Research Science Institute (RSI)

Mathematics Section

Abstracts of Final Research Papers

M.I.T.

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Rational Fixed Points of Polynomial Involutions

Kęstutis Česnavičius

A well-known theorem states that all polynomial involutions of $\mathbb{C}^n$ have at least one fixed point. A problem that Jean-Pierre Serre posed is whether the involution $F = (f_1, \ldots, f_n)$ where $f_i \in \mathbb{Q}[x_1, \ldots, x_n]$ has always a rational fixed point. We prove Serre's Conjecture for the affine case as well as the case when $\deg f_i = 1$ for $1 \leq i \leq n-1$ and $\deg f_n = d$ for any positive integer d. In addition, we also construct non-trivial examples of involutions for the case $n = 3$. Finally, we introduce the notion of *multidegree of a polynomial map* $F = (f_1, \ldots, f_n)$ as $M(d_1, \ldots, d_n)$ where $d_i = \deg f_i$, and we prove the following theorem:

Theorem. *Let a and b be positive integers. Then for $n = 1$ and also for all $n \geq \frac{(a-s)(b-s)}{s} - s$ and divisible by $s = \gcd(a, b)$ there exists a rational polynomial involution with multidegree $M(ab, ab^2, n)$.*

Megan Belzner

Emptying Sets: The Cookie Monster Problem

Wuttisak Trongsiriwat

Given a set of integers $S = \{k_1,\ k_2,\ \ldots,\ k_n\}$, the Cookie Monster Problem is the problem of making all elements of the set equal 0 in the minimum number of moves. Consider the analogy of cookie jars with distinct numbers of cookies, such that k_i is the number of cookies in the ith jar. The "Cookie Monster" wants to eat all the cookies, but at each move he must choose some subset of the jars and eat the same amount from each jar. The *Cookie Monster Number of* S, $CM(S)$, is the minimum number of such moves necessary to empty the jars. It has been shown previously that $\lceil \log_2(|S| + 1) \rceil \leq CM(S) \leq |S|$. In this paper we classify sets by determining what conditions are necessary for $CM(S)$ to equal 2 or 3 and what effect certain restrictions have on $CM(S)$. We also provide an alternative interpretation of the problem in the form of a combinatorial game and analyze the losing positions.

Rebecca Chen

A Computational Analysis of Triangle Subdivision

Prof. David Jerison and Mr. Wenzhe Wei

Subdivision schemes in computer-aided design are used to generate smoothly sculptured surfaces from coarse polygonal meshes. A determining factor in how smooth a refined mesh appears is the degree of regularity in the mesh's polygonal faces. However, to the best of our knowledge, there currently exists no method to test for regularity. This paper proposes the division of each polygonal face in a mesh into triangular sub-faces that can then be tested for flatness, a measure of irregularity, using a triangular shape z.

Sitan Chen

On the Rank Number of Grid Graphs

Jesse Geneson

A vertex k-ranking is a labeling of a graph with integers from 1 to k so any path between two vertices with the same label contains a vertex with a greater label. The minimum possible k for which a k-ranking exists is the rank number. For grid graphs, the rank number of $m \times n$ grid graphs has been found only for $m \leq 3$. In this paper, we determine its exact value for $4 \times n$ grid graphs and improve the upper bound for general grids. Finally, we improve lower bounds on the rank number for square and triangle grid graphs from logarithmic to linear. This new lower bound is key to characterizing the rank number for general grids. The new ideas on cut sets that we introduce can be used to study very large-scale integration (VLSI) circuits' graph separators, which have direct applications in optimizing area efficiency.

Sidharth Dhawan

Complexity of Interlocking Polyominoes

Zachary Abel

Polyominoes are a subset of polygons which can be constructed from integer-legnth squares fused at their edges. A system of polygons P is interlocked if no subset of the polygons in P can be removed arbitrarily far away from the rest. It is already known that polyominoes with four or fewer squares cannot interlock. It is also known that determining the interlockedness of polyominoes with an arbitrary number of squares is PSPACE hard. Here, we attempt determine which polyominoes are too simple to interlock and for which polyominoes determining interlockedness becomes PSPACE hard. We prove that polyominoes with five or fewer squares cannot interlock and that polyominoes with six or more squares can interlock. Also, we show that determining interlockedness of polyominoes with nine or more squares is PSPACE hard.

Eric Mannes

Bounds on Monotone Switching Networks for the Matching Problem

Aaron Potechin

Lower bounds on the space required for a computation are important results that demonstrate the limitations of computing. The problem studied here is to decide if a graph G has a k-matching. A k-matching is k distinct pairs of vertices such that each pair is joined by an edge.

The switching network model of computation can be used to find lower bounds, but so far, it has barely been explored. A switching network solving the k-matching problem is a second graph G' with two special vertices s' and t' that are connected by edges in G' if and only if the original graph G has a k-matching. We examine the special case of monotone switching networks in which only the presence of edges in G not their absence is used to help determine whether G has a k-matching.

We determine that for fixed k, the minimum size of a monotone switching network solving k-matching on a graph with n vertices is of order of magnitude $\log n$. We use a probabilistic argument for the upper bound and apply Potechin's technique of states of knowledge for the lower bound. We study certain-knowledge switching networks, a special class of monotone switching networks, determine that for fixed k their minimum size is of order of magnitude n^{2k-2}, and find other lower bounds in case k increases with n.

Todor Markov

On Extremal Degrees of Minimal Ramsey Graphs

Wuttisak Trongsiriwat

Let F, G and H be simple graphs. We say that F is (G, H)-Ramsey if any coloring of the edges of F in red and blue contains either a red subgraph isomorphic to G or a blue subgraph isomorphic to H. Furthermore, if the above property is not retained after removing some edge or vertex from F, then F is called (G, H)-minimal. We define $s(G, H)$ to be the minimal degree of any vertex of a (G, H)-minimal graph, and $\bar{s}(G, H)$ to be the minimum possible maximum degree of any (G, H)-minimal graph. We prove that $\bar{s}(G, H) \geq \Delta(G) + \Delta(H) - 1$ if $\Delta(G)$ and $\Delta(H)$ are both odd and $\bar{s}(G, H) \geq \Delta(G) + \Delta(H) - 2$ otherwise, where $\Delta(G)$ is the maximum degree of any vertex of G. We also prove that $s(K_{1,m}, T) = 1$ for any positive integer m and tree T, where $K_{1,m}$ is a star with $m + 1$ vertices.

Jessica Oehrlein

Book Thickness of Graphs and their Subdivisions

Aaron Potechin

A book embedding of a graph is an arrangement of vertices and edges with applications to very large scale integration (VLSI) design. A page is formed by labeled vertices arranged on a straight line called the spine and a half-plane of edges in which no edges cross. A book is the union of these pages. The minimum number of pages necessary to embed a graph G in a book is known as the book thickness of G, denoted $sn(G)$. We show that the book thickness of any subdivision of G is at most twice $sn(G)$. We also explore the number of pages necessary to embed a graph G in a book given a fixed vertex ordering. We find that this number of pages is exactly the chromatic number of the graph whose vertices are edges of G and in which there is an edge connecting two vertices if the two edges of G cross in a book embedding.

Matthew Rauen

On Strongly Multiplicative Graphs

Jesse Geneson

We analyze the problem of finding the maximal number of edges on a strongly multiplicative graph on n vertices, a problem which has applications to network modeling. Such a graph has n vertices labeled by the integers $1, 2, \ldots, n$ such that if each edge is labeled with the product of adjacent vertices, no two edges have the same label. This value is denoted by $\lambda(n)$, and we construct an analogous function where the two factors, a and b, are chosen from sets of differing cardinalities, denoted by $f(x, y)$. We establish the difference function $\delta_f(x, y)$ which is equal to the number of products constructible for the first time as the cardinality of the set of cardinality $y - 1$ is increased by 1. We prove the periodicity and symmetry of $\delta_f(x, y)$ and use it to create a linear approximation for $f(x, y)$ for fixed x in terms of y, and prove this approximation is the least squares regression line.

Abraham Shin

Analysis of Discrete Gaussian Free Field in Random Surfaces on the Motion of Strings

Wenzhe Wei

The Liouville Quantum Gravity model proposed by Polyakov is used to study motion of strings. Quantum gravity measures are assigned to each vertex on a grid graph and the edges that connect vertices are weighted based on the height values assigned to vertices. We look for the relationship between the size of the grid and the distance a string travels. In a finite discrete Gaussian free field (GFF), we discover that the power of the number of sides in a GFF is proportional to the distance from the center of the GFF to its half-to-boundary points. We find the equation $d = .1993n^{.7299}$, where d and n are the distance from the center and the number of sides of a grid, respectively. We determine the contour curves have Hausdorff dimension 2.1.

Adam H. Su

Rank-Generating Functions for the Distributive Lattice of Order Ideals for Comb Posets

Benjamin Iriarte

Let P_n be the poset with elements $s_{i,1}, s_{i,2}$ for $i \in [n]$, and cover relations $s_{1,1} \lessdot s_{2,1} \lessdot \cdots \lessdot s_{n,1}$ and $s_{i,1} \lessdot s_{i,2}$ for all $i \in [n]$. The Hasse diagram of P_n resembles a comb. We derive the rank-generating function of $J(P_n)$, the distributive lattice of order ideals of P_n. We then generalize this function to $J(P_{n,m})$. We also prove bounds on the maximal rank of $J(P_n)$.

Let P be the disjoint union of n chains $C_1, C_2, \ldots, C_n$, where C_i contains elements $s_{i,1}, s_{i,2}, \ldots, s_{i,m}$. Let P' be a generalization of P where we also permit an arbitrary number of cover relations of the form $s_{i,j} \gtrdot s_{i-1,j}$ for $2 \leq i \leq n$ and $1 \leq j \leq m$. We describe a method to construct the Hasse diagram for all $J(P')$. This provides an efficient method to visualize and describe the properties of new posets we define.

Zacharias Tsampasidis

Non-divisibility of Binomial coefficients with a Given Set of Primes

Kęstutis Česnavičius

The problem we tackle here asks whether there exist infinitely many binomial coefficients of the form $\binom{2n}{n+k}$ coprime to a given set of odd primes which we denote by P. Erdős, Graham, Ruzsa and Straus proved the infinitude of such n for $k = 0$ and two primes. Here we work on the analogous result for $k = 1$ and two primes using similar ideas to their proof. We also observe that it can be modified to try to prove the result for other values of k, such as 0, 2 but the case analysis is much more involved when $k \geq 2$. For the moment we do not have a way to extend it for a greater number of primes.