On Lower Central Series of the r,q-polynomial algebra

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Abstract

We study the lower central series of an associative algebra, defined as follows: $L_1 = A$, $L_{i+1} = [L_i, A]$, where $[\ ,\ ]$ is the bilinear Lie bracket operation. We look at the successive quotients $B_i = L_i/L_{i+1}$ and $N_i = M_i/M_{i+1}$, where M_i is the two-sided ideal generated by L_i. We aim to study the decomposition of N_i and B_i into free and torsion components using the structure theorem of finitely generated abelian groups. Using the computational algebra system *Magma* we gather lots of data and observe and prove various interesting patterns about these ranks and torsion. We mainly concentrate on the algebra $\mathbb{Z}\langle x, y\rangle/(qxy - ryx)$ where $(q, r) = 1$, also known as the q, r–polynomial algebra. We completely describe and prove the pattern of N_i and B_i for this algebra. We give some conjectures for algebra $\mathbb{Z}\langle x, y\rangle/((f_1), (f_2))$ where f_1 and f_2 are two homogeneous polynomials of degree two and three.

Summary

We study the lower central series $\{L_i\}$ of some algebras over the integers. Furthermore we look into the successive quotients of the lower central series $\{B_i\}$ and the successive quotients of the ideals generated by $\{L_i\}$, called $\{N_i\}$. We study the ranks and torsion of the $\{N_i\}$ and $\{B_i\}$ for various different algebras. We consider specifically the r, q-polynomial algebras and prove that there are closed forms both for the ranks and torsion of $\{B_i\}$ and $\{N_i\}$. We give some conjectures about some algebras moded by two homogeneous relations of degree two and three.

1 Introduction

The algebraic approach to geometry is based on replacing geometric spaces by algebras of "nice" functions on them. These algebras are commutative. However, in the noncommutative geometry, we replace these commutative algebras with similar noncommutative ones, pretending that they correspond to imaginary "noncommutative spaces". It is more complicated to study these noncommutative algebras compared to their commutative analogs. This is one of the main reasons why we define and study the so called lower central series of an associative algebra used as a measurement of how "far" from commutative an algebra is.

Recall that the lower central series $(L_i(A))$ are the successive subspaces of an associative algebra A formed from the commutators of A. We consider the successive quotients of the lower central series $B_i = L_i/L_{i+1}$. Furthermore, we look into the successive quotients of the two-sided ideal M_i generated by L_i, and we call these quotients N_i. Studying the structure and the properties of N_i helps us understand the structure of our algebra.

The study of the LCS (Lower Central Series) began in 2007, when Feigin and Shoikhet [1] found out that there exists an isomorphism between the space $B_2(A_n)$ (where $A = A_n$ is the free algebra over $\mathbb{C}$ with n generators) and the space of closed differential forms of positive even degree on the space $\mathbb{C}^n$, i.e.

$$B_2(A_n) \cong \Omega^{even>0}_{closed}(\mathbb{C}^n).$$

Later on Dobrovoska, Kim and Ma [2] described the quotient $A/M_i(A)$ for $i = 4$. Next, Balagovic and Balasubramanian [3] continued the study of B_i. One of their main results is the complete description of $B_2(A_2/(x^d + y^d))$. After that, Kerchev [4] studied N_i for free algebras and computed $N_i(A_n)$ for several values of i and n.

Zhou [5] studied the behavior of N_i for an associative algebra $\mathbb{Z}\langle x, y\rangle$ with various relations. She computed the ranks and torsion of various N_i using the computer program

Magma. Cordwell, Fei and Zhou [6] studied B_k and N_k for $A = A_n(\mathbb{Z})/(f)$, which is an algebra with a single homogeneous relation over $\mathbb{Z}$.

In this paper, we study the behaviour and the properties of the B_i and N_i for various algebras. We give some proposition for the behaviour of this series for $A = \mathbb{Z}\langle x, y\rangle/((f_1), (f_2))$ where f_1 and f_2 are two homogeneous polynomial of degree two and three. The process of collecting data is explained in the paper. The main goal of the project was to give a complete description of the ranks and torsions of the N_i and B_i for the associative algebra $A = \mathbb{Z}\langle x, y\rangle/(rxy - qyx)$ where $(r, q) = 1$, also known as the r, q-polynomial algebra.

In section 2, we give the needed preliminaries for the project. In section 3, we outline how we collect and how we interpret data. In the following section 4, we give some conjectures about the patterns for the rank and torsion of various algebras. Section 5 is the most important section, because there we show the proof of our main task.

2 Preliminaries

We begin with preliminary background.

Definition 2.1. Let $\mathbb{R}$ be a fixed commutative ring. An *associative R-algebra* is an abelian group A that has the structure of both a ring and an R-module in such a way that ring multiplication is R-bilinear:

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$$

for all $r \in \mathbb{R}$ and $x, y \in A$. We say that A is *unital* if it contains a multiplicative identity element.

In this paper, unless otherwise stated, we work with algebras over $\mathbb{Z}$. Since by definition A as a $\mathbb{Z}$-module, is an abelian group.

Given an associative algebra A, we define a bilinear Lie bracket operation $[\ ,\ ] : A \times A \to A$ by $[a, b] = ab - ba$. One can see that this operation satisfies $[a, a] = 0$ and the Jacobi identity:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

An algebra for which such a bracket operation is defined is called a *Lie Algebra*.

Definition 2.2. Given a Lie algebra A, we construct the lower central series for A as follows.

$$L_1 = \text{A}$$

$$L_{i+1} = [A, L_i]$$

where the space $[A, L_i]$ is spanned by all the elements of A and L_i. These series of Lie ideals is known as the *lower central series of* A. We abbreviate $L_i(A)$ by L_i.

For the purpose of our project, however, we are more interested in the successive quotients of L_i and the successive quotients of the two-sided ideal generated by it. That is why we define the next three series.

Notation 2.3. We define $B_i = L_i/L_{i+1}$.

Notation 2.4. Denote the two-sided ideals generated by each L_i by M_i, i.e. $M_i = A.L_i.A$. One can prove that $M_i = A.L_i$.

Notation 2.5. We define $N_i = M_i/M_{i+1}$.

We continue with more basic definitions that are used trough the paper.

Definition 2.6. A module R over a commutative ring K is graded if it has a direct sum decomposition into submodules $\bigoplus_{i\geq 0} R_i$. If A is an algebra such that $A_iA_j \subset A_{i+j}$, then A is a *graded algebra*.

Example 1. A polynomial ring $k[x_1, ..., x_n]$ is an example of graded algebra, where the grading is given by the degree of the polynomials. We observe that N_i inherits its grading from A. The part of N_i at degree d will be denoted as $N_i[d]$, which is a finitely generated k

- module.

Definition 2.7. *The torsion subgroup* or just *torsion* of an abelian group G is the group of elements of finite order.

The idea of torsion becomes especially important due to the Structure Theorem of Finitely Generated Abelian Groups, which states that groups can be separated into their free and torsion components, which we use in Section 5. We will use this theorem because we will look at the two components separately.

Theorem 1. *(Structure Theorem of Finitely Generated Abelian Groups) Every finitely generated abelian group G is isomorphic to a finite direct sum of infinite cyclic groups and cyclic groups of order p^n, for various primes p. This decomposition is unique up to order of summands.*

!!!cite!!!

This theorem can be restated as

$$G \cong F \oplus T,$$

where F is the free component, which is isomorphic to $\mathbb{Z}^r$ for some $r \in \mathbb{Z}$, and T is the torsion component, consisting of a finite sum of cyclic groups of order p^n for various primes p. In this case, the r is called the **rank** of the free component, or just as the "*rank*".

3 Data collection

The conjectures and theorems in the paper come from the data obtained at the first stage of the project. In this section we explain how the data is collected and how we read it. We use the computer algebra system *Magma*, designed mainly to solve problems in algebra and number theory. We wrote a specific program in Magma which gives us the rank and torsion of N_i and B_i for various algebras. Given an algebra, the input of the program consists of the relations we mod by, and their degrees. The output of the program consists of several lines, each of which represents the rank and the torsion of $N_i[d]$ for some i and d. We interpret the information and we put it in tables. This process is illustrated in the following example, together wit an explanation how to read the resulting table.

Example 3. Input:

$K = 2$ (degree of the relation we mod by)

$F_1 = 5xy - 3yx$ (the relation we mod by)

Output:

N 3 3 2 (Abelian Group of order 1)

N 4 3 0 (Abelian Group of order 1)

N 3 4 0 (Abelian Group isomorphic to $\mathbb{Z}/2 + \mathbb{Z}/2 + \mathbb{Z}/16$)

N 4 4 3 (Abelian Group of order 1)

The first number stands for the index of N, while the second is for the d-part of N, or i.e. N 1 5 = $N_1[5]$. The last number shows the rank we are searching for, and the last terms represents the torsion group. We present this information clearer in a table.

	$d = 3$	$d = 4$
N_3	2	$0(2^2 16)$
N_4		3

Table 1: $\mathbb{Z}\langle x, y\rangle/(5xy - 3yx)$

The $i, d - th$ cell in the table shows the tank and the torsion of $N_i[d]$. In this particular example the rank of $N_3[4]$ is equal to zero, and the torsion subgroup is isomorphic to $Z_2 \oplus Z_2 \oplus Z_{16}$.

4 Conjectures

In this section make conjectures about the ranks of $N_i[d]$ and $B_i[d]$, based on the patterns in the data, obtained from *Magma*.

Observation 2. *For the* $Z\langle x,y\rangle/(ax^2+bxy+cy^2)(x^3+y^3)$ *algebra, we can observe that when the coefficients a,b and c are all different from each other and from* ± 1 *and* 0*, the ranks of* $N_i[d]$ *are not dependant on this coefficients. Moreover, the table that corresponds to those ranks is exactly Table 2.*

	2	3	4	5	6	7	8
N_2	1						
N_3		2					
N_4			1				
N_5							
N_6							
N_7							
N_8							

Table 2: $Z\langle x,y\rangle/((ax^2+bxy+cy^2)(x^3+y^3))$

However, it is interesting that when some of the coefficients are equal to each other or equal to ± 1 *or 0, the tables corresponding to this input are completely different. In the Tables 3 to 6???? in the appendix we give some examples showing this phenomenon.*

Observation 3. *For the* $Z\langle x,y\rangle/(ax^2+bxy+cy^2)(x^3+y^3)$ *algebra, we can observe that when the coefficients a,b,c, and are all different from each other and from* ± 1 *and* 0*, the ranks of* $N_i[d]$ *are not dependant on this coefficients. Moreover, the table that corresponds to those ranks is exactly Table 3.*

	2	3	4	5	6	7	8
N_2	1						
N_3		2	1				
N_4			1				
N_5				2	1		
N_6					1		
N_7						2	1
N_8							1

Table 3: $\mathbb{Z}\langle x, y\rangle/((x^2+y^2),(ax^3+bx^2y+cy^2x+dy^3))$

Observation 4. *For the $\mathbb{Z}\langle x, y\rangle/(rxy-qyx)$ where $(r,q)=1$ we see that the ranks of $B_i[d]$ and $N_i[d]$ are not dependant on q and r. Moreover we observe that for them we can form a nice closed formula for their torsion. This is explained in Section 5.*

5 Complete description of the N_i and B_i for the $r,q-$ *polynomial* algebra

In this section we consider the specific algebra $\mathbb{Z}\langle x, y\rangle$ with the relation $qxy = rxy$, also known as the q,r-polynomial algebra. Tables smth to smth, located in Appendix A.1.show that there are clear and interesting patterns in the rank and torsion of the N_i and B_i. We found that below the diagonal $i = d-1$ the torsion in B_i and N_i is $\mathbb{Z}_{(q-r)^{d-1}}$. However, the torsion along this diagonal is more interesting. We will need to use a finer grading in order to understand it. We define x to have degree $(1,0)$ and y to have degree $(0,1)$, i.e. $L_1(1,0) = \langle x\rangle$ and $L_1(0,1) = \langle y\rangle$, where for degree (k,l), we know that $d = k+l$. For B_i we still have the same torsion as below the diagonal, however for N_i this is not true. We have that

$$TorN_i[d] = \bigoplus_{u+v=d} \mathbb{Z}_{q^{(u,v)}-r^{(u,v)}}$$

.

Note: We should always keep in mind that $TorN_i[d] = 0$ and $TorB_i[d]$ if $i > d$.

We want to show a closed formula for the torsion of $B_i[d]$ and $N_i[d]$. We can use the idea of finer grading, therefore if we prove a closed formula for $B_i(k,l)$ and $N_i(k,l)$ where $k + l = d$, we will be just able to sum over all k and l with this sum and we will find the desired formula. In addition since $B_i[d]$ and $N_i[d]$ are successive quotients of $L_i[d]$ and $M_i[d]$ respectively, we can just show a closed formula for $L_i[d]$ and $M_i[d]$. In Lemma 3, we will form a formula for $L_2(k,l)$ and then in Lemma 4, we will prove by induction the form for $L_i(k,l)$. In Lemma 6, we will prove the formula for $M_i(k,l)$. After we have formulas for $L_i(k,l)$ and $M_i(k,l)$ we can sum over all k and l with sum d and then just divide by $L_{i+1}[d]$ and $M_{i+1}[d]$ and we will reach a closed formula for $B_i[d]$ and $N_i[d]$. We start by proving the lemmas we already talked about.

Theorem 5. *Let L_i be the lower central series of the associative algebra $\mathbb{Z}/(qxy = ryx)$. Let B_i be the successive quotients $B_i = L_i \backslash L_{i+1}$. The torsion of $B_i[d]$ when $i \leq d-1$ is equal to $\mathbb{Z}_{(q-r)^{d-1}}$.*

Since we have the relation $qxy = ryx$, we can express the result of any bracket operation as the sum of $mx^ky^l + ny^lx^k$ for some integers m and n. The following facts are easy to prove, but are crucial for the proof of the thorem.

1. $L_n(k,l) = [L_1(1,0), L_{n-1}(k-1,l)] + [L_1(0,1), L_{n-1}(k,l-1)]$;

2. $L_1(n,0) = \langle x^n \rangle$, $L_1(0,n) = \langle y^n \rangle$;

3. For every k and l: $q^{kl}x^ky^l = r^{kl}y^lx^k$;

4. Since $(q,r)=1$, then $(q^k,r^l)=1$ where $k,l\in N$, therefore using the extended Euclidean algorithm we can find such numbers A_{kl} and B_{kl} such that $B_{kl}q^{kl}-A_{kl}r^{kl}=1$. Furthermore $L_1(k,l)=\langle -A_{kl}x^ky^l+B_{kl}y^lx^k\rangle$.

Lemma 6. *Let L_i be the lower central series of the associative algebra $\mathbb{Z}/(qxy=ryx)$. Then for $k+l>1$, we have that $L_2(k,l)=(q^{(k,l)}-r^{(k,l)})L_1(k,l)$.*

Proof: From Fact 3 we know that $q^{(k-1)l}x^{k-1}y^l=r^{(k-1)l}y^lx^{k-1}$. Let A and B, be such that $Bq^{(k-1)l}-Ar^{(k-1)l}=1$. Analogically, let $Cq^{kl}-Dr^{kl}=1$.

From Fact 4 we know that

$$L_1(k-1,l)=-Ax^{k-1}y^l+By^lx^{k-1}$$

$$L_1(k,l)=-Cx^ky^l+Dy^lx^k$$

.

From *Fact 1* we know that

$$L_2(k,l)=[L_1(1,0),L_1(k-1,l)]+[L_1(0,1),L_1(k,l-1)].$$

Let us focus our attention on $K=[L_1(1,0),L_1(k-1,l)]$.

$$\begin{aligned} K &= [L_1(1,0),L_1(k-1,l)]=[x,-Ax^{k-1}y^l+By^lx^{k-1}]= \\ &= -Ax^ky^l+Bxy^lx^{k-1}+Ax^{k-1}y^lx-By^lx^k. \end{aligned}$$

Two of our terms are of the desired form. However, we should express the other two in the following form:

$$xy^lx^{k-1} = (Bq^{(k-1)l} - Ar^{(k-1)l})xy^lx^{k-1} = Bq^{(k-2)l}r^ly^lx^k - Aq^{(r-1)l}x^ky^l,$$

$$x^{k-1}y^lx = (Bq^{(k-1)l} - Ar^{(k-1)l})x^{k-1}y^lx = Br^{(k-1)l}y^lx^k - Aq^lr^{(k-2)l}x^ky^l.$$

Now, we see that

$$K = (-A - ABq^{(r-1)l} - A^2q^lr^{(k-2)l})x^ky^l + (-B + ABr^{(k-1)l} + B^2r^lq^{(k-2)l})y^lx^k.$$

We want to express K as a multiple of $L_1(k,l)$. In order to do so, we can use the fact that $K = a(-Cx^ky^l + Dy^lx^k) + b(q^{kl}x^ky^l - r^{kl}y^lx^k)$. We can form the following matrix equation:

$$\begin{pmatrix} A_2 & q^{ij} \\ -B_2 & -r^{ij} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -A - ABq^{(i-1)j} - A^2q^jr^{(i-2)j} \\ -B + ABr^{(i-1)j} + B^2r^jq^{(i-2)j} \end{pmatrix}.$$

We should note that it was important that we constructed our matrix with determinant equal to 1. Solving this matrix equation for one leads to:

$$\begin{aligned} a &= -Ar^{kl} - ABq^{(k-1)l}r^{kl} - A^2q^lr^{2kl-2l} - Bq^{kl} + ABr^{(r-1)l}q^{kl} + B^2r^lq^{2kl-2l} \\ &= r^l(-Ar^{(k-1)l} + Bq^{(k-1)l}(Bq^{(k-1)l} - Ar^{(k-1)l})) - \\ &\quad q^l(Bq^{(k-1)l} - Ar^{(k-1)l}(Bq^{(k-1)l} - Ar^{k-1}l)) = r^l - q^l. \end{aligned}$$

Analogically for $P = [L_1(0,1)L_1(k,l-1)]$ we get $a' = r^k - q^k$. We are looking for the greatest common divider of a and a', which is the gcd of $r^l - q^l$ and $r^k - q^k$. The last is exactly $(r^{(k,l)} - q^{(k,l)})$. Therefore $L_2(k,l) = (q^{(k,l)} - r^{(k,l)})L_1(k,l)$. $\square$

We conclude that $B_1(k,l) = L_1(k,l) \backslash L_2(k,l) = (q^{(k,l)} - r^{(k,l)})$.

Lemma 7. *Let L_i be the lower central series of the associative algebra $\mathbb{Z}/(qxy = ryx)$. Then if $i + j \geq k$ $L_k(i, j) = (q - r)^{k-2}(q^{(i,j)-r^{(i,j)}})L_1(i, j)$.*

Proof: We will prove the statement of the lemma by induction.

The base case of the infuction $L_2(i, j) = (q^{(i,j)} - r^{(i,j)})(q - r)^{2-2}L_1(i, j) = (q^{(i,j)} - r^{(i,j)})L_1(i, j)$ is given in Lemma 3.

Let us assume that for every i, j

$$L_n(i, j) = (q - r)^{n-2}(q^{(i,j)} - r^{(i,j)})L_1(i, j).$$

This also means that $L_n(i - 1, j) = (q^{((i-1),j)} - r^{((i-1),j)})(q - r)^{n-2}$.

Therefore we have that

$$L_{n+1}(i, j) = [L(1, 0), L_n(i - 1, j)] + [L(0, 1), L_n(i, j - 1)].$$

Lets focus on

$$[L(1, 0), L_n(i - 1, j)] = (q^{((i-1),j)} - r^{((i-1),j)})(q - r)^{n-2}[L(1, 0), L(i - 1, j)].$$

We already saw in Lemma 3 that the last term is $(q^j - r^j)L_1(i, j)$, so

$$[L(1, 0)L_n(i - 1, j)] = (q^{((i-1),j)} - r^{((i-1),j)})(q - r)^{n-2}(q^j - r^j)L_1(i, j).$$

Analogically, for $L(0, 1)L_n(i, j - 1)$, we have that

$$[L(1, 0), L_n(i - 1, j)] = (q^{i,(j-1)} - r^{(i,(j-1))})(q - r)^{n-2}(q^j - r^j)L_1(i, j).$$

Since the greatest common divisor of $((q^{i,(j-1)} - r^{(i,(j-1))}), (q^{((i-1),j)} - r^{((i-1),j)})) = 1$, we can see that the gcd of $L(1,0)L_n(i-1,j)$ and $L(0,1)L_n(i,j-1)$ is $(q-r)^{n-1}(q^{(i,j)}-r^{(i,j)})L_1(i,j)$, which was exactly what we wanted to prove. $\square$

The proof of Theorem 2 follows directly from the proof of Lemma 3 and Lemma 4.

So far we looked into the B_i, in the rest of the section we concentrate on the N_i.

Theorem 8. *Let L_i be the lower central series of the associative algebra $\mathbb{Z}/(qxy = ryx)$. With M_n we will denote the two-sided ideal generated by L_i. Then*

$$M_n(i+k, j+l) = \sum L_1(i,j)L_n(k,l) = (q^{(k,l)} - r^{(k,l)})(q-r)^{n-2}L_1(i,j)L_1(k,l).$$

Proof:

We are searching for the greatest common divisor of all the products that we are summing. We observe that $(q-r)^{n-2}$ divides this gcd. In fact, we will prove that it is equal to $(q-r)^{n-2}$ by finding it for two terms. is the gcd of all of the terms.

We see that $L_1(1,0)L_n(k,l)$ and $L_1(0,1)L_n(k+1,l-1)$ participate in the sum $M_n(k+1,l)$.

There exist numbers A and B, such that $Bq^{kl} - Ar^{kl} = 1$ and $L_1(k,l) = -Ax^ky^l + By^lx^k$. Therefore:

$L_1(1,0)L_n(k,l) = x(-Ax^ky^l+By^lx^k) - Ax^{k+1}y^l + Bxy^lx^k = (-A-ABq^{kl})x^{k+1y^l+B^2r^lq^{(k-1)l}}y^lx^{k+1}$.

We know that $L_1(1,0)L_n(k,l) = a(-A_2x^{k+1}y^l+B_2y^lx^{k+1})+b(q^{(k+1)l}x^{k+1}y^l-r^{(k+1)l})y^lx^{k+1}$, where A_2 and B_2 are such that $L_1(k+1,l) = -A_2x^{k+1}y^l + B_2y^lx^{k+2}$. To find a we need to solve analogical matrix equation to the one in Lemma 3. It yields to:

$$a = -Ar^{kl}r^l + Bq^{kl}r^l(-Ar^{kl} + Bq^kl) = -Ar^{kl}r^l + Bq^{kl}r^l = r^l.$$

Analogically for $L_1(0,1)L_1(k+1,l-1)$ we get that $a' = q^{k+1}$.

Moreover, $L_1(0,1)L_n(k,l) = (q^{(k,l)} - r^{(k,l)})(q-r)^{n-2}L_1(1,0)L_1(k,l) = (q^{(k,l)} - r^{(k,l)})(q-r)^{n-2}r^lL_1(k+1,l)$. Similarly $L_1(0,1)L_1(k+1,l-1) = (q^{(k+1,l-1)} - r^{(k+1,l-1)})(q-r)^{n-2}q^{k+1}L_1k+1,l$. We see that the gcd of this two terms is exactly $(q-r)^{n-2}$, therefore it is exactly the greatest common divisor of all terms.

Corollary 1. When $i < d-1$ we know that $TorN_i(k,l) = (q-r)$ and when $i = d-1$, $TorN_i(k,l) = (q^{(k,l)} - r^{(k,l)})$.

The difference occurs because when $i = d-1$,to calculate $N_i[d]$ we have to look both at $M_i[d]$ and $M_{i+1}[d]$. However in this case $M_i[d] = L_i[d]$.

6 Conclusion

Using a program in *Magma* we collected a large amount of data in the form of tables, about the rank and torsion of N_i and B_i for several specific algebras. We provided description of the r, q-polynomial algebra. We also gave some conjectures about the rank and torsion of N_i for $\mathbb{Z}\ (f_1, f_2)$ where f_1 and f_2 are two homogeneous polynomials of degree two and three.

There are many different ways to continue this project. One of them is to try generalizing the problem for $Z\langle x, y\rangle/(qxy - ryx)$ for n variables and $\frac{n(n-1)}{2}$ relations of the form $a_1xy = b_1xy$. Another path would be to prove the conjectures that were outlined in Section 4. We can also study different families of algebra using the same ideas..

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A Observation 4.1

	2	3	4	5	6	7	8
N_2	1						
N_3		2	1				
N_4			1				
N_5				2	1		
N_6					1		
N_7						2	1
N_8							1

Table 4: $Z < x, y > /((x^2 + y^2)(x^3 + y^3))$

	2	3	4	5	6	7	8
N_2	1	1					
N_3		2	2				
N_4			1	1			
N_5				2	2		
N_6					1	1	
N_7						2	2
N_8							1

Table 5: $Z < x, y > /((x^2 + xy + yx + y^2)(x^3 + y^3))$

	2	3	4	5	6	7	8	9
N_2	1							
N_3		2						
N_4			3					
N_5				3	1			
N_6					2			
N_7						3		
N_8							3	1
N_9								2

Table 6: $Z < x, y > /((x^2 + xy + y^2)(x^3 + y^3))$

	2	3	4	5	6	7	8
N_2	1	1					
N_3		1					
N_4			1				
N_5							
N_6							
N_7							
N_8							

Table 7: $Z < x, y > /((x^2 + 2xy + y^2)(x^3 + y^3))$

B Observation 2.

	2	3	4	5	6	7	8
N_2	1						
N_3		2	1				
N_4			1				
N_5				4	2		
N_6					3		
N_7						6	3
N_8							4

Table 8: $Z < x, y > /((x^3 + x^2y + y^2x + y^3)(x^2 + y^2))$

	2	3	4	5	6	7	8
N_2	1						
N_3		1					
N_4							
N_5							
N_6							
N_7							
N_8							

Table 9: $Z < x, y > /((x^3 + xyx + yxy + y^3)(x^2 + y^2))$

	2	3	4	5	6	7	8
N_2	1	1					
N_3		2	2				
N_4			2				
N_5				3			
N_6					1		
N_7							
N_8							

Table 10: $Z < x, y > /((x^3 + x^2y + y^3)(x^2 + y^2))$

C Observation 3.

	2	3	4	5	6	7	8	9
N_2	1	(2^2)	(2^3)	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)
N_3		2	$(2^2.16)$	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)
N_4			3	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)
N_5				4	$(2^3.16.784)$	(2^6)	(2^7)	(2^8)
N_6					5	(2^6)	(2^7)	(2^8)
N_7						6	$(2^4.16^2.544)$	(2^8)
N_8							7	$(2^6.98^2)$
N_9								8

Table 11: $Z < x, y > /(5xy - 3yx)$

	2	3	4	5	6	7	8	9
N_2	1	(2^2)	(2^3)	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)
N_3		2	$(2^2.40)$	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)
N_4			3	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)
N_5				4	$(2^3.16.12640)$	(2^6)	(2^7)	(2^8)
N_6					5	(2^6)	(2^7)	(2^8)
N_7						6	$(2^4.16^2.2320)$	(2^8)
N_8							7	$(2^6.316^2)$
N_9								8

Table 12: $Z < x, y > /(7xy - 3yx)$

	2	3	4	5	6	7	8	9
N_2	1	(2^2)	(2^3)	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)
N_3		2	$(2^2.24)$	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)
N_4			3	(2^4)	(2^5)	(2^6)	(2^7)	(2^8)
N_5				4	$(2^3.24.5232)$	(2^6)	(2^7)	(2^8)
N_6					5	(2^6)	(2^7)	(2^8)
N_7						6	$(2^4.24^2.1776)$	(2^8)
N_8							7	$(2^6.218^2)$
N_9								8

Table 13: $Z < x, y > /(7xy - 5yx)$