Cyclic Base Orderings of Matroids

Raymond Luo

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Abstract

A cyclic base ordering of a matroid $M = (E, \mathcal{I})$ is a cyclic ordering of the elements of E such that every $r(E)$ consecutive elements form a base, where r is the rank function of M. An area of research in matroid theory asks which matroid classes exhibit cyclic base orderings under certain conditions. In this paper, we provide several necessary conditions for matching and graphic matroids to have cyclic base orderings. We also provide graph operations that preserve the existence of cyclic base orderings on graphic matroids.

1 Introduction

In 1935, Whitney [8] introduced the notion of matroids, which are combinatorial objects that generalize both the notion of linear independence from linear algebra and the notion of spanning trees from graph theory. Matroids are of significance in various mathematical fields. For example, matroids characterize problems that can be solved by greedy algorithms in the sense that a maximum weight problem in a downward closed set system can be solved by the greedy algorithm if and only if the system is a matroid. Moreover, matroid partition and matroid intersection algorithms give rise to combinatorial algorithms in concrete problems as specific cases. Other applications of matroids may be found in topology, network theory, and coding theory.

In this paper, we study the notion of cyclic base orderings of matroids. We defer basic definitions from matroid theory to Section 2. A *cyclic base ordering* of a matroid $M = (E, \mathcal{I})$ is a cyclic ordering of the elements of E such that every $r(E)$ consecutive elements form a base, where r is the rank function of M.

In this paper, we provide several conditions for the existence and structure of cyclic base orderings on graphic and matching matroids. We also look at graph operations that preserve the existence of cyclic base orderings on graphic matroids.

1.1 Prior Results

In 1988, Kajitani et al. [4] initiated the study of cyclic base orderings. They proved the existence of cyclic base orderings on various matroid classes and conjectured a characterization for matroids that exhibited cyclic base orderings. In particular, they showed that any graphic matroid whose corresponding graph can be decomposed into two edge-disjoint spanning trees exhibits a cyclic base ordering. It is then natural to ask which matroid classes have cyclic base orderings, given that the ground set of the matroid can be partitioned by some number of bases.

Problem 1.1 (Kajitani et al. [4]). *Let $M = (E, \mathcal{I})$ be a matroid. Suppose E can be partitioned into k bases. Is there a cyclic ordering of the elements of E such that any $|E|/k$ consecutive elements in the ordering form a base?*

Cyclic base orderings give stronger conclusions than linear orderings of the elements of the ground set. In fact, replacing "cyclic ordering" with "linear ordering" in the statement of Problem 1.1 makes the problem considerably easier. Kajitani et al. [4] provided a complete characterization for any matroid $M = (E, \mathcal{I})$ with rank function r that exhibits a linear ordering of E such that every w consecutive elements form an independent set for all positive integers $w \leq r(E)$.

Some progress has been made on Problem 1.1. Kajitani et al. [4] proved the case when $k = 2$ and M is graphic.

Theorem 1.2 (Kajitani et al. [4]). *Suppose a graph $G = (V, E)$ can be decomposed into two edge-disjoint spanning trees. Then there is a cyclic ordering of the edges of G such that every $|V| - 1$ consecutive edges in the cyclic ordering induces a spanning tree.*

Despite this result from the very beginning of the quest to answer Problem 1.1, much is unknown. For instance, the case where $k \geq 3$ and M is graphic is still open. Another natural class of matroids is the class of linear matroids. Indeed, this special case is also open even when $k = 2$. Other attempts in the literature studied the case where the assumption in Problem 1.1 that the ground set can be partitioned into k bases is dropped or replaced with stronger assumptions. For instance, van den Heuvel and Thomassé [7] proved a necessary and sufficient condition for all matroids $M = (E, \mathcal{I})$ with $\gcd(|E|, r(E)) = 1$ to have cyclic base orderings, where r is the rank function of M.

Theorem 1.3 (van den Heuvel and Thomassé [7]). *Let M be a matroid on ground set E with rank function r such that $\gcd(|E|, r(E)) = 1$. Then there is a cyclic ordering of the elements of E such that any consecutive $r(E)$ elements form a base if and only if*

$$r(E) \cdot |X| \leq |E| \cdot r(X) \tag{1}$$

holds for every $X \subseteq E$.

In 2013, Bonin [2] proved that any sparse paving matroid $M = (E, \mathcal{I})$ with rank function r has a cyclic base ordering if and only if (1) holds for all $X \subseteq E$, and very recently, in 2023, McGuiness [6] proved that any paving matroid $M = (E, \mathcal{I})$ with rank function r has a cyclic base ordering if and only if (1) holds for all $X \subseteq E$.

Another line of research studied special classes of graphs whose associated graphic matroids have cyclic base orderings without the partition assumption. This started from the paper of Kajitani et al. [4]:

Theorem 1.4 (Kajitani et al. [4]). *The graphic matroid of each of the following classes of graphs has a cyclic ordering in which every w consecutive edges of the cyclic ordering form an independent set:*

- *graphs decomposable into two disjoint spanning trees for all positive integers $w \leq r$,*
- *simple graphs for $w = 3, 4$,*
- *complete graphs for all positive integers $w \leq r$,*
- *2-trees*[1] *for all positive integers $w \leq r$,*

where r denotes the rank of the corresponding graphic matroid.

In 2014, Gu et al. [3] constructed cyclic base orderings for the graphic matroids of complete bipartite graphs, k-maximal graphs, and 3-trees. In 2021, Li et al. [5] proved that the graphic matroids of squares of cycles, wheel graphs, double wheel graphs all have cyclic base orderings. In 2022, Xia et al. [9] proved that the existence of a cyclic base ordering is closed under taking the series composition of two graphs with the same numbers of vertices and edges; they also proved the cases of generalized theta graphs and small triangular grid graphs.

1.2 Organization of the Paper

This paper is structured as follows. In Section 2, we introduce definitions and results from matroid theory used in subsequent sections. In Section 3, we give conditions that are necessary for the existence of cyclic base orderings in matching matroids and graphic matroids. We also investigate graph operations that preserve the existence of a cyclic base ordering in graphic matroids. In Section 4, we revisit Theorem 1.2 and prove an extension of it. Finally, in Section 5, we discuss possible future directions.

2 Preliminaries

First, we introduce basic definitions and properties from matroid theory.

Definition 2.1. A *matroid* M is an ordered pair $(E, \mathcal{I})$, where E is a finite set which we call the *ground set*, and $\mathcal{I}$ is a collection of subsets of E, whose elements we call the *independent sets*, such that the following two axioms are satisfied:

(I_1) If $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

(I_2) If $X, Y \in \mathcal{I}$ and $|Y| > |X|$, then there exists $u \in Y \backslash X$ such that $X + u \in \mathcal{I}$.[2]

The *rank function* of M is a set function $r : 2^E \to \mathbb{Z}_{\geq 0}$ which is defined as follows for all $X \subseteq E$:

$$r(X) := \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

In addition, we define the *bases* of M to be its maximal independent sets.

[1] A graph G is a *k-tree* if it is either the complete graph on k vertices or if G has a vertex v with degree $k-1$ such that v and its neighbors form a k-clique and $G - v$ is also a k-tree. 1-trees are simply trees themselves, and 2-trees are maximal series-parallel graphs.

[2] For a set X and an element u, we write $X + u = X \cup \{u\}$.

Note that all bases of a matroid have the same cardinality. The following example illustrates the aforementioned definitions.

Example 2.2. Let E be a set and let $\mathcal{I}$ be the power set of E. Then, $M = (E, \mathcal{I})$ is said to be the *trivial matroid*. All subsets of E are independent and the only base of M is E itself.

Recall that a cyclic base ordering of a matroid $M = (E, \mathcal{I})$ with rank function r is an ordering of the elements of E such that every $r(E)$ consecutive elements form a base. Concretely, if $E = \{x_1, \ldots, x_{|E|}\}$, we say $\mathcal{O} = (x_{\pi(1)}, \ldots, x_{\pi(|E|)})$ is an *ordering*, where π is a permutation of $\{1, \ldots, |E|\}$, and we define $\mathcal{O}(i) = x_{\pi(i)}$, and $\mathcal{O}^{-1}(x_{\pi(i)}) = i$. Moreover, we say $\mathcal{O}$ is a *cyclic base ordering* if $\{x_{\pi(j)}, \ldots, x_{\pi(j)+r(E)-1}\}$ forms a base for $j = 1, \ldots, |E|$ and all indices are taken modulo $|E|$. In this paper, we focus on two matroid classes that are defined on graphs, specifically, graphic matroids and matching matroids.

Definition 2.3 (Graphic matroids)**.** Let $G = (V, E)$ be a graph. Define

$$\mathcal{I} := \{S \subseteq E : (V, S) \text{ does not contain a cycle}\}.$$

Then $M := (E, \mathcal{I})$ is called the *graphic matroid* associated with G. It can be checked that graphic matroids do indeed satisfy the matroid axioms.

It turns out that the independent sets of the graphic matroid associated with a graph are exactly the forests of the graph. Moreover, the bases of a graphic matroid are precisely the maximal acyclic subgraphs of the associated graph. In particular, if a graph is connected, then the bases of its graphic matroid are exactly the spanning trees of the graph. If $\kappa(G)$ denotes the number of connected components of G, then the size of a base is $|V| - \kappa(G)$.

Now, we introduce matching matroids.

Definition 2.4 (Matching matroids)**.** Let $G = (V, E)$ be a graph. Define

$$\mathcal{I} := \{S \subseteq V : S \text{ is covered by some matching of } G\}.$$

Then we call $M := (V, \mathcal{I})$ the *matching matroid* associated with G. It can be checked that matching matroids do indeed satisfy the matroid axioms, but the proof is nontrivial.

The bases of a matching matroid are the vertices of maximum matchings and the independent sets are the subsets of vertices that are covered by a matching.

3 Conditions for Existence of Cyclic Base Orderings

In this section, we provide several necessary conditions for the existence of cyclic base orderings on matching matroids and graphic matroids. We also provide several operations which preserve the existence of cyclic base orderings on graphic matroids.

3.1 Matching Matroids

In this subsection, we focus on matching matroids on bipartite graphs. First, it is easy to verify that there indeed exists a bipartite graph of arbitrarily large size whose matching matroid exhibits a cyclic base ordering.

Proposition 3.1. *The matching matroid associated with the complete bipartite graph $K_{n,n}$ has a cyclic base ordering.*

Indeed, this is true because the only base of $K_{n,n}$'s associated matching matroid is the entire vertex set. In the following two theorems, we give necessary conditions for matching matroids to have cyclic base orderings.

Theorem 3.2. *Let $G = (V, E)$ be a graph, and let $\nu(G)$ be the* matching number *of G, i.e., the maximum cardinality of a matching in G. The matching matroid M of G has no cyclic base ordering if $4\nu(G) \leq |V|$.*

Proof. Suppose, for the sake of contradiction, that $4\nu(G) \leq |V|$ and a cyclic base ordering of M exists. Note that the size of a base of M is $2\nu(G)$. If $4\nu(G) \leq |V|$, then the cyclic base ordering contains two disjoint cyclic intervals of length $2\nu(G)$. Hence, there exist two disjoint bases, V_1 and V_2. Let E_1 and E_2 be the two matchings that cover V_1 and V_2, respectively. Consider $E_1 \cup E_2$. Since the sets of vertices that E_1 and E_2 cover, respectively, are disjoint, it follows that $E_1 \cup E_2$ is also a matching. However, $E_1 \cup E_2$ covers vertices from both bases, which contradicts the maximality of bases. $\square$

Theorem 3.3. *Let G be a bipartite graph with vertex partition (A, B). Then the matching matroid of G does not have a cyclic base ordering if $|A| \neq |B|$.*

Proof. Suppose that $|A| \neq |B|$. Let M be the matching matroid of G, and let the size of a base in M be s. Suppose, for the sake of contradiction, that a cyclic base ordering exists. Note that every base in M contains an equal number of vertices in A and in B. We count the number of vertices from A that are included in the cyclic base ordering. There are $|A| + |B|$ different bases in the cyclic base ordering. Since each base has exactly $\frac{s}{2}$ vertices from A, there are $\frac{s}{2} \cdot (|A| + |B|)$ elements from A with overcounts. However, since each element is represented in s different bases, the number of vertices from A with no overcounts is $\frac{s}{2} \cdot (|A| + |B|)$ divided by s. Hence, we have

$$|A| = \frac{\frac{s}{2} \cdot (|A| + |B|)}{s} = \frac{|A| + |B|}{2}.$$

This implies that $|A| = |B|$, a contradiction. $\square$

Theorems 3.2 and 3.3 give necessary but not sufficient conditions for matching matroids to have cyclic base orderings. Figure 1 shows an example of a graph that satisfies both of the conditions stated in Theorems 3.2 and 3.3. However, as the graph is disconnected, the two isolated vertices can never be covered by an edge; thus, its matching matroid does not have a cyclic base ordering.

Next, we describe a necessary condition in the structure of cyclic base orderings of a class of matching matroids.

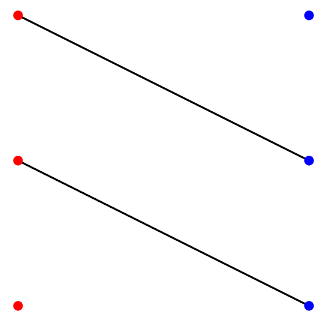

Figure 1: A bipartite graph with vertex partition (A, B) such that $|A| = |B|$ whose matching matroid has no cyclic base ordering.

Proposition 3.4. *Let $G = (V, E)$ be a bipartite graph with vertex partition (A, B). Let M be the matching matroid of G with rank function r. Suppose that $|A| = |B|$ and $r(V) = |A| + |B| - 2$. If M has a cyclic base ordering, then no two consecutive vertices in the cyclic base ordering can be both from A or both from B.*

Proof. Let $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{O} = \{t_1, t_2, \ldots, t_{2n}\}$ be a cyclic base ordering of M. First, since the set $\{t_1, t_2, \ldots, t_{2n-2}\}$ is a base, it must contain an equal number of vertices from A and from B. Thus, one of t_{2n-1} and t_{2n} is from A and the the other is from B. Hence, without loss of generality, t_i is from A if i is odd, and from B if i is even. Applying the same argument for all consecutive pairs of vertices shows that no two vertices in the cyclic base ordering can be both from A or both from B. In particular, the vertices in the cyclic base ordering must alternate between coming from A and coming from B. $\square$

This conclusion only follows with the assumption that $r(V) = |A| + |B| - 2$. If the rank was any less, then the conclusion would not hold in general. This is because if the rank was $r(V) = |A| + |B| - 2k$ for $k > 1$, then the cyclic base ordering could alternate vertices from A and B every k vertices. We hope these results on matching matroids can shed more light on which matching matroids contain cyclic base orderings.

3.2 Graphic Matroids

In this subsection, we study conditions for the existence of a cyclic base ordering in a graphic matroid. In particular, we are interested in graph operations that preserve the existence of a cyclic base ordering.

Theorem 3.5. *Let $G = (V, E)$ be a graph whose graphic matroid contains a cyclic base ordering. Then no vertex has degree less than $\left\lfloor \frac{|E|}{|V| - \kappa(G)} \right\rfloor$.*

Proof. Let M be the graphic matroid of G and let $\mathcal{O}$ be a cyclic base ordering for M. Since there are $|E|$ elements in $\mathcal{O}$ and the size of a base is $|V| - \kappa(G)$, there exist at least $\left\lfloor \frac{|E|}{|V| - \kappa(G)} \right\rfloor$ disjoint bases in $\mathcal{O}$. Let $v \in V$ be an arbitrary vertex. Since the edges in each base induce a spanning tree in every connected component of G, there exists an edge in each base that is connected to v. Hence, there exist at least $\left\lfloor \frac{|E|}{|V|-\kappa(G)} \right\rfloor$ edges connected to v, which means its degree is at least $\left\lfloor \frac{|E|}{|V|-\kappa(G)} \right\rfloor$, as desired. □

Now, we study operations that preserve the existence of cyclic base orderings on graphic matroids.

Theorem 3.6. *Let $G = (V, E)$ be a graph whose graphic matroid contains a cyclic base ordering and suppose $|V| - \kappa(G)$ divides $|E|$. Furthermore, assume there exists a vertex v of degree $k := \frac{|E|}{|V|-\kappa(G)}$. Then $G - v$ contains a cyclic base ordering.*

Proof. Let M be the graphic matroid associated with G and let $\mathcal{O}$ be a cyclic base ordering of M. First, partition $\mathcal{O}$ into k disjoint bases. Since each base must contain at least one edge that is connected to v and $\deg(v) = k$, it follows that each base contains *exactly* one edge connected to v. We claim that removing all edges connected to v in $\mathcal{O}$ results in a cyclic base ordering for $G - v$. Let B be a base consisting of arbitrary, consecutive k elements in $\mathcal{O}$. Let e be the only edge in B that is connected to v, and let T be the induced spanning tree of the connected component in which e is located in. There only exists one edge, namely e, connected to v in T, so removing e will result in a spanning tree. Hence, the new cyclic ordering is a cyclic base ordering for $G - v$. □

Theorem 3.7. *Let $G = (V, E)$ be a graph whose graphic matroid contains a cyclic base ordering and assume $|V| - \kappa(G)$ divides $|E|$. Let G' be a graph obtained by adding a new vertex v with $k := \frac{|E|}{|V|-\kappa(G)}$ edges connecting it to some connected component of G. Then G' contains a cyclic base ordering.*

Proof. Let M be the graphic matroid of G and let $\mathcal{O}$ be a cyclic base ordering of M. Partition $\mathcal{O}$ into k disjoint bases, and insert an edge connected to v in between every two consecutive disjoint bases. We claim that this creates a new cyclic base ordering. Let us show that every consecutive $|V| + 1$ edges creates a base. Note that every consecutive $|V| + 1$ edges contains exactly one edge, e, that is connected to v. Let H be the graph induced by the other $|V|$ vertices. If we include e in H, then the induced spanning tree in the connected component that includes e still stays an induced spanning tree. Hence, the new cyclic ordering is a cyclic base ordering. □

4 Revisiting the $k = 2$ Case of Graphic Matroids

Theorem 1.2 states that if a graph can be decomposed into two edge-disjoint spanning trees, its associated graphic matroid contains a cyclic base ordering. In this section, we prove that there exists a cyclic base ordering that must have a specific structure.

Theorem 4.1. *Suppose a graph $G = (V, E)$ can be decomposed into two edge-disjoint spanning trees, T_1 and T_2. Then there exists a cyclic base ordering where $|V| - 1$ consecutive elements are the edges of T_1 and the other $|V| - 1$ consecutive elements are the edges of T_2.*

Proof. Let $r = |V| - 1$ and note that $|E| = 2r$. We induct on r. If $r = 1$ (the graph has two vertices), then the theorem is vacuously true. Suppose $r \geq 2$. By Theorem 3.5, each vertex has degree at least 2. Moreover, we have

$$\sum_{v \in G} \deg(v) = 2|E| = 4(|V| - 1) = 4|V| - 4.$$

Thus, by the pigeonhole principle, there exists a vertex with degree less than 4. Hence, there exists a vertex v of degree 2 or 3.

Suppose $\deg(v) = 2$. Let its two neighbors be a and b. Assume, without loss of generality, that $(v, a) \in E(T_1), (v, b) \in E(T_2)$. Note that $G - v$ can be decomposed into two edge-disjoint spanning trees, $T_1 - (v, a)$ and $T_2 - (v, b)$. By the inductive hypothesis, there exists a cyclic base ordering $\mathcal{O}'$ for the graphic matroid of $G - v$ such that $\{\mathcal{O}'(1), \dots, \mathcal{O}'(r-1)\} = E(T_1 - (v, a))$ and $\{\mathcal{O}'(r), \dots, \mathcal{O}'(2r-2)\} = E(T_2 - (v, b))$. Now, we define a cyclic base ordering $\mathcal{O}$ for the graphic matroid of G as follows:

$$\mathcal{O}^{-1}(e) = \begin{cases} \mathcal{O}'^{-1}(e) & e \in E(G - v) \text{ and } \mathcal{O}'^{-1}(e) \leq r - 1 \\ r & e = (v, a) \\ \mathcal{O}'^{-1}(e) + 1 & e \in E(G - v) \text{ and } \mathcal{O}'^{-1}(e) \geq r \\ 2r & e = (v, b). \end{cases}$$

Let S be a set of r consecutive elements in $\mathcal{O}$. Note that there is exactly one edge in S that is connected to v. Let this edge be e. Moreover, note that the $r - 1$ edges in $S - e$ are consecutive elements in $\mathcal{O}'$ and induce a spanning tree in $G - v$. Adding e to the $r - 1$ edges induces a spanning tree in G. Thus, $\mathcal{O}$ is indeed a cyclic base ordering. Moreover, note that

$$\begin{aligned} E(T_1) &= \{\mathcal{O}(1), \dots, \mathcal{O}(r)\}, \\ E(T_2) &= \{\mathcal{O}(r+1), \dots, \mathcal{O}(2r)\}, \end{aligned}$$

completing the inductive step.

Suppose $\deg(v) = 3$. Let a, b, c be the three vertices that are adjacent to v. Without loss of generality, suppose $(v, a), (v, b) \in E(T_1)$ and $(v, c) \in E(T_2)$. Let $G' = G - v + (a, b)$. By the inductive hypothesis, G' can be decomposed into two edge-disjoint spanning trees so it has a cyclic base ordering $\mathcal{O}'$. Without loss of generality, set $\mathcal{O}'^{-1}((a, b)) = 1$. By the inductive hypothesis, we may also assume that $\{\mathcal{O}'(1), \dots, \mathcal{O}'(r-1)\} = E(T_1 - v + (a, b))$ and $\{\mathcal{O}'(r), \dots, \mathcal{O}'(2r-2)\} = E(T_2 - v)$. Thus, we can define an ordering $\mathcal{O}$ for the graphic matroid of G as follows:

$$\mathcal{O}^{-1}(e) = \begin{cases} 1 & e = (v, a) \\ \mathcal{O}'^{-1}(e) & e \in E(G') \text{ and } 2 \leq O'^{-1}(e) \leq r - 1 \\ r & e = (v, c) \\ \mathcal{O}'^{-1}(e) + 1 & e \in E(G') \text{ and } O'^{-1}(e) \geq r \\ 2r & e = (v, b). \end{cases}$$

Let S be a set of r consecutive elements from $\mathcal{O}$. If S contains only one edge connected to v, then the other $r - 1$ edges induce a spanning tree in $G - v + (a, b)$. Thus, adding the edge connected to v to the $r-1$ edges induces a spanning tree in G. If S contains both (v,a) and (v,b), then the edges in the set $S-(v,a)-(v,b)+(a,b)$ induce a spanning tree in $G-v+(a,b)$. Removing (a,b) and adding (v,a) and (v,b) induces a spanning tree in G. Thus, $\mathcal{O}$ is a cyclic base ordering. Moreover, note that

$$E(T_1)=\{\mathcal{O}(2r),\mathcal{O}(1),\dots,\mathcal{O}(r-1)\},$$
$$E(T_2)=\{\mathcal{O}(r),\dots,\mathcal{O}(2r-1)\},$$

completing the inductive step. □

Our hope is that Theorem 4.1 can potentially be useful in the $k=3$ case of Problem 1.1 when M is graphic. If we approach the $k=3$ case in the same way as the $k=2$ case, we start out by noting that there exists a vertex of degree $3,4$, or 5. When the degree is 3 or 4, the solution is analogous to the degree 2 and 3 cases when $k=2$, respectively. The more difficult case is when the degree is 5. This case is especially nontrivial because the inductive step does not work as expected; the operation preserving the cyclic base ordering from $r-1$ to r is not readily apparent. Perhaps Theorem 4.1 can be the missing piece in the puzzle and provide a potential way to complete the inductive step.

5 Concluding Remarks

In this paper, we first investigated necessary conditions for matching matroids and graphic matroids to exhibit a cyclic base ordering. We also looked at operations that preserve the existence of cyclic base orderings on graphic matroids. In addition, we were able to apply some of our results to obtain more specific structures on a result by Kajitani et al. [4]. We now discuss other open questions related to Problem 1.1.

Problem 5.1. *Do graphic matroids have an affirmative answer to the $k=3$ case of Problem 1.1?*

Problem 1.1 is connected to other important open problems in the literature. For instance, the following problem is related to the $k=2$ case of Problem 1.1.

Problem 5.2. *If the ground set E of a matroid can be partitioned into 2 bases, then for any set $X\subseteq E$, is there a base B such that $E\backslash B$ is also a base and $\lfloor |X|/2\rfloor \le |B\cap X|\le \lceil |X|/2\rceil$?*

Matroids with an affirmative answer to Problem 5.2 are said to be *equitable*. Indeed, equitability is a weaker notion than the existence of cyclic base orderings for the $k=2$ case.

Proposition 5.3. *If a matroid's ground set can be decomposed into 2 distinct bases and the matroid exhibits a cyclic base ordering, it is equitable.*

Proposition 5.3 is well-known, but its proof is not found in the literature so we include it for completeness below.

Proof. Let $M = (E, \mathcal{I})$ be a matroid that satisfies the aformentioned properties, and let $\mathcal{O} = (x_1, \dots, x_{|E|})$ be a cyclic base ordering. Clearly, the ground set can be partitioned into 2 bases: $\{x_1, \dots, x_{|E|/2}\}$ and $\{x_{|E|/2+1}, \dots, x_{|E|}\}$. Let $X \subseteq E$.

We first claim that there exists a base $B_1 = \{x_a, \dots, x_{a+|E|/2-1}\}$ such that $|B_1 \cap X| \leq \lceil |X|/2 \rceil$. Suppose, for the sake of contradiction, that $|B \cap X| > \lceil |X|/2 \rceil$ for all bases B. Then, summing $|B \cap X|$ over all bases $B = \{x_j, \dots, x_{j+|E|/2-1}\}$, where $1 \leq j \leq |E|$, we have

$$\sum_{B=\{x_j,\dots,x_{j+|E|/2-1}\}} |B \cap X| > |E| \cdot \left\lceil \frac{|X|}{2} \right\rceil.$$

Each element of X is counted $|E|/2$ times in the sum, so the inequality becomes

$$\frac{|E|}{2} \cdot |X| > |E| \cdot \left\lceil \frac{|X|}{2} \right\rceil.$$

This implies that $|X|/2 > \lceil |X|/2 \rceil$, a contradiction. Similarly, there exists a base $B_2 = \{x_b, \dots, x_{b+|E|/2-1}\}$ such that $|B_2 \cap X| \geq \lfloor |X|/2 \rfloor$. Without loss of generality, suppose $b \leq a$. Let $S = \{x_b, \dots, x_{b+|E|/2-1}\}$ be a base that is initially equal to B_2. Then $|S \cap X| \geq \lfloor |X|/2 \rfloor$. We want to gradually change S to be identical to B_1 by adding and removing one element at a time. More specifically, if $S = \{x_i, \dots, x_{i+|E|/2-1}\}$ for some i, then we will remove x_i from S and add $x_{i+|E|/2}$. We will repeat this process until $S = B_1$ at which point $|S \cap X| \leq \lceil |X|/2 \rceil$. It is clear that throughout this process, S remains a base because $\mathcal{O} = (x_1, \dots, x_{|E|})$ is a cyclic base ordering. Moreover, at each step, $|S \cap X|$ changes by at most 1. Thus, since $|S \cap X|$ is initially at least $\lfloor |X|/2 \rfloor$ and ends up at most $\lceil |X|/2 \rceil$, $|S \cap X|$ must equal either $\lfloor |X|/2 \rfloor$ or $\lceil |X|/2 \rceil$ at some point in the process, as desired. □

It should be noted that the technique used in the above proof can be regarded as a discrete analog of the intermediate value theorem, and is often used to prove the existence of something that satisfies two properties simultaneously. This problem and other variations of it have been studied in the literature. Aharoni et al. [1] studied a matroid intersection variant of Problem 5.2. Specifically, they proved the following weaker inequalities:

Theorem 5.4 (Aharoni et al. [1])**.** *Let M_1 and M_2 be two matroids on the same ground set E and suppose that E can be partitioned into two independent sets in both M_1 and M_2. Then for any set $X \subseteq E$, there is a set I that is independent in both M_1 and M_2 such that*

$$|I \cap X| \geq \left(\frac{1}{2} - \frac{1}{|E|}\right) |X| - 1,$$
$$|I \backslash X| \geq \left(\frac{1}{2} - \frac{1}{|E|}\right) |E \backslash X| - 1.$$

It would be interesting to improve the coefficient of $|X|$ and $|E \backslash X|$ in Theorem 5.4 to $1/2 - \alpha$ where $\alpha < 1/|E|$. Indeed, this could potentially shed more light on equitable matroids. Aharoni et al. [1] also conjectured the following, which is a strengthening of Theorem 5.4.

Conjecture 5.5 (Aharoni et al. [1]). *Let M_1 and M_2 be two matroids on the same ground set E and suppose that E can be partitioned into two independent sets in both M_1 and M_2. Then for any set $X \subseteq E$, there is a set I that is independent in both M_1 and M_2 such that*

$$|I \cap X| \geq \frac{|X|}{2} - 1,$$
$$|I \backslash X| \geq \frac{|E \backslash X|}{2} - 1.$$

This conjecture is very interesting by itself, and it can potentially lead to a better understanding of equitable matroids.

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