Homology and Brouwer's fixed point theorem

Evin Liang, Corey Zhao, Derek Zhao

MIT PRIMES

December 5, 2023

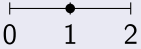
Simplices

Definition

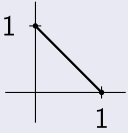
$$\Delta^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1, x_i \geq 0 \right\}$$

Example

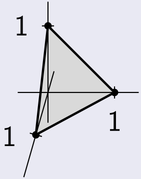
$n = 0$



$n = 1$



$n = 2$



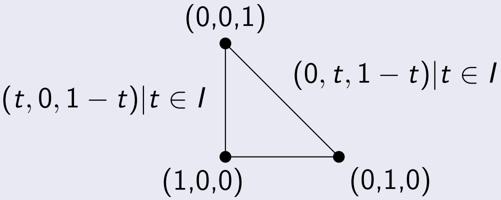
Simplicial complexes

Definition

A *face* of Δ^n is a subset $\{(x_0, x_1, \ldots, x_n) \in \Delta^n \mid x_{i_1}, \ldots, x_{i_j} = 0\}$.

Example

Triangle faces

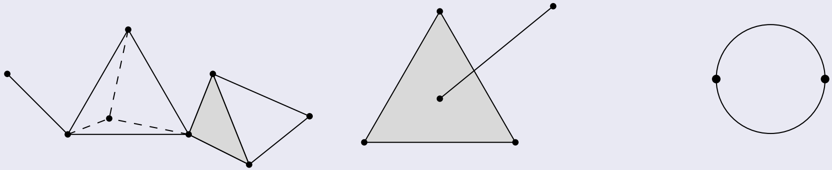


Simplicial complexes

Definition

A *simplicial complex* X is obtained by gluing together simplices along same dimensional faces such that every simplex in X is uniquely determined by its vertices.

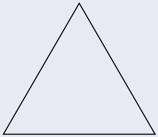
Example



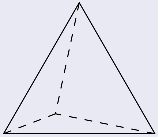
Simplicial complexes

Example

$S^1 = \partial\Delta^2$

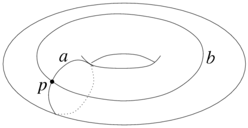


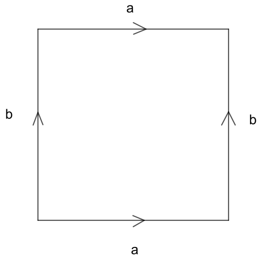
$S^2 = \partial\Delta^3$

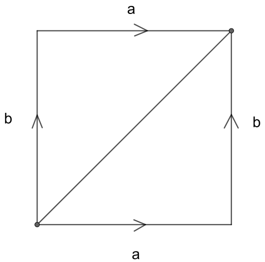


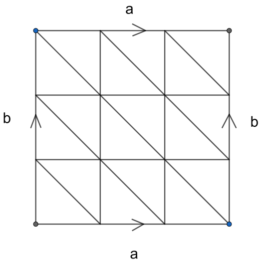
$S^n = \partial\Delta^{n+1}$

Simplicial complex of a torus T^2:









Motivation

Question

Can a simplicial complex of the torus be continuously and invertibly deformed to give a simplicial complex of the sphere?

Answer

No!

Idea

Associate algebraic objects to simplicial complexes to distinguish them.

Chain complexes

Denote the simplex in X with vertices $v_0, \ldots, v_n$ by $[v_0, \ldots, v_n]$.

Definition

The *pth chain group* of a simplicial complex X is

$$C_p(X) = \left\{ \sum_i a_i \cdot [v_{i_0}, \ldots, v_{i_p}] \mid a_i \in \mathbb{Q}, [v_{i_0}, \ldots, v_{i_p}] \text{ is a simplex of } X \right\}.$$

Definition

The *boundary operator* $\partial_p \colon C_p(X) \to C_{p-1}(X)$ is

$$\partial_p([v_0, \ldots, v_p]) = \sum_{i=0}^{p} (-1)^i \cdot [v_0, \ldots, \hat{v}_i, \ldots, v_p].$$

Homology

Proposition

$$\partial_{p-1} \circ \partial_p = 0$$

Proof.

$$\begin{aligned}
\partial(\partial([v_0, v_1, v_2])) &= \partial([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) \\
&= v_2 - v_1 - v_2 + v_0 + v_1 - v_0 \\
&= 0
\end{aligned}$$

□

Definition

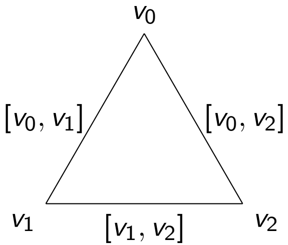
The *ith homology group* $H_i(K)$ is defined as

$$H_i(K) = \ker \partial_i / \operatorname{Im} \partial_{i+1}.$$

Remark

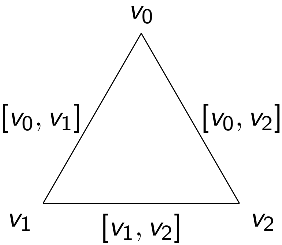
1. Quotient vector space:
   - V =vector space, $W \subset V$ a subspace.
   - $v, v' \in V$ are equivalent iff $v - v' \in W$.
   - V/W is the set of equivalence classes.
2. Since $\partial \circ \partial = 0$, $\operatorname{Im} \partial_{i+1} \subset \ker \partial_i$.

Homology of a circle



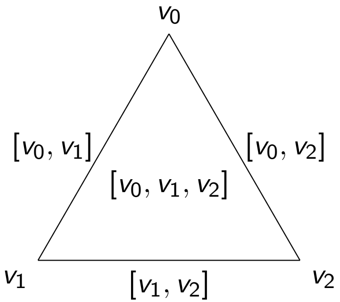
- $\ker \partial_1 = \langle [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \rangle$ and $\operatorname{Im} \partial_2 = 0$.
- $H_i(S^1) = \begin{cases} \mathbb{Q}, & i = 0, 1 \\ 0, & \text{else} \end{cases}$

Homology of a sphere



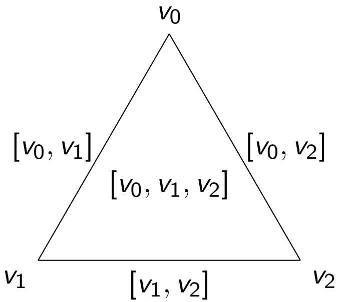
- $\ker \partial_1 = \langle [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \rangle$ and $\operatorname{Im} \partial_2 = 0$.
- $H_i(S^n) = \begin{cases} \mathbb{Q}, & i = 0, n \\ 0, & \text{else} \end{cases}$

Homology of a disc



- $\ker \partial_1 = \langle [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \rangle$ and $\operatorname{Im} \partial_2 = \langle [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \rangle$.
- $H_i(\Delta^2) = \begin{cases} \mathbb{Q}, & i = 0 \\ 0, & \text{else} \end{cases}$

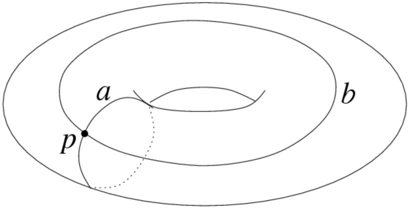
Homology of a disc



- $\ker \partial_1 = \langle [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \rangle$ and $\operatorname{Im} \partial_2 = \langle [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \rangle$.
- $H_i(\Delta^n) = \begin{cases} \mathbb{Q}, & i = 0 \\ 0, & \text{else} \end{cases}$

Homology of a Torus

$$H_i(T^2) = \begin{cases} \mathbb{Q} & i = 0 \\ \mathbb{Q}^2 & i = 1 \\ \mathbb{Q} & i = 2 \end{cases}$$



Properties

Theorem

If X is a space, then $H_\bullet(X)$ does not depend on the simplicial complex.

Theorem

$f: X \to Y$ a continuous map of simplicial complexes.

1. *We can produce a matrix $f_*: H_i(X) \to H_i(Y)$.*
2. *Given $g: Y \to Z$, $g_* \circ f_* = (g \circ f)_*$.*

Application of Homology

Definition

Two spaces X and Y are homeomorphic if there exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Application of Homology

Theorem

The sphere is not homeomorphic to the torus.

Proof.

- Suppose $S^2 \cong T^2$.
- There are continuous maps $f \colon S^2 \to T^2$ and $g \colon T^2 \to S^2$.
- These induce maps $f_* \colon H_1(S^2) \to H_1(T^2)$ and $g_* \colon H_1(T^2) \to H_1(S^2)$.
- $g_* \circ f_* = \mathrm{id}_{S^2}$ and $f_* \circ g_* = \mathrm{id}_{T^2}$, so f_* and g_* are invertible.
- Contradiction!

□

Brouwer

Theorem

Let $f\colon \Delta^n \to \Delta^n$ be a continuous mapping. Then there exists a point $x \in \Delta^n$ such that $f(x) = x$.

Notation

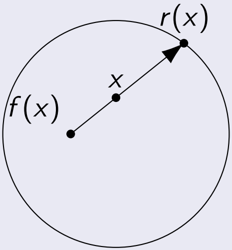
$Y \subset X$ a subspace, let $i\colon Y \to X$ denote the continuous inclusion.

Example

$S^{n-1} = \partial\Delta^n \subset \Delta^n$. We get an inclusion $i\colon S^{n-1} \to \Delta^n$.

Proof.

- Suppose that f has no fixed points.
- We get a map $r\colon \Delta^n \to \partial\Delta^n = S^{n-1}$:



- $r \circ i = \mathrm{id}_{S^{n-1}}$.

Brouwer

Proof.

$$\begin{array}{ccccc} & & \overset{r_* \circ i_* = \mathrm{id}_{*_{S^{n-1}}}}{\curvearrowright} & & \\ H_{n-1}(S^{n-1}) & \xrightarrow{i_*} & H_{n-1}(\Delta^n) & \xrightarrow{r_*} & H_{n-1}(S^{n-1}) \\ \| & & \| & & \| \\ \mathbb{Q} & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} \end{array}$$

Contradiction! □

Questions?

Acknowledgements

We thank our mentor Alex Pieloch for his guidance and assistance on this project.

We thank Slava Gerovitch, Pavel Etingof, and Tanya Khovanova as well as the entire PRIMES staff for organizing the PRIMES program.