2-\cos(\psi_2)\right)\\&\qquad\leq\sqrt{\alpha^2-2\alpha\beta\cos(\gamma)\cos(\psi_2)+\beta^2\cos(\gamma+\psi_2)\cos(\gamma-\psi_2)}.\end{aligned}$$

Squaring both sides, we obtain

$$\begin{aligned}&\alpha^2-2\alpha\beta\cos(\gamma)\left(2-\cos(\psi_2)\right)+\beta^2\cos^2(\gamma)\left(2-\cos(\psi_2)\right)^2\\&\qquad\leq\alpha^2-2\alpha\beta\cos(\gamma)\cos(\psi_2)+\beta^2\cos(\gamma+\psi_2)\cos(\gamma-\psi_2)\end{aligned}$$

Using the trigonometric identity

$$\cos(\gamma+\psi_2)\cos(\gamma-\psi_2)=\cos^2(\gamma)\cos^2(\psi_2)-\sin^2(\gamma)\sin^2(\psi_2),$$

we reduce the inequality to

$$\frac{4\alpha}{\beta}\cos(\gamma)\left(\cos(\psi_2)-1\right)\leq4\cos^2\left(\cos(\psi_2)-1\right)-\sin^2(\gamma)\sin^2(\psi_2).$$

Finally, using the trigonometric identity $\sin^2(\gamma)=1-\cos^2(\gamma)$, we arrive at

$$4\cos(\gamma)\left(\frac{\alpha}{\beta}-\cos(\gamma)\right)\geq\left(1-\cos^2(\gamma)\right)\left(1+\cos(\psi_2)\right),$$

as required. ■

We also present the following theorem that allows $x(r), u(r) \leq 0$.

Theorem 2.4. *Let Ω be bounded by the continuous parametrizations $\vec{x}(r) = \langle x(r), y(r)\rangle$ and $\vec{u}(r) = \langle u(r), v(r)\rangle$ for $y(r) \geq v(r)$ and all $r \in [r_0, r_f]$. Let $\psi(r)$ be the angle of the center of $S_r(0) \cap \Omega$ to the x-axis. Then*

$$\lim_{r_2 \to r_1} |\psi(r_2) - \psi(r_1)| = 0. \tag{3}$$

Proof. The arcs under consideration are $S_{r,\psi(r),\varphi(r)}$. To prove (3), it suffices to show that the functions $r \mapsto \psi(r)$ and $r \mapsto \varphi(r)$ are continuous functions of r. Let $\theta(r)$ be the angle subtended by $S_{r,\psi(r),\varphi(r)}$ and let $\theta_t(r)$ be the angle of the line segment connecting the origin to $\vec{x}(r)$ from the x-axis. By definition, $\theta(r) = 2\varphi(r)$. Then we have

$$\cos(\theta(r)) = \frac{\vec{x}(r) \cdot \vec{u}(r)}{r^2} = \frac{x(r)u(r) + y(r)v(r)}{r^2} \quad \text{and} \quad \cos(\theta_t(r)) = \frac{\vec{x}(r) \cdot \vec{e_1}}{r} = \frac{x(r)}{r}.$$

Because $\psi(r) = \theta_t(r) - \frac{\theta(r)}{2} = \theta_t(r) - \varphi(r)$, we have

$$\psi(r) = \arccos\left(\frac{x(r)}{r}\right) - \frac{1}{2}\arccos\left(\frac{x(r)u(r) + y(r)v(r)}{r^2}\right) \quad \text{and}$$

$$\varphi(r) = \frac{1}{2}\arccos\left(\frac{x(r)u(r) + y(r)v(r)}{r^2}\right),$$

which are continuous because

$$\left|\frac{x(r)u(r) + y(r)v(r)}{r^2}\right| \leq 1 \quad \text{and} \quad \left|\frac{x(r)}{r}\right| \leq 1.$$

Hence the result. ■

Theorem 2.4 allows $x(r), u(r) \leq 0$ because $\psi(r_1) - \psi(r_2)$ becomes arbitrarily close for $r_1 - r_2$ sufficiently close.

2.3 Necessary and Sufficient Conditions from Forbidden Pictures

If the line segments of length m and l intersect the circle of radius r_1, as shown in Figure 5, then the region R is ambiguous. This ambiguity comes from whether the part of the green line segment contained in $S_{r_1}(0)$ is a part of the boundary of R. Let η be the angle of the green line segment from the horizontal line, and let τ be the angle of the tangent vector of the circle at the point from the horizontal line. To constrain the line segments, the size of η needs to be less than the size of τ. That is, $|\eta| \leq |\tau|$. We have two cases to consider: the case for the bottom green line segment of length l and the top green line segment of length m, as shown in Figure 6.

Lemma 2.5. *The inequality*

$$r_1 \leq r_2 \cos(\gamma) \cos(\psi_2) \tag{4}$$

implies that the line segments of lengths m *and* l *does not intersect* $S_{r_1,0,\varphi_1}$ *twice.*

Proof. We first consider the line segment of length l. We begin with $|\eta| \leq |\tau|$. To get the inequality in terms of angles, we observe that

$$\tan(\eta) = \frac{r_2 \sin(\psi_2 - \varphi_2) + r_1 \sin(\varphi_1)}{r_2 \cos(\psi_2 - \varphi_2) + r_1 \cos(\varphi_1)} \quad \text{and} \quad \tan(\tau) = \frac{\cos(\varphi_1)}{\sin(\varphi_1)}.$$

Thus, $\tan(\eta) \leq \tan(\tau)$, and after simplification, we obtain

$$r_1 \leq r_2 \cos\left(\psi_2 + \gamma\right). \tag{5}$$

The line segment of length m is similar. We begin with $|\eta| \leq |\tau|$, and we obtain

$$r_1 \leq r_2 \cos(\psi_2 - \gamma). \tag{6}$$

The result follows from adding Inqualities (5) and (6). ■

We now present a sufficient condition for Inequality (2).

Lemma 2.6. *The inequality*

$$2\cos^2(\gamma)\cos^2(\psi_2) + 2 - 4\cos^2(\gamma)\cos(\psi_2) \geq \cos(\psi_2)\left(1 - \cos^2(\gamma)\right)\left(1 + \cos(\psi_2)\right) \tag{7}$$

is sufficient for Inequality (2) *to be satisfied.*

Proof. Let $\rho = r_2/r_1$. Then

$$\frac{\alpha}{\beta} = \frac{1+\rho^2}{2\rho}.$$

If we define $f(\rho) = \alpha/\beta$, then $f(\rho)$ is increasing for $\rho > 1$ because $1 + \rho^2 = 2\rho$ when $\rho = 1$ and $\rho^2 > \rho$ for $\rho > 1$. Also,

$$\rho \geq \rho_{\min} := \frac{1}{\cos(\gamma)\cos(\psi_2)}$$

after manipulating Inequality (4). Substituting $f(\rho)$ into Inequality (2) results in

$$4\cos(\gamma)\left(f(\rho) - \cos(\gamma)\right) \geq \left(1 - \cos^2(\gamma)\right)\left(1 + \cos(\psi_2)\right).$$

Because $f(\rho)$ is increasing for $\rho > 1$, it suffices to show that the minimum of the left-hand side is at least as big as the right-hand side. That is,

$$4\cos(\gamma)\left(f(\rho_{\min}) - \cos(\gamma)\right) \geq \left(1 - \cos^2(\gamma)\right)\left(1 + \cos(\psi_2)\right).$$

Substituting this minimum value of $\rho_{\min}$ yields the result. ■

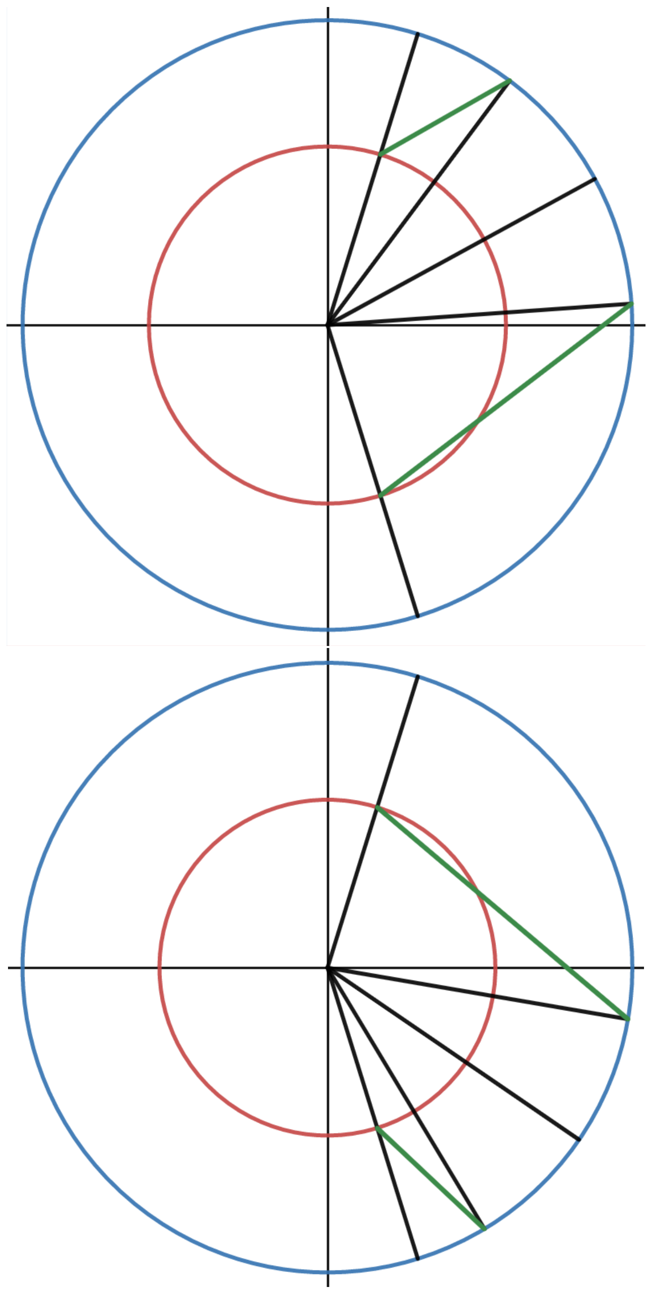


Figure 5: In both cases, a green line segment intersects the red arc twice. This makes the region R between the two arcs and line segments ambiguous.

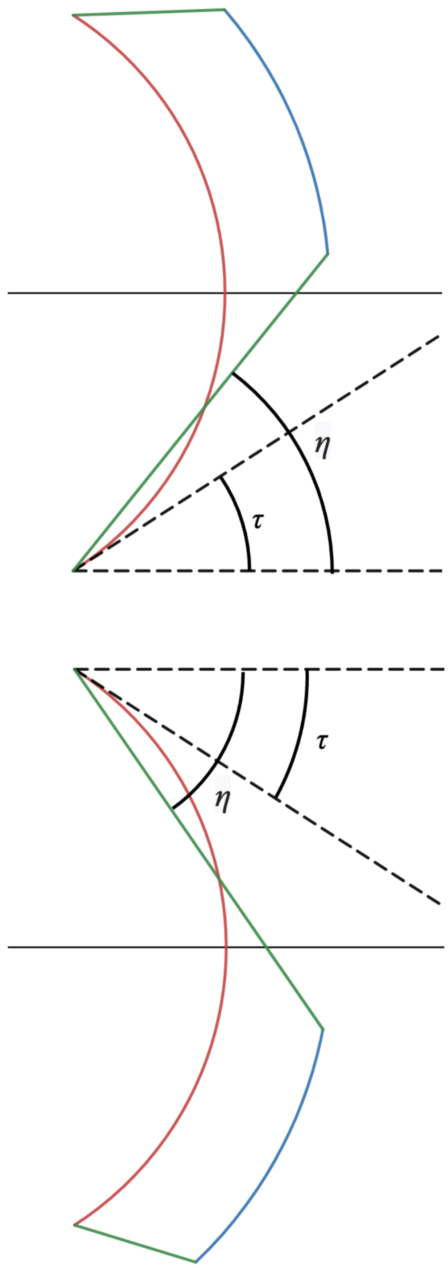


Figure 6: The constraint $|\eta| \leq |\tau|$ is necessary to avoid the green line intersecting the arc of radius r_1 twice.

2.4 Proof of Theorem 2.1

We now present the following lemma.

Lemma 2.7. *If* $r_1 < r_2 \cos(\gamma)\cos(\psi_2)$ *and* $\cos(\gamma), \cos(\psi_2) > 0$, *then* $m + l \geq m' + l'$.

Proof. We need to prove that Inequality (7) is true. Then by Lemma 2.6, we also show Lemma 2.3 which implies the result. Inequality (7) simplifies to

$$\left(3\cos^2(\gamma) - 1\right)\cos^2(\psi_2) - \left(3\cos^2(\gamma) + 1\right)\cos(\psi_2) + 2 \geq 0. \tag{8}$$

We need to show that Inequality (8) is true. This is true when $\cos(\psi_2) = 1$. Then assume that $\cos(\psi_2) \neq 1$. We start with the inequalities $\cos(\psi_2) < 1$ and $3\cos^2(\psi_2) - 1 < 2$. This results in $2 > \cos(\psi_2)\left(3\cos^2(\gamma) - 1\right)$. This is equivalent to $2 + \cos(\psi_2) > 3\cos^2(\gamma)\cos(\psi_2)$. The following inequalities are equivalent.

$$\frac{1 - \cos(\psi_2)}{1 - \cos(\psi_2)} + (1 + \cos(\psi_2)) > 3\cos^2(\gamma)\cos(\psi_2).$$

$$\frac{1}{1 - \cos(\psi_2)} + (1 + \cos(\psi_2)) > \frac{\cos(\psi_2)}{1 - \cos(\psi_2)} + 3\cos^2(\gamma)\cos(\psi_2).$$

$$1 + \left(1 - \cos^2(\psi_2)\right) > \cos(\psi_2) + 3\cos^2(\gamma)\cos(\psi_2)\left(1 - \cos(\psi_2)\right).$$

Moving all the terms in the right-hand side to the left-hand side yields Inequality (8). ■

With Lemma 2.7, we are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\vec{x}(r) = \langle x(r), y(r)\rangle$ and $\vec{u}(r) = \langle u(r), v(r)\rangle$ for $y(r) \geq v(r)$ and $r \in [r_0, r_f]$. Given any partition $\mathscr{P}([r_0, r_f]) = \{r_0 < r_1 < \cdots < r_N\}$, let

$$F(\mathscr{P}) = \sum_{j=1}^{N} \left(\|\vec{x}(r_j) - \vec{x}(r_{j-1})\| + \|\vec{u}(r_j) - \vec{u}(r_{j-1})\|\right).$$

Also, let $\vec{X}(r)$ and $\vec{U}(r)$ be the parametrizations of the curves $\vec{x}(r)$ and $\vec{u}(r)$ after circular symmetrization. Let

$$G(\mathscr{P}) = \sum_{j=1}^{N} \left(\left\|\vec{X}(r_j) - \vec{X}(r_{j-1})\right\| + \left\|\vec{U}(r_j) - \vec{U}(r_{j-1})\right\|\right).$$

By Lemma 2.7, each term in the sum of $F(\mathscr{P})$ is at least as big as the respective term in the sum of $G(\mathscr{P})$ for $j = 1, \ldots, N$. Thus, $F(\mathscr{P}) \geq G(\mathscr{P})$. From [5], given a partition $\mathscr{P} = \{a = t_0 < t_1 < \cdots < t_k = b\}$, the arc length of a continuous parametrized curve $\alpha : [a, b] \to \mathbb{R}^2$ is given by

$$\sup_{\mathscr{P}} \left\{\sum_{i=1}^{k} \|\alpha(t_i) - \alpha(t_{i-1})\|\right\}.$$

From [4], suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. If $f \geq g$, then $\sup f \geq \sup g$. Then we have

$$\sup F(\mathscr{P}) \geq \sup G(\mathscr{P}),$$

which proves the result. ■

2.5 Proof of Theorem 2.2

Now suppose we have a region with multiple arcs of intersections. That is, suppose an arc of radius r_1 intersects the region in n separate arcs of intersections $S_{r_1,\psi_1(r_1),\varphi_1(r_1)}, S_{r_1,\psi_2(r_1),\varphi_2(r_1)}, \ldots, S_{r_1,\psi_n(r_1),\varphi_n(r_1)}$. Also, suppose an arc of radius r_2 intersects the region in n separate arcs of intersections $S_{r_2,\psi_1(r_2),\varphi_1(r_2)}, S_{r_2,\psi_2(r_2),\varphi_2(r_2)}, \ldots, S_{r_2,\psi_n(r_2),\varphi_n(r_2)}$. This is shown in Figures 7 and 8.

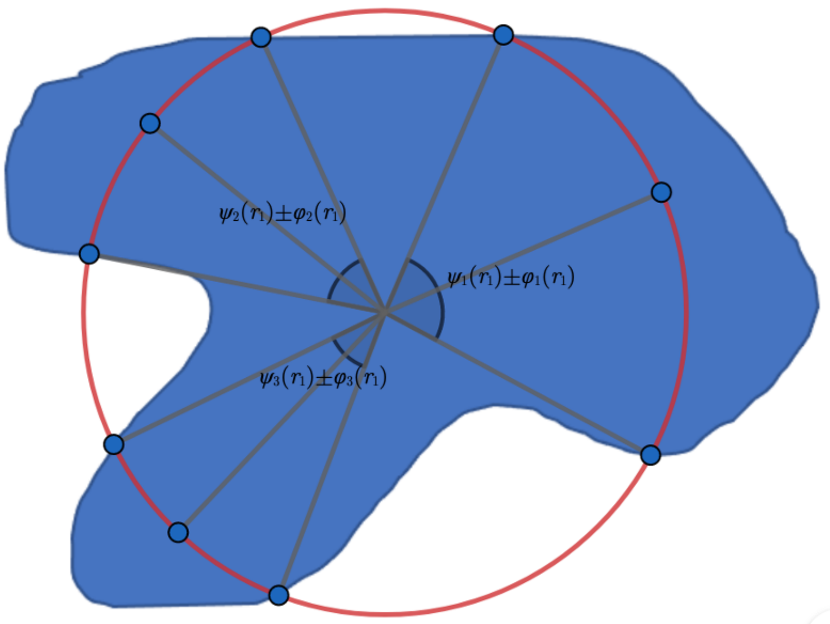


Figure 7: The arcs of intersections for a circle of radius r_1.

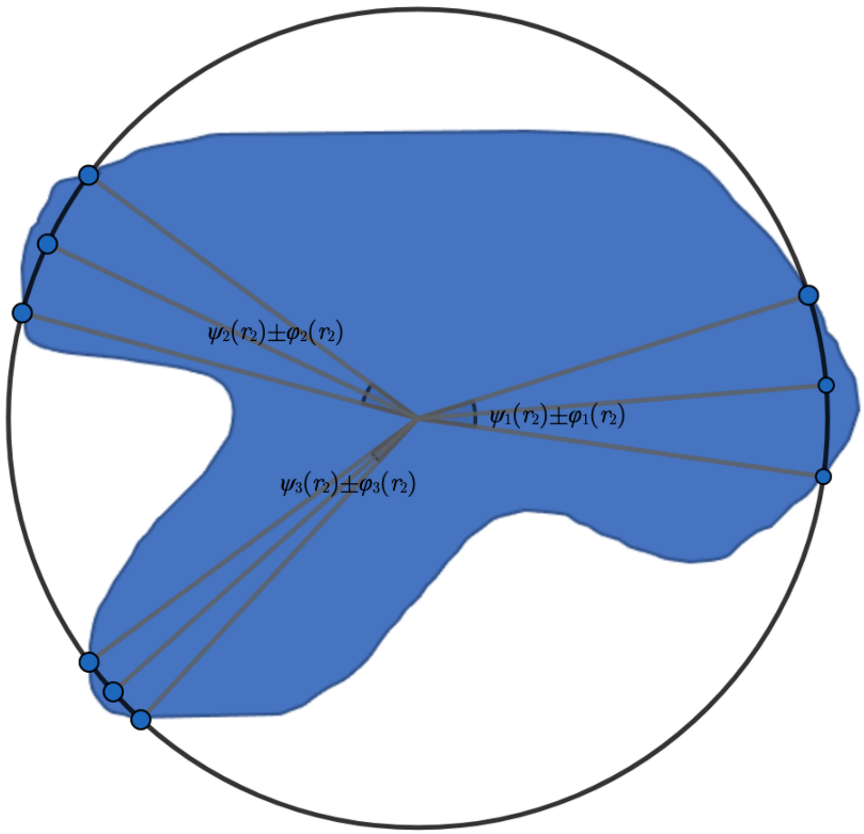


Figure 8: The arcs of intersection for a circle of radius r_2.

We rotate these arcs of radius r_2 such that the centers of these arcs are at $\psi_n(r_1)$. Thus, the arcs of radius r_2 are now $S_{r_2,\psi_1(r_1),\varphi_1(r_2)}, S_{r_2,\psi_2(r_1),\varphi_2(r_2)}, \dots,$ and $S_{r_2,\psi_n(r_1),\varphi_n(r_2)}$. By Theorem 3.1, the sum of the perimeters of the arcs of intersection $S_{r_2,\psi_1(r_1),\varphi_1(r_2)}, \dots, S_{r_2,\psi_n(r_1),\varphi_n(r_2)}$ is less or equal to the perimeter of the arcs of intersection $S_{r_2,\psi_1(r_2),\varphi_1(r_2)}, \dots, S_{r_2,\psi_n(r_2),\varphi_n(r_2)}$.

Then, we replace the n arcs of radius r_1 with a single arc $S_{r_1,0,\sum_{i=1}^n \varphi_i(r_1)}$. Moreover, we replace the new arcs of radius r_2 with a single arc $S_{r_1,0,\sum_{i=1}^n \varphi_i(r_2)}$. With this, we can now present the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $f : X \to \mathbb{R}$, where X is a convex subset of a real vector space. Then f is convex if and only if for all $0 \le t \le 1$ and for all $x_1, x_2 \in X$,

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2). \tag{9}$$

The length of a line segment of the symmetrized region between two arcs of radius r_1 and r_2 with difference between the aperture γ is

$$f(\gamma) = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\gamma)}. \tag{10}$$

To show that the perimeter decreases after symmetrization for n arcs, we have to prove

$$\sum_{j=1}^{n} f(\gamma_j) \geq f\left(\sum_{j=1}^{n} \gamma_j\right). \tag{11}$$

Since

$$nf\left(\sum_{j=1}^{n} \frac{\gamma_j}{n}\right) \geq f\left(\sum_{j=1}^{n} \gamma_j\right),$$

it suffices to show that

$$\sum_{j=1}^{n} f(\gamma_j) \geq nf\left(\sum_{j=1}^{n} \frac{\gamma_j}{n}\right). \tag{12}$$

To prove (12), we want to show that $f(\gamma)$ is convex. Our basis of induction is that $f(\gamma)$ is convex for two variables. We find $d^2f/d\gamma^2$ to be

$$\frac{d^2f}{d\gamma^2} = \frac{r_1r_2}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\gamma)}}\left(\cos(\gamma) - \frac{\sin^2(\gamma)}{r_1^2 + r_2^2 - 2r_1r_2\cos(\gamma)}\right) \geq 0. \tag{13}$$

From our basis of induction, we know that

$$f\left(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2\right) \leq \frac{f(\gamma_1) + f(\gamma_2)}{2}.$$

Now assume that $f(t_1x_1 + \cdots + t_{n-1}x_{n-1}) < t_1f(x_1) + \cdots + t_{n-1}f(x_{n-1})$. We aim to show that $f(t_1x_1 + \cdots + t_nx_n) < t_1f(x_1) + \cdots + t_nf(x_n)$. Let $s = t_1 + \cdots + t_{n-1}$ and

$$y = \frac{t_1x_1 + \cdots + t_{n-1}x_{n-1}}{s}.$$

Then we have

$$\begin{aligned}
f(t_1x_1 + \cdots + t_nx_n) &= f(sy + t_nx_n) \\
&< sf(y) + t_nf(x_n) \\
&= sf\left(\frac{t_1}{s}x_1 + \cdots + \frac{t_{n-1}}{s}x_{n-1}\right) + t_nf(x_n) \\
&< s\left(\frac{t_1}{s}f(x_1) + \cdots + \frac{t_{n-1}}{s}f(x_{n-1})\right) + t_nf(x_n) \\
&= t_1f(x_1) + \cdots + t_nf(x_n).
\end{aligned}$$

Thus, f is convex. By the convexity of f, the perimeter of $S_{r_1,0,\sum_{i=1}^{n}\varphi_i(r_1)}$ and $S_{r_1,0,\sum_{i=1}^{n}\varphi_i(r_2)}$ is less than or equal to the sum of the perimeters of $S_{r_2,\psi_i(r_1),\varphi_i(r_2)}$ and $S_{r_1,\psi_i(r_1),\varphi_i(r_1)}$ for $i = 1, \ldots, n$. Therefore, the perimeter of any region with any arcs of intersection does not increase after circular symmetrization. ■

3 Diameter Decreases

In this section, we present a proof that circular symmetrization decreases the diameter of a region entirely to the right of the y-axis. We assume that the region is composed of single arcs for $r > 0$. To do this, we present two lemmas concerning the distance between two arcs. First, we note that the diameter of Ω is the maximum distance between two points contained in Ω. Even though the two points attaining this distance must be in Ω, the line segment connecting the two points does not necessarily have to be contained in Ω if Ω is not convex.

Lemma 3.1. *Let $0 < r_1 < r_2$ and $0 < \varphi_1, \varphi_2 < \pi/2$, the maximum distance $d(S_{r_1,0,\varphi_1}, S_{r_2,0,\varphi_2})$ is attained by pairs of opposite endpoints, which are the pair $(r_1\cos(\varphi_1), r_1\sin(-\varphi_1))$ and $(r_2\cos(\varphi_2), r_2\sin(\varphi_2))$ and the pair $(r_1\cos(\varphi_1), r_1\sin(\varphi_1))$ and $(r_2\cos(\varphi_2), r_2\sin(-\varphi_2))$.*

Proof. By definition, we have

$$S_{r_1,0,\varphi_1} = \{(r_1\cos(t), r_1\sin(t)) : |t| \le \varphi_1\} \quad \text{and}$$

$$S_{r_2,0,\varphi_2} = \{(r_2\cos(s), r_2\sin(s)) : |s| \le \varphi_2\}.$$

For any $t \in [-\varphi_1, \varphi_1]$ and $s \in [-\varphi_2, \varphi_2]$, the distance between $(r_1\cos(t), r_1\sin(t))$ and $(r_2\cos(s), r_2\sin(s))$ satisfy

$$|(r_1\cos(t), r_1\sin(t)) - (r_2\cos(s), r_2\sin(s))|^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(t-s).$$

To maximize this distance, we must minimize $\cos(t-s)$. Since $t \in [-\varphi_1, \varphi_1]$ and $s \in [-\varphi_2, \varphi_2]$ with $0 < \varphi_1, \varphi_2 < \pi/2$, we have the difference $s - t \in (-\pi, \pi)$, where cosine is decreasing. Thus, the distance is maximized when $t = -\varphi_1$ and $s = \varphi_2$ or when $t = \varphi_1$ and $s = -\varphi_2$. ∎

Lemma 3.2. *Suppose $0 < r_1 < r_2$. For angles $0 < \varphi_1, \varphi_2 < \pi/2$ and angles $|\psi_1 - \psi_2| < \pi/2$ we have that*

$$d(S_{r_2,\psi_2,\varphi_2}, S_{r_1,\psi_1,\varphi_1}) \ge d(S_{r_2,0,\varphi_2}, S_{r_1,0,\varphi_1}). \tag{14}$$

Thus, if two arcs of different radii are aligned such that the centers of the arcs make the same angle to the x-axis, then the distance will not increase.

Proof. We use a similar set-up to that of Section 2.1. Without loss of generality, assume $\psi_1 = 0$. Then the two arcs S_{r_1,ψ_1,φ_1} and S_{r_2,ψ_2,φ_2} along with the line segments connecting the endpoints form an annular region. Instead of considering the green line segments in Figure 4, we construct the diagonal line segments which connects the top of each arc to the bottom of the other, as shown in Figure 9. For simplicity, let $\sigma = \varphi_1 + \varphi_2$. The lengths of these segments are $\sqrt{\alpha - \beta\cos(\sigma + \psi_2)}$ and $\sqrt{\alpha - \beta\cos(\sigma - \psi_2)}$. When we symmetrize, let $\psi_2 = 0$, so the length of the diagonal segments are $\sqrt{\alpha - \beta\cos(\sigma)}$. By our assumption that $0 < \varphi_1, \varphi_2 < \pi/2$, we know $|\sigma| \ne \pi$. It follows that one of $\cos(\sigma + \psi_2)$ and $\cos(\sigma - \psi_2)$ is smaller than $\cos(\sigma)$. This means that one of d_1 and d_2 is smaller.

The maximum distance $d(S_{r_2,\psi_2,\varphi_2}, S_{r_1,0,\varphi_1})$ exceeds both of these lengths, so we have

$$\begin{aligned} d(S_{r_2,\psi_2,\varphi_2}, S_{r_1,0,\varphi_1}) &\geq \max\left(\sqrt{\alpha - \beta\cos(\sigma + \psi_2)}, \sqrt{\alpha - \beta\cos(\sigma - \psi_2)}\right) \\ &\geq \sqrt{\alpha - \beta\cos(\sigma)} \\ &= d(S_{r_2,0,\varphi_2}, S_{r_1,0,\varphi_1}). \end{aligned}$$

In the last equality, we use Lemma 3.1. ■

With these lemmas, we present the following theorem.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a compact region to the right of the y-axis such that for any radius $r > 0$, the circle $S_r(0)$ intersects Ω in a single arc if the intersection is nonempty. Then* $\operatorname{diam}(\operatorname{Circ}(\Omega)) \leq \operatorname{diam}(\Omega)$.

Proof. Let $x, y \in \operatorname{Circ}(\Omega)$ be points that attain the maximum distance. In other words, $d(x, y) = \operatorname{diam}(\operatorname{Circ}(\Omega))$. We consider two cases according to whether or not x and y are at the same distance from the origin.

First, suppose that x and y are not the same distance from the origin. Say, without loss of generality, that $\|x\| = r_1 < \|y\| = r_2$. Let $S_{r_1,\psi_1,\varphi_1} = S_{r_1}(0) \cap \Omega$ and $S_{r_2,\psi_2,\varphi_2} = S_{r_2}(0) \cap \Omega$. These arcs are rotated to $S_{r_1,0,\varphi_1}$ and $S_{r_2,0,\varphi_2}$. Because x and y attain the maximum distance between any two points in $\operatorname{Circ}(\Omega)$, they also attain the maximum distance between any two points on $S_{r_1,0,\varphi_1}$ and $S_{r_2,0,\varphi_2}$. By Lemma 3.2, we have

$$d(S_{r_2,0,\varphi_2}, S_{r_1,0,\varphi_1}) \leq d(S_{r_2,\psi_2,\varphi_2}, S_{r_1,\psi_1,\varphi_1}).$$

Finally, by definition of diameter, we have $\operatorname{diam}(\Omega) \geq d(S_{r_2,\psi_2,\varphi_2}, S_{r_1,\psi_1,\varphi_1})$. It follows that

$$\begin{aligned} \operatorname{diam}(\operatorname{Circ}(\Omega)) = d(x, y) &= d(S_{r_2,0,\varphi_2}, S_{r_1,0,\varphi_1}) \\ &\leq d(S_{r_2,\psi_2,\varphi_2}, S_{r_1,\psi_1,\varphi_1}) \\ &\leq \operatorname{diam}(\Omega). \end{aligned}$$

Then, consider the case that $\|x\| = \|y\| = r$. In this case, x and y are on the arc $S_{r,0,\varphi}$, which was rotated during circular symmetrization from an arc $S_{r,\psi,\varphi}$. If we denote $\tilde{x}$ and $\tilde{y}$ as the preimages of x and y under this rotation, respectively, then we have

$$\operatorname{diam}(\operatorname{Circ}(\Omega)) = d(x, y) = d(\tilde{x}, \tilde{y}) \leq \operatorname{diam}(\Omega),$$

as required. ■

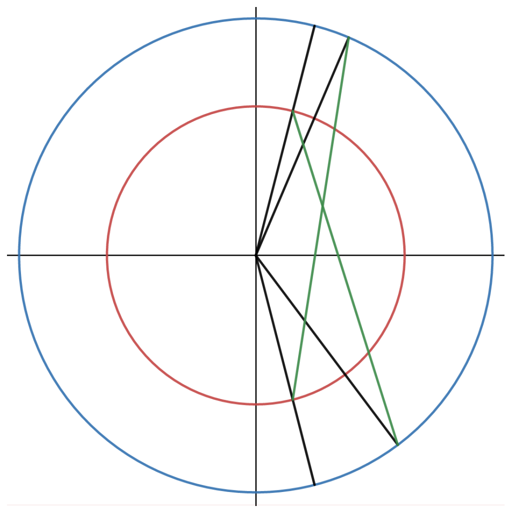


Figure 9: For the case of diameter of a region, we connect the segments connecting pairs of opposite endpoints.

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