Research Science Institute (RSI)

Mathematics Section

Abstracts of Final Research Papers

M.I.T.

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Pawel Burzyński

Estimating Sums of Independent Random Variables

under the direction of Chiheon Kim

Abstract

The paper deals with a problem proposed by Uriel Feige in 2005: if $X_1, \dots, X_n$ is a set of independent nonnegative random variables with expectations equal to 1, is it true that $\mathbb{P}\left(\sum_{i=1}^n X_i < n+1\right) \geqslant \frac{1}{e}$? He proved that $\mathbb{P}\left(\sum_{i=1}^n X_i < n+1\right) \geqslant \frac{1}{13}$. In this paper we prove that infimum of the $\mathbb{P}\left(\sum_{i=1}^n X_i < n+1\right)$ can be achieved when all random variables have only two possible values, and one of them is 0. We also give a partial solution to the case when all random variables are equally distributed. We prove that the inequality holds when n goes to infinity and provide numerical evidence that the probability decreases when n increases.

Benjamin Yuhang Chen

Distinct Distances Between Sets of Points on a Line and a Hyperplane in $\mathbb{R}^d$

under the direction of Thao Do

Abstract

A variant of Erdős's distinct distances problem considers two sets of points in Euclidean space $\mathcal{P}_1$ and $\mathcal{P}_2$, both of cardinality n, and asks whether we can find a superlinear bound on the number of distinct distances between all pairs of points with one in $\mathcal{P}_1$ and the other in $\mathcal{P}_2$. In 2013, Sharir, Sheffer, and Solymosi showed a lower bound of $\Omega(n^{4/3})$ when $\mathcal{P}_1$ and $\mathcal{P}_2$ are both collinear point sets in $\mathbb{R}^2$, where the two lines defined by $\mathcal{P}_1$ and $\mathcal{P}_2$ are not orthogonal or parallel. Here, we contain $\mathcal{P}_1$ in a line l and $\mathcal{P}_2$ in a hyperplane in $\mathbb{R}^d$. We prove that the number of distinct distances in this case has a lower bound of $\Omega(n^{6/5})$ given some restrictions on l and $\mathcal{P}_2$.

Jenning Chen

Finding α-Hölder Continuous Curves through Points in the Unit Square

under the direction of Thao Do

Abstract

The Erdős-Szekeres theorem states that given a sequence of N real numbers, there exists a monotonic subsequence of length at least $N^{1/2}$. By applying this theorem to N points in the unit square, we observe that there exists a function f which passes through at least $N^{1/2}$ of the given points that is Lipschitz continuous, or that has a bounded first derivative. We extend this result by determining the maximum number of the given points $N^{\beta}, 0 < \beta < 1$, that a function can contain while maintaining α-Hölder continuity, a more general form of Lipschitz continuity.

Angela Deng

Growth of Module Dimensions in Auslander-Reiten Quivers of $\tilde{D}_n$-type and $\tilde{E}_{6,7,8}$-type Quivers

under the direction of Guangyi Yue

Abstract

Representations of quivers are frequently used to classify algebras and describe their structure, and so they have a wide range of applications across mathematics and theoretical science. A quiver is a set of vertices connected by arrows, similar to a directed graph, and a representation of a quiver assigns a vector space to each vertex and a map to each arrow. For a quiver Q, the Auslander-Reiten quiver of Q is a quiver with each vertex corresponding to a unique indecomposable module of the path algebra of Q. We study the dimensions of the indecomposable modules assigned to each vertex of the infinite Auslander-Reiten quivers of $\tilde{D}_n$ and $\tilde{E}_{6,7,8}$ type quivers. We prove that the dimensions are bounded linearly for both $\tilde{D}_n$ and $\tilde{E}_{6,7,8}$ type quivers.

Caleb He

On the Spectral Invariance of Ellipses in Convex, Planar Domains

under the direction of Ethan Yale Jaffe

Abstract

Consider Ω, an open subset of $\mathbb{R}^2$ with a convex and smooth boundary. The set of eigenvalues λ for the Dirichlet problem on Ω with clamped boundary is known as the *spectrum* of Ω. We investigate the question posed by M. Kac, which asks whether the shape of a drum Ω is *spectrally invariant*, or determined by the spectrum. Specifically, we ask the question when Ω is an ellipse. Marvizi and Melrose found a family of spectral invariants I_k that were able to prove the spectral invariance of the disk and are believed to determine the spectral invariance of ellipses. They also showed that these invariants can be calculated from the coefficients in the asymptotic expansion of $\lim\limits_{n\to\infty} \mathcal{L}_n$, where $\mathcal{L}_n$ is the length of a *closed geodesic* of length n, or a path governed by the billiard ball map β. We compute β explicitly on the ellipse and provide a precise description of calculating the asymptotic expansion of $\mathcal{L}_n$.

Dona-Maria Ivanova

On the Distortion of Embedding Perfect Binary Trees into Low-dimensional Euclidean Spaces

under the direction of Zhenkun Li

Abstract

The paper considers the problem of embedding binary trees into $\mathbb{R}^d$ for a fixed positive integer d. This problem is part of a more general question concerning distortions of embedding of finite metric spaces into another metric spaces studied by Bourgain. Matousek's observation, that for embedding trees into an infinite-dimensional Euclidean space the upper bound of the distortion is achieved by binary trees motivated us to study the embeding of this particular class of trees into $\mathbb{R}^d$. We extend a result of Kumamoto and Miyano by showing that the distortion of an optimal embedding of binary tree with n vertices in $\mathbb{R}^d$ is $\Theta\left(\frac{n^{1/d}}{\log_2 n}\right)$ and provide a construction achieving it.

Asha Ramanujam

Properties Of Triangles When They Undergo The Curve-Shortening Flow

under the direction of Ao Sun

Abstract

In this paper, we study the properties of triangles when they undergo the curve-shortening flow. We first take the case of the isosceles triangle and prove that the legs of the triangle always decrease and the vertex angle will decrease if it is lesser than $\frac{\pi}{3}$. Our main result is that there exists a time T such that $\alpha(T) = 0$ (where $\alpha(t)$ is the vertex angle at time t) and $x(T) > 0$ (where $x(t)$ is the legs of isosceles triangle at time t) when $0 \leq \alpha(0) < \frac{\pi}{3}$, showing that the triangle becomes a straight line rather than a single point. The equilaterel triangle turns out to be a self-shrinker and hence converges to a single point. We also find expressions for the change in side lengths and angles in any general triangle and prove that the length of each side reduces during the flow.

Nolan Reilly

Collective Instabilities of Linearly Coupled Parametric Oscillators

under the direction of Mason Biamonte

Abstract

The Faraday instability is the formation of waves on the surface of a vibrated liquid [Faraday, 1830]. While this phenomenon is well-understood in the setting of a fluid container with uniform depth [Benjamin and Ursell, 1954; Krishna & Tuckerman, 1994; Miles & Henderson, 1990], little is known about the instability in the presence of variable topography [Milewski, Galeanos-Rios, 2015; Milewski, Galeanos-Rios, Nachbin, Bush, 2015]. Motivated by studies of bouncing drops over nonuniform bottoms, we study a toy model of the Faraday instability in the presence of non-uniform topography in which a Klein-Gordon type equation reduces to a system of coupled parametric oscillators.

We find that even small coupling causes qualitatively different stability behavior in many-oscillator systems. In particular, whereas single parametric oscillators are unstable for all nonzero forcing at certain parameter values, in N coupled oscillators, for any nonzero value of coupling, these areas each split into $\frac{N(N+1)}{2}$ regions, each slightly displaced from the original position. The analytic formulae for these points of instability for two and three oscillators then serve as springboards for studying the particular structure of instability around each of these points, and investigating if there is any significant difference among the points. This, in turn, should reflect fundamental properties of the manifestation of the Faraday instability over spatially-varying container topography.

Sonia Reilly

Bounding the Deviation of the Valuation Property of Quermassintegrals for Non-convex Sets

under the direction of Vishesh Jain

Abstract

The first quermassintegral of a set $C \subset \mathbb{R}^d$ is a characteristic of C closely related to its surface area. Quermassintegrals have traditionally been viewed in the context of convex geometry, and their definition relies on a classic theorem in the field, Steiner's Formula for Parallel Bodies. Nonetheless, the concept can apply to any set, and in this paper we look at how the properties of quermassintegrals of non-convex sets differ from those of convex sets. Specifically, the first quermassintegral, denoted by W_1, satisfies the so-called valuation property, $W_1(C \cup D) = W_1(C) + W_1(D) - W_1(C \cap D)$, when C, D, and their union are convex. Here we determine how far nonconvex sets deviate from the valuation property of quermassintegrals by placing bounds on the value $\eta(C, D) = W_1(C) + W_1(D) - W_1(C \cup D) - W_1(C \cap D)$, for $C \cup D$ non-convex. Using both set-theoretic approaches and fundamental results of integral geometry, we show that in $\mathbb{R}^2$, $0 \leq \eta(C, D) \leq W_1(\text{conv}(C \cup D) \setminus (C \cup D))$, provided C and D are convex and their union is connected. Furthermore, we prove that the lower bound holds for all $C, D \subset \mathbb{R}^d$.

Dhruv Rohatgi

When Two-Holed Torus Graphs are Hamiltonian

under the direction of Chiheon Kim

Abstract

Trotter and Erdös found conditions for when a directed $m \times n$ grid graph on a torus is Hamiltonian. We consider the analogous graphs on a two-holed torus, and study their Hamiltonicity. We find an $\mathcal{O}(n^4)$ algorithm to determine the Hamiltonicity of one of these graphs and an $\mathcal{O}(\log(n))$ algorithm to find the number of diagonals. We also show that for a fixed n, the binary sequence $\{x_m\}$ where $x_m = 1$ if the m and n are coprime and the $m \times n$ grid is Hamiltonian, and $x_m = 0$ otherwise, has periodicity $12n$; and we completely classify which graphs are Hamiltonian in the cases where $n = m$, $n = 2$, the $m \times n$ graph has 1 diagonal, or the $\frac{m}{2} \times \frac{n}{2}$ graph has 1 diagonal.