CLASSIFICATION OF NON-DEGENERATE SYMMETRIC BILINEAR FORMS IN THE VERLINDE CATEGORY Ver_4^+

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Abstract. Although Deligne's theorem classifies all symmetric tensor categories (STCs) with moderate growth over algebraically closed fields of characteristic zero, the classification does not extend to positive characteristic. At the forefront of the study of STCs is the search for an analog to Deligne's theorem in positive characteristic, and it has become increasingly apparent that the Verlinde categories are to play a significant role. Moreover, these categories are largely unstudied, but have already shown very interesting phenomena as both a generalization of and a departure from superalgebra and supergeometry. In this paper, we study Ver_4^+, the simplest non-trivial Verlinde category in characteristic 2. In particular, we classify all isomorphism classes of non-degenerate symmetric bilinear forms and study the associated Witt semi-ring that arises from the direct sum and tensor product operations on bilinear forms.

Contents

1. Introduction

1.1. **The broader picture: the quest for Deligne's theorem in positive characteristic.** While the study of the representation theory of groups initially started by finding and classifying individual representations, the modern perspective is to consider the category of all representations in totality. The notion of a *symmetric tensor category* (always assumed to be of *moderate growth* [1] in this paper) arises by axiomatizing the fundamental properties of representation categories of groups (see [EGNO; EK21] for basic details). A symmetric tensor category (STC) can be thought of as a "home" to do commutative algebra and algebraic geometry without the language of vectors and vector spaces. One implication is that given an STC $\mathcal{C}$, we can construct affine group schemes over $\mathcal{C}$, whose representation categories give us other STCs. These are all said to *fiber* over $\mathcal{C}$. Because it is shown in

[1] A symmetric tensor category has *moderate growth* if the lengths of tensor powers of every object are bounded by an exponential function. Although we will assume all STCs are of moderate growth, the study of STCs of non-moderate growth has also attracted attention (see [DM82; Del02; Del07; Eti16; HS22] for examples of such categories).

[CEO23] that every STC fibers over a so-called *incompressible* STC, it remains to classify the incompressible STCs.

The STCs defined over an algebraically closed field $\mathbb{K}$ of characteristic $p = 0$ are well-understood thanks to Deligne's theorem (see [Del02; Del07]). This theorem states that, up to parity action, all manifestations of such STCs are simply representation categories of supergroup schemes, i.e. they fiber over $\mathrm{sVec}_{\mathbb{K}}$. This means $\mathrm{Vec}_{\mathbb{K}}$ and $\mathrm{sVec}_{\mathbb{K}}$ are the only incompressible STCs in characteristic zero, and therefore, characteristic zero affords only ordinary and super algebra and geometry.

As is par for the course, the story is completely different in positive characteristic. The most basic counterexample when the characteristic p is larger than 3 is the *Verlinde category* Ver_p, which contains $\mathrm{sVec}_{\mathbb{K}}$ as a subcategory (see [GM94; GK92; Ost20]). This STC arises as the *semisimplification* of the representation category $\mathrm{Rep}\,\boldsymbol{\alpha}_p = \mathrm{Rep}\,\mathbb{K}[t]/(t^p)$ of the first Frobenius kernel $\boldsymbol{\alpha}_p$ of the additive group scheme $\mathbb{G}_a$ (cf. [EO21]). It can be thought of as the positive-characteristic analog to $\mathrm{Rep}\,SL_2\mathbb{C}$ with some truncation involved when taking tensor products. For instance, when $p = 5$, there is an object $X \in \mathrm{Ver}_5$ (which can be thought of as the analog of the adjoint representation of $SL_2\mathbb{C}$) that satisfies $\mathbb{1} \oplus X = X \otimes X$, where $\mathbb{1}$ is the unit object in the category. If this category were to fiber over supervector spaces, then X would need to have integral dimension; this is impossible because there is no integral solution to $1 + \dim X = (\dim X)^2$.

With Deligne's theorem failing in positive characteristic, much work has been done in recent years to find a suitable analog. The category Ver_p has served as a reasonable starting point: first, Ostrik proved in [Ost20] that every semisimple STC fibers over Ver_p, and this was later strengthened in [CEO22] to say that an STC fibers over Ver_p if and only if it is *Frobenius exact*. Indeed, the Verlinde category Ver_p sits in a larger sequence

$$\mathrm{Ver}_p \subseteq \mathrm{Ver}_{p^2} \subseteq \cdots \subseteq \mathrm{Ver}_{p^\infty}$$

of incompressible STCs called the Verlinde categories. These were first discovered for $p = 2$ in [BE19] and then generalized for all $p > 0$ in [BEO23]. Therein, it is conjectured that the correct replacement for $\mathrm{sVec}_{\mathbb{K}}$ in Deligne's theorem is Ver_{p^∞}, which is to say that every STC fibers over Ver_{p^∞}.

1.2. **Content of this paper.** Although they arise out of the search for Deligne's theorem in positive characteristic, the Verlinde categories seem to be interesting objects in their own right as they exhibit new phenomena all the while generalizing the classical theory. For instance, in [Ven22], the finite-length representations of the group scheme $GL(X)$ for an object $X \in \mathrm{Ver}_p$ are classified. Therein, the corresponding generalization of a torus no longer has one-dimensional representations, yet its representation theory is still semisimple.

However, for the most part, these Verlinde categories have barely been studied. In this paper, we consider the simplest example in characteristic 2, which is Ver_4^+, a subcategory of $\mathrm{Ver}_4 = \mathrm{Ver}_{2^2}$ that was first shown to not fiber over the category of vector spaces in [Ven15] (note that Ver_2 is just the category of vector spaces). We usually cannot use the language of vector spaces to describe objects in STCs, but as a tensor category, Ver_4^+ is just $\mathrm{Rep}\,\mathbb{K}[t]/(t^2)$ (and is therefore not semisimple). The symmetric structure, however, is different and arises from equipping the Hopf algebra $\mathbb{K}[t]/(t^2)$ with a triangular structure (see [EGNO, §8.3]) with R-matrix given by

$$R \coloneqq 1 \otimes 1 + t \otimes t.$$

In this category, we classify all alternating bilinear and all symmetric bilinear forms, up to isomorphism. We also describe how different isomorphism classes of bilinear forms interact when we take their direct sum and tensor product.

Here, we say a form $B : U \otimes U \to \mathbb{K}$ on an object $U \in \mathrm{Ver}_4^+$ is *alternating* (resp. *symmetric*) if it vanishes on the kernel (resp. image) of the map $1_{U\otimes U} - c_{U,U}$, where $c_{U,U} : U \otimes U \to U \otimes U$ is the braiding in this category given by

$$c_{U,U}(u \otimes u') = u' \otimes u + (t.u') \otimes (t.u)$$

for $u, u' \in U$. In semisimple STCs like Ver_p, the classification reduces to the vector space setting. In Ver_4^+, the presence of the two-dimensional indecomposable representation P of $\mathbb{K}[t]/(t^2)$ makes the classification more challenging.

We find that there are ultimately six families of non-degenerate symmetric bilinear forms, two of which are indexed by a parameter. We also calculate the Witt semi-ring, which is the semi-ring structure imposed on the set of isomorphism classes where addition is given by direct sum and multiplication is given by tensor product.

In §2, we define the Verlinde category Ver_4^+, state some basic properties of symmetric bilinear forms, and establish the existence of a semi-ring structure on the isomorphism classes in our classification. In Section §3, we first classify non-degenerate symmetric bilinear forms on the object nP, then use this to recover the complete classification for an arbitrary object in Ver_4^+. Finally, we describe the structure of the Witt semi-ring in Section §4.

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2. Basic Properties of the Verlinde Category Ver_4^+

In this section, we define the Verlinde category Ver_4^+ and state its basic properties. Throughout this paper, we define $\mathbb{K}$ as an algebraically closed field of characteristic $p = 2$. We will also assume a cursory familiarity with the language of Hopf algebras and tensor categories (cf. [EGNO; EK21]) and suppress associativity morphisms in our notation.

2.1. **The Hopf Algebra $\mathbb{K}[t]/(t^2)$.** The unital algebra $A \coloneqq \mathbb{K}[t]/(t^2)$ admits the structure of a Hopf algebra with comultiplication $\Delta : A \to A \otimes A$, counit $\epsilon : A \to \mathbb{K}$, and antipode $S : A \to A$ uniquely determined by

$$\begin{aligned}\Delta(t) = 1 \otimes t + t \otimes 1;\\ \epsilon(t) = 0;\\ S(t) = t.\end{aligned}$$

By the theory of Jordan canonical forms, A has two indecomposable modules up to isomorphism: the trivial representation, denoted $\mathbb{1}$, which is simple, and a two-dimensional module P, which is an extension of $\mathbb{1}$ by itself. The Krull-Schmidt theorem tells us that any module U over A is (non-uniquely) isomorphic to $m\mathbb{1} \oplus nP$, with m and n invariants of U. We will often fix such a decomposition and let the sets

$$\begin{aligned}\{v_1, v_2, \ldots, v_m\}\\ \{w_1, x_1, \ldots, w_n, x_n\}\end{aligned} \tag{2.1}$$

denote a basis of $m\mathbb{1}$ and a basis of nP, respectively, where $t.v_j = 0$ for all $1 \le j \le m$ and $t.w_k = x_k$ for all $1 \le k \le n$. Moreover, we write $U = V \oplus W \oplus X$, where V is the span of the vectors $\{v_j\}_{j=1}^m$, W is the span of the vectors $\{w_k\}_{k=1}^n$, and X is the span of the vectors $\{x_k\}_{k=1}^n$. The vector space of morphisms $\mathrm{Hom}_A(M, N)$ between two representations M, N is simply the collection of linear maps that respect the t-action, meaning that $t.\phi(\mu) = \phi(t.\mu)$ for all $\mu \in M$ and $\phi \in \mathrm{Hom}_A(M, N)$.

Note that the linear map $\varphi \in \mathrm{Hom}_{\mathrm{Ver}_4^+}(U, U)$ given by $\varphi(u) = t.u$ is a morphism in the category Ver_4^+ because it commutes with the t-action. With respect to the decomposition of U described above, $\mathrm{im}(\varphi) = X$ and $\ker(\varphi) = V \oplus X$. Thus, X and $V \oplus X$ are fixed, while V and W are dependent on a choice of basis because the decomposition of U into $m\mathbb{1} \oplus nP$ is not unique.

Given an A-module U, there is a (left) dual module U^* with the t-action defined by

$$(t.f)(u) = f(S(t).u) = f(t.u)$$

for all $f \in U^*$. With respect to the basis of U given by (2.1), U^* has a dual basis given by the union of the following two sets:

$$\begin{aligned}\{v_1^*, v_2^*, \ldots, v_m^*\}\\ \{x_1^*, w_1^*, \ldots, x_n^*, w_n^*\}.\end{aligned} \tag{2.2}$$

Here, $t.v_j^* = 0$ for all $1 \le j \le m$, and $t.x_k^* = w_k^*$ for all $1 \le k \le n$. Finally, given any two A-modules M and N, the tensor product $M \otimes N$ admits the structure of an A-module via the comultiplication map. It is determined by

$$t.(\mu \otimes \nu) = (t.\mu) \otimes \nu + \mu \otimes (t.\nu)$$

for all $\mu \in M$ and $\nu \in N$. Explicitly, if two copies of P have a fixed bases $\{w, x\}$ and $\{\omega, \chi\}$, respectively, then their tensor product is $P \otimes P = P \oplus P$. A basis for the first summand is $\{w \otimes \chi, x \otimes \chi\}$, and a basis for the second summand is $\{w \otimes \omega, x \otimes \omega + w \otimes \chi\}$.

We can then define the representation category $\mathrm{Rep}\, A$ to be the category whose objects are A-modules and whose morphisms between two A-modules M, N are the maps $\mathrm{Hom}_A(M, N)$. These structures endow $\mathrm{Rep}\, A$ with the structure of a tensor category.

2.2. **Triangular Structure on $\mathbb{K}[t]/(t^2)$ and the Verlinde Category Ver_4^+.** The Hopf algebra A is said to have a *triangular structure with R-matrix R* if there exists an invertible element R in the algebra $A \otimes A$ such that the following identities hold:

$$\begin{aligned}(\Delta \otimes 1_A)(R) &= R^{13}R^{23};\\ (1_A \otimes \Delta)(R) &= R^{13}R^{12};\\ (\sigma_{A,A} \circ \Delta)(a) &= R\Delta(a)R^{-1} \quad \forall a \in A;\\ R^{-1} &= R^{21},\end{aligned}$$

where $\sigma_{X,Y}$ is the permutation of components on $X \otimes Y$. The term $R^{i_1,\dots,i_k}$ is given by permuting $R \otimes 1^{l-2}$ so that the component of R along the j-th tensor is now along i_j-th component and where the value of l is determined by the left-hand side. For example, if $R = a \otimes b + c \otimes d$ and $l = 3$, then $R^{13} = a \otimes 1 \otimes b + c \otimes 1 \otimes d$. Given a triangular structure on A, we can endow $\mathrm{Rep}\, A$ with a symmetric structure to construct the symmetric tensor category $\mathrm{Rep}(A, R)$. We define the braiding c, a natural transformation between the bifunctors $- \otimes - : \mathrm{Rep}\, A \times \mathrm{Rep}\, A \to \mathrm{Rep}\, A$ and $\sigma_{-,-} \circ (- \otimes -) : \mathrm{Rep}\, A \times \mathrm{Rep}\, A \to \mathrm{Rep}\, A$, by

$$c_{V,W}(v \otimes w) = \sigma_{V,W}(R.(v \otimes w))$$

for all $V, W \in \mathrm{Rep}\, A$ and $v \in V, w \in W$. In the case $R = 1 \otimes 1$, we recover the usual symmetric structure on the category $\mathrm{Rep}\, A$.

Lemma 2.1. *There is a triangular structure on A with R-matrix given by $R = 1 \otimes 1 + t \otimes t$.*

Proof. This is a straightforward verification of the axioms. For instance, to see that R is invertible, we notice that

$$\begin{aligned}R^2 &= (1 \otimes 1 + t \otimes t)(1 \otimes 1 + t \otimes t)\\ &= 1 \otimes 1 + 2(t \otimes t) + t^2 \otimes t^2 = 1 \otimes 1,\end{aligned}$$

so R is its own inverse. We can also check that

$$\begin{aligned}(\Delta \otimes 1_A)(R) &= (\Delta \otimes 1_A)(1 \otimes 1 + t \otimes t)\\ &= \Delta(1) \otimes 1 + \Delta(t) \otimes t\\ &= 1 \otimes 1 \otimes 1 + 1 \otimes t \otimes t + t \otimes 1 \otimes t\\ &= 1 \otimes 1 \otimes 1 + 1 \otimes t \otimes t + t \otimes 1 \otimes t + t \otimes t \otimes t^2\\ &= (1 \otimes 1 \otimes 1 + t \otimes 1 \otimes t)(1 \otimes 1 \otimes 1 + 1 \otimes t \otimes t)\\ &= R^{13}R^{23}.\end{aligned}$$

□

Therefore, we have the following definition:

Definition 2.2. The *Verlinde category* Ver_4^+ is the representation category $\mathrm{Rep}(A, R)$, where $A = \mathbb{K}[t]/(t^2)$ and $R = 1 \otimes 1 + t \otimes t$ is the R-matrix imposing the triangular structure on A.

The braiding c is explicitly given by

$$c_{V,W}(v \otimes w) = w \otimes v + (t.w) \otimes (t.v)$$

for all $V, W \in \operatorname{Rep} A$ and $v \in V, w \in W$. It is shown in [Ven15] that Ver_4^+ does not fiber over the category of vector spaces [2]. For more information on triangular Hopf algebras, see [EGNO, §8.3].

2.3. **Bilinear Forms in Ver_4^+.** A *bilinear form* on an object $U \in \mathrm{Ver}_4^+$ is any element of $\mathrm{Hom}_{\mathrm{Ver}_4^+}(U \otimes U, \mathbb{1})$. A bilinear form $\beta : U \otimes U \to \mathbb{1}$ must satisfy, for all $u, u' \in U$,

$$\begin{aligned} 0 &= t.(\beta(u \otimes u')) = \beta(t.(u \otimes u')) \\ &= \beta((t.u) \otimes u' + u \otimes (t.u')) \\ &\implies \beta(t.u, u') = \beta(u, t.u') \end{aligned}$$

because β is also an A-module homomorphism. We will freely identify β with the corresponding bilinear map $U \times U \to \mathbb{1}$, so we sometimes write $\beta(u, u')$ instead of writing $\beta(u \otimes u')$. By tensor-hom adjunction, there is an isomorphism between $\mathrm{Hom}_{\mathrm{Ver}_4^+}(U \otimes U, \mathbb{1})$ and $\mathrm{Hom}_{\mathrm{Ver}_4^+}(U, U^*)$. We say a bilinear form $\beta : U \otimes U \to \mathbb{1}$ is *non-degenerate* if the image of β under this isomorphism is an invertible map in $\mathrm{Hom}_{\mathrm{Ver}_4^+}(U, U^*)$. We will often denote this image by β'.

This paper primarily focuses on non-degenerate bilinear forms that are symmetric. A bilinear form $\beta : U \otimes U \to \mathbb{1}$ is said to be *symmetric* if it vanishes on the image of the map $1_{U\otimes U} - c_{U,U}$ (or equivalently, if $\beta = \beta \circ c_{U,U}$). Special cases of symmetric bilinear forms are alternating and super-alternating bilinear forms: β is *alternating* if it also vanishes on the kernel of the map $1_{U\otimes U} - c_{U,U}$, and it is *super-alternating* if for all $u \in U$, we have $\beta(u \otimes u) = 0$. All alternating bilinear forms are symmetric because $(1_{U\otimes U} - c_{U,U})^2 = 0$ and therefore $\ker(1_{U\otimes U} - c_{U,U}) \supseteq \operatorname{im}(1_{U\otimes U} - c_{U,U})$. It turns out that symmetric bilinear forms in Ver_4^+ reduce to symmetric bilinear forms in the underlying category $\operatorname{Rep} A$:

Lemma 2.3. *Let $\beta : U \otimes U \to \mathbb{1}$ be a bilinear form in Ver_4^+. Then, β is symmetric if and only if $\beta(u \otimes u') = \beta(u' \otimes u)$ for all $u, u' \in U$. This also means that all super-alternating bilinear forms are symmetric.*

Proof. Suppose β is symmetric. Then,

$$\begin{aligned} \beta(u \otimes u') &= \beta(u' \otimes u) + \beta((t.u') \otimes (t.u)) \\ &= \beta(u' \otimes u) + \beta(u' \otimes (t^2.u)) \\ &= \beta(u' \otimes u). \end{aligned}$$

The reverse direction follows by running these steps backwards. Finally, if β is super-alternating, then

$$\begin{aligned} 0 &= \beta((u + u') \otimes (u + u')) \\ &= \beta(u \otimes u) + \beta(u \otimes u') + \beta(u' \otimes u) + \beta(u' \otimes u') \\ &\implies \beta(u \otimes u') = \beta(u' \otimes u), \end{aligned}$$

so β is symmetric. $\square$

[2]There is no category of supervector spaces in characteristic 2, but in loc. cit., it is argued that Ver_4^+ could be viewed as the analog in characteristic 2.

We can also identify the additional criteria that symmetric bilinear forms must satisfy to be alternating.

Proposition 2.4. *Let $\beta : U \otimes U \to \mathbb{1}$ be a symmetric bilinear form in Ver_4^+. Fix a decomposition of U by $U = m\mathbb{1} \oplus nP = V \oplus W \oplus X$ with respect to the basis given by (2.1). Then, β is alternating if and only if $\beta(v_j \otimes v_j) = 0$ for all $1 \leq j \leq m$. Equivalently, β is alternating if and only if $\beta(u \otimes u) = 0$ for all $u \in V \oplus X$.*

Proof. With respect to the basis given by (2.1), a basis for $U \otimes U$ is given by

$$\begin{aligned} &v_j \otimes v_{j'}, v_j \otimes w_k, v_j \otimes x_k, \\ &w_k \otimes v_j, w_k \otimes w_{k'}, w_k \otimes x_{k'}, \\ &x_k \otimes v_j, x_k \otimes w_{k'}, x_k \otimes x_{k'}, \end{aligned} \tag{2.3}$$

where $1 \leq j, j' \leq m$, $1 \leq k, k' \leq n$. Using this basis, we can construct another basis of $U \otimes U$, given by the vectors below.

$$\begin{array}{ll} v_j \otimes v_j & * \\ v_j \otimes v_{j'} \quad (j < j') & \\ v_j \otimes v_{j'} + v_{j'} \otimes v_j \quad (j \neq j') & * \\ v_j \otimes x_k + x_k \otimes v_j & * \\ v_j \otimes x_k & \\ v_j \otimes w_k & \\ v_j \otimes w_k + w_k \otimes v_j & * \\ x_k \otimes x_k & * \\ w_k \otimes x_{k'} + x_{k'} \otimes w_k & * \\ w_k \otimes w_{k'} + w_{k'} \otimes w_k + x_{k'} \otimes x_k \quad (k \neq k') & * \\ x_k \otimes w_{k'} & \\ w_k \otimes w_{k'} \quad (k \neq k') & \\ w_k \otimes w_k & \end{array} \tag{2.4}$$

To see that these vectors form a basis of $U \otimes U$, observe that we can recover all vectors in the basis described by (2.3) and that the number of vectors in (2.4) is $m^2+4mn+4n^2 = (m+2n)^2$, which is the dimension of $U \otimes U$.

The starred vectors in (2.4) vanish under $1_{U\otimes U} - c_{U,U}$. The unstarred vectors are sent as follows:

$$\begin{aligned} v_j \otimes v_{j'} &\to v_j \otimes v_{j'} + v_{j'} \otimes v_j, \\ v_j \otimes x_k &\to v_j \otimes x_k + x_k \otimes v_j, \\ v_j \otimes w_k &\to v_j \otimes w_k + w_k \otimes v_j, \\ x_k \otimes w_{k'} &\to x_k \otimes w_{k'} + w_{k'} \otimes x_j, \\ w_k \otimes w_{k'} &\to w_k \otimes w_{k'} + w_{k'} \otimes w_k + x_{k'} \otimes x_k, \\ w_k \otimes w_k &\to x_k \otimes x_k. \end{aligned} \tag{2.5}$$

We can show that no linear combination of these unstarred vectors is in the kernel of the map $1_{U\otimes U} - c_{U,U}$. For each unstarred vector u, there exists a vector b_u in the basis given by (2.3) such that the coefficient of b_u is nonzero in $1_{U\otimes U} - c_{U,U}(u)$ and zero in the image of all other unstarred vectors in (2.3):

u	b_u
$v_j \otimes v_{j'}$ $(j < j')$	$v_j \otimes v_{j'}$
$v_j \otimes x_k$	$v_j \otimes x_k$
$v_j \otimes w_k$	$v_j \otimes w_k$
$x_k \otimes w_{k'}$	$x_k \otimes w_{k'}$
$w_k \otimes w_{k'}$ $(k \neq k')$	$x_{k'} \otimes x_k$
$w_k \otimes w_k$	$x_k \otimes x_k$

Thus, the starred vectors form a basis of $\ker(1_{U\otimes U} - c_{U,U})$. By definition, β must vanish on the image of $1_{U\otimes U} - c_{U,U}$. As shown in (2.5), $\operatorname{im}(1_{U\otimes U} - c_{U,U})$ includes all starred vectors in (2.4) except for those of the form $v_j \otimes v_j$. Therefore, we obtain alternating bilinear forms from symmetric bilinear forms by imposing the additional condition that $\beta(v_j \otimes v_j) = 0$ for all $1 \leq j \leq m$.

Now, we prove that this requirement is equivalent to β vanishing on $u \otimes u$ for all vectors $u \in V \oplus X$. Notice that $\beta(x_k \otimes x_k) = \beta(t.w_k, t.w_k) = \beta(w_k, t^2.w_k) = 0$ for all $1 \leq k \leq n$. Thus, β is alternating if $\beta(\mu \otimes \mu) = 0$ for all vectors μ in the basis $\{v_1, v_2, \ldots v_m, x_1, x_2, \ldots x_n\}$ of $V \oplus X$. Given vectors $u_1, u_2 \in U$ such that $\beta(u_1 \otimes u_1) = 0$ and $\beta(u_2 \otimes u_2) = 0$, we have

$$\beta((u_1 + u_2) \otimes (u_1 + u_2)) = \beta(u_1 \otimes u_1) + \beta(u_2 \otimes u_2) = 0,$$

and for any scalar $k \in \mathbb{K}$,

$$\beta(ku_1 \otimes ku_1) = k^2 \beta(u_1, u_1) = 0.$$

Therefore, if $\beta(\mu \otimes \mu) = 0$ for all vectors μ in a basis of $V \oplus X$, then that $\beta(u \otimes u) = 0$ for all $u \in V \oplus X$. □

Note that $U = nP$ when $\dim(V) = 0$, so the proposition above proves that a non-degenerate symmetric bilinear form β on the direct sum of P objects is necessarily alternating.

We can now provide a basis-invariant description of alternating bilinear forms.

Proposition 2.5. *Let $\beta : U \otimes U \to \mathbb{1}$ be a symmetric bilinear form in Ver_4^+. Then, β is alternating if and only if $\beta(u \otimes u) = 0$ for all $u \in U$ such that $t.u = 0$. In particular, all super-alternating bilinear forms are alternating.*

Proof. With respect to the decomposition $U = m\mathbb{1} \oplus nP = V \oplus W \oplus X$, we have $t.u = 0$ if and only if $u \in V \oplus X$. The claim follows from Proposition 2.4. □

As in the ordinary vector space setting, decomposing a bilinear form into the sum of smaller forms by way of orthogonal complements will be a key idea. If β is a bilinear form on U and S is a subobject of U, we define the *orthogonal complement* $S^\perp$ of S (in U and with respect to β) to be

$$S^\perp := \ker(U \xrightarrow{\beta'} U^* \xrightarrow{\pi} S^*),$$

where the map π is the usual projection map.

Here are some well known-properties about bilinear forms that extend to our setting:

Proposition 2.6. *Let β be a non-degenerate symmetric bilinear form on $U \in \mathrm{Ver}_4^+$, and let S be a subobject of U. If the restriction of β to S is non-degenerate, then $U = S \oplus S^\perp$, and moreover, the restriction of U to $S^\perp$ is also non-degenerate.*

Proof. The proofs in the classical setting extend to our setting ([Con08, Theorem 3.12]). □

We can also define how to take direct sums and tensor products of bilinear forms to produce new bilinear forms. Given two non-degenerate symmetric bilinear forms $\beta : U \otimes U \to \mathbb{1}$ and $\eta : R \otimes R \to \mathbb{1}$ in Ver_4^+, we can define their direct sum $\beta \oplus \eta : (U \oplus R)^{\otimes 2} \to \mathbb{1}$ in the usual way, given by

$$(\beta \oplus \eta)(u_1 \oplus r_1, u_2 \oplus r_2) = \beta(u_1, u_2) + \eta(r_1, r_2)$$

for all $u \in U$ and $r \in R$. The tensor product $\beta \hat{\otimes} \eta : (U \otimes R)^{\otimes 2} \to \mathbb{1}$ of two forms is slightly different because it involves the braiding. For all $u_1, u_2 \in U$ and $r_1, r_2 \in R$, we have

$$(\beta \hat{\otimes} \eta)(u_1 \otimes r_1, u_2 \otimes r_2) = \beta(u_1, u_2)\eta(r_1, r_2) + \beta(u_1, t.u_2)\eta(t.r_1, r_2).$$

More generally, these definitions arise from the following composition of maps:

$$(U \otimes R)^{\otimes 2} \xrightarrow{1_U \otimes c_{U,R} \otimes 1_R} (U \otimes U) \otimes (R \otimes R) \xrightarrow{\beta \otimes \eta} \mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}.$$

Given two bilinear forms β, η on the same object $U \in \mathrm{Ver}_4^+$, we say that β and η are in the same *isomorphism class* of bilinear forms if there exists an invertible map $\phi \in \mathrm{Hom}_{\mathrm{Ver}_4^+}(U, U)$ such that $\beta = \eta \circ (\phi \otimes \phi)$. This is an equivalence relation on the set of all (non-degenerate symmetric) bilinear forms.

We are ultimately only interested in isomorphism classes of bilinear forms, so for convenience, we will often write that two forms are equal to each other if they lie in the same isomorphism class. We will also freely identify a representative of an isomorphism class with the class itself. As the next subsection demonstrates, we can establish a semi-ring structure by taking the direct sum and tensor product on the set of isomorphism classes of non-degenerate symmetric bilinear forms.

2.4. **Witt Semi-Ring.** Let $\mathcal{W}$ denote the set of isomorphism classes of non-degenerate symmetric bilinear forms in Ver_4^+. The operations $(\oplus, \otimes)$ endow $\mathcal{W}$ with the structure of a semi-ring (where $\oplus$ defines addition and $\otimes$ defines multiplication), which we call the *Witt semi-ring*. Below, we prove some basic properties about this semi-ring, including the fact that it is commutative. We will fully describe it in §4. Proving that the Witt semi-ring is a commutative monoid under addition and satisfies distributivity is fully classical, so we do not present proofs of these properties. We will prove the rest of the axioms, starting with closure under multiplication:

Lemma 2.7. *Let β and η be symmetric bilinear forms on objects U and R in Ver_4^+, respectively. Then, the tensor product $\beta \hat{\otimes} \eta$ is a symmetric bilinear form* [3]. *Moreover, the equivalence class of $\beta \hat{\otimes} \eta$ does not depend on the choice of representative from the equivalence class of β or η.*

[3] This statement is true in any STC and can be proven using the coherence diagrams for braidings that arise from the symmetric structure.

Proof. Since $\beta\hat{\otimes}\eta$ is a morphism in Ver_4^+, proving the first part of the claim reduces to establishing symmetry. The set of vectors $u \otimes r$ where $u \in U, r \in R$ contains a basis of $U \otimes R$, so it suffices to prove that $\beta\hat{\otimes}\eta(u_1 \otimes r_1, u_2 \otimes r_2) = \beta\hat{\otimes}\eta(u_2 \otimes r_2, u_1 \otimes r_1)$ for all vectors $u_1, u_2 \in U$ and $r_1, r_2 \in R$. This follows directly from properties of β and η:

$$\begin{aligned}\beta\hat{\otimes}\eta(u_1 \otimes r_1, u_2 \otimes r_2) &= \beta(u_1, u_2)\eta(r_1, r_2) + \beta(u_1, t.u_2)\beta(t.r_1, r_2)\\ &= \beta(u_2, u_1)\eta(r_2, r_1) + \beta(u_2, t.u_1)\eta(t.r_2, r_1)\\ &= \beta\hat{\otimes}\eta(u_2 \otimes r_2, u_1 \otimes r_1).\end{aligned}$$

Finally, suppose that the the non-degenerate symmetric bilinear form β_1 is in the same isomorphism class as β via the morphism $\phi : U \to U$. Then, $\beta\hat{\otimes}\eta$ and $\beta_1\hat{\otimes}\eta$ are in the same isomorphism class via the morphism $\phi \otimes 1_R$. □

Proposition 2.8. *Let β and η be non-degenerate symmetric bilinear forms on objects U and R in Ver_4^+, respectively. The tensor product $\beta\hat{\otimes}\eta$ is also non-degenerate.*

Proof. Suppose for the sake of contradiction that $\beta\hat{\otimes}\eta$ is degenerate. Then, there exists a nonzero vector $\mu \otimes \rho \in U \otimes R$ such that $\beta\hat{\otimes}\eta(\mu \otimes \rho, u \otimes r) = 0$ for all $u \in U, r \in R$. Let B_U and B_R denote bases for U and R, respectively. We can express $\mu \otimes \rho$ as $\sum\limits_{i,j} k_{i,j} \cdot u_i \otimes r_j$, with $k_{i,j} \in \mathbb{K}, u_i \in B_u, r_j \in B_R$ for all i, j. Since $\beta\hat{\otimes}\eta$ is bilinear, $\beta\hat{\otimes}\eta(\mu \otimes \rho, u \otimes r)$ is equivalent to

$$\sum_{i,j} \beta\hat{\otimes}\eta(k_{i,j} \cdot u_i \otimes r_j, u \otimes r) = \sum_{i,j} k_{i,j} \cdot \beta(u_i, u)\eta(r_j, r) + \beta(u_i, t.u)\eta(r_j, t.r) = 0 \tag{2.6}$$

for all vectors $u \in U, r \in R$. We can also write $\beta\hat{\otimes}\eta(\mu \otimes \rho, t.u \otimes t.r)$ as

$$\sum_{i,j} k_{i,j} \cdot \beta(u_i, t.u)\eta(r_j, t.r) + k_{i,j} \cdot \beta(u_i, t.(t.u))\eta(r_j, t.(t.r)) = \sum_{i,j} k_{i,j} \cdot \beta(u_i, t.u)\eta(r_j, t.r) = 0.$$

(2.6) now simplifies to $\sum_{i,j} k_{i,j} \cdot \beta(u_i, u)\eta(r_j, r) = 0$, which is purely classical. We can finish this proof using ideas from the ordinary setting. Since $\mu \otimes \rho$ is nonzero, $k_{i,j}$ must be nonzero for some i, j. Without loss of generality, we can assume $k_{1,1} \neq 0$.

Now, let S_U be the span of $B_U - \{u_1\}$. Because $\dim(S_U) < \dim(U)$, there must exist a vector $u' \in U$ such that $u' \perp S_U$. Similarly, we can define S_R as the span of $B_R - \{r_1\}$ and let r' be a vector in R such that $r' \perp S_R$. By the non-degeneracy of β and η, the quantities $\beta(u', u_1)$ and $\eta(r', u_1)$ must be nonzero. Then,

$$\sum_{i,j} k_{i,j} \cdot \beta(u_i, u')\eta(r_j, r') = k_{1,1}\beta(u_1, u')\beta(r_1, r') \neq 0,$$

which is a contradiction. □

Now, we verify the remaining axioms, including commutativity of multiplication.

(1) **Associativity of multiplication.** Let β, η, and ζ be non-degenerate symmetric bilinear forms on the objects U, R, Z in Ver_4^+, respectively. The set of vectors of the form $u \otimes r \otimes z$ where $u \in U, r \in R, z \in Z$ contains a basis for $U \otimes R \otimes Z$. It is sufficient to prove that

$$(\beta\hat{\otimes}\eta)\hat{\otimes}\zeta((u_1 \otimes r_1) \otimes z_1, (u_2 \otimes r_2) \otimes z_2 = (\beta\hat{\otimes}(\eta\hat{\otimes}\zeta))(u_1 \otimes (r_1 \otimes z_1), u_2 \otimes (r_2 \otimes z_2))$$

for all vectors $u_1 \otimes r_1 \otimes z_1, u_2 \otimes r_2 \otimes z_2$ in this basis:

$$\begin{aligned}
&((\beta\hat{\otimes}\eta)\hat{\otimes}\zeta)((u_1 \otimes r_1) \otimes z_1, (u_2 \otimes r_2) \otimes z_2) \\
&\qquad = (\beta\hat{\otimes}\eta)(u_1 \otimes r_1, u_2 \otimes r_2)\zeta(z_1, z_2) + (\beta\hat{\otimes}\eta)(u_1 \otimes r_1, t.(u_2 \otimes r_2))\zeta(t.z_1, z_2) \\
&\qquad = (\beta\hat{\otimes}\eta)(u_1 \otimes r_1, u_2 \otimes r_2)\zeta(z_1, z_2) \\
&\qquad\quad + (\beta\hat{\otimes}\eta)(u_1 \otimes r_1, t.u_2 \otimes r_2 + u_2 \otimes t.r_2)\zeta(t.z_1, z_2) \\
&\qquad = (\beta(u_1, u_2)\eta(r_1, r_2) + \beta(u_1, t.u_2)\eta(t.r_1, r_2))\zeta(z_1, z_2) \\
&\qquad\quad + (\beta(u_1, t.u_2)\eta(r_1, r_2) + \beta(u_1, u_2)\eta(r_1, t.r_2))\zeta(t.z_1, z_2) \\
&\qquad = \beta(u_1, u_2)\eta(r_1, r_2)\zeta(z_1, z_2) + \beta(u_1, t.u_2)\eta(t.r_1, r_2)\zeta(z_1, z_2) \\
&\qquad\quad + \beta(u_1, t.u_2)\eta(r_1, r_2)\zeta(t.z_1, z_2) + \beta(u_1, u_2)\eta(r_1, t.r_2)\zeta(t.z_1, z_2) \\
&\qquad = \beta(u_1, u_2)(\eta(r_1, r_2)\zeta(z_1, z_2) + \eta(r_1, t.r_2)\zeta(t.z_1, z_2)) \\
&\qquad\quad + \beta(u_1, t.u_2)(\eta(t.r_1, r_2)\zeta(z_1, z_2) + \eta(r_1, r_2)\zeta(t.z_1, z_2)) \\
&\qquad = \beta(u_1, u_2)(\eta(r_1, r_2)\zeta(z_1, z_2) + \eta(r_1, t.r_2)\zeta(t.z_1, z_2)) \\
&\qquad\quad + \beta(u_1, t.u_2)(\eta\hat{\otimes}\zeta)(t.r_1 \otimes z_1 + r_1 \otimes t.z_1, r_2 \otimes z_2)) \\
&\qquad = \beta(u_1, u_2)(\eta\hat{\otimes}\zeta)(r_1 \otimes z_1, r_2 \otimes z_2) + \beta(u_1, t.u_2)(\eta\hat{\otimes}\zeta)(t.(r_1 \otimes z_1), r_2 \otimes z_2) \\
&\qquad = (\beta\hat{\otimes}(\eta\hat{\otimes}\zeta))(u_1 \otimes (r_1 \otimes z_1), u_2 \otimes (r_2 \otimes z_2)).
\end{aligned}$$

(2) **Commutativity of multiplication.** The set of vectors expressible as $u \otimes r$ for some $u \in U, r \in R$ includes a basis of $U \otimes R$. Therefore, we only need to show commutativity holds for all vectors $u_1 \otimes r_1, u_2 \otimes r_2 \in U \otimes R$. We claim that $\beta\hat{\otimes}\eta$ and $\eta\hat{\otimes}\beta$ are isomorphic via the braiding, meaning

$$\beta\hat{\otimes}\eta(u_1 \otimes r_1, u_2 \otimes r_2) = \eta\hat{\otimes}\beta(c_{U,R}(u_1 \otimes r_1), c_{U,R}(u_2 \otimes r_2)).$$

First, we can show that

$$\begin{aligned}
\beta\hat{\otimes}\eta(u_1 \otimes r_1, u_2 \otimes r_2) &= \beta(u_1, u_2)\eta(r_1, r_2) + \beta(u_1, t.u_2)\eta(t.r_1, r_2) \\
&= \eta(r_1, r_2)\beta(u_1, u_2) + \eta(r_1, t.r_2)\beta(t.u_1, u_2) \\
&= \eta\hat{\otimes}\beta(r_1 \otimes u_1, r_2 \otimes u_2).
\end{aligned}$$

Furthermore, we can determine that

$$\begin{aligned}
&\eta\hat{\otimes}\beta(t.r_1 \otimes t.u_1, r_2 \otimes u_2) = \eta(t.r_1, r_2)\beta(t.u_1, u_2) + \eta(t.r_1, t.r_2)\beta(t^2.u_1, u_2) \\
&= \eta(t.r_1, r_2)\beta(t.u_1, u_2) = \eta(r_1, t.r_2)\beta(u_1, t.u_2) = \eta\hat{\otimes}\beta(r_1 \otimes u_1, t.r_2 \otimes t.u_2)
\end{aligned}$$

and

$$\eta\hat{\otimes}\beta(t.r_1 \otimes t.u_1, t.r_2 \otimes t.u_2) = \eta(t.r_1, t.r_2)\beta(t.u_1, t.u_2) + \eta(t^2.r_1, t.r_2)\beta(t^2.u_1, t.u_2) = 0.$$

Together, these equations prove our claim because we can now write

$$\begin{aligned}
\eta\hat{\otimes}\beta(c_{U,R}(u_1 \otimes r_1), c_{U,R}(u_2 \otimes r_2)) &= \eta\hat{\otimes}\beta(r_1 \otimes u_1 + t.r_1 \otimes t.u_1, r_2 \otimes u_2 + t.r_2 \otimes t.u_2) \\
&= \eta\hat{\otimes}\beta(r_1 \otimes u_1, r_2 \otimes u_2) \\
&= \beta\hat{\otimes}\eta(u_1 \otimes r_1, u_2 \otimes r_2).
\end{aligned}$$

(3) **Multiplicative identity.** Consider an object $R \cong \mathbb{1} \in \mathrm{Ver}_4^+$. We can take a nonzero vector $r_1 \in \mathbb{1}$, which must satisfy $t.r_1 = 0$, and fix a basis of $\mathbb{1}$ by $\{r_1\}$. We claim that the multiplicative identity is the isomorphism class that has a representative $\eta : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ given by $\eta(r_1, r_1) = 1$. Given a vector $u_1 \in U$ and scalars $k_1, k_2 \in \mathbb{K}$, we can write $k_1 u_1 \otimes k_2 r_1$ as $k_1 k_2 u_1 \otimes r_1$. We can thus express any vector in $U \otimes \mathbb{1}$ as $u \otimes r_1$ for some $u \in U$. For all $u_1, u_2 \in U$,

$$\beta \hat{\otimes} \eta(u_1 \otimes r_1, u_2 \otimes r_1) = \beta(u_1, u_1)\eta(r_1, r_1) + \beta(u_1, t.u_1)\beta(t.r_1, r_1) = \beta(u_1, u_1).$$

Therefore, $\beta \hat{\otimes} \eta$ and β belong to the same isomorphism class, which shows by commutativity that $\beta \hat{\otimes} \eta = \beta = \eta \hat{\otimes} \beta$.

3. Classification of Non-Degenerate Symmetric Bilinear Forms in Ver_4^+

We have now set the stage to classify the non-degenerate symmetric bilinear forms in Ver_4^+.

3.1. **Classifying forms on objects of the form $m\mathbb{1}$ and of the form nP.** Before we can approach the general case, it is easier to classify forms on objects of the form $m\mathbb{1}$ and on objects of the form nP. The former is the well-known classification of symmetric bilinear forms in the ordinary vector space setting:

Theorem 3.1 ([Gla05])**.** *Let β be a non-degenerate symmetric bilinear form on a vector space Z. Then, there exists a basis for Z in which the associated matrix of β is either the identity matrix or direct sums of the 2×2 matrix given by*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For each dimension, these two classes of forms are non-isomorphic. If $\dim Z = m$, let us denote some representative of the first isomorphism class as α_1^m and denote some representative of the second isomorphism class as α_2^m (which exists only for even m).

Changing basis amounts to conjugation by an invertible map in $\mathrm{Hom}(Z, Z)$. However, the endomorphism spaces in Ver_4^+ are considerably more restrictive, and therefore, we find more isomorphism classes of non-degenerate symmetric bilinear forms. We start our classification with the following straightforward lemma:

Lemma 3.2. *Let β a symmetric bilinear form on an object $U \in \mathrm{Ver}_4^+$ with the decomposition $U = m\mathbb{1} \oplus nP = V \oplus W \oplus X$ arising from the basis described by* (2.1)*. Then, β must satisfy the following for all $1 \leq i \leq m$ and $1 \leq j, k \leq n$:*

(1) $\beta(v_i, x_j) = 0$, meaning $\beta|_{V \otimes X} = 0$ and $\beta|_{X \otimes V} = 0$;
(2) $\beta(w_j, x_k) = \beta(x_j, w_k)$;
(3) $\beta(x_j, x_k) = 0$, meaning $\beta|_{X \otimes X} = 0$.

Proof. This is a direct consequence of the fact that $\beta(t.u, u') = \beta(u, t.u')$ for all $u, u' \in U$. □

The following motivates why we first consider the classification of $m\mathbb{1}$ and nP separately.

Proposition 3.3. *Let β a non-degenerate symmetric bilinear form on an object $U \in \mathrm{Ver}_4^+$ with the decomposition $U = m\mathbb{1} \oplus nP = V \oplus W \oplus X$ arising from the basis described by* (2.1)*. Then, the restriction of β to V is also non-degenerate.*

Proof. Suppose for the sake of contradiction that β is degenerate on V. Then, there exists a nonzero vector $v \in V$ such that $\beta|_{\mathbb{K}v\otimes V} = 0$. By Lemma 3.2, we know that $\beta|_{\mathbb{K}v\otimes X} = 0$ and $\beta|_{X\otimes(V\oplus X)} = 0$. Therefore, $\beta|_{(\mathbb{K}v\oplus X)\otimes(V\oplus X)} = 0$, and the adjunct map $\beta' : U \to U^*$ must map any $u \in \mathbb{K}v \oplus X$ to a vector in W^*, where we decompose $U^* = V^* \oplus W^* \oplus X^*$. However, $\dim(\mathbb{K}v \oplus X) = n+1$ and $\dim(W^*) = n$, so there exists a nonzero vector u in $\mathbb{K}v \oplus X$ such that $\beta'(u) = 0$, contradicting the non-degeneracy of β on U. □

An object $U \in \mathrm{Ver}_4^+$ can be decomposed into $V \cong m\mathbb{1}$ and $V^\perp \cong nP$. If β is a non-degenerate symmetric bilinear form on U, then by Propositions 3.3 and 2.6, we can choose V such that both $\beta|_V$ and $\beta|_{V^\perp}$ are non-degenerate symmetric bilinear forms. Because V is an ordinary vector space, we already know that $\beta|_V$ belongs to one of the two classes in Theorem 3.1. In the remainder of this section, we will classify isomorphism classes of forms on $V^\perp \cong nP$.

We will first show that on the object P, there exist infinitely many isomorphism classes of bilinear forms, each indexed by an element of $\mathbb{K}$. We will denote suitable representatives for these isomorphism classes as $\beta_P(y) : P \otimes P \to \mathbb{1}$, where $y \in \mathbb{K}$. Similarly, on the object $2P$, there exist two isomorphism classes not arising from $\beta_P(y) \oplus \beta_P(z)$, which we will call $\beta_{2P}(i) : 2P \otimes 2P \to \mathbb{1}$ for $i = 0, 1$.

Lemma 3.4. *Let η be a non-degenerate symmetric bilinear form on the object P. There exists a basis of P such that the associated matrix of η is given by*

$$\begin{bmatrix} y & 1 \\ 1 & 0 \end{bmatrix} \tag{3.1}$$

for suitable $y \in \mathbb{K}$. These forms are pairwise non-isomorphic.

Proof. Let p, q be basis vectors of P such that $t.p = q$. The quantity $\eta(p, q)$ is nonzero as otherwise, q would be in the kernel of η, and the form would be degenerate. Moreover, $\eta(q, q) = \eta(t.p, t.p) = \eta(p, t^2.p) = 0$. Therefore, we can scale the basis vectors by $1/\sqrt{\eta(p, q)}$ (which is a valid base change), and the associated matrix of η with respect to this new basis is given by

$$\begin{bmatrix} \frac{\eta(p,p)}{\eta(p,q)} & 1 \\ 1 & 0 \end{bmatrix}.$$

Now, any map $P \to P$ is determined by where it sends p, so it follows immediately that these forms are pairwise non-isomorphic. □

The isomorphism class arising from the form in Lemma 3.4 will be represented by $\beta_P(y)$ for $y \in \mathbb{K}$. We can also classify some forms on the object $2P$.

Definition 3.5. We say a bilinear form β on an object $U \in \mathrm{Ver}_4^+$ is *oscillating* if for all $u \in U$, we have $\beta(u, t.u) = 0$.

With this definition, we have the following lemma:

Lemma 3.6. *Let η be a non-degenerate symmetric oscillating bilinear form on the object nP (with $n > 1$). Then, there is a subobject $S \cong 2P$ of nP such that the restriction of η to S is non-degenerate, and moreover, there exists a basis of S for which the associated matrix* *of $\eta|_S$ is given by one of the following two matrices:*

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.3}$$

The first form will be denoted as $\beta_{2P}(0)$, and the second will be denoted as $\beta_{2P}(1)$. These forms are not isomorphic (and are also not isomorphic to $\beta_P(y) \oplus \beta_P(z)$ for any $y, z \in \mathbb{K}$).

Proof. Let p be a vector in nP such that $t.p \neq 0$ (such a vector necessarily exists). The non-degeneracy of η means there must exist a vector $q \in nP$ such that $\eta(t.p, q) \neq 0$. By the assumption that η is oscillating, $\eta(u, t.u) = 0$ for all $u \in nP$. Therefore, $q \neq p$. Since $0 \neq \eta(t.p, q) = \eta(p, t.q)$, we have $t.q \neq 0$. Let S be the subobject of nP spanned by the basis vectors $\{p, t.p, q, t.q\}$. The matrix associated to $\eta|_S$ on this basis is of the form

$$\begin{bmatrix} * & 0 & * & \lambda \\ 0 & 0 & \lambda & 0 \\ * & \lambda & * & 0 \\ \lambda & 0 & 0 & 0 \end{bmatrix}$$

for some nonzero $\lambda \in \mathbb{K}$ and with $*$ denoting suitable entries such that the matrix is symmetric. Once we rescale each basis vector by $\frac{1}{\sqrt{\lambda}}$, the matrix with respect to this basis becomes

$$\begin{bmatrix} b & 0 & c & 1 \\ 0 & 0 & 1 & 0 \\ c & 1 & a & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

where $a, b, c \in \mathbb{K}$. Then, we replace q by $q' = q + c(t.q)$, which is a valid change of basis because $t.(q + c(t.q)) = t.q$. The associated matrix of η is now given by

$$\begin{bmatrix} b & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix above has determinant 1, so this basis change preserves non-degeneracy.

If $a = b = 0$, we get the isomorphism class $\beta_{2P}(0)$, as claimed. Now, suppose $b \neq 0$ but $a = 0$. We can define $p' = \frac{1}{\sqrt{b}}p$ and $q'' = \sqrt{b}q'$. Then, with respect to the basis $\{p', t.p', q'', t.q''\}$, the associated matrix of η is given by

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which is the associated matrix of the form $\beta_{2P}(1)$ representing our second isomorphism class. Similarly, if $a = 0$ and $b \neq 0$, we can interchange the order of $p, t.p$ with $q', t.q'$ in our basis and then apply the same process, which will give us the same matrix. Therefore, suppose that both a and b are nonzero. We can find $d \in \mathbb{K}$ such that $k := \sqrt{b} + d\sqrt{a} \neq 0$. We define a new basis $\{p', t.p', q'', t.q''\}$ of $2P$ given by $p' = \frac{1}{k}(p + dq' + da(t.p) + b(t.q'))$ and $q'' = \sqrt{a}p + \sqrt{b}q'$. We have:

- $\eta(p', p') = \frac{1}{k^2}(b + d^2 a) = \frac{1}{k^2}(k^2) = 1$,
- $\eta(p', t.p') = \frac{1}{k^2}(2d) = 0$,
- $\eta(p', q'') = \frac{1}{k}(\sqrt{w}y + b\sqrt{y}w + \sqrt{a}b + d\sqrt{b}a) = 0$,
- $\eta(t.p', q'') = \frac{1}{k}(\sqrt{b} + d\sqrt{a}) = \frac{1}{k}(k) = 1$, and
- $\eta(q'', q'') = (\sqrt{a})^2 b + (\sqrt{b})^2 a = 0$.

Therefore, with respect to this new basis, the associated matrix of the form is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which we have already seen. Thus, we obtain the form $\beta_{2P}(0)$ when $y = w = 0$ and $\beta_{2P}(1)$ otherwise.

To see that $\beta_{2P}(0)$ and $\beta_{2P}(1)$ give rise to distinct isomorphism classes, notice that the first form is super-alternating and the second form is not. Moreover, these two forms are oscillating, so they are non-isomorphic to the forms $\beta_P(k) \oplus \beta_P(l)$ where $k, l \in \mathbb{K}$, which are not oscillating. □

The forms arising in Lemma 3.4 and Lemma 3.6 serve as the building blocks for all forms on nP, as the next lemma demonstrates.

Lemma 3.7. *Any non-degenerate symmetric bilinear form β on the object nP admits one of the following two direct sum decompositions:*

$$\beta = \bigoplus_{i=1}^{n} \beta_P(y_i)$$

$$\beta = \bigoplus_{j=1}^{n/2} \beta_{2P}(a_j)$$

for suitable $y_i \in \mathbb{K}$ and $a_j \in \{0, 1\}$.

Proof. Suppose that we can find a vector $u \in nP$ such that $\beta(u, t.u) \neq 0$. Then, β restricted to the subobject Z of nP spanned by $\{u, t.u\}$ is non-degenerate, and therefore, by Lemma 3.4, $\beta|_Z$ is in the isomorphism class as $\beta_P(y)$ for some $y \in \mathbb{K}$.

Otherwise, we have $\beta(u, t.u) = 0$ for all $u \in nP$ (i.e. the form is oscillating). In this case, Lemma 3.6 applies, and we can find a subobject Y of nP for which the restriction of β gives the form $\beta_{2P}(a_j)$.

In either case, once we find such a subobject Z or Y, we can take its orthogonal complement and proceed inductively by way of Proposition 2.6. This proves that β is of the form

$$\beta = \bigoplus_i \beta_P(y_i) \oplus \bigoplus_j \beta_{2P}(a_j)$$

for suitable $y_i \in \mathbb{K}$ and $a_j \in \{0,1\}$. Now, given this decomposition, suppose that both isomorphism classes are present. Then, there is a basis $\{p, t.p, q, t.q, r, t.r\}$ of a subobject $S \cong 3P$ of nP such that the associated matrix of $\beta|_S$ relative to this basis is given by

$$\begin{bmatrix} y & 1 & & & & \\ 1 & 0 & & & & \\ & & a & 0 & 0 & 1 \\ & & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \end{bmatrix},$$

with $y \in \mathbb{K}$ and $a \in \{0,1\}$. Let $p' = p+q+r+(y+a)(t.p)$ and $q' = p+q$. Then, let $\tilde{S}$ denote the subobject of S spanned by $\{p', t.p', q', t.q'\}$. With respect to this basis, the associated matrix of $\beta|_{\tilde{S}}$ is given by

$$\begin{bmatrix} * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & * & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $*$ are suitable entries. Hence, the restriction of β to $\tilde{S}$ is the direct sum $\beta_P(\tilde{y}) \oplus \beta_P(\tilde{z})$ for suitable $\tilde{y}, \tilde{z} \in \mathbb{K}$. Moreover, we can write $S = \tilde{S} \oplus \tilde{S}^\perp$. By Lemma 3.4, the restriction of β to $\tilde{S}^\perp$ will be of the form $\beta_P(\tilde{a})$ for suitable $\tilde{a} \in \mathbb{K}$. Thus, the direct sum of $\beta_P(y)$ with $\beta_{2P}(a)$ can be rewritten as the direct sum $\beta_P(\tilde{y}) \oplus \beta_P(\tilde{z}) \oplus \beta_P(\tilde{a})$. From here, the statement of the lemma follows.

□

Lemma 3.7 shows that any non-degenerate symmetric bilinear form on nP is either the sum of $n/2$-copies of irreducible forms on $2P$ or the sum of n-copies of irreducible forms on P. We will show that in the former case, there are two distinct isomorphism classes that arise, whereas in the latter, there are infinitely many. We begin with the first case, which is easier to prove:

Lemma 3.8. *Suppose β is a non-degenerate symmetric bilinear form on nP such that*

$$\beta = \bigoplus_{j=1}^{n/2} \beta_{2P}(a_j)$$

for $a_j \in \{0,1\}$. Then, β is in the same isomorphism class as one of the following two forms:

$$\beta^n_{2P;0} := \beta_{2P}(0)^{\oplus \frac{n}{2}}$$
$$\beta^n_{2P;1} := \beta_{2P}(1) \oplus \beta_{2P}(0)^{\oplus \frac{n-2}{2}}.$$

The two forms are not isomorphic.

Proof. We are done if for at most one value of j, we have $a_j = 1$. So let us suppose there are least two such values of j. Without loss of generality, we can assume they are they are the first two indices, i.e. $a_1 = a_2 = 1$. Now, we will consider the direct summand $\beta_{2P}(a_1) \oplus \beta_{2P}(a_2)$ of β, with basis $\{u_1, t.u_1, u_2, t.u_2\}$ for the first copy of $2P$ and $\{u_3, t.u_3, u_4, t.u_4\}$ a basis for the second copy of $2P$. We claim that this form can be written as $\beta_{2P}(0) \oplus \beta_{2P}(1)$ by suitably changing basis.

Let $u_5 = u_1 + u_3$, and let $u_6 = u_2$. The associated matrix of β restricted to the subobject S_1 spanned by $\{u_5, t.u_5, u_6, t.u_6\}$ (with respect to this basis) is given by (3.2). Similarly, define $u_7 = u_3 + t.u_2$ and $u_8 = u_2 + u_4$. The associated matrix of β restricted to the subobject S_2 spanned by $\{u_7, t.u_7, u_8, t.u_8\}$ (with respect to this basis) is given by (3.3). Moreover, we can see that S_1 and S_2 are orthogonal complements. This shows that $\beta_{2P}(1) \oplus \beta_{2P}(1) = \beta_{2P}(0) \oplus \beta_{2P}(1)$; the claim follows by induction. The two forms are not isomorphic because the the form $\beta^n_{2P;0}$ is super-alternating, whereas the form $\beta^n_{2P;1}$ is not. □

We now consider the second case, where the non-degenerate symmetric bilinear form is the sum of forms on the object P. The procedure for doing so is more complicated than that of the first case. To start, we have the following lemma.

Lemma 3.9. *For any $y \neq z \in \mathbb{K}$, the form $\beta = \beta_P(y) \oplus \beta_P(z)$ is in the same isomorphism class as $\beta_P(a) \oplus \beta_P(y + z + a)$ for all $a \in \mathbb{K}$.*

Proof. Let $\{u_1, t.u_1\}$ be a basis of the first P object such that the associated matrix of $\beta_P(y)$ is given by (3.2), and let $\{u_2, t.u_2\}$ be a basis of the second P object such that the associated matrix of $\beta_P(z)$ is given by (3.3). For some arbitrary $a \in \mathbb{K}$, let $k = \sqrt{\frac{z+a}{z+y}}$, which is well-defined because $y \neq z$. Define $c = ky$ and $d = (1 + k)x$. Then, $k(1 + k)y + (1 + k)kz + c(1 + k) + dk = k((1 + k)y + d) + (1 + a)(az + c) = 0$. Now, let $u_3 = ku_1 + (1 + k)u_2 + ct.u_1 + dt.u_2$, and let $u_4 = (1 + k)u_1 + ku_2$. We have

- $\beta(u_3, u_3) = k^2 y + (1 + k)^2 z = k^2(y + z) + z = a$,
- $\beta(u_3, t.u_3) = k^2 + (1 + k)^2 = 1$,
- $\beta(u_3, u_4) = k(1 + k)y + (1 + k)kz + c(1 + k) + dk = 0$,
- $\beta(u_3, t.u_4) = k(1 + k) + (1 + k)k = 0$,
- $\beta(u_4, u_4) = (1 + k)^2 y + k^2 z = k^2(y + z) + y = y + z + a$, and
- $\beta(u_4, t.u_4) = (1 + k)^2 + k^2 = 1$.

Therefore, with respect to the basis $\{u_3, t.u_3, u_4, t.u_4\}$, the associated matrix of β is

$$\begin{array}{cccc} u_3 & t.u_3 & u_4 & t.u_4 \end{array}$$
$$\begin{bmatrix} a & 1 & & \\ 1 & 0 & & \\ & & y+z+a & 1 \\ & & 1 & 0 \end{bmatrix}.$$

This proves the claim. □

Now, our strategy will be to repeatedly use Lemma 3.9 to convert a form that is the direct sum of forms described in Lemma 3.4 into a canonical form. For simplicity, we will refer to the process of identifying $\beta_P(y) \oplus \beta_P(z)$ with $\beta_P(a) \oplus \beta_P(y + z + a)$ as "replacing y, z by $a, y + z + a$". Given a form $\beta_P(y)$, we will refer to y as the *assigned scalar* of $\beta_P(y)$.

Lemma 3.10. *Let β be a non-degenerate symmetric bilinear form on the object nP with $n > 1$ such that*

$$\beta = \bigoplus_{i=1}^{n} \beta_P(y_i)$$

for suitable $y_i \in \mathbb{K}$. If not all values of y_i are the same, then we can write

$$\beta = \beta_P(k) \oplus \beta_P(1) \oplus \beta_P(0)^{\oplus(n-2)}$$

for some suitable $k \in \mathbb{K}$. If $n = 2$, then $k \neq 1$.

Proof. First of all, let us suppose that $n = 2$. Then, we have $\beta = \beta_P(y_1) \oplus \beta_P(y_2)$. We can replace y_1, y_2 with $1, y_1 + y_2 + 1$ and let $k = y_1 + y_2 + 1 \neq 1$.

Now, suppose that $n \geq 3$. If $n - 1$ of the assigned scalars are zero and the remaining scalar is 1, then we are done. If instead the remaining scalar is some $\lambda \neq 0 \in \mathbb{K}$, then we can do the replacement $\lambda, 0 \mapsto 1, \lambda + 1$, and we are done again. If $n - 2$ of the assigned scalars are zero and the remaining two are $\lambda, \mu \in \mathbb{K} - \{0\}$, then we can do the substitution $\lambda, \mu \mapsto 1, \lambda + \mu + 1$ if $\lambda \neq \mu$. If $\lambda = \mu$, we can first do the substitution $0, \lambda \mapsto 1, \lambda + 1$, then do the substitution $\lambda + 1, \mu \mapsto 1, 0$ (converting the three assigned scalars $\lambda, \mu, 0$ into $1, 1, 0$). This covers the case where $n - 2$ assigned scalars are zero.

Therefore, let us assume that at most $n - 3$ of the assigned scalars are zero. If no assigned scalars are zero, we can find y_a and y_b with $y_a \neq y_b$ and do the replacement $y_a, y_b \mapsto 0, y_a + y_b$. Hence, we can ensure that least one of the assigned scalars is zero. If $n = 3$, this returns us to the case where $n - 2$ assigned scalars are zero. When $n > 3$, we can find three additional assigned scalars y_a, y_b, and y_c with $y_a \neq 0$. We can then perform the following iterative procedure until we arrive at a form that has $n - 2$ zeroes as assigned scalars. Let d be a nonzero scalar satisfying $d \neq y_b$ and $d \neq y_a + y_c$. We can do the replacements

$$0, y_a, y_b, y_c \mapsto d, y_a + d, y_b, y_c \mapsto 0, y_a + d, y_b + d, y_c \mapsto 0, 0, y_b + d, y_c + y_a + d,$$

where the notation is extended with two assigned scalars replaced in each step. These replacements give us an additional zero as an assigned scalar. The above process can be repeated until we have $n - 2$ zeroes as assigned scalars, which is a case we have already considered. This proves the lemma. ☐

We combine our previous work to get the following theorem.

Theorem 3.11. *Any non-degenerate symmetric bilinear form β on nP lies in the isomorphism class of one of the following types of forms:*

$$\begin{aligned}
\beta^n_{2P;0} &= \beta_{2P}(0)^{\oplus \frac{n}{2}} \quad (2 \mid n) \\
\beta^n_{2P;1} &= \beta_{2P}(1) \oplus \beta_{2P}(0)^{\oplus \frac{n-2}{2}} \quad (2 \mid n; n > 0) \\
\beta_{y,1;0} &\coloneqq \beta_P(y) \oplus \beta_P(1) \quad (y \neq 1 \in \mathbb{K}; n = 2) \\
\beta_{y,1;n-2} &\coloneqq \beta_P(y) \oplus \beta_P(1) \oplus \beta_P(0)^{\oplus(n-2)} \quad (y \in \mathbb{K}; n \geq 3) \\
\beta^n_y &\coloneqq \beta_P(y)^{\oplus n} \quad (y \in \mathbb{K}; n > 0).
\end{aligned}$$

These forms are pairwise non-isomorphic, except some of the $\beta_{y,1;n-2}$ may represent the same isomorphism class for different y (which we will see is not the case in Lemma 3.25).

Proof. This follows from Lemma 3.7, Lemma 3.8, and Lemma 3.10. To that see we have distinct isomorphism classes, we will observe some properties about the forms. The first form is oscillating and super-alternating. The second form is not super-alternating but is oscillating. The remaining forms are not oscillating. Notice that $y\beta_y^n(u,t.u) = \beta_y^n(u,u)$ for all $u \in nP$, whereas for no $y \in \mathbb{K}$ does there exist $z \in \mathbb{K}$ such that $z\beta_{y,1;n-2}(u,u) = \beta_{y,1;n-2}(u,t.u)$ for all $u \in nP$. Therefore, we deduce that the β_y^n are pairwise non-isomorphic and not isomorphic to anything else on the list. This proves the claim. □

3.2. **Classifying non-degenerate bilinear forms in the general case.** We now have classifications for the non-degenerate symmetric bilinear forms on objects of the form $m\mathbb{1}$ (Theorem 3.1) and for those on objects of the form nP (Theorem 3.11). In this section, we will use these results to provide the classification for any object $U \in \mathrm{Ver}_4^+$ with decomposition $U = m\mathbb{1} \oplus nP = V \oplus W \oplus X$ arising from the basis given by (2.1).

Lemma 3.12. *Let β be a non-degenerate symmetric bilinear form on $U \in \mathrm{Ver}_4^+$, and suppose that $U = V \oplus V^\perp$, where $V \cong m\mathbb{1}$, $V^\perp \cong nP$, and $\beta|_V = \alpha_1^m$. Then, either $\beta = \alpha_1^m \oplus \beta_{2P,0}^n$ or $\beta = \alpha_1^m \oplus \beta_0^n$.*

Proof. By Lemma 3.7, we know that β is either in the same isomorphism class as

$$\alpha_1^m \oplus \bigoplus_{i=1}^{n} \beta_P(y_i)$$

or

$$\alpha_1^m \oplus \bigoplus_{j=1}^{n/2} \beta_{2P}(a_j).$$

Let us deal with the former case first. We claim that

$$\alpha_1^1 \oplus \beta_P(y_i) = \alpha_1^1 \oplus \beta_P(0)$$

for all values of y_i. The associated matrix of the left-hand side is given by

$$\begin{array}{c} \begin{matrix} u_1 & u_2 & t.u_2 \end{matrix} \\ \begin{bmatrix} 1 & & \\ & y_i & 1 \\ & 1 & 0 \end{bmatrix} \end{array}$$

in some suitable basis $\{u_1, u_2, t.u_2\}$. Let $u_3 = u_1 + \sqrt{y_i}t.u_2$ and $u_4 = \sqrt{y_i}u_1 + u_2$. Then, we can see that

- $\beta(u_3, u_3) = 1$,
- $\beta(u_3, u_4) = \sqrt{y_i} + \sqrt{y_i} = 0$,
- $\beta(u_3, t.u_4) = 0$,
- $\beta(u_4, u_4) = y_i + y_i = 0$,
- $\beta(u_4, t.u_4) = 1$, and
- the space spanned by u_3 is perpendicular to the space spanned by $\{u_4, t.u_4\}$.

In the basis $\{u_3, u_4, t.u_4\}$, the associated matrix is given by

$$\begin{array}{ccc} u_3 & u_4 & t.u_4 \end{array}$$
$$\begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix},$$

which shows the claim. Since $m > 0$, after iterating this procedure for each i, we see that

$$\beta = \alpha_1^m \oplus \bigoplus_{i=1}^{n} \beta_P(y_i) = \alpha_1^m \oplus \beta_0^n.$$

Now, let us move to the second case, where

$$\beta = \alpha_1^m \oplus \bigoplus_{j=1}^{n/2} \beta_{2P}(a_j).$$

We want to show that $\beta = \alpha_1^m \oplus \beta_{2P,0}^n$; this will follow if we can show that

$$\alpha_1^1 \oplus \beta_{2P}(1) = \alpha_1^1 \oplus \beta_{2P}(0).$$

In other words, we need to find a change of basis so that we can go from the first matrix below to the second matrix below:

$$\begin{array}{ccccc} u_1 & u_2 & t.u_2 & u_3 & t.u_3 \end{array} \qquad \begin{array}{ccccc} u_4 & u_5 & t.u_5 & u_6 & t.u_6 \end{array}$$
$$\begin{bmatrix} 1 & & & & \\ & 1 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & & & & \\ & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Such a basis change is given by letting $u_4 = u_1 + t.u_3$, $u_5 = u_1 + u_2$, and $u_6 = u_3$. Iterating this for each value of j such that $a_j = 1$ proves the second case. $\square$

Using the previous lemma and our classifications on $V \cong m\mathbb{1}$ and $V^\perp \cong nP$, we obtain a classification of the non-degenerate symmetric bilinear forms on an object $U \cong m\mathbb{1} \oplus nP$. In the following theorem, we represent our forms using their corresponding associated matrices, writing $I_m = \begin{bmatrix} 1 & & \\ & \ddots^{m} & \\ & & 1 \end{bmatrix}$ and $A_z(y) = \begin{bmatrix} y & & 1 \\ & \iddots_{z} & \\ 1 & & \end{bmatrix}$.

Theorem 3.13. *Let $U \cong m\mathbb{1} \oplus nP$. For any non-degenerate symmetric bilinear form β, there exists a basis of U such that the associated matrix of β is one of the 6 forms below. We present the matrices block-diagonally, with the first block representing $\beta|_V$ and the second block representing $\beta|_{V^\perp}$, where $V \cong m\mathbb{1}$ and $V^\perp \cong nP$ is some suitable subobject of U.*

(A) $$\left[\begin{array}{c|ccc} I_m & & & \\ \hline & A_4(0) & & \\ & & \ddots^{\frac{n}{2}} & \\ & & & A_4(0) \end{array}\right] \quad (m > 0; 2 \mid n)$$

$$\text{(B)}\qquad \left[\begin{array}{c|ccc} I_m & & & \\ \hline & A_2(0) & & \\ & & \ddots^{\,n} & \\ & & & A_2(0) \end{array}\right] \quad (m>0; n>0)$$

$$\text{(C)}\qquad \left[\begin{array}{ccc|ccc} A_2(0) & & & & & \\ & \ddots^{\,\frac{m}{2}} & & & & \\ & & A_2(0) & & & \\ \hline & & & A_4(0) & & \\ & & & & \ddots^{\,\frac{n}{2}} & \\ & & & & & A_4(0) \end{array}\right] \quad (2 \mid m; 2 \mid n)$$

$$\text{(D)}\qquad \left[\begin{array}{ccc|cccc} A_2(0) & & & & & & \\ & \ddots^{\,\frac{m}{2}} & & & & & \\ & & A_2(0) & & & & \\ \hline & & & A_4(0) & & & \\ & & & & \ddots^{\,\frac{n}{2}-1} & & \\ & & & & & A_4(0) & \\ & & & & & & A_4(1) \end{array}\right] \quad (2 \mid m; 2 \mid n; n \geq 2)$$

$$\text{(E}(y)\text{)}\qquad \left[\begin{array}{ccc|ccc} A_2(0) & & & & & \\ & \ddots^{\,\frac{m}{2}} & & & & \\ & & A_2(0) & & & \\ \hline & & & A_2(y) & & \\ & & & & \ddots^{\,n} & \\ & & & & & A_2(y) \end{array}\right] \quad (y \in \mathbb{K}; 2 \mid m; n>0)$$

$$\text{(F}(1+y)\text{)}\qquad \left[\begin{array}{ccc|cccc} A_2(0) & & & & & & \\ & \ddots^{\,\frac{m}{2}} & & & & & \\ & & A_2(0) & & & & \\ \hline & & & A_2(0) & & & \\ & & & & \ddots^{\,n-2} & & \\ & & & & & A_2(1) & \\ & & & & & & A_2(y) \end{array}\right] \quad \left(\begin{array}{l} y \in \mathbb{K}; \\ 2 \mid m; n \geq 2; \\ (1+y, n) \neq (0,2) \end{array}\right)$$

Proof. Write $U = V \oplus V^\perp$ for some $V \cong m\mathbb{1}$. If the restriction of β to V decomposes as α_1^m, then Lemma 3.12 shows that β is either in the isomorphism class A or the isomorphism class B. Otherwise, Theorem 3.11 gives a form belonging to one of the isomorphism classes C through F. Since all alternating bilinear forms are symmetric, we have also classified all non-degenerate alternating bilinear forms on objects in Ver_4^+ (we will specify which forms are alternating in Theorem 3.26). In the next subsection, we will prove that forms in these isomorphism classes are pairwise non-isomorphic. □

3.3. **Proving Non-Isomorphism.** We start by describing basis-invariant properties of non-degenerate symmetric bilinear forms on objects $U \in \mathrm{Ver}_4^+$. Without loss of generality, assume that the basis on which β is represented in Theorem 3.13 is the basis given by (2.1). Recall that this gives rise to the decomposition $U = m\mathbb{1} \oplus nP = V \oplus W \oplus X$.

Definition 3.14. Given a symmetric bilinear form β on U, we define a *good pair* as an ordered pair of scalars $(k, l) \in \mathbb{K}^2$ satisfying $k\beta(u, t.u) = l\beta(u, u)$ for all $u \in U$.

Proposition 3.15. *Let β be a symmetric bilinear form on U, and let k, ℓ be scalars in $\mathbb{K}$. If $k\beta(u_1, t.u_1) = \ell\beta(u_1, u_1)$ for all vectors u_1 in a basis of U, then $k\beta(u, t.u) = \ell\beta(u, u)$ for all $u \in U$.*

Proof. If $u_1, u_2 \in U$ satisfy $k\beta(u_1, t.u_1) = \ell\beta(u_1, u_1)$ and $k\beta(u_2, t.u_2) = \ell\beta(u_2, u_2)$, then

$$\begin{aligned}
k\beta(u_1 + u_2, t.(u_1 + u_2)) &= k\beta(u_1 + u_2, t.u_1 + t.u_2) \\
&= k\beta(u_1, t.u_1) + k\beta(u_2, t.u_2) + k\beta(u_1, t.u_2) + k\beta(u_2, t.u_1) \\
&= \ell\beta(u_1, u_1) + \ell\beta(u_2, u_2) + 2k\beta(u_1, t.u_2) \\
&= \ell\beta(u_1, u_1) + \ell\beta(u_2, u_2) + 0 \\
&= \ell\beta(u_1, u_1) + \ell\beta(u_2, u_2) + 2\ell\beta(u_1, u_2) \\
&= \ell\beta(u_1 + u_2, u_1 + u_2),
\end{aligned}$$

and for any scalar j,

$$k\beta(ju_1, t.(ju_1)) = k\beta(ju_1, jt.u_1) = kj^2\beta(u_1, t.u_1) = \ell j^2\beta(u_1, u_1) = \ell\beta(ju_1, ju_1).$$

□

In the case that a symmetric bilinear form has the good pair $(1, 0)$, we recover the definition of an oscillating bilinear form. In the case that a symmetric bilinear form has the good pair $(0, 1)$, we recover the definition of a super-alternating bilinear form. Alternating forms in our classification have additional invariant properties:

Proposition 3.16. *Let β be a non-degenerate alternating bilinear form on U, and suppose x is a vector in X. For all $u \in U$ such that $t.u = x$, the quantity $\beta(u, u)$ is fixed.*

Proof. Suppose u_1 and u_2 are vectors in U such that $t.u_1 = t.u_2 = x$. Then, $t.(u_1 + u_2) = 0$, which implies $\beta(u_1 + u_2, u_1 + u_2) = 0$ by Proposition 2.5. We have

$$\begin{aligned}
\beta(u_1, u_1) &= \beta(u_1, u_1) + \beta(u_1, u_2) + \beta(u_2, u_1) \\
&= \beta(u_1, u_1 + u_2) + \beta(u_2, u_1 + u_2) + \beta(u_2, u_2) \\
&= \beta(u_1 + u_2, u_1 + u_2) + \beta(u_2, u_2) \\
&= \beta(u_2, u_2).
\end{aligned}$$

□

Proposition 3.17. *Let β be a non-degenerate alternating form on U, and suppose x_1, x_2 are vectors in X. For all $u_2 \in U$ such that $t.u_2 = x_2$, the quantity $\beta(x_1, u_2)$ is fixed.*

Proof. Let $u_1 \in U$ be a vector such that $t.u_1 = x_1$, and suppose u_3, u_4 are vectors in U such that $t.u_3 = x_2$ and $t.u_4 = x_2$. There must exist some vector $u_5 \in U$ satisfying $t.u_5 = 0$ such that $u_3 = u_4 + u_5$. Then, $\beta(x_1, u_3) = \beta(x_1, u_4 + u_5) = \beta(x_1, u_4) + \beta(x_1, u_5) = \beta(x_1, u_4) + \beta(t.u_1, u_5) = \beta(u_1, u_4) + \beta(u_1, t.u_5) = \beta(x_1, u_4) + \beta(u_1, 0) = \beta(x_1, u_4)$, as desired. □

Given $x \in X$, $u_1 \in U$, and $u \in U$ such that $t.u = x$, the propositions above prove that $\beta(u, u)$ and $\beta(u_1, u)$ do not depend on the choice of representative from the preimage of x under the map of the t-action. This motivates the following definitions:

Definition 3.18. Let β be a non-degenerate alternating bilinear form on U. The *X-function* $f : X \to \mathbb{K}$ of β is defined by $f_\beta(x) = \beta(u, u)$, where $x \in X$ and $u \in U$ is in the preimage of x under the map of the t-action.

Definition 3.19. Let β be a non-degenerate alternating bilinear form on U. The *X-form* $g : X \otimes X \to \mathbb{K}$ of β is defined by $g(x_1, x_2) = \beta(x_1, u_2)$, where $x_1, x_2 \in X$, and $u_2 \in U$ is in the pre-image of x_2 under the map of the t-action.

Proposition 3.20. *Let β be a non-degenerate alternating bilinear form on U. The X-form of β is non-degenerate, symmetric, and bilinear.*

Proof. Denote the X-form of β by g. First, suppose for the sake of contradiction that g is degenerate. Then, there exists a vector $x \in X$ such that $g(x, x') = 0$ for all $x' \in X$. Thus, for any vector u' such that $t.u' \in X$, $\beta(x, u') = 0$. However, X is the image of U under the t-action, so $\beta(x, u') = 0$ for all $u' \in U$, which is impossible because β is non-degenerate.

Now, we prove that g is symmetric and bilinear. Let x_1, x_2, x_3 be arbitrary vectors in X. There exist vectors $u_1, u_2, u_3 \in U$ such that $t.u_1 = x_1, t.u_2 = x_2$, and $t.u_3 = u_3$. Symmetry holds because $g(x_1, x_2) = \beta(t.u_1, u_2) = \beta(u_1, t.u_2) = \beta(t.u_2, u_1) = \beta(x_2, u_1) = g(x_2, x_1)$. To verify bilinearity, we can check that $g(x_1, x_2) + g(x_1, x_3) = \beta(u_1, x_2) + \beta(u_1, x_3)$ $= \beta(u_1, x_2 + x_3) = g(x_1, x_2 + x_3)$, and $g(x_1, kx_2) = \beta(u_1, kx_2) = k\beta(u_1, x_2) = kg(x_1, x_2)$ for any scalar k. By symmetry, these relations also hold on the left side of g. □

Definition 3.21. Let β be a non-degenerate alternating bilinear form on U. Given a basis of X, the *X-matrix* of β is the associated matrix of the X-form of β.

Because the X-form is non-degenerate for any non-degenerate alternating bilinear form, we know that the X-matrix is always invertible. Next, we introduce the basis-invariant notion of the *form invariant* to distinguish between isomorphism classes of forms.

Definition 3.22. Suppose that β is a non-degenerate alternating bilinear form on U. Let $\{\chi_1, \ldots, \chi_n\}$ be a basis of X, and denote the X-matrix of β with respect to this basis by M. The *form invariant* of $\mathcal{I}_\beta$ of β is the sum $\sum_{i=1}^n f_\beta(\chi_i)(M^{-1})_{ii}$.

Remark 3.23. Let η be a non-degenerate alternating bilinear form on an object R with decomposition $R = p\mathbb{1} \oplus qP$. The formula for $\mathcal{I}_\eta$ is only dependent on the restriction of η to qP, so $\mathcal{I}_\eta = \mathcal{I}_{\eta|_{qP}}$.

Theorem 3.24. *Let β be a non-degenerate alternating bilinear form on U. The form invariant of β is basis-invariant.*

Proof. Denote the X-function and X-form of β by f and g, respectively, and with respect to the basis $\{x_1, x_2, \ldots, x_m\}$ of X, define M to be the X-matrix of β. Given an invertible linear transformation $A : X \to X$, we want to show that when evaluated on the basis

$\{Ax_1, Ax_2, \ldots, Ax_n\}$, the form invariant remains unchanged. First, we show that the associated matrix of g with respect to this basis is $A^\top MA$. Using the property that g is bilinear, we can rewrite each entry of this associated matrix as follows:

$$g(Ax_i, Ax_j) = \sum_{1\le k,\ell\le n} A_{ki}A_{\ell j}g(x_k, x_\ell) = \sum_{1\le k,\ell\le n} A_{ki}A_{\ell j}M_{k\ell} = \sum_{1\le k,\ell\le n} A^\top_{ik}M_{k\ell}A_{\ell j} = (A^\top MA)_{ij}.$$

Additionally, we have

$$f(Ax_i) = \beta\left(\sum_{j=1}^{n} A_{ji}w_j, \sum_{k=1}^{n} A_{ki}w_k\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} A_{ji}A_{ki}\beta(w_j, w_k).$$

For each pair (a, b) where $1 \le a, b \le n$, we have $A_{ai}A_{bi}\beta(w_a, w_b) = A_{bi}A_{ai}\beta(w_b, w_a)$, which implies $A_{ai}A_{bi}\beta(w_a, w_b) + A_{bi}A_{ai}\beta(w_b, w_a) = 0$ in characteristic 2. Therefore, we can simplify $f(Ax_i)$ to

$$\sum_{j=1}^{n} A_{ji}^2\beta(w_j, w_j) = \sum_{j=1}^{n} A_{ji}^2 f(x_j).$$

We want to prove

$$\sum_{i=1}^{n} f(x_i)(M^{-1})_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ji}^2 f(x_j)(A^\top MA)^{-1}_{ii},$$

and it suffices to show that

$$(M^{-1})_{ii} = \sum_{k=1}^{n} A_{ik}^2 (A^\top MA)^{-1}_{kk}.$$

The matrix M^{-1} can be written as $A(A^\top MA)^{-1}A^\top$. Thus,

$$M^{-1}_{ii} = \sum_{1\le j,k\le n} A_{ij}(A^\top MA)^{-1}_{jk}A^\top_{ki} = \sum_{1\le j,k\le n} A_{ij}A_{ik}(A^\top MA)^{-1}_{jk}.$$

Since $A^\top MA$ is symmetric, $(A^\top MA)^{-1}$ must also be symmetric.

For each pair (a, b) where $1 \le a, b \le n$, we have $A_{ia}A_{ib}(A^\top MA)^{-1}_{ab} = A_{ib}A_{ia}(A^\top MA)^{-1}_{ba}$, which means that $A_{ia}A_{ib}(A^\top MA)^{-1}_{ab} + A_{ib}A_{ia}(A^\top MA)^{-1}_{ba} = 0$. Then,

$$\sum_{1\le j,k\le n} A_{ij}A_{ik}(A^\top MA)^{-1}_{jk} = \sum_{1\le k\le n} A_{ik}A_{ik}(A^\top MA)^{-1}_{kk} = \sum_{k=1}^{n} A_{ik}^2(A^\top MA)^{-1}_{kk},$$

as desired. □

We are now ready to prove non-isomorphism.

Lemma 3.25. *For all $a, b \in \mathbb{K}$, forms in the class $F(1 + a)$ and forms in the isomorphism class $F(1 + b)$ are isomorphic only if $a = b$.*

Let β be a form in F$(1+a)$. We will use the basis given by (2.1) to represent the associated matrix of β in Theorem 3.13. With respect to the basis $\{x_1, x_2, \ldots, x_n\}$, the X-matrix M of β is the identity matrix I_n. Then, $\sum_{i=1}^{n} f(x_i)(M^{-1})_{ii} = \sum_{i=1}^{n} f(x_i)$, which is the sum of the diagonal entries of M. The form invariant of β thus evaluates to $\mathcal{I}_\beta = 1 + a$. Since $1 + a = 1 + b$ only if $a = b$, this proves the lemma.

Theorem 3.26. *The forms described in Theorem 3.13 are pairwise non-isomorphic.*

Proof. By Proposition 2.4, the alternating bilinear forms in our classification are those that vanish on $v_j \otimes v_j$ for $1 \le j \le m$. We deduce that forms in the isomorphism classes A and B are not alternating, while forms of the remaining four classes are. Thus, forms in A and B are not isomorphic to forms in the other classes.

By Proposition 3.15, we can determine the good pairs of forms in our classification by examining the properties of vectors in a basis of U. Forms belonging to B and F have a single good pair $(0, 0)$, whereas the good pairs of forms in A and D are $(k, 0)$ for all scalars k, Forms in E(a) where $a \in \mathbb{K}$ have the good pairs (ka, k) for all scalars k. For all $k, l \in \mathbb{K}, u \in U$, $\beta(u, t.u) = \beta(u, u) = 0$ for all forms β in C. Therefore, forms in C have the good pair (k, l) for all scalars k, l.

We can use the criterion of distinct good pairs to conclude that forms in A and B are not isomorphic and forms belonging to the classes C, D, E, and F are pairwise non-isomorphic. Finally, we proved in Lemma 3.25 that the forms in $F(1 + a)$ and forms in $F(1 + b)$ with $a \ne b \in \mathbb{K}$ are distinct. □

We finish this section with calculating the form invariants of the forms described by C, D, and E. This information becomes useful in the next section, where we determine the direct sum and tensor product on bilinear forms described by our isomorphism classes.

Proposition 3.27. *The form invariants of forms in C and D are zero, and for $a \in \mathbb{K}$, the form invariant of forms in E(a) is na.*

Suppose β is a non-degenerate symmetric bilinear form in E(a). Again, we use the basis given by (2.1) to represent the associated matrix of β in Theorem 3.13. The X-matrix of β with respect to this basis is the identity matrix I_n, and for $1 \le i \le n$, $f_\beta(x_i) = a$. The form invariant of β evaluates to $\mathcal{I}_\beta = na$.

Now, suppose β is a form in C or D. With respect to the same basis, the X-matrix of β, which we will once again denote M, is direct sums of the 2×2 matrix given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since M is its own inverse, $M_{ii}^{-1} = 0$ for $1 \le i \le n$. Thus, $\mathcal{I}_\beta = 0$.

4. Witt Semi-Ring Structure

In this section, we describe the structure of the Witt semi-ring of isomorphism classes of non-degenerate symmetric bilinear forms in Ver_4^+ (see §2.4). Our results are provided in the table at the end of each subsection. As a set, the elements of the Witt semi-ring are the isomorphism classes of the non-degenerate symmetric bilinear forms described in Theorem 3.13. Recall that addition is given by direct sum and multiplication is given by tensor product.

Throughout this section, we let β and η denote non-degenerate symmetric bilinear forms on objects $U, R \in \mathrm{Ver}_4^+$, respectively. We fix a basis of $U = m\mathbb{1} \oplus nP$ as given by (2.1), and we fix a basis of $R = p\mathbb{1} \oplus qP$ by

$$\{\nu_1, \nu_2, \ldots, \nu_p, \omega_1, \chi_1, \ldots, \omega_q, \chi_q\},$$

where $t.v_j = 0$ for all $1 \leq j \leq p$ and $t.w_k = x_k$ for all $1 \leq k \leq q$. The direct sum $\beta \oplus \eta$ acts on the object $U \oplus R = (m+p)\mathbb{1} \oplus (n+q)P$. The tensor product $\beta \hat{\otimes} \eta$ acts on the object $U \otimes R = mp\mathbb{1} \oplus mqP \oplus npP \oplus nq(P \otimes P)$, which is equivalent to $mp\mathbb{1} \oplus (mq + np + 2nq)P$ because $P \otimes P = P \oplus P$. Given β and η, we determine which isomorphism classes their direct sum and tensor product belong to (denoted A through F, as labeled in Theorem 3.13).

4.1. **Direct Sum.** In this section, we describe the invariant properties of $\beta \oplus \eta$, which will enable us to classify the form up to isomorphism.

Lemma 4.1. *The good pairs of $\beta \oplus \eta$ are the intersection of the good pairs of β and the good pairs of η.*

Proof. Let k, ℓ be scalars in $\mathbb{K}$. If $k\beta(u, t.u) = \ell\beta(u, u)$ for all $u \in U$ and $k\eta(r, t.r) = \ell\eta(r, r)$ for all $r \in R$, we have

$$\begin{aligned} k\beta(u, t.u) + k\eta(r, t.r) &= \ell\beta(u, u) + \ell\eta(r, r) \\ \implies k\beta \oplus \eta(u \oplus r, t.(u \oplus r)) &= \ell\beta \oplus \eta(u \oplus r, u \oplus r). \end{aligned}$$

For the converse, we suppose (k, ℓ) is a good pair of $\beta \oplus \eta$, meaning

$$k\beta \oplus \eta(u \oplus r, t.(u \oplus r)) = \ell\beta \oplus \eta(u \oplus r, u \oplus r) \tag{4.1}$$

for all $u \oplus r \in U \oplus R$. We have $\ell\beta \oplus \eta(u \oplus r, u \oplus r) = \ell\beta(u, u) + \ell\eta(r, r)$, and the left-hand side of (4.1) evaluates to

$$k\beta \oplus \eta(u \oplus r, t.(u \oplus r)) = k\beta \oplus \eta(u \oplus r, t.u \oplus t.r) = k\beta(u, t.u) + k\eta(r, t.r).$$

Thus, we can rewrite (4.1) as

$$k\beta(u, t.u) + k\eta(r, t.r) = \ell\beta(u, u) \oplus \ell\eta(r, r).$$

Setting $r = 0$ in the equation above yields $k\beta(u, t.u) = \ell\beta(u, u)$, and setting $u = 0$ yields $k\eta(r, t.r) = \ell\eta(r, r)$. □

Lemma 4.2. *The direct sum $\beta \oplus \eta$ is alternating if and only if both β and η are alternating.*

Proof. Decompose $U = V_U \oplus W_U \oplus X_U$ and $R = V_R \oplus W_R \oplus X_R$. If β and η are alternating, then by Proposition 2.4, $\beta(a, a) = 0$ for all $a \in V_U \oplus X_U$, and $\eta(b, b) = 0$ for all $b \in V_R \oplus X_R$. Then, $\beta \oplus \eta(a \oplus b, a \oplus b) = \beta(a, a) + \eta(b, b) = 0$ for all $a \in V_U \oplus X_U, b \in V_R \oplus X_R$, which proves by Proposition 2.4 that $\beta \oplus \eta$ is alternating.

To prove the converse, we will show that $\beta \oplus \eta$ is not alternating when at least one of β and η is not alternating. If β is not alternating, then Proposition 2.4 implies the existence of a vector $v_1 \in V_U$ such that $\beta(v_1, v_1) \neq 0$. For any vector χ in X_R, $t.(v_1 + \chi) = t.v_1 + t.\chi = 0$, and $\eta(\chi, \chi) = 0$. Consequently, $\beta \oplus \eta(v_1 + \chi, v_1 + \chi) = \beta(v_1, v_1) + \eta(\chi, \chi) \neq 0$, and it follows from Proposition 2.5 that $\beta \oplus \eta$ is not alternating. □

Lemma 4.3. *If both β and η are alternating, then $\mathcal{I}_{\beta \oplus \eta} = \mathcal{I}_\beta + \mathcal{I}_\eta$.*

Proof. First, let us establish our notation for this proof. The bases of X_U and X_R are given by $\{x_1, x_2, \ldots, x_n\}$ and $\{\chi_1, \chi_2, \ldots, \chi_q\}$, respectively. We denote the X-function of β by f_β, the X-function of η by f_η, and the X-function of $\beta \oplus \eta$ by $f_{\beta \oplus \eta}$. Additionally, X-matrices of β, η, and $\beta \oplus \eta$ are denoted by M_β, M_η, and M, respectively.

Define a basis of $\beta \oplus \eta$ by $\{b_1, \ldots, b_{n+q}\}$ where the vectors $b_1, \ldots, b_n$ are given by $x_1, \ldots, x_n$ and the vectors $b_{n+1}, \ldots, b_{n+q}$ are given by $\chi_1, \ldots, \chi_q$. For any $1 \leq i \leq n$, $f_{\beta \oplus \eta}(x_i + 0) =$

$\beta(x_i, x_i) = f_\beta(x_i)$. We also have $f_{\beta\oplus\eta}(0+\chi_i) = f_\eta(\chi_i)$ for all $1 \le i \le q$. There is a similar relationship between the X-matrices of our forms: $M = M_\beta \oplus M_\eta = \begin{bmatrix} M_\beta & 0 \\ 0 & M_\eta \end{bmatrix}$, so $M^{-1} = \begin{bmatrix} M_\beta^{-1} & 0 \\ 0 & M_\eta^{-1} \end{bmatrix}$. Thus,

$$\begin{aligned} \mathcal{I}_{\beta\oplus\eta} &= \sum_{i=1}^{n+q} f_{\beta\oplus\eta}(b_i)(M^{-1})_{ii} \\ &= \sum_{i=1}^{n} f_{\beta\oplus\eta}(x_i+0)(M^{-1})_{ii} + \sum_{i=n+1}^{n+q} f_{\beta\oplus\eta}(0+\chi_{i-n})(M^{-1})_{ii} \\ &= \sum_{i=1}^{n} f_\beta(x_i)(M_\beta^{-1})_{ii} + \sum_{i=1}^{q} f_\eta(\chi_i)(M_\eta^{-1})_{ii}. \end{aligned}$$

□

We can now apply our work from the previous section on good pairs and alternating forms (Theorem 3.26) and form invariants (Lemma 3.25, Proposition 3.27) to determine the direct sum of isomorphism classes in our Witt semi-ring.

$\oplus$	A	B	C	D	E(a)	F(a)
A	A	B	A	A	B	B
B		B	B	B	B	B
C			C	D	E(a)	F(a)
D				D	F(na)	F(a)
E(b)					$a = b \to$ E(a); $a \neq b \to$ F($na + qb$)	F($a + qb$)
F(b)						F($a + b$)

In the table above, a and b represent arbitrary scalars. We list the isomorphism classes of β and η in the top row and the leftmost column, respectively (the blank entries are given by commutativity).

4.2. **Tensor Product.** To determine the tensor product on bilinear forms in our setting, we will employ a similar strategy as the one we used to find the direct sum. Recall that we fixed a basis of $U = m\mathbb{1} \oplus nP$ by

$$\{v_1, v_2, \ldots, v_m, w_1, x_1, \ldots, w_n, x_n\}$$

and a basis of $R = p\mathbb{1} \oplus qP$ by

$$\{\nu_1, \nu_2, \ldots, \nu_q, \omega_1, \chi_1, \ldots, \omega_q, \chi_q\}.$$

Remark 4.4. Some statements in this section assume properties for at least one of β and η or assume different properties for β and η. By commutativity, these claims are also true when we interchange the assumptions for β and the assumptions for η.

First, we will determine the good pairs of $\beta\hat{\otimes}\eta$. By Proposition 3.15, it suffices to consider the pairs $(k, \ell) \in \mathbb{K}^2$ that satisfy the property

$$k\beta\hat{\otimes}\eta(b_1 \otimes b_2, t.(b_1 \otimes b_2)) = \ell\beta\hat{\otimes}\eta(b_1 \otimes b_2, b_1 \otimes b_2)$$

for all vectors $b_1 \otimes b_2$ in a basis of $U \otimes R$. It is easier for us to instead consider the pairs $(k, \ell) \in \mathbb{K}$ that satisfy this property for vectors of the form $u \otimes r \in U \otimes R$. This will give us all of the good pairs of $\beta \hat{\otimes} \eta$ because set of all vectors in $U \otimes R$ expressible as $u \otimes r$ contains a basis for $U \otimes R$. For vectors of this form, we have

$$\begin{aligned} \beta\hat{\otimes}\eta(u \otimes r, u \otimes r) &= \beta(u,u)\eta(r,r) + \beta(u,t.u)\eta(r,t.r), \\ \beta\hat{\otimes}\eta(u \otimes r, t.(u \otimes r)) &= \beta\hat{\otimes}\eta(u \otimes r, t.u \otimes r + u \otimes t.r) \\ &= \beta\hat{\otimes}\eta(u \otimes r, t.u \otimes r) + \beta\hat{\otimes}\eta(u \otimes r, u \otimes t.r), \\ &= \beta(u,t.u)\eta(r,r) + \beta(u,u)\eta(r,t.r). \end{aligned} \tag{4.2}$$

We begin with the cases where at least one of β and η lies in the isomorphism classes C or E(1).

Proposition 4.5. *If β lies in C, then $\beta\hat{\otimes}\eta$ must also belong to C.*

Proof. Since β is in C, $\beta(u, t.u) = 0$ and $\beta(u, u) = 0$ for all $u \in U$. For all $r \in R$, we thus have $\beta\hat{\otimes}\eta(u \otimes r, u \otimes r) = 0$ and $\beta\hat{\otimes}\eta(u \otimes r, t.(u \otimes r)) = 0$ by the equations in (4.2). These properties are only exhibited by forms in C. □

Proposition 4.6. *Suppose that η lies in E(1) and β does not belong to the isomorphism classes C or E(1). Then, $\beta\hat{\otimes}\eta$ is in E(1).*

Proof. The equation $\beta\hat{\otimes}\eta(u \otimes r, t.(u \otimes r)) = \beta\hat{\otimes}\eta(u \otimes r, u \otimes r)$ holds for all vectors of the form $u \otimes r$ in $U \otimes R$. We can see that $(1, 1)$ is a good pair of $\beta\hat{\otimes}\eta$, which is only true for forms belonging to the classes C and E(1). Since β is not in C or E(1), there exists a vector $u_1 \in U$ such that $\beta(u_1, u_1) \neq \beta(u_1, t.u_1)$. Furthermore, since η is in E(1), there exists a vector $r_1 \in \eta$ such that $\eta(r_1, r_1) = \eta(r_1, t.r_1) \neq 0$. Then, $\beta\hat{\otimes}\eta(u_1 \otimes r_1, u_1 \otimes r_1)$ must be nonzero, which cannot be true for forms in C. □

Proposition 4.7. *If β and η are both in E(1), then $\beta\hat{\otimes}\eta$ belongs to C.*

Proof. If β and η are both in E(1), then they must each have the good pair $(1, 1)$. In other words, $\beta(u, t.u) = \beta(u, u)$ for all $u \in U$, and $\eta(r, t.r) = \eta(r, r)$ for all $r \in R$. For all values of $u \otimes r \in U \otimes R$, we thus have

$$\beta\hat{\otimes}\eta(u \otimes r, u \otimes r) = \beta(u,u)\eta(r,r) + \beta(u,t.u)\eta(r,t.r) = 2 \cdot \beta(u,u)\eta(r,r) = 0,$$
$$\beta\hat{\otimes}\eta(u \otimes r, t.(u \otimes r)) = \beta(u,t.u)\eta(r,r) + \beta(u,u)\eta(r,t.r) = 2 \cdot \beta(u,u)\eta(r,r) = 0.$$

These equations only hold for forms in C. □

The remaining cases occur when neither β nor η belongs to C or E(1). To address these cases, we start with the following proposition.

Proposition 4.8. *Suppose β has a single good pair $(0, 0)$. For any scalars k, ℓ, there exists a solution to the system of equations*

$$\beta(u,u) = k,$$
$$\beta(u,t.u) = \ell.$$

Proof. Since $(0, 0)$ is the only good pair of β, there exists a vector $\mu_1 \in U$ such that at least one of $\beta(\mu_1, \mu_1)$ and $\beta(\mu_1, t.\mu_1)$ is nonzero. Let k_1, ℓ_1 be the scalars given by $k_1 := \beta(\mu_1, \mu_1)$

and $\ell_1 := \beta(\mu_1, t.\mu_1)$. Then, $(k_1, \ell_1) \neq (0,0)$. If $\beta(\mu_1,\mu_1)\beta(\mu, t.\mu) = \beta(\mu_1, t.\mu_1)\beta(\mu,\mu)$ for all $\mu \in U$, then (k_1, ℓ_1) would be a good pair of β. Therefore, since $(0,0)$ is the only good pair of β, there must exist some vector $\mu_2 \in U$ such that

$$k_1\beta(\mu_2, t.\mu_2) \neq \ell_1\beta(\mu_2,\mu_2).$$

Defining $k_2 := \beta(\mu_2,\mu_2)$ and $\ell_2 := \beta(\mu_2, t.\mu_2)$, we have $k_1\ell_2 \neq k_2\ell_1$. The pairs $(k_1,\ell_1),(k_2,\ell_2)$ are linearly independent vectors over $\mathbb{K}^2$, so $(k_1,\ell_1),(k_2,\ell_2)$ span $\mathbb{K}^2$. Thus, there exist scalars c, d such that $c(k_1,\ell_1) + d(k_2,\ell_2) = (k,\ell)$.

Let $u = \sqrt{c}\mu_1 + \sqrt{d}\mu_2$. We have

$$\begin{aligned}\beta(u,u) &= \beta(\sqrt{c}\mu_1 + \sqrt{d}\mu_2, \sqrt{c}\mu_1 + \sqrt{d}\mu_2)\\ &= c\beta(\mu_1,\mu_1) + d\beta(\mu_2,\mu_2) + 2\cdot\sqrt{cd}\beta(\mu_1,\mu_2))\\ &= c\beta(\mu_1,\mu_1) + d\beta(\mu_2,\mu_2) = k\end{aligned}$$

and

$$\begin{aligned}\beta(u,t.u) &= \beta(\sqrt{c}\mu_1 + \sqrt{d}\mu_2, t.(\sqrt{c}\mu_1 + \sqrt{d}\mu_2))\\ &= \beta(\sqrt{c}\mu_1 + \sqrt{d}\mu_2, \sqrt{c}t.\mu_1 + \sqrt{d}t.\mu_2)\\ &= c\beta(\mu_1,t.\mu_1) + d\beta(\mu_2,t.\mu_2) + \sqrt{cd}(\beta(\mu_1,t.\mu_2) + \beta(t.\mu_1,\mu_2))\\ &= c\beta(\mu_1,t.\mu_1) + d\beta(\mu_2,t.\mu_2) + \sqrt{cd}(\beta(\mu_1,t.\mu_2) + \beta(\mu_1,t.\mu_2))\\ &= c\beta(\mu_1,t.\mu_1) + d\beta(\mu_2,t.\mu_2) + 2\cdot\sqrt{cd}\beta(\mu_1,t.\mu_2)\\ &= c\beta(\mu_1,t.\mu_1) + d\beta(\mu_2,t.\mu_2) = \ell,\end{aligned}$$

which shows that u is a solution to the system. ☐

Lemma 4.9. *Suppose that the only good pair of β is $(0,0)$ and that η does not belong to the classes C or E(1). Then, the only good pair of $\beta\hat{\otimes}\eta$ is $(0,0)$.*

Proof. Since η is not in C or E(1), there must exist a vector $r \in R$ such that $\eta(r,r) \neq \eta(r,t.r)$. Then, for any scalars $a, b \in \mathbb{K}$, the system of equations

$$\begin{aligned}a &= c\eta(r,r) + d\eta(r,t.r),\\ b &= c\eta(r,t.r) + d\eta(r,r)\end{aligned}$$

has a solution in some scalars c and d. By Proposition 4.8, there exists a vector $u \in U$ such that $\beta(u,u) = c, \beta(u,t.u) = d$. We obtain

$$\begin{aligned}\beta\hat{\otimes}\eta(u\otimes r, u\otimes r) &= \beta(u,u)\eta(r,r) + \beta(u,t.u)\eta(r,t.r) = c\eta(r,r) + d\eta(r,t.r) = a,\\ \beta\hat{\otimes}\eta(u\otimes r, t.(u\otimes r)) &= \beta(u,t.u)\eta(r,r) + \beta(u,u)\eta(r,t.r) = d\eta(r,r) + c\eta(r,t.r) = b.\end{aligned}$$

For $(k,l) \in \mathbb{K}^2$ to be a good pair of $\beta\hat{\otimes}\eta$, the equation $kb = la$ must hold for all values of a, b. This is only true when $(k,l) = (0,0)$. ☐

Lemma 4.10. *Let k_1, k_2, ℓ_1, and ℓ_2 be elements of $\mathbb{K}$. Suppose that the good pairs of β are the multiples of (k_1,ℓ_1) and the good pairs of η are the multiples of (k_2,ℓ_2). Suppose further that β and η are not in C or E(1). Then, the good pairs of $\beta\hat{\otimes}\eta$ are the multiples of $(k_1k_2 + \ell_1\ell_2, k_1\ell_2 + \ell_1k_2)$.*

Proof. First, we observe that for all $u \in U, r \in R$,

$$\begin{aligned}
&(k_1k_2 + \ell_1\ell_2)\beta\hat{\otimes}\eta(u \otimes r, t.(u \otimes r)) \\
&\quad = (k_1k_2 + \ell_1\ell_2)(\beta(u,u)\eta(r,t.r) + \beta(u,t.u)\eta(r,r)) \\
&\quad = k_1k_2\beta(u,t.u)\eta(r,r) + k_1k_2\beta(u,u)\eta(r,t.r) + \ell_1\ell_2\beta(u,t.u)\eta(r,r) + \ell_1\ell_2\beta(u,u)\eta(r,t.r) \\
&\quad = k_2\ell_1\beta(u,u)\eta(r,r) + k_1\ell_2\beta(u,u)\eta(r,r) + k_2\ell_1\beta(u,t.u)\eta(r,t.r) + k_1\ell_2\beta(u,t.u)\eta(r,t.r) \\
&\quad = (k_1\ell_2 + \ell_1k_2)(\beta(u,u)\eta(r,r) + \beta(u,t.u)\eta(r,t.r)) \\
&\quad = (k_1\ell_2 + \ell_1k_2)\beta\hat{\otimes}\eta(u \otimes r, u \otimes r),
\end{aligned}$$

which shows that the multiples of $(k_1k_2 + \ell_1\ell_2, k_1\ell_2 + \ell_1k_2)$ are good pairs of $\beta\hat{\otimes}\eta$. It remains to prove that they are the only good pairs of $\beta\hat{\otimes}\eta$.

If $k_1\ell_2 = \ell_1k_2$, then the multiples of $(1,0)$ are good pairs of $\beta\hat{\otimes}\eta$. If $k_1\ell_2 \neq \ell_1k_2$, then the multiples of $(\frac{k_1k_2+\ell_1\ell_2}{k_1\ell_2+\ell_1k_2}, 1)$ are good pairs of $\beta\hat{\otimes}\eta$. In either case, $\beta\hat{\otimes}\eta$ will not have other good pairs unless it belongs to C. We will prove that this cannot occur.

Because β does not belong to C or E(1), there exists a vector $u' \in U$ such that at least one of $\beta(u',u'), \beta(u',t.u')$ is nonzero. Similarly, because η does not belong to C or E(1), there exists a vector $r' \in R$ such that at least one of $\eta(r',r'), \eta(r',t.r')$ is nonzero. The quantities $\beta(u',u') + \beta(u',t.u')$ and $\eta(r',r') + \eta(r',t.r')$ are therefore both nonzero, and their product

$$\begin{aligned}
&(\beta(u',u') + \beta(u',t.u'))(\beta(r',r') + \beta(r',t.r')) \\
&\quad = (\beta(u',u')\beta(r'r') + \beta(u',t.u')\eta(r',t.r')) + (\beta(u',t.u')\eta(r',r') + \beta(u',u')\eta(r',t.r')) \\
&\quad = \beta\hat{\otimes}\eta(u' \otimes r', u' \otimes r') + \beta\hat{\otimes}\eta(u' \otimes r', t.(u' \otimes r'))
\end{aligned}$$

must also be nonzero. At least one of $\beta\hat{\otimes}\eta(u' \otimes r', u' \otimes r')$ and $\beta\hat{\otimes}\eta(u' \otimes r', t.(u' \otimes r'))$ is nonzero; this cannot be the case for forms in C. Hence, $\beta\hat{\otimes}\eta$ has no other good pairs, which proves the claim. ☐

Our work above fully determines the good pairs of $\beta\hat{\otimes}\eta$ in the remaining cases. Now, we will find when $\beta\hat{\otimes}\eta$ is alternating.

Lemma 4.11. *The form $\beta\hat{\otimes}\eta$ is alternating if and only if at least one of β and η is alternating.*

Proof. The the object $U \otimes R$ can be decomposed as $U \otimes R = (m\mathbb{1} \oplus nP) \otimes (p\mathbb{1} \oplus qP) = mp\mathbb{1} \oplus (2nq + mq + np)P$. A basis for $mp\mathbb{1}$ is given by the vectors $v_i \otimes \nu_j$ where $1 \leq i \leq m, 1 \leq j \leq p$. By Proposition 2.4, the form $\beta\hat{\otimes}\eta$ is alternating when $\beta\hat{\otimes}\eta(v_i \otimes \nu_j, v_i \otimes \nu_j) = 0$ for all $1 \leq i \leq m, 1 \leq j \leq p$.

Expanding, we have

$$\beta\hat{\otimes}\eta(v_i \otimes \nu_j, v_i \otimes \nu_j) = \beta(v_i,v_i)\eta(\nu_j,\nu_j) + \beta(v_i,t.v_i)\beta(t.\nu_j,\nu_j) = \beta(v_i,v_i)\eta(\nu_j,\nu_j).$$

By Proposition 2.4, $\beta(v_i,v_i) = 0$ for all $1 \leq i \leq m$ if and only if β is alternating, and $\eta(\nu_j,\nu_j) = 0$ for all $1 \leq j \leq p$ if and only if η is alternating. Thus, $\beta\hat{\otimes}\eta$ is alternating if and only if β is alternating, η is alternating, or both β and η are alternating. ☐

We will now describe the form invariant $\mathcal{I}_{\beta\hat{\otimes}\eta}$ when $\beta\hat{\otimes}\eta$ is alternating. By Propositions 2.6 and 3.3, we can choose decompositions of U and R such that $m\mathbb{1} \perp nP$ and $p\mathbb{1} \perp qP$. This results in a decomposition $U \otimes R = mp\mathbb{1} \oplus mqP \oplus npP \oplus 2nqP$ where the subobjects $mp\mathbb{1}$, mqP, npP, and $2nqP$ are mutually orthogonal.

By Remark 3.23, the form invariant of $\beta\hat{\otimes}\eta$ is equal to the form invariant of $\beta\hat{\otimes}\eta$ restricted to $mqP \oplus npP \oplus 2nqP$. The restrictions of $\beta\hat{\otimes}\eta$ to mqP, nqP, and $2nqP$ are all alternating, so we can apply Lemma 4.3 to write

$$\mathcal{I}_{\beta\hat{\otimes}\eta} = \mathcal{I}_{\beta\hat{\otimes}\eta|_{mqP}} + \mathcal{I}_{\beta\hat{\otimes}\eta|_{npP}} + \mathcal{I}_{\beta\hat{\otimes}\eta|_{2nqP}}. \tag{4.3}$$

Therefore, our approach will be to determine the form invariants of the restrictions of $\beta\hat{\otimes}\eta$ to the objects mqP, npP, and $2nqP$.

Proposition 4.12. *If $\beta\hat{\otimes}\eta$ is alternating, then the form invariant of $\beta\hat{\otimes}\eta$ restricted to $nP \otimes qP = 2nqP$ is zero.*

Proof. The object $2nqP$ contains the $2nq$ linearly independent vectors given by $w_i\otimes\chi_j, x_i\otimes\chi_j$ for $1 \le i \le n, 1 \le j \le q$. Observe that $t.(w_i\otimes\chi_j) = x_i\otimes\chi_j$. Now, consider X-function and X-form of $\beta\hat{\otimes}\eta$, which we will denote by f and g, respectively. For all $1 \le i,k \le n, 1 \le j,\ell \le q$,

$$\begin{aligned} g(x_i \otimes \chi_j, x_k \otimes \chi_\ell) &= \beta\hat{\otimes}\eta(w_i \otimes \chi_j, x_k \otimes \chi_\ell) \\ &= \beta(w_i, x_k)\eta(\chi_j, \chi_\ell) + \beta(w_i, t.x_k)\eta(t.\chi_j, \chi_\ell) \\ &= \beta(w_i, x_k)\eta(\chi_j, \chi_\ell) + \beta(w_i, 0)\eta(0, \chi_\ell) = 0. \end{aligned}$$

Furthermore, for all $1 \le i \le n, 1 \le j \le q$,

$$f(x_i \otimes \chi_j) = \beta\hat{\otimes}\eta(w_i \otimes \chi_j, w_i \otimes \chi_j) = \beta(w_i, w_i)\eta(\chi_j, \chi_j) + \beta(w_i, x_i)\eta(\chi_j, 0) = 0.$$

A basis $\{b_1, b_2, \ldots, b_{2nq}\}$ of the image of $2nqP$ under the map of the t-action can be constructed such that the vectors $b_{nq+1}, \ldots b_{2nq}$ are given by $x_i \otimes \chi_j$, where $1 \le i \le n, 1 \le j \le q$. Using this basis, we construct the X-matrix of $\beta\hat{\otimes}\eta$ restricted to $2nqP$. It is of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

where A, B, C are matrix blocks and 0 represents the zero matrix. We know by the non-degeneracy of the X-form (proved in Proposition 3.20) that M is invertible, so B and C must also be invertible. We calculate that M^{-1} is equal to

$$\left[\begin{array}{c|c} 0 & C^{-1} \\ \hline B^{-1} & B^{-1}AC^{-1} \end{array}\right].$$

Thus, $M^{-1}_{kk} = 0$ for $1 \le k \le nq$ and $f(b_k) = 0$ for $nq < k \le 2nq$. The form invariant of $\beta\hat{\otimes}\eta$ restricted to $2nqP$ evaluates to $\mathcal{I}_{\beta\hat{\otimes}\eta|_{2nqP}} = \sum_{k=1}^{2nq} f(b_k)M^{-1}_{kk} = 0$. ☐

Proposition 4.13. *Suppose $\beta\hat{\otimes}\eta$ is alternating. If β is not alternating, then the form invariant of $\beta\hat{\otimes}\eta$ restricted to $m\mathbb{1} \otimes qP = mqP$ is $m\mathcal{I}_{\eta|_{qP}}$.*

Proof. The object $m\mathbb{1}$ is the direct sum of m $\mathbb{1}$ objects, for each of which the restriction of β is non-degenerate. The object mqP is the direct sum of m copies of $\mathbb{1}\otimes qP$. Each $\mathbb{1}\otimes qP$ object is alternating, so applying Lemma 4.3 reduces the claim to proving that $\mathcal{I}_{\beta\hat{\otimes}\eta|_{\mathbb{1}\otimes qP}} = \mathcal{I}_{\eta|_{qP}}$. This is true because $\beta\hat{\otimes}\eta|_{\mathbb{1}\otimes qP} \cong \eta|_{qP}$. □

Proposition 4.14. *Suppose $\beta\hat{\otimes}\eta$ is alternating. If β is alternating, the form invariant of $\beta\hat{\otimes}\eta$ restricted to $m\mathbb{1}\otimes qP = mqP$ is zero.*

Proof. The object $m\mathbb{1}$ is the direct sum of $\frac{m}{2}$ $2\mathbb{1}$ objects, each of which has a basis $\{u_1, u_2\}$ such that $\beta(u_1, u_1) = 0$, $\beta(u_2, u_2) = 0$, and $\beta(u_1, u_2) = 1$. The object $2\mathbb{1}\otimes qP$ is alternating, and mqP is the direct sum of $\frac{m}{2}$ copies of $2\mathbb{1}\otimes qP$. Applying Lemma 4.3 to these $\frac{m}{2}$ objects, we only need to show that $\mathcal{I}_{\beta\hat{\otimes}\eta|_{2\mathbb{1}\otimes qP}} = 0$. We will do so by directly calculating this form invariant.

The object $2\mathbb{1}\otimes qP$ contains the $2q$ linearly independent vectors given by $u_1\otimes\omega_i$ and $u_1\otimes\chi_i$, where $t.(u_1\otimes\omega_i) = u_1\otimes\chi_i$ for $1 \le i \le q$. Denote the X-function and the X-form of $\beta\hat{\otimes}\eta$ by f and g, respectively. For $1 \le i \le q$, we have

$$f(u_1\otimes\chi_i) = \beta\hat{\otimes}\eta(u_1\otimes\omega_i, u_1\otimes\omega_i) = \beta(u_1, u_1)\eta(\omega_i, \omega_i) = 0,$$

and for all $1 \le i, j \le q$, we have

$$g(u_1\otimes\chi_i, u_1\otimes\chi_j) = \beta\hat{\otimes}\eta(u_1\otimes\omega_i, u_1\otimes\chi_j) = \beta(u_1, u_1)\eta(\omega_i, \chi_j) = 0.$$

We can construct a basis $\{b_1, b_2, \dots b_{2q}\}$ of the image of $2\mathbb{1}\otimes qP$ under the map of the t-action such that the vectors $b_{q+1}, \dots, b_{2q}$ are given by $u_1\otimes\chi_i$ for $1 \le i \le q$. Let M be the X-matrix of $\beta\hat{\otimes}\eta$ on this basis. By the same reasoning used for the case in Lemma 4.12, $M^{-1}_{kk} = 0$ for $1 \le k \le q$ and $f(b_k) = 0$ for $q < k \le 2q$, which proves that

$$\sum_{k=1}^{2q} f(b_k)M^{-1}_{kk} = 0.$$

Having shown that the form invariant of $\beta\hat{\otimes}\eta$ restricted to each $2\mathbb{1}\otimes qP$ object is zero, we also have $\mathcal{I}_{\beta\hat{\otimes}\eta|_{m\mathbb{1}\otimes qP}} = 0$. □

By commutativity, the previous two lemmas prove that $\mathcal{I}_{\beta\hat{\otimes}\eta|_{npP}} = p\mathcal{I}_{\beta|nP}$ when η is not alternating and $\mathcal{I}_{\beta\hat{\otimes}\eta|_{npP}} = 0$ when η is alternating.

Given an alternating form $\beta\hat{\otimes}\eta$, we can now find the form invariant of $\beta\hat{\otimes}\eta$ using (4.3). At least one of β and η must be alternating by Lemma 4.11. By Propositions 4.12, 4.13, and 4.14, $\mathcal{I}_{\beta\hat{\otimes}\eta} = 0$ when both β and η are alternating, $\mathcal{I}_{\beta\hat{\otimes}\eta} = m\mathcal{I}_{\eta|_{qP}} = m\mathcal{I}_\eta$ when β is not alternating, and $\mathcal{I}_{\beta\hat{\otimes}\eta} = p\mathcal{I}_{\beta|_{nP}} = p\mathcal{I}_\beta$ when η is not alternating.

Our work in this section determines the good pairs of $\beta\hat{\otimes}\eta$, when $\beta\hat{\otimes}\eta$ is alternating, and the form invariant of $\beta\hat{\otimes}\eta$ when the form is alternating. This enables us to calculate the tensor product on our isomorphism classes.

$\otimes$	A	B	C	D	E(1)	E(a)	F(a)
A	A	B	C	D	E(1)	E(a)	F(pa)
B		B	C	F(0)	E(1)	F(pna)	F(pa)
C			C	C	C	C	C
D				D	E(1)	E(a)	F(0)
E(1)					C	E(1)	E(1)
E(b)						$a = b \to$ D; $a \neq b \to$ E($(ab+1)/(a+b)$)	F(0)
F(b)							F(0)

In the table above, we again use a and b to denote arbitrary scalars. The top row describes the isomorphism class of β (on $m\mathbb{1} \oplus nP$), and the leftmost column describes the isomorphism class of η (on $p\mathbb{1} \oplus qP$).

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